STAT GU4261/GR5261 Statistical Methods in Finance

Fall 2018

Homeowork 6 Suggested Solution

If you have questions on the solution, please contact Brian at hl2902@columbia.edu

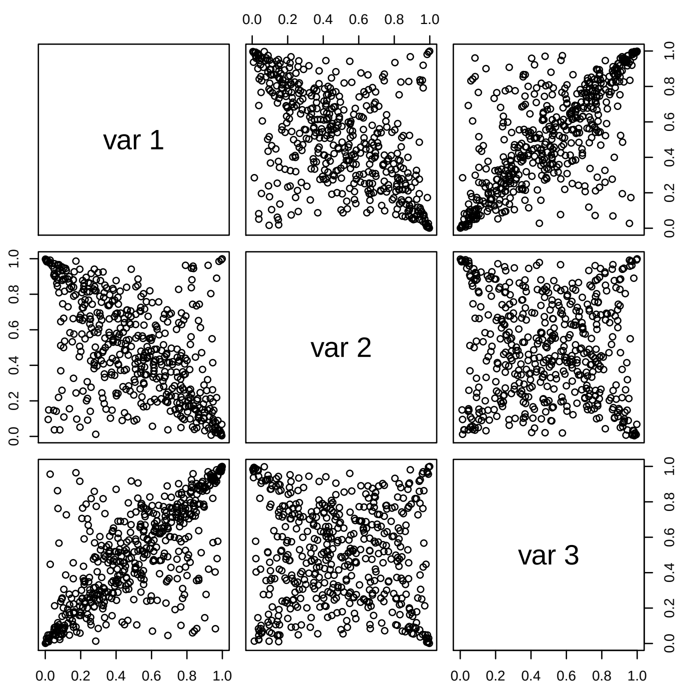
Answers for Q1-Q2 are from the book solution.

1. (a) $t$-copula. The correlation matrix is

$$\begin{pmatrix} 1 & -0.6 & 0.75 \\ -0.6 & 1 & 0 \\ 0.75 & 0 & 1 \end{pmatrix}.$$

The degrees of freedom parameter is 1.

   (b) The sample size is 500.




2. (a) Components 2 and 3 would be uniformly scattered over the unit square if they were independent. Clearly, the scatter is not uniform, so they do not appear independent.

   (b) The non-uniformity mentioned in (a) is that there are more data in the corners, which shows that extreme values tend to occur together, although because of the zero correlation, a positive extreme value of one component is equally likely to be paired with a positive or negative extreme value of the other component.

(c) The effects of tail dependence is the tendency of extreme values to pair. The negative correlation of components 1 and 2 shows in the concentration of the data along the diagonal from upper left to lower right. Positive extreme values in one component tend to pair with negative extreme values of the other component. The positive correlation of components 2 and 3 shows in the concentration of the data along the diagonal from lower left to upper right. Positive extreme values in one component tend to pair with positive extreme values of the other component.

(d) The output is below and the confidence interval is (0.6603, 0.7484) which does not quite include 0.75. This is not surprising. 0.75 is the correlation between the t-distributed random variables that define the copula and need not be the same as the uniformly-distributed variables in the copula itself.

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cor.test(rand_t_cop[,1],rand_t_cop[,3])

Pearson's product-moment correlation

data:  rand_t_cop[, 1] and rand_t_cop[, 3]
t = 22.314, df = 498, p-value < 2.2e-16
alternative hypothesis: true correlation is not equal to 0
95 percent confidence interval:
0.6603249 0.7483624
sample estimates:
cor
0.707073
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3. (a) Exponential distribution. Expected values are 1/2, 1/3, 1/4.

(b) Let $X_2$ and $X_3$ denote the second and third components. Denote $F_2$ and $F_3$ to be the corresponding marginal distributions. We have

$$\begin{aligned}\mathbb{P}(X_2 \le u, X_3 \le v) &= C_{\text{Gaussian}}(F_2(u), F_3(v)) \\ &= P(\Phi(Y_2) \le F_2(u), \Phi(Y_3) \le F_3(v)),\end{aligned}$$

where $(Y_2, Y_3)$ is a bivariate normal random vector with 0 correlation. Recall that if two components in a multivariate normal distribution are uncorrelated then they are independent. Therefore, we have

$$P(\Phi(Y_2) \le F_2(u), \Phi(Y_3) \le F_3(v)) = P(\Phi(Y_2) \le F_2(u))P(\Phi(Y_3) \le F_3(v)) = F_2(u)F_3(v).$$

Hence, we have shown that, for any $u, v$,

$$\mathbb{P}(X_2 \le u, X_3 \le v) = F_2(u)F_3(v).$$

So, $X_2$ and $X_3$ are <u>independent</u>.

4. (a) We know that

$$\begin{aligned}\text{Payoff} &= \begin{cases} 2 & \text{if both bonds default before the end of the first year} \\ 1 & \text{if both survive the first year but they default before the end of the second year} \end{cases} \\ &= \begin{cases} 2 & \text{if } T_A < 1 \text{ and } T_B < 1, \\ 1 & \text{if } 1 < T_A \le 2 \text{ and } 1 < T_B \le 2. \end{cases}\end{aligned}$$

Let $T$ be the time where the payment (if any) will be received. The fair price is the expected discounted payoff:

$$\begin{aligned}\text{Price} &= e^{-rT}\mathbb{E}(\text{Payoff}) \\ &= \mathbb{E}(\text{Payoff}) \quad (\because r=0) \\ &= 2\mathbb{P}(T_A<1, T_B<1)+\mathbb{P}(1<T_A\le 2, 1<T_B\le 2).\end{aligned}$$

To obtain these probabilities, the key point is to use the following relationship:

$$\mathbb{P}(T_A<a, T_B<b)=\mathbb{P}(F_A(T_A)<F_A(a), F_B(T_B)<F_B(b))=C(F_A(a), F_B(b)). \qquad (1)$$

Since the marginal distribution of $T_A$ and $T_B$ are exponential,

$$F_A(1)=1-e^{-\lambda_A}, F_B(1)=1-e^{-\lambda_B}, F_A(2)=1-e^{-2\lambda_A} \text{ and } F_B(2)=1-e^{-2\lambda_B}.$$

Now,

$$\mathbb{P}(T_A<1, T_B<1)=C(F_A(1), F_B(1))=0.01191161.$$

For $\mathbb{P}(1<T_A\le 2, 1<T_B\le 2)$, we shall write it in terms of several terms that could be using (1):

$$\begin{aligned}& \mathbb{P}(1<T_A\le 2, 1<T_B\le 2) \\ =\;& \mathbb{P}(T_A\le 2, 1<T_B\le 2)-\mathbb{P}(T_A\le 1, 1<T_B\le 2) \\ =\;& \mathbb{P}(T_A\le 2, T_B\le 2)-\mathbb{P}(T_A\le 2, T_B\le 1)-\Big\{\mathbb{P}(T_A\le 1, T_B\le 2)-\mathbb{P}(T_A\le 1, T_B\le 1)\Big\} \\ =\;& C(F_A(2), F_B(2))-C(F_A(2), F_B(1))-\Big\{C(F_A(1), F_B(2))-C(F_A(1), F_B(1))\Big\} \\ =\;& 0.008855882.\end{aligned}$$

Therefore, the price is \$0.0326791 million.

(b) The price is now

$$\text{Price}=2\mathbb{P}(T_A<1, T_B<1)+\mathbb{P}(\{1\le T_A, 1\le T_B\}\cap(\{T_A\le 2\}\cup\{T_B\le 2\})).$$

Note that

$$\begin{aligned}& \mathbb{P}(\{1\le T_A, 1\le T_B\}\cap(\{T_A\le 2\}\cup\{T_B\le 2\})) \\ =\;& \mathbb{P}\Big(\{1\le T_A\le 2, 1\le T_B\}\cup\{1\le T_A, 1\le T_B\le 2\}\Big) \\ =\;& \mathbb{P}(1\le T_A\le 2, 1\le T_B)+\mathbb{P}(1\le T_A, 1\le T_B\le 2)-\mathbb{P}(1\le T_A\le 2, 1\le T_B\le 2) \\ =\;& \Big[\mathbb{P}(1\le T_A\le 2)-\mathbb{P}(1\le T_A\le 2, T_B<1)\Big]+\Big[\mathbb{P}(1\le T_B\le 2)-\mathbb{P}(T_A<1, 1\le T_B\le 2)\Big] \\ & -\mathbb{P}(1\le T_A\le 2, 1\le T_B\le 2) \\ =\;& \Big[\mathbb{P}(1\le T_A\le 2)-\mathbb{P}(T_A\le 2, T_B<1)+\mathbb{P}(T_A\le 1, T_B<1)\Big] \\ & +\Big[\mathbb{P}(1\le T_B\le 2)-\mathbb{P}(T_A<1, T_B\le 2)+\mathbb{P}(T_A<1, T_B<1)\Big] \\ & -\mathbb{P}(1\le T_A\le 2, 1\le T_B\le 2).\end{aligned}$$

The terms in the last line could be computed using the marginal distributions and the copula function. The price is \$0.1437758 million.