Lagrange multipliers: an example

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Example

- Consider the function of two variables

$$f(x,y) = x^2 + y^2$$

and our goal is to solve

$$\min_{x,y} f(x,y)$$

- the first order necessary conditions are

$$\frac{\partial}{\partial x} f(x,y) = 2x = 0 \qquad \text{and} \qquad \frac{\partial}{\partial \mathrm{y}} \mathrm{f}(\mathrm{x},\mathrm{y}) = 2\mathrm{y} = 0$$

- This give $x = 0$ and $y = 0$
- Since this function is convex (why?), the solution is (0, 0).

Example

- Next, consider the same function

$$f(x,y) = x^2 + y^2$$

and our goal is to solve

$$\min_{x,y} f(x,y)$$

subject to $x + y = 1$.

- Solving for y we get y = 1-x. Therefore we seek x that minimizes

$$g(x) = x^2 + (1-x)^2$$

- the first order necessary conditions are

$$\frac{\partial}{\partial x} g(x) = 2x - 2((1-x) = 0.$$

- This give $x = 1/2$
- Since this function is convex (why?), the solution is (1/2, 1/2).

- Another method to solve
$$\min_{x,y} f(x, y)$$
subject to $x + y = 1$ is to use Lagrange multipliers.
- This method augments the function to be minimized with a linear function of the constraint in homogeneous form. The constraint in homogenous form is
$$x + y = 1.$$
- The augmented function to be minimized is called the Lagrangian and is given by
$$L(x, y, \lambda) = x^2 + y^2 + \lambda(x + y - 1)$$
- The coefficient on the constraint in homogeneous form, λ, is called the Lagrange multiplier. It measures the cost of imposing the constraint relative to the unconstrained problem.

- The problem we consider now is

$$\min_{x,y,\lambda} L(x, y, \lambda)$$

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- the first order necessary conditions are

$$\begin{aligned} \frac{\partial}{\partial x} L(x, y, \lambda) &= 2x + \lambda = 0 \\ \frac{\partial}{\partial y} L(x, y, \lambda) &= 2y + \lambda = 0, \\ \frac{\partial}{\partial y} L(x, y, \lambda) &= x + y - 1 = 0 \end{aligned}$$

- These equations imply that

$$x = y = -\lambda/2$$

  Therefore $x = y = 1/2$