Fast Iterative Solvers. Project 2.

Summer Semester 2025

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Due: Friday 8th August, 2025, 23:59 hrs.

The project may be completed individually or in pairs. However, if it is developed in pairs, ***each participant must submit their own copy*** *of the project to Moodle, including both names on the submission.*

Overview

The main objective of this project is to implement and compare two iterative methods, Power Iteration and the Lanczos Method, to approximate the largest eigenvalue of a symmetric positive definite (s.p.d.) matrix. Make sure to write an efficient implementation of both schemes. [1].

General Instructions

Download the matrix

- `project2_spd_matrix.txt` (symmetric positive definite)

from the project page on Moodle. These matrices are stored in *modified compressed sparse row* (MSR) format. Some annotations to help read the files are provided along with the assignment.

Specific Instructions

Step 1

Using the matrix given, implement the power iteration considering the following:

- Note that this matrix is symmetric positive definite.
- Use the initial guess $x = \frac{1}{\sqrt{n}}(1, 1, \ldots, 1)^T$
- Stop the iteration when $|\lambda^{(k)} - \lambda^{(k-1)}| < 10^{-8}$
- Report the approximation for the eigenvalue that you obtained.
- Plot convergence $|\lambda^{(k)} - \lambda^{(k-1)}|$ against the iteration index k on a semilog scale (linear in k).

[1] In particular, it should be noted that the algorithms given in class are not necessarily meant to be copied verbatim. They were optimized for readability, not for efficiency. Here, the power iteration should take no more that one matrix-vector product per iteration.

Step 2

Implement a Lanczos method to find the largest eigenvalue of the matrix `cg_test_msr.txt` from the first project.

Note: As part of the Lanczos method you need to numerically compute the eigenvalues of the small tridiagonal matrix that you generate as part of the algorithm. It is ok to use a power iteration to get the maximum eigenvalue of that matrix. However, you should use a lower tolerance, depending on the size of the Krylov space. (See below.)

Use the vector $\mathbf{x} = \frac{1}{\sqrt{n}}(1, 1, \ldots, 1)^T$ as initial guess, where n is the dimension of the problem.

We compare the maximum eigenvalue that we obtain with the maximum eigenvalue obtained from the pure power iteration (see step 1). For that purpose, you should re-run the power iteration with the matrix `cg_matrix_msr_1`, and `cg_matrix_msr_2`. For timing (see below) you should run both the Lanczos method and the power iteration with the same compilation options, and possibly suppress output to make results as comparable as possible. You should also exclude the setup time, e.g. I/O operations, from the timing.

Consider to do:

- For the pure power iteration, plot the error against both iteration index and runtime on a semi-log scale (linear in time).
- For the matrix:
  - `cg_matrix_msr_1`, the largest eigenvalue is $\lambda_1 = 1.59710384442648 \times 10^8$
  - `cg_matrix_msr_2`, the largest eigenvalue is $\lambda_1 = 4.36322641800025 \times 10^8$
- Run the Lanczos method for $m = 30, 50, 75, 100$, where m is the dimension of the Krylov space.
- Use a power iteration to compute the maximum eigenvalue of the triangular Lanczos matrix.
- For the power iteration you can use a convergence criterion

  $$|\theta^{(k)} - \theta^{(k-1)}| < \text{tol}$$

  where $\theta^{(k)}$ is the approximation to the maximum eigenvalue of the Lanczos matrix at iteration k. For the tolerance `tol` you can use:
  - 10^{-2} for $m = 30$
  - 10^{-4} for $m = 50$
  - 10^{-6} for $m = 75$
  - 10^{-10} for $m = 100$
- You may optionally try to optimize the tolerances for the Lanczos method yourself.
- Record the final error and overall runtime for each m and generate a plot on a semi-log scale (linear in time). Compare to the pure power iteration.

Report and submittion

You should write a short report that addresses all the points raised in the previous section. **This report should not exceed four pages (excluding figures).**

In addition, please submit all the documents, if you submit more than one, in a compressed folder (.zip, .rar, etc) with the following file name **lastname_firstname_P2**. In the case that you have made the project in pairs, use the following:

ownlastname_ownfirstname_partnerlastname_partnerfirstname_P2.