1 Short Answers

Note 5

In each part below, provide the number/equation and brief justification.

(a) A connected planar simple graph has 5 more edges than it has vertices. How many faces does it have?

(b) How many edges need to be removed from a 3-dimensional hypercube to get a tree?

(c) The Euler's formula $v - e + f = 2$ requires the planar graph to be connected. What is the analogous formula for planar graphs wth k connected components?

Solution:

(a) **7.**
Use Euler's formula $v + f = e + 2$.

(b) **5.**
The 3-dimensional hypercube has $3(2^3)/2 = 12$ edges and $2^3 = 8$ vertices. A tree on 8 vertices has 7 edges, so one needs to remove 5 edges.

(c) Let v_i, e_i, f_i be the number of vertices, edges, and faces respectively for the ith connected component. Let V, E, F be the analogous quantities for the entire graph. For each connected component i, Euler's equation gives $v_i - e_i + f_i = 2$. Summing over this for all i, we get

$$\sum_{i=1}^{k}(v_i - e_i + f_i) = 2k$$

When we add up the vertices and edges, the total count doesn't change. Only the number of faces changes when we consider multiple connected components. When we 'combine' two connected components (aka consider them to be the same graph), they end up sharing the 'infinite' face, so if we were to add the number of faces, we have to subtract 1 from our count. Thus,

$$V - E + \sum_{i=1}^{k} f_i = 2k$$

When we combine together k components, we end up overcounting the infinite face $k - 1$ times, so $F = \sum_i f_i - (k - 1)$ Equivalently,

$$V - E + F + (k - 1) = 2k$$

, or $V - E + F = k + 1$.

2 Always, Sometimes, or Never

In each part below, you are given some information about a graph G. Using only the information in the current part, say whether G will always be planar, always be non-planar, or could be either. If you think it is always planar or always non-planar, prove it. If you think it could be either, give a planar example and a non-planar example.

(a) G can be vertex-colored with 4 colors.

(b) G requires 7 colors to be vertex-colored.

(c) $e \le 3v - 6$, where e is the number of edges of G and v is the number of vertices of G.

(d) G is connected, and each vertex in G has degree at most 2.

(e) Each vertex in G has degree at most 2.

Solution:

(a) Either planar or non-planar. By the 4-color theorem, any planar graph can provide the planar example. The easiest non-planar example is $K_{3,3}$, which can be 2-colored because it is bipartite. (Certainly, any graph which can be colored using only 2 colors can also be colored using 4 colors.)

(b) Always non-planar. The 4-color theorem tells us that if a graph is planar, it can be colored using only 4 colors. The contrapositive of this is that if a graph requires more than 4 colors to vertex-color, it must be non-planar. (Using the 5- or 6-color theorem would also work.)

(c) Either planar or non-planar. From the notes, we know that every planar graph follows this formula, so any planar graph is a valid planar example. The easiest non-planar example is again $K_{3,3}$, which has $e = 9$ and $v = 6$, meaning our formula becomes $9 \le 3(6) - 6 = 12$, which is certainly true.

(d) Always planar. There are two cases to deal with here: either G is a tree, or G is not a tree and so contains at least one cycle. In the former case, we're immediately done, since all trees are planar. In the latter case, consider any cycle in G. We know that every vertex in that cycle is adjacent to the vertex to its left in the cycle and to the vertex to its right in the cycle. But we also know that no vertex can be connected to more than two other vertices, so the cycle isn't connected to anything else. But G is a connected graph, so we must have that G is just a single large cycle. And we can certainly draw a simple cycle on a plane without crossing any edges, so even in this case G is still planar.

Alternatively, we can use Kuratowski's theorem; since each vertex has a degree of at most 2, it is impossible for G to contain K_5 or $K_{3,3}$. This means that G must be planar.

(e) Always planar. Each of G's connected components is connected and has no vertex of degree more than 2, so by the previous part, each of them must be planar. Thus, each of G's connected components must be planar, so G itself must be planar.

Alternatively, we can follow the same procedure as the previous alternate solution; each vertex still has a degree of at most 2, so it is impossible for G to contain K_5 or $K_{3,3}$. This means that G must be planar.
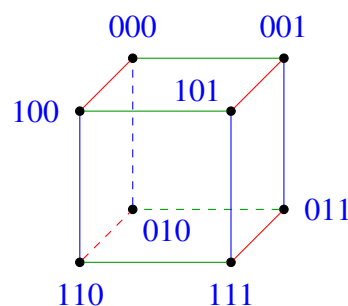
3 Hypercubes

The vertex set of the n-dimensional hypercube $G = (V, E)$ is given by $V = \{0, 1\}^n$ (recall that $\{0, 1\}^n$ denotes the set of all n-bit strings). There is an edge between two vertices x and y if and only if x and y differ in exactly one bit position.

(a) Draw 1-, 2-, and 3-dimensional hypercubes and label the vertices using the corresponding bit strings.

(b) Show that the edges of an n-dimensional hypercube can be colored using n colors so that no pair of edges sharing a common vertex have the same color.

(c) Show that for any $n \geq 1$, the n-dimensional hypercube is bipartite.

Solution:

(a) The three hypercubes are a line, a square, and a cube, respectively. See also note 5 for pictures.

(b) Consider each edge that changes the ith bit for some $i \leq n$. Every vertex touches exactly one of these edges, because there is exactly one way to change the ith bit in any bitstring. Coloring each of these edges color i ensures that each vertex will then be adjacent to n differently colored edges, since there are n different bits to change, and no two edges representing bit changes on different bits have the same color.

An example for the three dimensional case is shown below (red is the first bit, blue is the second bit, and green is the third bit):
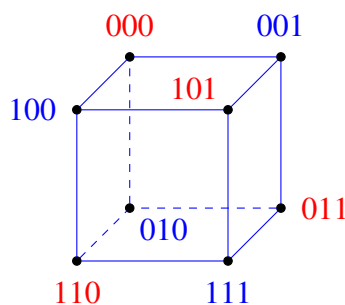

Alternate solution (using induction):

In the base case of $n = 1$, the hypercube of only one line can be edge colored with 1 color. Next, suppose that the n dimensional hypercube can be colored with n colors. Recall that the

$n+1$ dimensional hypercube is composed of two n dimensional hypercubes; each of these hypercubes can be colored with n colors by the inductive hypothesis.

We can connect the two n dimensional hypercubes with edges colored with a different color; this will be our $(n+1)$th color. Since these new edges will always be between distinct pairs of vertices, one from each subcube, none of these new edges will share a vertex, giving a valid coloring of the $n+1$ dimensional hypercube with $n+1$ colors.

(c) Consider the vertices with an even number of 0 bits and the vertices with an odd number of 0 bits. Each vertex with an even number of 0 bits is adjacent only to vertices with an odd number of 0 bits, since each edge represents a single bit change (either a 0 bit is added by flipping a 1 bit, or a 0 bit is removed by flipping a 0 bit). Let L be the set of the vertices with an even number of 0 bits and let R be the vertices with an odd number of 0 bits, then no two adjacent vertices will belong to the same set.

An example for the three dimensional case is shown below (L are blue vertices, and R are red vertices):

Alternate solution (using induction and coloring):

It may be simpler to that a graph being 2-colorable is the same as being bipartite. Now, the argument is easier to state. First the base case is a hypercube with two vertices which is clearly two-colorable. Then notice, switching the colors in a two-coloring is still valid as if endpoints are differently colored, switching leaves them differently colored. Now, recursively one two colors the two subcubes the same, and then switches the colors in one subcube. The internal to subcube edges are fine by induction. The edges across are fine as the corresponding vertices are differently colored due to the switching.

4 Triangular Faces

Suppose we have a connected planar graph G with v vertices and e edges such that $e = 3v - 6$. Prove that in any planar drawing of G, every face must be a triangle; that is, prove that every face must be incident to exactly three edges of G.

Solution:

Suppose for the sake of contradiction that we have found a planar drawing of G such that one of the faces is incident on more than three edges.

Choose an arbitrary vertex on that face to call v_0, and number the other vertices around the face $v_1, v_2, \ldots, v_k$ proceeding clockwise from v_0. Since this face has at least 4 sides, we know that v_0 and v_2 do not have an edge between them. Furthermore, we know that we can add this edge to the planar drawing of G without having it cross any existing edges by just letting it cross the face. Thus, adding an edge between v_0 and v_2 results in a planar graph with v vertices and $e + 1 = 3v - 5$ edges.

However, we know that a planar graph can have at most $3v - 6$ edges, so this is a contradiction. Thus, we must have that no such face exists; that is, we must have that every face in G is incident on exactly 3 edges.

Alternate solution: Similar to the previous approach, suppose for contradiction that we have a face with at more than three sides. Following a similar procedure as the notes when deriving the $e \leq 3v - 6$ inequality, we can sum over all the sides for each face.

This gives us

$$2e = \sum_{i=1}^{f} s_i > 3f.$$

The first equality is because we count each edge twice when adding up the sides for each face, and the last inequality is because we know there is some face with more than 3 sides, and all other faces have exactly 3 sides. This makes the summation *strictly* greater than $3f$.

Simplifying, we have $f < \frac{2}{3}e$. Euler's formula for planar graphs says that $v + f = e + 2$; the LHS is upper bounded by $v + \frac{2}{3}e$ (as $\frac{2}{3}e$ is an upper bound for f), so we have $v + \frac{2}{3}e > e + 2$. Rearranging, we have $v - 2 > \frac{1}{3}e \implies e < 3v - 6$.

However, we assumed that $e = 3v - 6$, but just showed that $e < 3v - 6$, a contradiction. This means that there does not exist any face with more than 3 sides; all faces must be triangles.