STAT GU4261/GR5261 Statistical Methods in Finance

Fall 2018

Homeowork 1 Suggested Solution

If you have any questions about the solution, please contact Brian at hl2902@columbia.edu

R lab section. Problem 4 on page 13:
Just copy the code in the book and run it. The answer is 0.50988.
R lab section. Problem 5, 6 on page 14:

(a) 0.3982

(b) 0.5833

R code:

```
# Problem 5
# parameter setting
P0 = 1e6
mu = 0.05/253
sigma = 0.23/sqrt(253)
sell_price = 1100000
cut_loss_price = 950000

# simulation parameter
n = 10000
profit = rep(0,n)
loss =rep(0,n)
set.seed(1999)
for (i in 1:n) {
R = rnorm(100, mu, sigma)
P = P0*exp(cumsum(R))

# the first day of having loss
loss_day = which(P < cut_loss_price)[1]

# the first day of having proft
profit_day = which(P > sell_price)[1]

if (max(P) >= sell_price & min(P) < cut_loss_price) {
profit[i] = as.numeric(loss_day > profit_day)
loss[i] = as.numeric(loss_day < profit_day)
}

if ( max(P) >= sell_price & min(P) >= cut_loss_price) {
profit[i] = 1
```

```
}

if ( max(P) < sell_price & min(P) < cut_loss_price) {
loss[i] = 1
}

if (max(P) < sell_price & min(P) >= cut_loss_price) {
loss[i] = as.numeric(P[100] < 1e6)
}
}
mean(profit)
mean(loss)
```

<u>Problem 3.11.1 in textbook:</u>

(a) $y_{20} = \frac{1}{20}\int_0^{20}(0.028 + 0.00042t)dt = 0.0322.$

(b) $P = 1000e^{-\int_0^{15}(0.028+0.00042t)dt} = 626.7.$

<u>Problem 3.11.3 in textbook:</u>

(a) Coupon rate > current yield if and only if price > par. Therefore, it is selling above par.

(b) Price > par if and only if copun rate > current yield > yield to maturity. Hence, yield to maturity is below 2.8%.

Remark: Let y be the yield. The bond price is

$$P = \sum_{t=1}^{T} \frac{C}{(1+y)^t} + \frac{\text{PAR}}{(1+y)^T}.$$

Divided both sides by P and rearranging the terms, we obtain

$$\frac{C}{P} = \frac{1 - \frac{\text{PAR}}{P}\frac{1}{(1+y)^T}}{\sum_{t=1}^{T}\frac{1}{(1+y)^t}}.$$

Since

$$\sum_{t=1}^{T}\frac{1}{(1+y)^t} = \frac{\frac{1}{(1+y)^{T+1}} - \frac{1}{1+y}}{\frac{1}{1+y} - 1} = \frac{1 - \frac{1}{(1+y)^T}}{y},$$

we have

$$\frac{C}{P} = y \times \frac{1 - \frac{\text{PAR}}{P}\frac{1}{(1+y)^T}}{1 - \frac{1}{(1+y)^T}}.$$

As $\frac{C}{P}$ is the current yield, we see from the above equation that current yield > yield to maturity if and only if price > PAR.

<u>Problem 3.11.4 in textbook:</u>

(a) $y_5 = \frac{1}{5}\int_0^5(0.032 + 0.001t + 0.0002t^2)dt = 0.0362$

(b) $P = \text{PAR}e^{-5y_5} = 0.834\text{PAR}.$

Problem 3.11.7 in textbook:

(a) $C = \frac{0.085}{2} \times 1000 = 42.5.$

(b)

$$P = \sum_{t=1}^{38} \frac{42.5}{(1+\frac{0.076}{2})^t} + \frac{1000}{(1+\frac{0.076}{2})^{38}} = 1089.717908.$$

(c)

$$P = 42.5 + \sum_{t=1}^{38} \frac{42.5}{(1+\frac{0.076}{2})^t} + \frac{1000}{(1+\frac{0.076}{2})^{38}} = 1132.217908.$$

Problem 3.11.13 in textbook:

$$\begin{aligned} y_{20} &= \frac{1}{20}\int_0^{20} r(t)dt \\ &= \frac{1}{20}\int_0^{10} 0.03 + 0.001t\ dt + \frac{1}{20}\int_{10}^{20} 0.03 + 0.001t - 0.00021(t-10)dt \\ &= \frac{1}{20}\left[0.03t + \frac{0.001}{2}t^2\right]_0^{20} - \frac{1}{20}\left[\frac{0.00021}{2}(t-10)^2\right]_{10}^{20} \\ &= 0.039475. \end{aligned}$$

Problem 3.11.15 in textbook:
First,

$$\frac{d}{d\delta}\sum_{i=1}^N C_i e^{-T_i(y_{T_i}+\delta)} = -\sum_{i=1}^N T_i C_i e^{-T_i(y_{T_i}+\delta)}.$$

Setting $\delta = 0$,

$$\begin{aligned} \left.\frac{d}{d\delta}\sum_{i=1}^N C_i e^{-T_i(y_{T_i}+\delta)}\right|_{\delta=0} &= -\sum_{i=1}^N T_i C_i e^{-T_i y_{T_i}} = -\left(\sum_{i=1}^N \frac{C_i e^{-T_i y_{T_i}}}{\sum_{j=1}^N C_j e^{-T_j y_{T_j}}} T_i\right)\sum_{j=1}^N C_j e^{-T_j y_{T_j}} \\ &= -\mathrm{DUR}\sum_{j=1}^N C_j e^{-T_j y_{T_j}}. \end{aligned}$$

Note that

$$\left.\frac{d}{d\delta}\sum_{i=1}^N C_i e^{-T_i(y_{T_i}+\delta)}\right|_{\delta=0} = \lim_{\delta\to 0}\frac{\sum_{i=1}^N C_i e^{-T_i(y_{T_i}+\delta)} - \sum_{i=1}^N C_i e^{-T_i y_{T_i}}}{\delta}$$

Hence, when δ is small,

$$\frac{\sum_{i=1}^N C_i e^{-T_i(y_{T_i}+\delta)} - \sum_{i=1}^N C_i e^{-T_i y_{T_i}}}{\delta} \approx -\mathrm{DUR}\sum_{i=1}^N C_i e^{-T_i y_{T_i}}.$$

Finally, as $\sum_{i=1}^N C_i e^{-T_i y_{T_i}}$ is the bond price, we have

$$\frac{\text{change bond price}}{\text{bond price}} \approx -\mathrm{DUR}\times\delta.$$

Problem 3.11.21 in textbook:
We need to solve y in

$$1015 = \frac{25}{y} + (1000 - \frac{25}{y})(1+y)^{-8}.$$

It can be found that $y = 0.0229$. R code:

```
f = function(y) {
return(25/y + (1000- 25/y)*(1+y)^(-8) - 1015)
}
uniroot(f, interval = c(0.001,0.05))
```

Other questions:

(6) $\mathbb{P}(U \le u) = \mathbb{P}(F(X) \le u) = \mathbb{P}(X \le F^{-1}(u)) = F(F^{-1}(u)) = u$. Hence, $U \sim \text{Unif}(0,1)$.

(7) (a) By chain rule and fundamental theorem of calculus,

$$f_Y(y) = \frac{d}{dy}\mathbb{P}(Y \le y) = \frac{d}{dy}\mathbb{P}(X \le \log Y) = \frac{d \log y}{dy}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(\log y - \mu)^2}{2\sigma^2}} = \frac{1}{y\sqrt{2\pi\sigma^2}}e^{-\frac{(\log y-\mu)^2}{2\sigma^2}}.$$

(b) We first prove the claim in the hint. Note that

$$\begin{aligned}
\mathrm{E}(e^{Xt}) &= \int_{-\infty}^{\infty} e^{xt}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}dx \\
&= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2-2x\mu+\mu^2-2x\sigma^2 t}{2\sigma^2}\right\}dx \\
&= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{[x-(\mu+\sigma^2 t)]^2-2\mu\sigma^2 t-\sigma^4t^2}{2\sigma^2}\right\}dx \\
&= \exp\left\{\mu t+\frac{1}{2}\sigma^2t^2\right\}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{[x-(\mu+\sigma^2t)]^2}{2\sigma^2}\right\}dx \\
&= \exp\left\{\mu t+\frac{1}{2}\sigma^2t^2\right\}.
\end{aligned}$$

Hence, the mean is $\mathbb{E}(e^X) = e^{\mu+\frac{1}{2}\sigma^2}$ and

$$\begin{aligned}
\mathrm{Var}(Y) &= \mathrm{Var}(e^X) \\
&= \mathrm{E}(e^X)^2 - [\mathrm{E}(e^X)]^2 \\
&= E(e^{2X}) - [\mathrm{E}(e^X)]^2 \\
&= e^{2\mu+2\sigma^2} - [e^{\mu+\sigma^2/2}]^2 \\
&= e^{2\mu+\sigma^2}(e^{\sigma^2}-1).
\end{aligned}$$

(8) (a) $\mathbb{E}(F_n(x)) = \frac{1}{n}\sum_{i=1}^n \mathbb{P}(X_i \le x) = F(x)$.

(b) $\mathrm{Var}(F_n(x)) = \frac{1}{n^2}\sum_{i=1}^n \mathrm{Var}(I(X_i \le x))) = \frac{F(x)(1-F(x))}{n}$.

(c) Note that $F_n(x) = \frac{1}{n}\sum_{i=1}^n (X_i \le x)$, which is the average of iid random variables with mean $F(x)$ and variance $F(x)(1-F(x))$. Hence, by central limit theorem, the term converges (in distribution) to the standard normal distribution $\mathcal{N}(0,1)$ as $n \to \infty$.