Copulas

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- Understanding and quantifying dependence is at the core of all modeling efforts in financial econometrics.
- The linear correlation coefficient, which is the most widely used measure of dependence is only a measure of linear dependence.
- It is a meaningful measure of dependence if asset returns are well represented by an elliptical distribution.
- Outside the world of elliptical distributions, however, using the linear correlation coefficient as a measure of dependence may lead to misleading conclusions.
- As a result we need alternative methods to capture codependency
- One class of alternatives are copula-based dependence measures.

Introduction

- A copula is a multivariate distribution function whose univariate marginal distributions are all Uniform(0,1)
- Recall that if X is continuous random variable with CDF F_X then $F_X(X)$ has Uniform(0,1)
- Suppose that $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_d)^T$ has a multivariate CDF $F_{\mathbf{Y}}$ with continuous marginals $F_{Y_1}, F_{Y_2}, \ldots, F_{Y_d}$, then the CDF of $(F_{Y_1}(Y_1), F_{Y_2}(Y_2), \ldots, F_{Y_d}(Y_d))^T$ is a copula.
- This CDF is called the copula of $\mathbf{Y}$ and is denoted by $C_{\mathbf{Y}}$.
- We have

$$C_{\mathbf{Y}}(u_1, u_2, \ldots, u_d) = P(F_{Y_1}(Y_1) \leq u_1, F_{Y_2}(Y_2) \leq u_2, \ldots, F_{Y_d}(Y_d) \leq u_d)$$

- That is

$$\begin{aligned} C_{\mathbf{Y}}(u_1, u_2, \ldots, u_d) &= P(F_{Y_1}(Y_1) \leq u_1, F_{Y_2}(Y_2) \leq u_2, \ldots, F_{Y_d}(Y_d) \leq u_d) \\ &= P(Y_1 \leq F_{Y_1}^{-1}(u_1), Y_2 \leq F_{Y_2}^{-1}(u_2), \ldots, Y_d \leq F_{Y_d}^{-1}(u_d)) \\ &= F_{\mathbf{Y}}(F_{Y_1}^{-1}(u_1), F_{Y_2}^{-1}(u_2), \ldots, F_{Y_d}^{-1}(u_d)) \end{aligned}$$

- Letting $u_j = F_{Y_j}(y_j)$, we get

$$F_{\mathbf{Y}}(y_1, y_2, \ldots, y_d) = C_{\mathbf{Y}}(F_{Y_1}(y_1), F_{Y_2}(y_2), \ldots, F_{Y_d}(y_d))$$

- let

$$
\begin{aligned}
f_{Y_j}(y_j) &= \frac{d}{dy_j} F_{Y_j}(y_j), \quad j = 1, 2, \ldots, d, \\
f_{\mathbf{Y}}(y_1, y_2, \ldots, y_d) &= \frac{\partial^d}{\partial y_1 \partial y_2 \ldots \partial y_d} F_{\mathbf{Y}}(y_1, y_2, \ldots, y_d) \\
c_{\mathbf{Y}}(u_1, u_2, \ldots, u_d) &= \frac{\partial^d}{\partial u_1 \partial u_2 \ldots \partial u_d} C_{\mathbf{Y}}(u_1, u_2, \ldots, u_d)
\end{aligned}
$$

- then

$$
f_{\mathbf{Y}}(y_1, y_2, \ldots, y_d) = c_{\mathbf{Y}}(F_{Y_1}(y_1), F_{Y_2}(y_2), \ldots, F_{Y_d}(y_d)) f_{Y_1}(y_1) f_{Y_2}(y_2) \ldots f_{Y_d}(y_d)
$$

- Notice that i $Y_1, Y_2, \ldots, Y_d$ are independent if and only if

$$
c_{\mathbf{Y}}(u_1, u_2, \ldots, u_d) = 1, \forall (u_1, u_2, \ldots, u_d)^T
$$

- $f_{\mathbf{Y}}$ is therefore decomposed into the copula $C_{\mathbf{Y}}$ which contains all the information about the dependencies among $Y_1, Y_2, \ldots, Y_d$ and the univariate marginal pdfs.

Notice that

$$c_{\mathbf{Y}}(u_1, u_2, \ldots, u_d) = \frac{f_{\mathbf{Y}}(F_{Y_1}^{-1}(u_1), F_{Y_2}^{-1}(u_2), \ldots, F_{Y_d}(u_d))}{\prod_{i=1}^{d} f_{Y_i}(F_{Y_i}^{-1}(u_i))}$$

Examples of copulas

- The d-dimentional independence copula

$$C^{ind}(u_1, u_2, \ldots, u_d) = u_1 u_2 \ldots u_d$$

  and has density

$$c^{ind}(u_1, u_2, \ldots, u_d) = 1$$

  on $[0, 1]^d$.

- The d-dimensional co-monotonicity copula C^M has perfect positive dependence. Let $U \sim$ Uniform(0,1). The co-monotonicity copula is the CDF of $\mathbf{U} = (U, U, \ldots, U)^T$ and is given by

$$C^M(u_1, u_2, \ldots, u_d) = P(U \leq u_1, U \leq u_2, \ldots, U \leq u_d) = min(u_1, u_2, \ldots, u_d)$$

- The tow-dimensional counter-monotonicity copula C^{CM} and is the CDF of $\mathbf{U} = (U, 1 - U)^T$ and is given by

$$C^{CM}(u_1, u_2) = P(U_1 \leq u_1, 1-U \leq u_2) = P(1-u_2 \leq U \leq u_1) = max(u_1+u_2-1, 0)$$

- Elliptical copulas have become very popular in finance and risk management because of their easy implementation.
- Actual elliptical copula classes implemented in the package are normalCopula for normal copula tCopula for t-copula, specified by multivariate normal and multivariate t distribution.
- Both copulas has a dispersion matrix, inherited from the elliptical distributions, and t-copula has one more parameter, the degrees of freedom (df)
- Commonly used dispersion structures are implemented: autoregressive of order 1 (ar1), exchangeable (ex), Toeplitz (toep), and unstructured (un). The corresponding correlation matrices are, for example, in the case of dimension $p = 3$,

$$\begin{pmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_1 \\ \rho_2 & \rho_1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_3 \\ \rho_2 & \rho_3 & 1 \end{pmatrix}$$

Examples of copulas

- Gaussian Copula. In the bivariate case, the Gaussian copula is given by

$$C_\rho(u_1, u_2) = \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi(1-\rho^2)} \exp\left\{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right\} dxdy$$

  where ρ is the parameter of the copula, and Φ^{-1} is the inverse of the standard univariate normal distribution function. This can be generalized to d-dimension and the parameter of the copula is the correlation matrix denoted by Ω.

- The Students t-copula: this copula allows for joint fat tails and an increased probability of joint extreme events compared with the Gaussian copula and is given by

$$C_{\rho,\nu}(u_1, u_2) = \int_{-\infty}^{t_\nu^{-1}(u_1)} \int_{-\infty}^{t_\nu^{-1}(u_2)} \frac{1}{2\pi(1-\rho^2)^{1/2}} \left\{1 + \frac{x^2 - 2\rho xy + y^2}{\nu(1-\rho^2)}\right\} dxdy$$

  where ρ and ν are the parameter of the copula, and t_ν^{-1} is the inverse of the standard univariate student-t-distribution with ν degrees of freedom

Figure: Normal-Copula

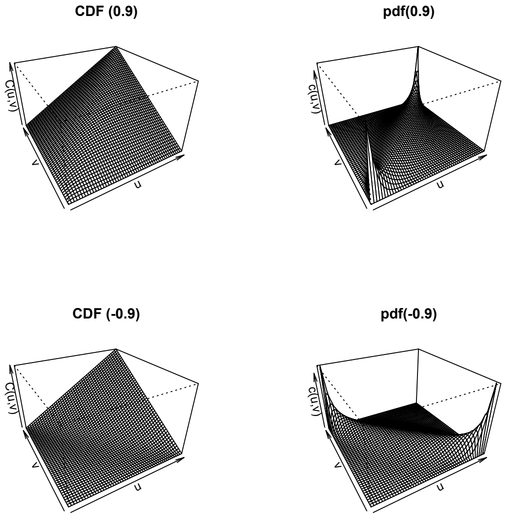
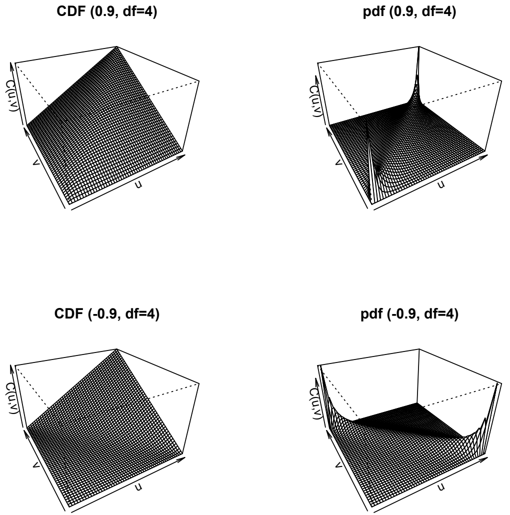


Figure: t-Copula



- The Students t-dependence structure introduces an additional parameter compared with the Gaussian copula, namely the degrees of freedom ν.
- There are also a number of copulas which are not derived from multivariate distribution functions, but do have simple closed forms. Next we will consider two explicit copulas; the Clayton, the Gumbel copulas. They the so-called Archimedian copulas.

An Archimedean copula with a strict generator has the form

$$C(u_1, u_2, \ldots, u_d) = \phi^{-1}(\phi(u_1) + \phi(u_2) + \ldots + \phi(u_d))$$

where the function ϕ is the generator of the copula and satisfies

1. ϕ is a continuous, strictly decreasing, and convex function mapping $[0, 1]$ onto $[0, \infty]$
2. $\phi(0) = \infty$
3. $\phi(1) = 0$.

- Notice that the value of $C(u_1, u_2, \ldots, u_d)$ is unchanged if we permute $u_1, u_2, \ldots, u_d$.
- A distribution with this property is called exchangeable.
- One consequence of exchangeability is that both Kendalls and Spearmans rank cor relation (to be introduced later) are the same for all pairs of variables.
- As a results, Archimedean copulas are most useful in the bivariate case or in applications where we expect all pairs to have similar dependencies.

- if we take

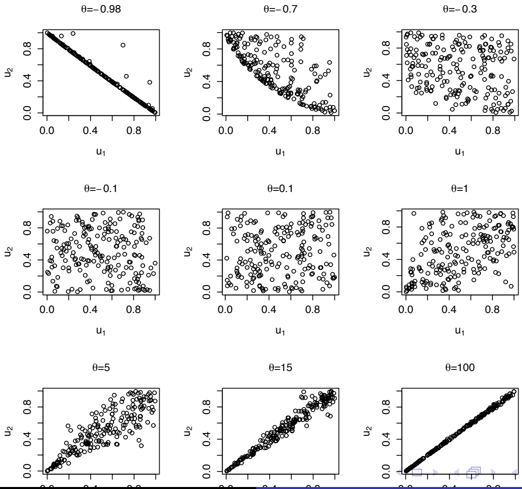
$$\phi(t) = \frac{t^{-\theta} - 1}{\theta}, \quad \theta > 0$$

  we get the Clayton copula
- The Clayton copula, which is an asymmetric copula, exhibiting greater dependence in the negative tail than in the positive. This copula is given by

$$C^{Cl}(u_1, u_2, \ldots, u_d) = \begin{cases} (u_1^{-\theta} + u_2^{-\theta} + \ldots + u_d^{-\theta} - d + 1)^{-1/\theta}, & \theta > 0 \\ u_1 u_2 \ldots u_d, & \theta = 0 \end{cases}$$

- The copula can be extended to the case where $-1 < \theta < 0$.

Figure: Clayton-Copula



- The Gumbel copula has generator

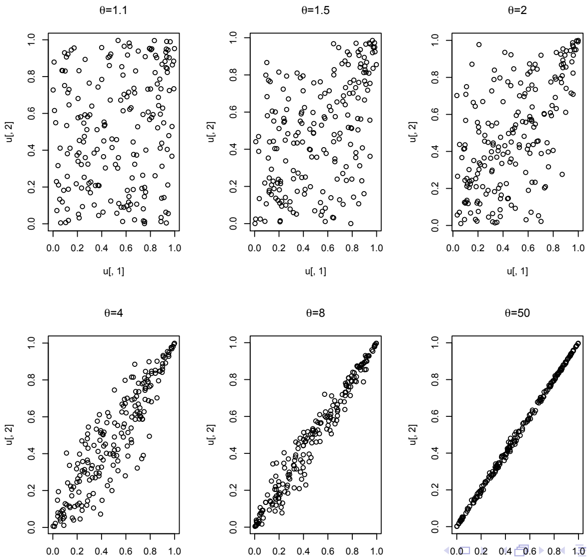
$$\phi(t) = \{-\log(t)\}^{\theta}, \quad \theta \geq 1$$

- The Gumbel copula is also an asymmetric copula, but it is exhibiting greater dependence in the positive tail than in the negative. This copula is given by

$$C^{Gu}(u_1, u_2, \ldots, u_d) = \exp\left[-\left\{(-\log(u_1))^{\theta} + (-\log(u_2))^{\theta} + \ldots + (-\log(u_d))^{\theta}\right\}^{1/\theta}\right]$$

- The Gumbel copula is the independence copula when $\theta = 1$ and converges to the co-monotonicity copula as $\theta \to \infty$ but the Gumbel copula cannot have negative dependence

Figure: Gumbel-Copula



- The following code creates a normal-Copula object and a t-Copula object,
  > myCop.norm = ellipCopula(family = "normal", dim = 3, dispstr = "ex", param = 0.4)
  > myCop.t = ellipCopula(family = "t", dim = 3, dispstr = "toep", param = c(0.8, 0.5), df = 8)
  > myCop.clayton = archmCopula(family = "clayton", dim = 3, param = 2
- The mvdc class is designed to construct multivariate distributions with given margins using copulas. For example
  myMvd = mvdc(copula = myCop.clayton, margins = c("norm", "norm", "norm"), paramMargins = list(list(mean = 0, sd = 2), list(mean = 0, sd = 1), list(mean = 0, sd = 2)))
  creates a clayton copula when the marginal distribution are normal

- To generate 4 numbers from the myCop.t created created above, we use:
  u = rcopula(myCop.t, 4)
- To generate 4 numbers from MyMvd created above we use
  u = rmvdc(myMvd, 4)

- Since the copula of a multivariate distribution describes its dependence structure, it might be appropriate to use measures of dependence which are copula-based.
- The bivariate concordance measures Kendalls tau and Spearmans rho, as well as the coefficient of tail dependence, can, as opposed to the linear correlation coefficient, be expressed in terms of the underlying copula alone.

- Let (Y_1, Y_2) be a bivariate random vector and let (Y_1^*, Y_2^*) be an independent copy of (Y_1, Y_2). Then $(Y_1, Y_2))$ and (Y_1^*, Y_2^*) are called a concordant pair if the ranking of Y_1 relative to Y_1^* is the same as the ranking of Y_2 relative to Y_2^*, that is, either $Y1 > Y_1^*$ and $Y_2 > Y_2^*$ or $Y_1 < Y_1^*$ and $Y_2 < Y_2^*$
- In either case $(Y_1 - Y_1^*)(Y_2 - Y_2^*) > 0$. Similarly, (Y_1, Y_2) and (Y_1^*, Y_2^*) are called a discordant pair if $(Y_1 - Y_1^*)(Y_2 - Y_2^*) < 0$.
- Kendalls tau is the probability of a concordant pair minus the probability of a discordant pair. Therefore, Kendalls tau for (Y_1, Y_2) is

$$\begin{aligned} \rho_\tau(Y_1, Y_2) &= P((Y_1 - Y_1^*)(Y_2 - Y_2^*) > 0) - P((Y_1 - Y_1^*)(Y_2 - Y_2^*) < 0) \\ &= E(sign((Y_1 - Y_1^*)(Y_2 - Y_2^*)) \end{aligned}$$

where

$$sign(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \\ 0, & x = 0 \end{cases}$$

Kendall's tau

- Kendall's tau of two variables Y_1 and Y_1 can be expressed as

$$\rho_\tau(Y_1, Y_2) = 4\int_0^1\int_0^1 C(u,v)dC(u,v) - 1$$

  where $C(u,v)$ is the copula of the bivariate distribution function of Y_1 and Y_2
- For the Gaussian copula

$$cor(Y_1, Y_2) = \rho = sin(\frac{\pi}{2}\rho_\tau)$$

- For the Clayton copula

$$\rho_\tau = \frac{\theta}{\theta+2}$$

- For the Gumbel copula

$$\rho_\tau = 1 - \frac{1}{\theta}$$

- If we have a bivariate sample $(Y_{i1}, Y_{i,2})^T, i = 1, 2, \ldots, n$ then the sample Kendalls tau is

$$\hat{\rho}_\tau(Y_1, Y_2) = \frac{1}{\binom{n}{2}} \sum_{1 \le i < j \le n} sign\left\{(Y_{i1} - Y_{j1})(Y_{i,2} - Y_{j2})\right\}$$

- Spearmans rho of two variables Y_1 and Y_2 is defined as

$$\rho_S(Y_1, Y_2) = cor(F_{Y_1}(Y_1), F_{Y_2}(Y_2))$$

- Since the distribution of $(F_{Y_1}(Y_1), F_{Y_2}(Y_2))$ is the copula of (Y_1, Y_2), Spearmans rho, like Kendalls tau, depends only on the copula
- Spearman's tau of two variables Y_1 and Y_1 can be expressed as

$$\rho_S(Y_1, Y_2) = 12 \int_0^1 \int_0^1 C(u, v) du dv - 3$$

- If we have a bivariate sample $(Y_{i1}, Y_{i,2})^T, i = 1, 2, \ldots, n$ then the sample Spermans tau is

$$\hat{\rho}_S(Y_1, Y_2) = \frac{12}{n(n^2-1)} \sum_{i=1}^{n} \left\{ rank(Y_{i1}) - \frac{n+1}{2} \right\} \left\{ rank(Y_{i2}) - \frac{n+1}{2} \right\}$$

- There is a saying in finance that in times of stress, correlations will increase.
- Bivariate tail dependence measures the amount of dependence in the upper and lower quadrant tail of a bivariate distribution.
- This is of great interest for the risk manager trying to guard against concurrent bad events.
- Next we discuss the coefficient of tail dependence
- We will discuss
  1. Coefficient of lower tail dependence is denoted by λ_l
  2. Coefficient of upper tail dependence is denoted by λ_u

Tail Dependence

- The coefficient of lower tail dependence is denoted by λ_l is defined for $\mathbf{Y} = (Y_1, Y_2)^T$ as

$$\begin{aligned}\lambda_l &= \lim_{q \downarrow 0} P\{Y_2 \leq F_{Y_2}^{-1}(q) | Y_1 \leq F_{Y_1}^{-1}(q)\} \\ &= \lim_{q \downarrow 0} \frac{C_{\mathbf{Y}}(q,q)}{q}\end{aligned}$$

- The coefficient of upper tail dependence is denoted by λ_u is defined as

$$\begin{aligned}\lambda_u &= \lim_{q \uparrow 1} P\{Y_2 \geq F_{Y_2}^{-1}(q) | Y_1 \geq F_{Y_1}^{-1}(q)\} \\ &= 2 - \lim_{q \uparrow 1} \frac{1 - C(q,q)}{1-q}\end{aligned}$$

- For the Gaussian copula, the coefficients of lower tail and upper tail dependence are

$$\lambda_l = \lambda_u = 2 \lim_{x \to -\infty} \Phi\left(x\sqrt{\frac{1-\rho}{1+\rho}}\right) = 0.$$

  This means, that regardless of high correlation $\rho \neq \pm 1$ we choose, if we go far enough into the tail, extreme events appear to occur independently in Y_1 and Y_2 .
- For the Students t-copula, the coefficients of lower and upper tail dependence are

$$\lambda_l = \lambda_u = 2F_{t,\nu+1}\left(-\sqrt{\nu+1}\sqrt{\frac{1-\rho}{1+\rho}}\right)$$

  where $F_{t,\nu+1}$ is the CDF of a t-distribution with $\nu + 1$ degrees of freedom

- The Clayton copula is lower tail dependent. That is, the coefficient of the upper tail dependence $\lambda_u = 0$ and the coefficient of the lower tail dependence is

$$\lambda_l = 2^{-1/\delta}$$

- The Gumbel copula is upper tail dependent. That is, the coefficient of the lower tail dependence $\lambda_l = 0$ and the coefficient of the upper tail dependence is

$$\lambda_u = 2 - 2^{1/\delta}$$

- Suppose we have parametric models $F_{Y_1}(.|\boldsymbol{\theta}_1), F_{Y_2}(.|\boldsymbol{\theta}_1), \ldots, F_{Y_d}(.|\boldsymbol{\theta}_d)$ for the marginal CDFs as well as a parametric model $c_{\mathbf{Y}}(.|\boldsymbol{\theta}_C)$ for the copula density. The log-likelihood is

$$\begin{aligned} L(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_d, \boldsymbol{\theta}_C) &= \sum_{i=1}^{n} \log\left[c_{\mathbf{Y}}\left(F_{Y_1}(Y_{i1}|\boldsymbol{\theta}_1), \ldots, F_{Y_d}(Y_{id}|\boldsymbol{\theta}_d)|\boldsymbol{\theta}_C\right)\right] \\ &+ \sum_{i=1}^{n}\left(\log\{f_{Y_1}(Y_{i1}|\boldsymbol{\theta}_1)\} + \ldots + \log\{f_{Y_d}(Y_{id}|\boldsymbol{\theta}_d)\}\right) \end{aligned}$$

- Maximum likelihood estimation finds the maximum of $L(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_d, \boldsymbol{\theta}_C)$ over the entire set of $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_d, \boldsymbol{\theta}_C)$.

- There are two potential problems with maximum likelihood estimation.
  1. Maximizing $L(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_d, \boldsymbol{\theta}_C)$ can be a challenging numerical problem because of the large number of parameters
  2. maximum likelihood estimation requires parametric models for both the copula and the marginal distributions. If any of the marginal distributions are not well fit by a convenient parametric family, this may cause biases in the estimated parameters of both the marginal distributions and the copula.
- Alternative approach: Use the semiparametric approach to pseudo-maximum likelihood estimation, where the marginal distributions are estimated nonparametrically.

- Pseudo-maximum likelihood estimation is a two-step process.
  1. In the first step, each of the d marginal distribution functions is estimated, one at a time. Let $\hat{F}_{Y_j}$ be the estimate of the jth marginal CDF, $j = 1, 2, \ldots, d$.
  2. In the second step,

$$\sum_{i=1}^{n} \log \left[c_{\mathbf{Y}} \left(\hat{F}_{Y_1}(Y_{i1}), \ldots, \hat{F}_{Y_d}(Y_{id}) | \boldsymbol{\theta}_C \right) \right]$$

  is maximized over $\boldsymbol{\theta}_C$ (ignoring the rest of the likelihood function).

- There are two approaches to step 1: parametric and nonparametric
  1. In the parametric approach, parametric models $F_{Y_1}(.|\boldsymbol{\theta}_1), \ldots, F_{Y_d}(.|\boldsymbol{\theta})$ for the marginal CDFs are assumed and in the maximum likelihood estimation and the data $Y_{1j}, Y_{2j}, \ldots, Y_{nj}$ for the jth variable are used to estimate $\boldsymbol{\theta}_j$. The $\hat{F}_{Y_j} = F_{Y_j}(.|\hat{\boldsymbol{\theta}}_j)$.
  2. In the nonparametric case, $\hat{F}_{Y_j}$ is estimated by the empirical CDF of $Y_{1j}, \ldots, Y_{nj}$, except that the divisor is $n+1$ instead of n so that

$$\hat{F}_{Y_j}(y) = \frac{\sum_{i=1}^{n} I(Y_{ij} \leq y)}{n+1}$$

suppose n is the sample size and there are 4 variables X_1, X_2, X_3 and X_4. First we transform the data using the empiricals (we actually divide by $n+1$ instead of n) the new data set is called edata
edata= cbind(rank(X_1)/($n+1$), rank(X_1)/($n+1$), rank(X_1)/($n+1$), rank(X_1)/($n+1$))
to fit the normal copula, you need to create the following copula object:
ncop=normalCopula(param=c(0,0,0, 0,0,0), dim=4, dispstr="un")
Now we fit the copula:
fn=fitCopula(data=edata, copula=ncop, method="ml")
the results are stored in fn. Next we fit the t -opula and other copulas.
tcop=fitCopula(param=c(0,0,0, 0,0,0), dim=4, dispstr="un", df=5)
ft=fitCopula(data=edata, copula=tcop, method="ml")
clcop=archmCopula(family="clayton", dim=4, param=2)
fclayton = fitCopula(data=edata,method="ml", copula=clcop)
gcop=archmCopula(family="gumbel", dim=4, param=2)
fgumbel = fitCopula(data=edata,method="ml", copula=gcop)
You need to compare the fits using AIC or BIC. In the output you will find the value of the likelihood function.