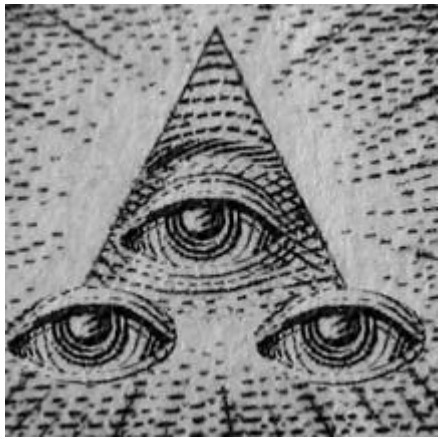

# CS61B

Lecture 35: Radix Sort

- Counting Sort
- LSD Radix Sort
- LSD Radix Sort vs. Comparison Sorting
- MSD Radix Sort

# Comparison Based Sorting

The key idea from our previous sorting lecture: Sorting requires Ω(N log N) compares in the worst case.

- Thus, the ultimate comparison based sorting algorithm has a worst case runtime of Θ(N log N).

From an asymptotic perspective, that means no matter how clever we are, we can never beat Merge Sort's worst case runtime of Θ(N log N).

- ...but what if we don't compare at all?

# Example #1: Sleep Sort (for Sorting Integers) (not actually good)

For each integer x in array A, start a new program that:

- Sleeps for x seconds.
- Prints x.

All start at the same time.

Runtime:

- N + max(A)

The catch: On real machines, scheduling execution of programs must be done by an operating system. In practice requires list of running programs sorted by sleep time.

**Genius sorting algorithm: Sleep sort**

1 Name: **Anonymous** 2011-01-20 12:22

Man, am I a genius. Check out this sorting algorithm I just invented.

```
#!/bin/bash
function f() {
    sleep "$1"
    echo "$1"
}
while [ -n "$1" ]
do
    f "$1" &
    shift
done
wait
```

example usage:

```
./sleepsort.bash 5 3 6 3 6 3 1 4 7
```

Invented by 4chan.

# Example #2: Counting Sort: Exploiting Space Instead of Time

| # | | | |
|---|---|---|---|
| 5 | Sandra | Vanilla | Grimes |
| 0 | Lauren | Mint | Jon Talabot |
| 11 | Lisa | Vanilla | Blue Peter |
| 9 | Dave | Chocolate | Superpope |
| 4 | JS | Fish | The Filthy Reds |
| 7 | James | Rocky Road | Robots are Supreme |
| 3 | Edith | Vanilla | My Bloody Valentine |
| 6 | Swimp | Chocolate | Sef |
| 1 | Delbert | Strawberry | Ronald Jenkees |
| 2 | Glaser | Cardamom | Rx Nightly |
| 8 | Lee | Vanilla | La(r)va |
| 10 | Bearman | Butter Pecan | Extrobophile |

Assuming keys are unique integers 0 to 11.

Idea:

- Create a new array.
- Copy item with key i into ith entry of new array.

# Example #2: Counting Sort: Exploiting Space Instead of Time

| # | | | |
|---|---|---|---|
| 5 | Sandra | Vanilla | Grimes |
| 0 | Lauren | Mint | Jon Talabot |
| 11 | Lisa | Vanilla | Blue Peter |
| 9 | Dave | Chocolate | Superpope |
| 4 | JS | Fish | The Filthy Reds |
| 7 | James | Rocky Road | Robots are Supreme |
| 3 | Edith | Vanilla | My Bloody Valentine |
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| 1 | Delbert | Strawberry | Ronald Jenkees |
| 2 | Glaser | Cardamom | Rx Nightly |
| 8 | Lee | Vanilla | La(r)va |
| 10 | Bearman | Butter Pecan | Extrobophile |

| # | | | |
|---|---|---|---|
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| 5 | Sandra | Vanilla | Grimes |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |

# Example #2: Counting Sort: Exploiting Space Instead of Time

| # | | | |
|---|---|---|---|
| 5 | Sandra | Vanilla | Grimes |
| 0 | Lauren | Mint | Jon Talabot |
| 11 | Lisa | Vanilla | Blue Peter |
| 9 | Dave | Chocolate | Superpope |
| 4 | JS | Fish | The Filthy Reds |
| 7 | James | Rocky Road | Robots are Supreme |
| 3 | Edith | Vanilla | My Bloody Valentine |
| 6 | Swimp | Chocolate | Sef |
| 1 | Delbert | Strawberry | Ronald Jenkees |
| 2 | Glaser | Cardamom | Rx Nightly |
| 8 | Lee | Vanilla | La(r)va |
| 10 | Bearman | Butter Pecan | Extrobophile |

| # | | | |
|---|---|---|---|
| 0 | Lauren | Mint | Jon Talabot |
| | | | |
| | | | |
| | | | |
| | | | |
| 5 | Sandra | Vanilla | Grimes |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |

# Example #2: Counting Sort: Exploiting Space Instead of Time

| # | | | |
|---|---|---|---|
| 5 | Sandra | Vanilla | Grimes |
| 0 | Lauren | Mint | Jon Talabot |
| 11 | Lisa | Vanilla | Blue Peter |
| 9 | Dave | Chocolate | Superpope |
| 4 | JS | Fish | The Filthy Reds |
| 7 | James | Rocky Road | Robots are Supreme |
| 3 | Edith | Vanilla | My Bloody Valentine |
| 6 | Swimp | Chocolate | Sef |
| 1 | Delbert | Strawberry | Ronald Jenkees |
| 2 | Glaser | Cardamom | Rx Nightly |
| 8 | Lee | Vanilla | La(r)va |
| 10 | Bearman | Butter Pecan | Extrobophile |

| # | | | |
|---|---|---|---|
| 0 | Lauren | Mint | Jon Talabot |
| | | | |
| | | | |
| | | | |
| | | | |
| 5 | Sandra | Vanilla | Grimes |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| 11 | Lisa | Vanilla | Blue Peter |

# Example #2: Counting Sort: Exploiting Space Instead of Time

| # | | | |
|---|---|---|---|
| 5 | Sandra | Vanilla | Grimes |
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| 7 | James | Rocky Road | Robots are Supreme |
| 3 | Edith | Vanilla | My Bloody Valentine |
| 6 | Swimp | Chocolate | Sef |
| 1 | Delbert | Strawberry | Ronald Jenkees |
| 2 | Glaser | Cardamom | Rx Nightly |
| 8 | Lee | Vanilla | La(r)va |
| 10 | Bearman | Butter Pecan | Extrobophile |

| # | | | |
|---|---|---|---|
| 0 | Lauren | Mint | Jon Talabot |
| 1 | Delbert | Strawberry | Ronald Jenkees |
| 2 | Glaser | Cardamom | Rx Nightly |
| 3 | Edith | Vanilla | My Bloody Valentine |
| 4 | JS | Fish | The Filthy Reds |
| 5 | Sandra | Vanilla | Grimes |
| 6 | Swimp | Chocolate | Sef |
| 7 | James | Rocky Road | Robots are Supreme |
| 8 | Lee | Vanilla | La(r)va |
| 9 | Dave | Chocolate | Superpope |
| 10 | Bearman | Butter Pecan | Extrobophile |
| 11 | Lisa | Vanilla | Blue Peter |

# Generalizing Counting Sort

We just sorted N items in Θ(N) worst case time.

- Avoiding yes/no questions lets us dodge our lower bound based on puppy, cat, dog!

Simplest case:

- Keys are unique integers from 0 to N-1.

More complex cases:

- Non-unique keys.
- Non-consecutive keys.
- Non-numerical keys.

# Counting Sort: http://yellkey.com/carry

Alphabet case: Keys belong to a finite ordered alphabet.

- Example: {♣, ♠, ♥, ♦} (in that order)

| | |
|---|---|
| ♠ | Lauren |
| ♥ | Delbert |
| ♦ | Glaser |
| ♣ | Edith |
| ♠ | JS |
| ♦ | Sandra |
| ♥ | Swimp |
| ♥ | James |
| ♣ | Lee |
| ♥ | Dave |
| ♣ | Bearman |
| ♦ | Lisa |

| | | |
|---|---|---|
| 0 | | |
| 1 | | |
| 2 | | |
| 3 | | |
| 4 | | |
| 5 | | |
| 6 | | |
| 7 | | |
| 8 | | |
| 9 | | |
| 10 | | |
| 11 | | |

Sorted

Question: What will be the index of the first ♥?

# Counting Sort

Alphabet case: Keys belong to a finite ordered alphabet.

- Example: {♣, ♠, ♥, ♦} (in that order)

| | |
|---|---|
| ♠ | Lauren |
| ♥ | Delbert |
| ♦ | Glaser |
| ♣ | Edith |
| ♠ | JS |
| ♦ | Sandra |
| ♥ | Swimp |
| ♥ | James |
| ♣ | Lee |
| ♥ | Dave |
| ♣ | Bearman |
| ♦ | Lisa |

| | | |
|---|---|---|
| 0 | ♣ | |
| 1 | ♣ | |
| 2 | ♣ | |
| 3 | ♠ | |
| 4 | ♠ | |
| 5 | | |
| 6 | | |
| 7 | | |
| 8 | | |
| 9 | | |
| 10 | | |
| 11 | | |

Sorted

Question: What will be the index of the first ♥?

# Implementing Counting Sort with Counting Arrays

Counting sort:

- Count number of occurrences of each item.
- Iterate through list, using count array to decide where to put everything.
- Interactive Demo

Bottom line, we can use counting sort to sort N objects in $\Theta(N)$ time.

# Counting Sort Runtime

# Counting Sort vs. Quicksort: http://yellkey.com/slack

For sorting an array of the 100 largest cities by population, which sort do you think has a better expected worst case runtime in seconds?

A. Counting Sort (as described in our demo)
B. Quicksort

First question to ask yourself: What is the alphabet for counting sort here?

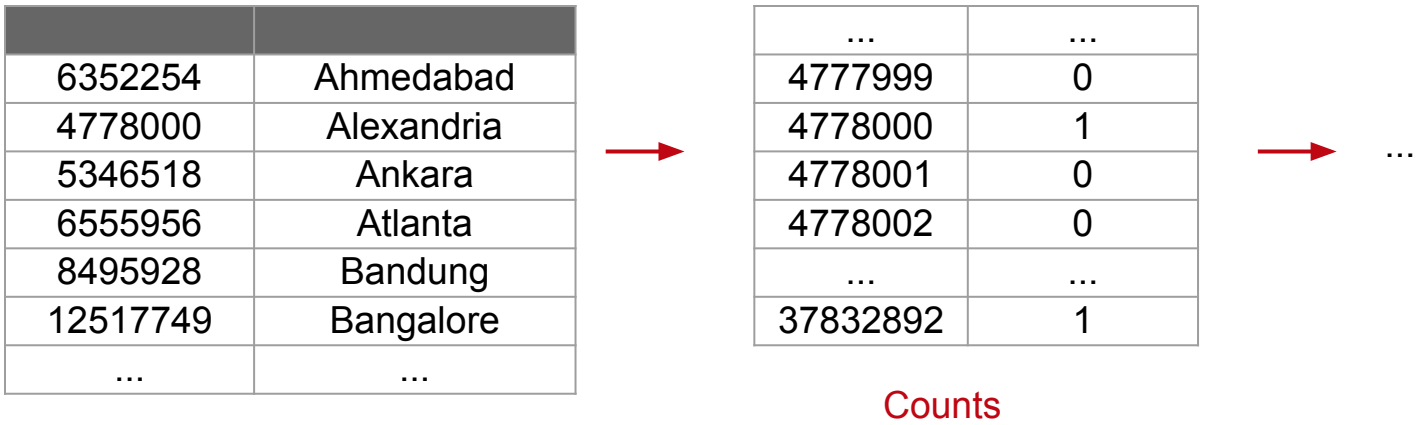
# Counting Sort vs. Quicksort

For sorting an array of the 100 largest cities by population, which sort do you think has a better expected worst case runtime in seconds?

A. Counting Sort (as described in our demo)

**B. Quicksort**

Counting sort requires building an array of size 37,832,892 (population of Tokyo).



| | |
|---|---|
| 6352254 | Ahmedabad |
| 4778000 | Alexandria |
| 5346518 | Ankara |
| 6555956 | Atlanta |
| 8495928 | Bandung |
| 12517749 | Bangalore |
| ... | ... |

| ... | ... |
|---|---|
| 4777999 | 0 |
| 4778000 | 1 |
| 4778001 | 0 |
| 4778002 | 0 |
| ... | ... |
| 37832892 | 1 |

# Counting Sort Runtime Analysis

What is the runtime for counting sort on N keys with alphabet of size R?

- Treat R as a variable, not a constant.

# Counting Sort Runtime Analysis

Total runtime on N keys with alphabet of size R: $\Theta(N+R)$

- Create an array of size R to store counts: $\Theta(R)$
- Counting number of each item: $\Theta(N)$
- Calculating target positions of each item: $\Theta(R)$
- Creating an array of size N to store ordered data: $\Theta(N)$
- Copying items from original array to ordered array: Do N times:
  - Check target position: $\Theta(1)$
  - Update target position: $\Theta(1)$
- Copying items from ordered array back to original array: $\Theta(N)$

For ordered array.

For counts and starting points.

Memory usage: $\Theta(N+R)$

Empirical experiments needed to compare vs. Quicksort on practical inputs.

Bottom line: If N is $\geq$ R, then we expect reasonable performance.

# Counting Sort vs. Quicksort: http://yellkey.com/improve

For sorting really really really big collections of items from some alphabet, which algorithm will be fastest?

A. Counting Sort
B. Quicksort

# Counting Sort vs. Quicksort

For sorting really really really big collections of items from some alphabet, which algorithm will be fastest?

**A. Counting Sort: Θ(N+R)**

B. Quicksort: Θ(N log N)

For sufficiently large collections, counting sort will simply be faster.

# Sort Summary

| | Memory | Runtime | Notes | Stable? |
|---|---|---|---|---|
| Heapsort | $\Theta(1)$ | $\Theta(N \log N)$ | Bad caching (61C) | No |
| Insertion | $\Theta(1)$ | $\Theta(N^2)$ | Small N, almost sorted | Yes |
| Mergesort | $\Theta(N)$ | $\Theta(N \log N)$ | Fastest stable | Yes |
| Random Quicksort | $\Theta(\log N)$ | $\Theta(N \log N)$ expected | Fastest compare sort | No |
| Counting Sort | $\Theta(N+R)$ | $\Theta(N+R)$ | Alphabet keys only | Yes |

N: Number of keys. R: Size of alphabet.

Counting sort is nice, but alphabetic restriction limits usefulness.

- No obvious way to sort hard-to-count things like Strings.

# LSD Radix Sort

# Radix Sort

Not all keys belong to finite alphabets, e.g. Strings.

- However, Strings consist of characters from a finite alphabet.

| | |
|---|---|
| horse | Lauren |
| elf | Delbert |
| cat | Glaser |
| crab | Edith |
| monkey | JS |
| rhino | Sandra |
| raccoon | Swimp |
| cat | James |
| fish | Lee |
| tree | Dave |
| virus | Bearman |
| human | Lisa |

| | |
|---|---|
| ♠♠ | Lauren |
| ♥♦ | Delbert |
| ♦♣ | Glaser |
| ♣♥ | Edith |
| ♠♥ | JS |
| ♦♣ | Sandra |
| ♥♠ | Swimp |
| ♥♦ | James |
| ♣♠ | Lee |
| ♥♣ | Dave |
| ♣♠ | Bearman |
| ♦♠ | Lisa |

| | |
|---|---|
| 4238xz | Lauren |
| 34163 | Delbert |
| 123 | Glaser |
| 43415 | Edith |
| 9918 | JS |
| 767 | Sandra |
| 3 | Swimp |
| 634 | James |
| 724 | Lee |
| 2346 | Dave |
| 457 | Bearman |
| 312 | Lisa |

# LSD (Least Significant Digit) Radix Sort -- Using Counting Sort

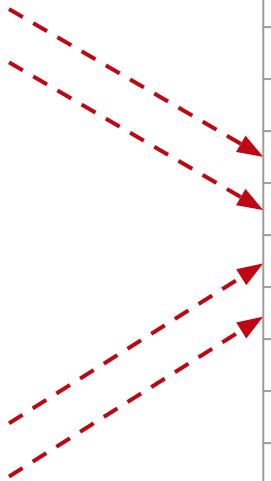
Sort each digit independently from rightmost digit towards left.

- Example: Over {♣, ♠, ♥, ♦}

| | |
|---|---|
| ♠♠ | Lauren |
| ♥♦ | Delbert |
| ♦♣ | Glaser |
| ♣♥ | Edith |
| ♠♥ | JS |
| ♦♣ | Sandra |
| ♥♠ | Swimp |
| ♥♦ | James |
| ♣♠ | Lee |
| ♥♣ | Dave |
| ♣♠ | Bearman |
| ♦♠ | Lisa |

→

| | |
|---|---|
| ♦♣ | Glaser |
| ♦♣ | Sandra |
| ♥♣ | Dave |
| ♥♠ | Swimp |
| ♠♠ | Lauren |
| ♣♠ | Lee |
| ♣♠ | Bearman |
| ♦♠ | Lisa |
| ♠♥ | JS |
| ♣♥ | Edith |
| ♥♦ | James |
| ♥♦ | Delbert |



| | |
|---|---|
| ♣♠ | Lee |
| ♣♠ | Bearman |
| ♣♥ | Edith |
| ♠♠ | Lauren |
| ♠♥ | JS |
| ♥♣ | Dave |
| ♥♠ | Swimp |
| ♥♦ | James |
| ♥♦ | Delbert |
| ♦♣ | Glaser |
| ♦♣ | Sandra |
| ♦♠ | Lisa |

# LSD (Least Significant Digit) Radix Sort -- Using Counting Sort

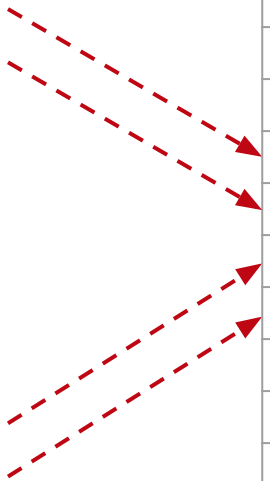
Sort each digit independently from rightmost digit towards left.

- Example: Over {1, 2, 3, 4}

Note: Lauren and Swimp are backwards in middle column, same with Delbert and James, same with Edith and JS

| | |
|---|---|
| 22 | **Lauren** |
| 34 | Delbert |
| 41 | Glaser |
| 13 | Edith |
| 23 | JS |
| 41 | Sandra |
| 32 | **Swimp** |
| 34 | James |
| 12 | Lee |
| 31 | Dave |
| 12 | Bearman |
| 42 | Lisa |

| | |
|---|---|
| 41 | Glaser |
| 41 | Sandra |
| 31 | Dave |
| 32 | Swimp |
| 22 | Lauren |
| 12 | Lee |
| 12 | Bearman |
| 42 | Lisa |
| 23 | JS |
| 13 | Edith |
| 34 | James |
| 34 | Delbert |



| | |
|---|---|
| 12 | Lee |
| 12 | Bearman |
| 13 | Edith |
| 22 | Lauren |
| 23 | JS |
| 31 | Dave |
| 32 | Swimp |
| 34 | James |
| 34 | Delbert |
| 41 | Glaser |
| 41 | Sandra |
| 42 | Lisa |

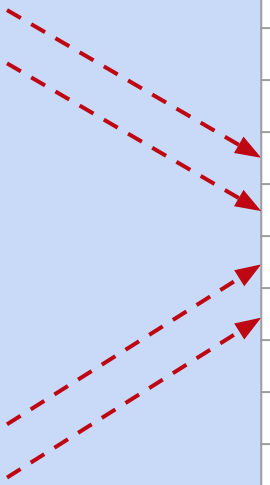
# LSD Runtime

What is the runtime of LSD sort?

- Pick appropriate letters to represent non-constant terms.

| | |
|---|---|
| 22 | Lauren |
| 34 | Delbert |
| 41 | Glaser |
| 13 | Edith |
| 23 | JS |
| 41 | Sandra |
| 32 | Swimp |
| 34 | James |
| 12 | Lee |
| 31 | Dave |
| 12 | Bearman |
| 42 | Lisa |

| | |
|---|---|
| 41 | Glaser |
| 41 | Sandra |
| 31 | Dave |
| 32 | Swimp |
| 22 | Lauren |
| 12 | Lee |
| 12 | Bearman |
| 42 | Lisa |
| 23 | JS |
| 13 | Edith |
| 34 | James |
| 34 | Delbert |



| | |
|---|---|
| 12 | Lee |
| 12 | Bearman |
| 13 | Edith |
| 22 | Lauren |
| 23 | JS |
| 31 | Dave |
| 32 | Swimp |
| 34 | James |
| 34 | Delbert |
| 41 | Glaser |
| 41 | Sandra |
| 42 | Lisa |

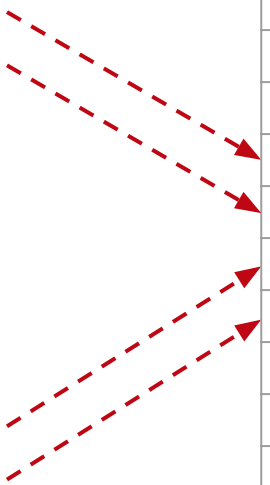
# LSD Runtime

What is the runtime of LSD sort?

- Θ(WN+WR)
- N: Number of items, R: size of alphabet, W: Width of each item in # digits

| | |
|---|---|
| 22 | Lauren |
| 34 | Delbert |
| 41 | Glaser |
| 13 | Edith |
| 23 | JS |
| 41 | Sandra |
| 32 | Swimp |
| 34 | James |
| 12 | Lee |
| 31 | Dave |
| 12 | Bearman |
| 42 | Lisa |

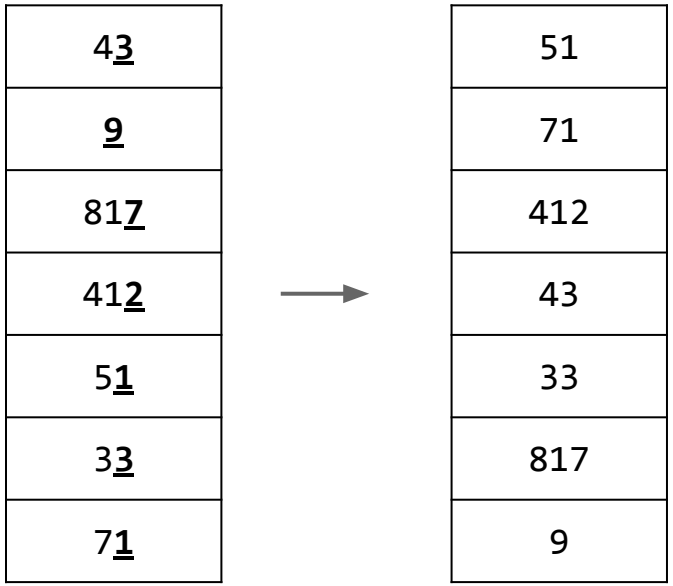
| | |
|---|---|
| 41 | Glaser |
| 41 | Sandra |
| 31 | Dave |
| 32 | Swimp |
| 22 | Lauren |
| 12 | Lee |
| 12 | Bearman |
| 42 | Lisa |
| 23 | JS |
| 13 | Edith |
| 34 | James |
| 34 | Delbert |



| | |
|---|---|
| 12 | Lee |
| 12 | Bearman |
| 13 | Edith |
| 22 | Lauren |
| 23 | JS |
| 31 | Dave |
| 32 | Swimp |
| 34 | James |
| 34 | Delbert |
| 41 | Glaser |
| 41 | Sandra |
| 42 | Lisa |

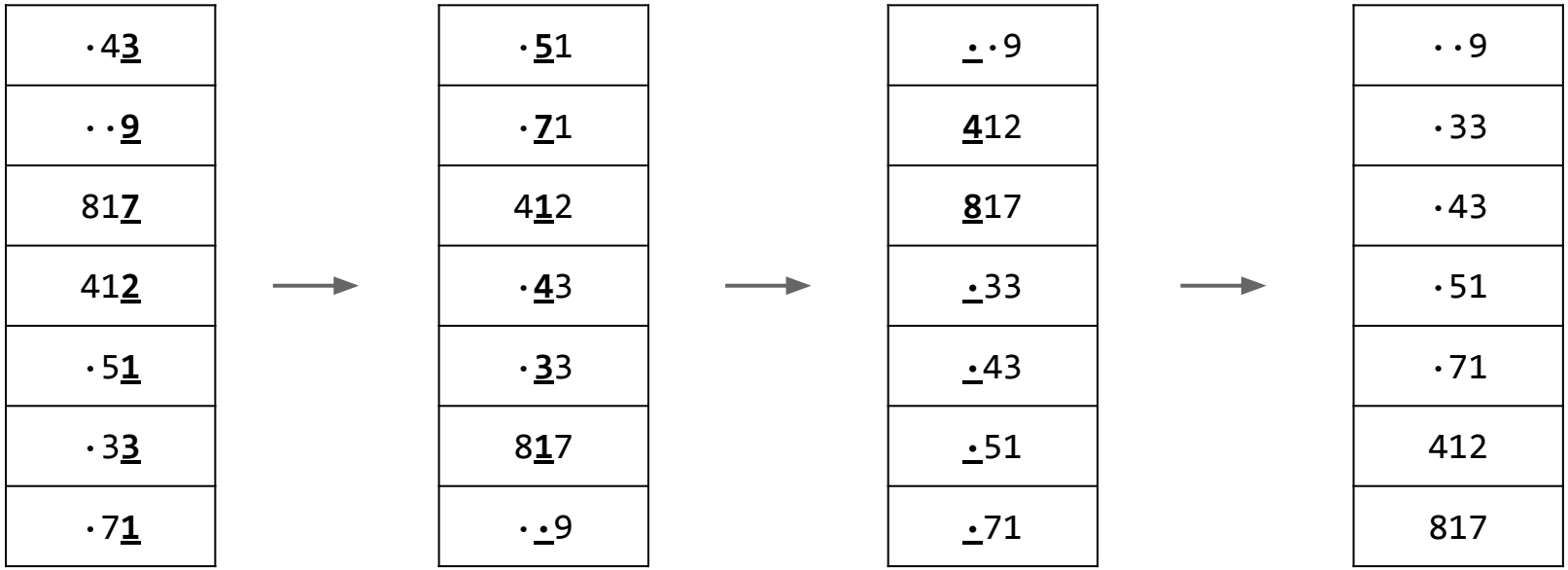
# Non-equal Key Lengths

After processing least significant digit, we have array shown below. Now what?



# Non-equal Key Lengths

When keys are of different lengths, can treat empty spaces as less than all other characters.



# Sorting Summary

W passes of counting sort: Θ(WN+WR) runtime.

- Annoying feature: Runtime depends on length of longest key.

| | Memory | Runtime | Notes | Stable? |
|---|---|---|---|---|
| Heapsort | Θ(1) | Θ(N log N)* | Bad caching (61C) | No |
| Insertion | Θ(1) | $\Theta(N^2)$* | Small N, almost sorted | Yes |
| Mergesort | Θ(N) | Θ(N log N)* | Fastest stable sort | Yes |
| Random Quicksort | Θ(log N) | Θ(N log N)* expected | Fastest compare sort | No |
| Counting Sort | Θ(N+R) | Θ(N+R) | Alphabet keys only | Yes |
| LSD Sort | Θ(N+R) | Θ(WN+WR) | Strings of alphabetical keys only | Yes |

N: Number of keys. R: Size of alphabet. W: Width of longest key.

*: Assumes constant compareTo time.

# MSD Radix Sort

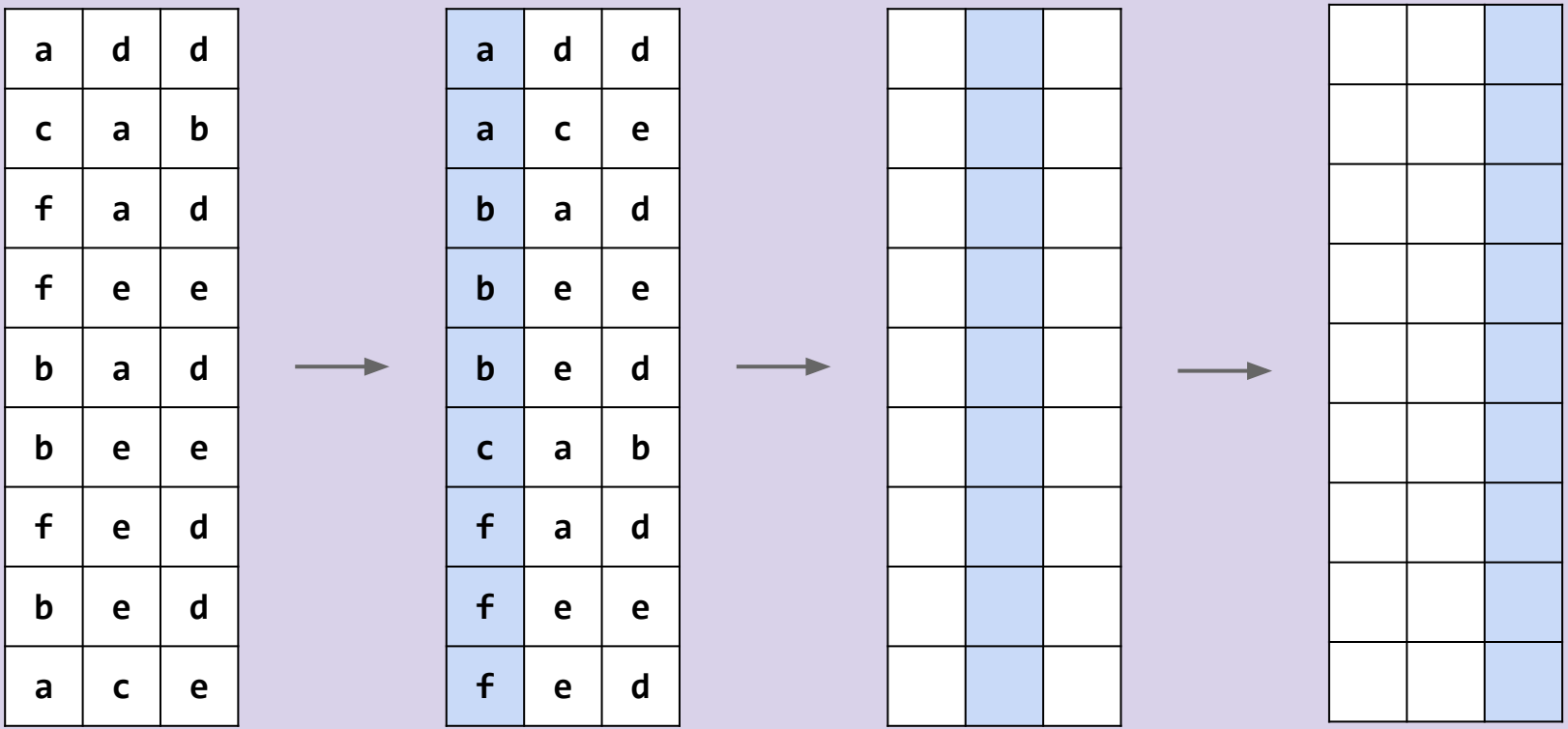
# MSD (Most Significant Digit) Radix Sort

Basic idea: Just like LSD, but sort from leftmost digit towards the right.

| |
|---|
| Pseudopseudohypoparathyroidism |
| Floccinaucinihilipilification |
| Antidisestablishmentarianism |
| Honorificabilitudinitatibus |
| Pneumonoultramicroscopicsilicovolcanoconiosis |

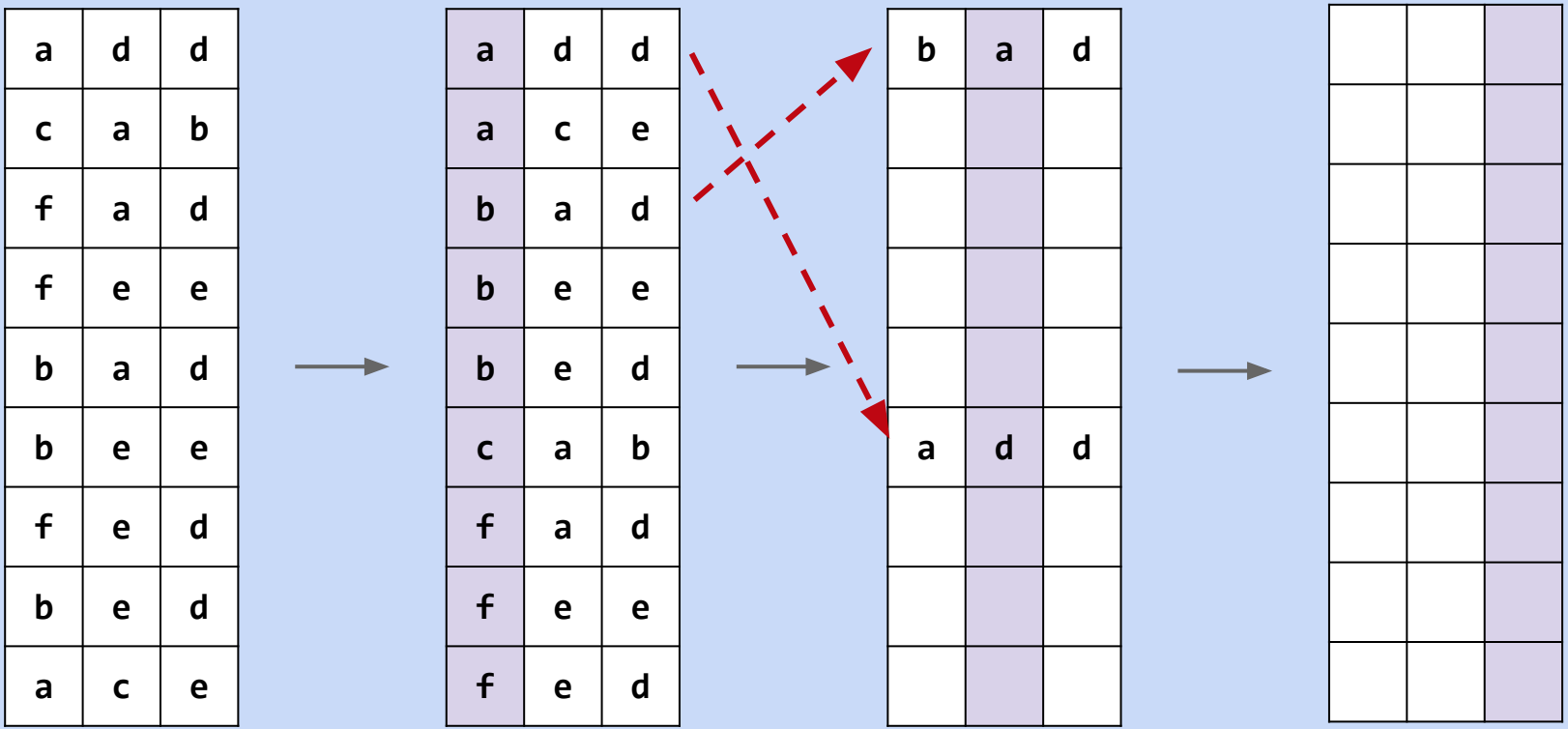
# MSD Sort Question: http://yellkey.com/yourself

Suppose we sort by topmost digit, then middle digit, then rightmost digit. Will we arrive at the correct result? A. Yes, B. No



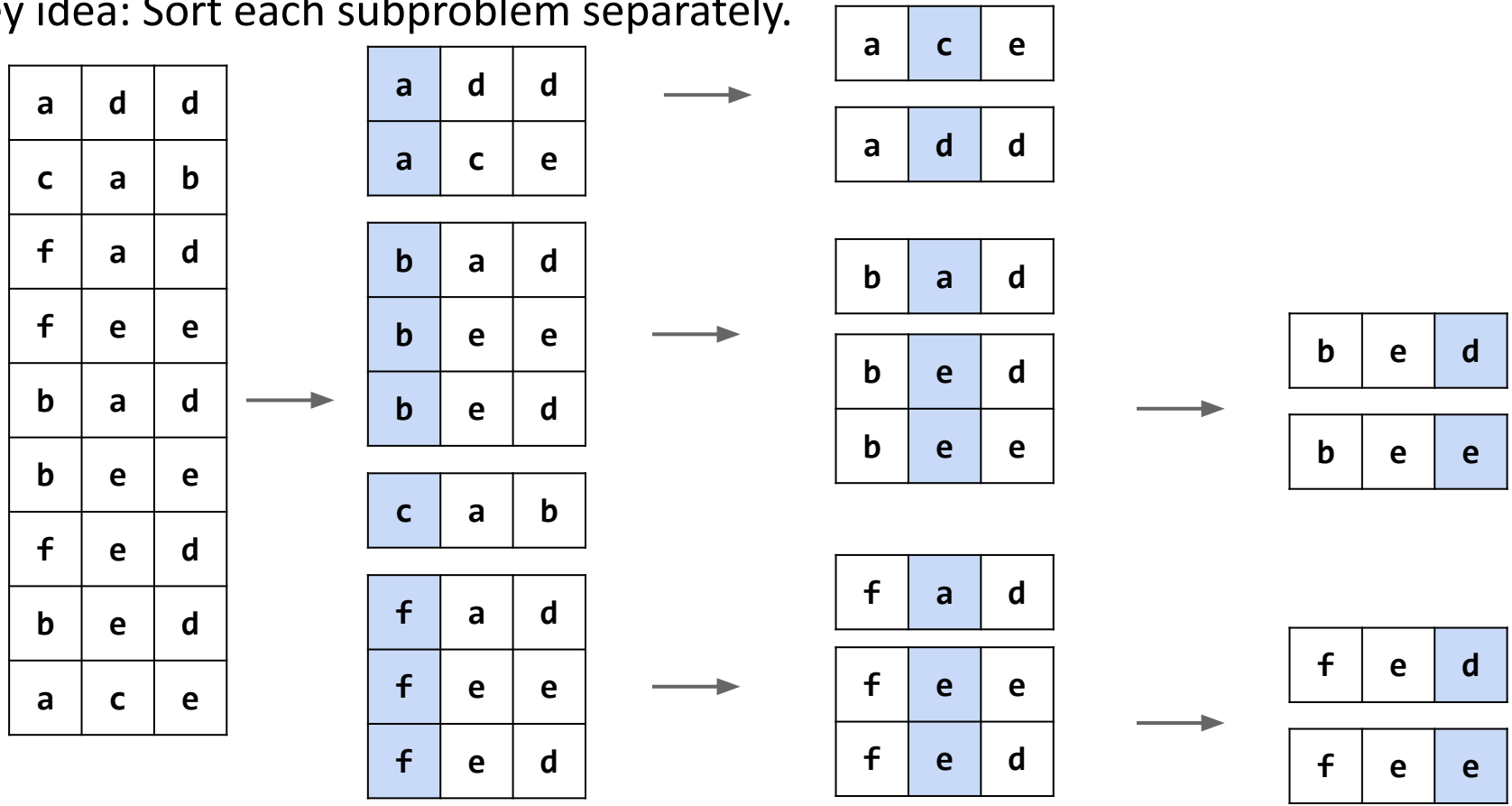
# MSD Sort Question

Suppose we sort by topmost digit, then middle digit, then rightmost digit. Will we arrive at the correct result? A. Yes, **B. No.** How do we fix?



# MSD Radix Sort (correct edition)

Key idea: Sort each subproblem separately.



# Runtime of MSD

What is the Best Case of MSD sort (in terms of N, W, R)?

What is the Worst Case of MSD sort (in terms of N, W, R)?

# Runtime of MSD

**Best Case.**

- We finish in one counting sort pass, looking only at the top digit: $\Theta(N + R)$

**Worst Case.**

- We have to look at every character, degenerating to LSD sort: $\Theta(WN + WR)$

# Sorting Runtime Analysis

| | Memory | Runtime (worst) | Notes | Stable? |
|---|---|---|---|---|
| Heapsort | Θ(1) | Θ(N log N)* | Bad caching (61C) | No |
| Insertion | Θ(1) | $\Theta(N^2)$* | Fastest for small N, almost sorted data | Yes |
| Mergesort | Θ(N) | Θ(N log N)* | Fastest stable sort | Yes |
| Random Quicksort | Θ(log N) | Θ(N log N)* expected | Fastest compare sort | No |
| Counting Sort | Θ(N+R) | Θ(N+R) | Alphabet keys only | Yes |
| LSD Sort | Θ(N+R) | Θ(WN+WR) | Strings of alphabetical keys only | Yes |
| MSD Sort | Θ(N+WR) | Θ(N+R) (best)<br>Θ(WN+WR) (worst) | Bad caching (61C) | Yes |

N: Number of keys. R: Size of alphabet. W: Width of longest key.

*: Assumes constant compareTo time.

# Sounds of Sorting Algorithms

Starts with selection sort: https://www.youtube.com/watch?v=kPRA0W1kECg

Insertion sort: https://www.youtube.com/watch?v=kPRA0W1kECg&t=0m9s

Quicksort: https://www.youtube.com/watch?v=kPRA0W1kECg&t=0m38s

Mergesort: https://www.youtube.com/watch?v=kPRA0W1kECg&t=1m05s

Heapsort: https://www.youtube.com/watch?v=kPRA0W1kECg&t=1m28s

LSD sort: https://www.youtube.com/watch?v=kPRA0W1kECg&t=1m54s

MSD sort: https://www.youtube.com/watch?v=kPRA0W1kECg&t=2m10s

Shell's sort: https://www.youtube.com/watch?v=kPRA0W1kECg&t=3m37s

Questions to ponder (later… after class):

- How many items are sorted in the video for selection sort?
- Why does insertion sort take longer / more compares than selection sort?
- At what time stamp does the first partition complete for Quicksort?
- Could the size of the input used by mergesort in the video be a power of 2?
- What do the colors mean for heapsort?
- How many characters are in the alphabet used for the LSD sort problem?
- How many digits are in the keys used for the LSD sort problem?

# Citations

Creepy eye thing, title slide:
http://photos3.meetupstatic.com/photos/event/a/3/f/4/highres_335141972.jpeg