Fixed Income Securities

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Introduction

- When you own a share of a stock, you have partial ownership. In which case you share in both the profits and losses. Nothing is guaranteed.
- When you buy a bond, you make a loan to the company. The corporation is then obligated (has an obligation) to pay back the principle plus interest (unless it defaults). You will receive a fixed stream of income. Bonds are called fixed-income securities
- Bonds might appear risk-free but actually this is not true.
  - For long term bond, your income is guaranteed only if you keep the bond to maturity.
  - Suppose you buy a bond paying 5% and the rate of interest increases to 6% . Your bond is inferior to the new bonds offering 6% . Consequently, the price of your bond will decrease. If you sell the bond you would lose money

Introduction

The interest rate of a bond depends on the maturity of the bond.
On March 28, 2001, the interest rate of Treasury's:

- 4.23% for 3-month bills
- 4.41% on 2-year notes
- 5.46% on 30-year bonds

On March 22, 2006 (oddly, there is little dependence)

- 4.69% on 3-month
- 4.69% on 5-year
- 4.70% on 10-year
- 4.91% on 20-year
- 4.73% on 30-year

Zero Coupon Bonds

- zero-coupon bonds pay no principle or interest until maturity
- zero-coupon bond = pure discount bond
- par value (face value) is the payment made to the bond holder at maturity
- a zero coupon bond sells for less than par
- any bond selling for less than par is a discount bond

Zero Coupon Bonds

Pricing of a zero coupon bond is related to the risk free interest rate.

- For a 10 year zero coupon bond with a face value of \$1000 and interest rate at r =4%, the (fair) market price (present value, PV) is

$$\frac{1000}{(1+0.04)^{10}}$$

  if the interest is compounded annually.
- The price of a 20-year zero with par value of \$1000 when the interest rate is r =6% and compounded annually is

$$\frac{1000}{(1+r)^{20}} = \frac{1000}{1.06^{20}} = 311.80.$$

- If however interest r =6% and compounded every six months , the price is

$$\frac{1000}{(1+r/2)^{40}} = \frac{1000}{1.03^{40}} = 305.56.$$

- If interest rate $r = 6\%$ is compounded continuously , the price is

$$\frac{1000}{e^{20r}} = \frac{1000}{e^{20(0.06)}} = 301.19$$

Zero Coupon Bonds

The price fluctuates with the interest rate
Case 1:

- assume semi-annual compounding
- you just bought the zero for \$306.56
- six months later the interest rate increased to 7% The price would now be

$$\frac{1000}{1.035^{39}} = 261.41$$

- investment would drop by

$$306.56 - 261.41 = 45.15$$

- You still get your \$1000 if you keep the bond for 20 years if you sell it now you will lose \$45.15, a return of

$$\frac{-45.15}{306.56} = -14.73\%$$

  for a half-year or -29.46% per year

Zero Coupon Bonds

Case 2:

- assume semi-annual compounding
- you just bought the zero for \$306.56
- six months later the interest rate increased to 5% The price would now be

$$\frac{1000}{1.025^{39}} = 381.74$$

- investment would go up by

$$381.74 - 306.56 = 75.18$$

- You still get your \$1000 if you keep the bond for 20 years if you sell it now you will lose \$45.15, a return of

$$\frac{75.18}{306.56} = 24.5\%$$

for a half-year or 49% per year

Zero Coupon Bonds

Case 3:

- assume semi-annual compounding
- you just bought the zero for \$306.56
- six months later the interest rate increased to 6% The price would now be

$$\frac{1000}{1.03^{39}} = 315.75$$

- investment would go up by

$$315.75 - 306.56 = 75.18$$

- You still get your \$1000 if you keep the bond for 20 years if you sell it now you will lose \$45.15, a return of

$$\frac{9.19}{306.56} = 3\%$$

for a half-year or 6% per year

Zero Coupon Bonds

General formula: The price of a zero coupon bond is given by

$$\text{PRICE} = \text{PAR}(1 + r)^{-T}$$

if we assume that

- T is the time to maturity in years
- the annual rate of interest is r with annual compounding. r is called the yield to maturity.

If we assume the interest rate r is per year with semi-annual compounding, the price is

$$\text{PRICE} = \text{PAR}(1 + r/2)^{-2T}$$

For a T-year zero coupon bond with interest compounded n times a year, the is price is

$$\text{Price} = \text{PAR}(1 + r/n)^{-nT}$$

Coupon Bonds

coupon bonds make regular interest payments and are usually sold at the par value

consider a 20-year coupon bond with a par value of \$1000 and 6% annual coupon rate with semi-annual coupon payments

- coupon payment will be \$30
- bond holder receives 40 payments of \$30
- plus a principle payment of \$1000 after 20 years

Coupon Bonds

- assume discounting at the coupon rate
- then present value of all payments, with discounting at the 6% annual rate (3% semi-annual), equals \$1000:

$$\sum_{t=1}^{40} \frac{30}{1.03^t} + \frac{1000}{1.03^{40}} = 1000$$

- interest (discount) rate unchanged six months later then the bond (including first coupon) worth

$$\sum_{t=0}^{39} \frac{30}{1.03^t} + \frac{1000}{1.03^{39}} = 1030$$

which results in a 6% annual return

Coupon Bonds

- interest (discount) rate increases to 7%, then after six months the bond (plus the interest due) worth

$$\sum_{t=0}^{39} \frac{30}{1.035^t} + \frac{1000}{1.035^{39}} = 924$$

and the annual return is

$$2\left(\frac{924.48 - 1000}{1000}\right) = -15.1\%$$

- interest (discount) rate drops to 5% after six months then investment is worth

$$\sum_{t=0}^{39} \frac{30}{1.025^t} + \frac{1000}{1.025^{39}} = 1153.70$$

which results in a 6% annual return

$$2\left(\frac{1153.70 - 1000}{1000}\right) = -30.72\%$$

Coupon Bonds

The general formula is given by

$$
\begin{aligned}
\text{bond price} &= \sum_{t=1}^{2T} \frac{C}{(1+r)^t} + \frac{PAR}{(1+r)^{2T}} \\
&= \frac{C}{r} + \left\{ \text{PAR} - \frac{C}{r} \right\} (1+r)^{-2T}
\end{aligned}
$$

where

- PAR = par value
- T = maturity (in years)
- r = interest rate per half-year

This formula is derived using the fact that for $a \neq 1$

$$1 + a + a^2 + \ldots + a^n = \frac{1 - a^{n+1}}{1 - a}.$$

(use $a = 1/(1 + r)$)

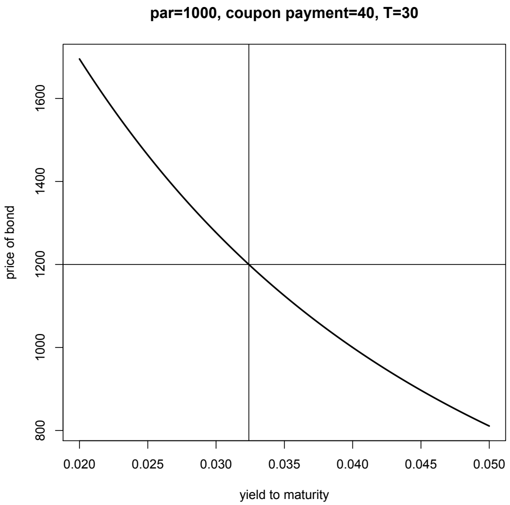
Yield to Maturity

- Suppose a bond with $T = 30$, $C = 40$, semi-annual compounding and PAR$= 1000$ is selling for \$1200, 200 above the par value.
- Bond selling at par value implies interest rate $= 0.04 = 4\%/$ half year $(= 0.08 = 8\%$ /year).
- 8% is called the coupon rate
- Since the bond is not selling at par, if you purchase it at \$ 1200, the rate will be less than 8%.
- Coupon payments are \$ 40 or $40/1200$ =3.33% per half-year of 6.66% per year. 6.67% is called the current rate.
- At maturity, you will get only \$ 1000 of \$ 1200 invested, for this bond the rate solves

$$1200 = \frac{40}{r} + \left\{1000 - \frac{40}{r}\right\}(1 + r)^{-60}$$

the rate r is called yield to maturity. In this case this rate is 0.0324 per half year.

Yield to Maturity



Comparison of the three rates

$$\text{price} > \text{par} \Longrightarrow \text{coupon rate} > \text{current rate} > \text{yield to maturity}$$

Bond selling below par

In this case everything is reversed.

$$\text{price} < \text{par} \Longrightarrow \text{coupon rate} < \text{current rate} < \text{yield to maturity}$$

The general formula for yield to maturity is

$$\begin{aligned} \text{PRICE} &= \sum_{t=1}^{2T} \frac{C}{(1+y)^t} + \frac{PAR}{(1+y)^{2T}} \\ &= \frac{C}{y} + \left\{ \text{PAR} - \frac{C}{y} \right\} (1+y)^{-2T} \end{aligned}$$

where

- PRICE is the price of the bond.
- PAR is the par value.
- C is the semi-annual coupon payment.
- T is the time to maturity in years

Yield to Maturity

For a zero-coupon bond, $C = 0$ and the equation above becomes

$$\text{PRICE} = \text{PAR}(1+y)^{-2T}$$

This implies that for a zero-coupon bond, the yield to maturity is the interest rate.

Spot rates

- Interest rates vary through time.
- This occurs primarily because inflation rates are expected to differ through time.
- Consider two zero coupon bonds.
  - Bond A is a one-year bond with par value of \$ 1000 and interest rate $r_A = 0.08$ or 8% per year
  - Bond B is a two-year bond with par value \$1,000 and interest rate $r_B = 0.10$ or 10%.

Spot rates

- These two rates of interest are examples of spot rates.
- Inequality in interest rates occurs because inflation is expected to be higher over the second year than over the first year.
- 8% and 10% are examples of spot rates.
- The yield to maturity of a zero coupon bond of maturity n years is called the n year spot rate
- A coupon bond is a bundle of zeros, each with a different maturity and therefore a different spot rate

Spot rates

Example 1: Given the spot rates r_1 equals 8% and r_2 equals 10% what should a 5% percent coupon, two-year bond cost? Assume yearly compounding.

$$\text{PRICE} = \frac{50}{1+0.08} + \frac{1050}{(1+0.10)^2} = 914.06$$

Example 2: A one year coupon bond with

- semi-annual coupon payments of \$ 40
- a par of \$ 1000

Spot rates

Suppose that

- the one half year spot rate is 5% per year
- the one year spot rate is 6% per year

We can think of the coupon as being composed of two zero coupon bonds

- one with $T = 1/2$ and par value of 40
- one with $T = 1$ and par value of 1040
- the price of the bond is the sum of the prices of these two zero coupons and is given by

$$\frac{40}{1 + 0.025} + \frac{1040}{(1 + 0.03)^2} = 1019.$$

- the yield to maturity on this coupon is the value of y that satisfies

$$\frac{40}{1 + y} + \frac{1040}{(1 + y)^2} = 1019.$$

In this case $y = 0.0299$.

General Formula

Suppose that a coupon bond

- pays semi-annual payments of C
- has par value of PAR
- has T years until maturity

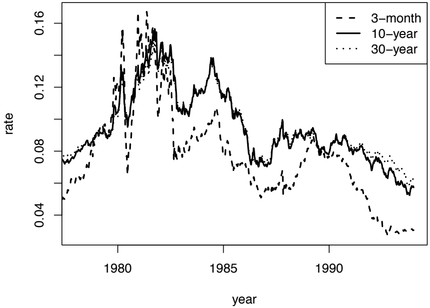
General Formula

Suppose $r_1, r_2, \ldots, r_{2T}$, be the spot rate for 1/2, 1, 1.5, ..., T years, the yield to maturity is the value of y that satisfies

$$\frac{C}{1+r_1} + \frac{C}{(1+r_2)^2} + \ldots + \frac{C}{(1+r_{2T-1})^{2T-1}} + \frac{C+PAR}{(1+r_{2T})^{2T}} = \frac{C}{1+y} + \frac{C}{(1+y)^2} + \ldots + \frac{C}{(1+y)^{2T-1}} + \frac{PAR+C}{(1+y)^{2T}}$$

- The relationship between interest rates or bond yields and different terms or maturities.
- The term structure of interest rate is therefore a description of how, at a given time, yield to maturity depends on maturity

Yield to Maturity



Tterm structure for all maturities up to n years can be described by any one of the following sets:

- prices of zero coupon bonds of maturities 1-year, 2-years, ..., n-years denoted here by $P(1), P(2), \ldots, P(n)$
- spot rates (yields of maturity of zero coupon bonds) of maturities 1-year, 2-years, ..., n-years denoted by $y_1, y_2, \ldots, , y_n$
- forwards rates $r_1, r_2, \ldots, r_n$ where r_k is the interest for the kth year only.

- Each of the above sets can be computed from either of the other sets.
- Break down the time interval between the present time and the maturity time into short time segments with constant interest rate within each segment and varying interest rates between segments

Example (yield to maturity from forward interest rate example):

Year (i)	Interest rate (r_i)
1	6%
2	7%
3	8%

Term Structure of Interest Rates

- par value \$1000. One year zero coupon would sell for

$$P(1) = \frac{1000}{1+r_1} = \frac{1000}{1+0.06} = 943.40$$

- par value \$1000. Two year zero coupon would sell for

$$P(2) = \frac{1000}{(1+r_1)(1+r_2)} = \frac{1000}{(1+0.06)(1+0.07)} = 881.68$$

- par value \$1000. Three year zero coupon would sell for

$$P(3) = \frac{1000}{(1+r_1)(1+r_2)(1+r_3)} = \frac{1000}{(1+0.06)(1+0.07)(1+0.08)} = 816.37$$

- In general, the present value of \$ 1 paid n periods from now is

$$\frac{1}{(1+r_1)(1+r_2)\dots(1+r_n)}$$

where $r_1, r_2, \dots, r_n$ are the forward interest rates during the periods $1, 2, \dots, n$, respectively.

- Since the yield to maturity rate y_n satisfies

$$\frac{1}{(1+r_1)(1+r_2)\dots(1+r_n)} = \frac{1}{(1+y)^n},$$

we have

$$y_n = \sqrt[n]{(1+r_1)(1+r_2)\dots(1+r_n)} - 1$$

($1 + y_n$ is the geometric mean of $(1+r_1), (1+r_2), \dots, (1+r_n)$.)
In our example, we have

$$y_1 = r_1 = 0.06, \quad y_2 = \sqrt{(1+r_1)(1+r_2)} - 1 = \sqrt{(1.06)(1.07)} - 1 = 0.0649$$

and

$$y_3 = \sqrt{(1+r_1)(1+r_2)(+r_3)} - 1 = \sqrt{(1.06)(1.07)(1.08)} - 1 = 0.0700$$

Example (forward rates from yields to maturity):

- $r_1 = y_1$ and

$$r_n = \frac{(1+y_n)^n}{(1+y_{n-1})^{n-1}} - 1$$

- the yield to maturity from bond prices:

$$y_n = \sqrt[n]{\frac{1000}{P(n)}} - 1$$

- the yield to maturity from forward rates

$$y_n = \sqrt[n]{(1+r_1)(1+r_2)\ldots(1+r_n)} - 1$$

- bond prices from yields to maturity

$$P(n) = \frac{1000}{(1+y_n)^n}.$$

- r_n from the prices of zero coupon bonds

$$r_n = \frac{P(n-1)}{P(n)} - 1$$

Continuous Compounding

Continuous compounding simplifies the relationships between forward rates, yields to maturity of zeros (spots rates) and prices of zeros.

- Prices from forward rates

$$P(1) = \frac{1000}{exp(r_1)} = 1000e^{-r_1}$$

  and in general

$$P(n) = 1000e^{-(r_1+r_2+\ldots+r_n)}$$

- Forward rates from prices

$$\frac{P(n-1)}{P(n)} = \frac{e^{r_1+r_2+\ldots+r_n}}{e^{r_1+r_2+\ldots+r_{n-1}}},$$

  therefore

$$r_n = \log\left\{\frac{P(n-1)}{P(n)}\right\}$$

Continuous Compounding

- Yield to maturity from forward rates: since

$$P(n) = 1000e^{-ny_n}$$

we have

$$y_n = (r_1 + r_2 + \ldots + r_n)/n$$

- $r_1, r_2, \ldots, r_n$ are found from $y_1, y_2, \ldots, y_n$ by

$$r_1 = y_1 \quad \text{and} \quad r_n = ny_n - (n-1)y_{n-1}$$

for $n > 1$.

Continuous Forward Rates

We have assumed that forward interest rates

- are constant within each year.
- have a fixed starting time with all maturities some integer number of years from this date.

Continuous Compounding

To be realistic, we assume that

$$P(T) = \exp\{-\int_0^T r(t)dt\}$$

where

- $r(t)$ called the forward rate function
- par value $=1$

Continuous Compounding

- if $r(t) = r_k$ for $k-1 \leq t \leq k$ then

$$\int_0^T r(t)dt = r_1 + r_1 + \ldots + r_T$$

so that for Par=1, the price is

$$P(T) = \exp\{-(r_1 + r_1 + \ldots + r_T)\}$$

- the yield to maturity of a bond with maturity T is defined as

$$y_T = \frac{1}{T}\int_0^T r(t)dt$$

- As a consequence for Par=1, the price is

$$P(T) = \exp\{-Ty_T\}$$

Continuous Compounding

- Example: Suppose the forward rate is

$$r(t) = 0.03 + 0.0005t$$

find $r(10), y_{10}$ and $P(10)$.
Answer:

  - $r(10) = 0.03 + 0.0005(10) = 0.035$
  - $$y_{10} = \frac{1}{10}\int_0^{10}(0.03+0.0005t)dt = \frac{1}{10}\left(0.03(10) + 0.0005\frac{100}{2}\right) = 0.0325$$
  - $$P(10) = \exp\{-10(0.0325)\} = 0.7725$$

Duration of a coupon bond

How sensitive is a bond price to changes in yield?

- The Discount function is

$$D(T) = \exp(-\int_0^T r(t)dt) = \exp(-Ty_T)$$

- Suppose that y_T changes to $y_T + \delta$. The change in $D(T)$ is approximately δ times

$$\frac{d}{dy_T}\exp(-Ty_T) \approx -T\ exp(-Ty_T) = -TD(T).$$

- Therefore

$$\frac{\text{change in bond price}}{\text{bond price}} \approx -T \times \text{change in yield}$$

- The minus sign on the right hand side shows us that the bond prices move in the opposite direction to interest rates.