Name: Key

University of Colorado at Boulder
Math 3510 Exam 3
April 20, 2011

Problem	Points	Score
1	6	
2	6	
3	6	
4	6	
5	6	
6	6	
Total	36	

Instructions: This is a 50 minute exam. You should show enough of your work to justify your answers. All of your work should be done on the exam pages: no scratch paper. The exam is closed book and closed notes.

1. (a) What are the parameters, the mean, the variance and the probability mass function for a Negative Binomial Distribution?

parameters: r, p where $r \in \{1, 2, 3, \ldots\}$ and $p \in [0, 1]$.

mean: r/p variance: $r(1-p)/p^2$

pmf: $P(X = x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}$, for $x \in \{r, r+1, r+2, \ldots\}$

(b) What are the parameter, the mean, and the probility density function for an Exponential Distribution?

parameter: λ mean: $1/\lambda$ where $\lambda \in (0, \infty)$.

$$\text{pdf: } f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{for } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

2. State the definitions for

(a) a *continuous random variable.*

A random variable X is continuous if $P(X = x) = 0$ for every real number x.

(b) an *absolutely continuous random variable.*

A random variable X is called absolutely continuous if there is a nonnegative function $f: \mathbb{R} \to \mathbb{R}$ such that for all real numbers a: $P(X \leq a) = \int_{-\infty}^{a} f(x)\,dx$.

3. State the definition for *the expected value of an absolutely continuous random variable.*

Let X be an absolutely continuous random variable with probability density function $f: \mathbb{R} \to \mathbb{R}$. Then the expected value of X is $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$, provided $\int_{-\infty}^{\infty} |x| f(x)\,dx < \infty$.

4. (a) Derive the mean and variance for a random variable that has a Bernoulli distribution.

pmf

x	0	1
$P(X=x)$	$1-p$	p

$$\mu = E(X) = 0\cdot(1-p) + 1\cdot p = p$$

$$E(X^2) = 0^2\cdot(1-p) + 1^2\cdot p = p$$

$$\sigma^2 = var(X) = E(X^2) - \mu^2 = p - p^2 = p(1-p).$$

$$\mu = p, \quad \sigma^2 = p(1-p)$$

(b) Use the properties of expected value and variance to derive the mean and variance for a random variable that has a Binomial distribution.

Let $Y_1, \ldots, Y_n$ be independent random variables having identical Bernoulli distributions with common parameter p. Then $X = Y_1 + \cdots + Y_n$ has a Binomial distribution with parameters n and p.

$$E(X) = E(Y_1 + \cdots + Y_n) = E(Y_1) + \cdots + E(Y_n) = p + \cdots + p = np$$

$$var(X) = var(Y_1 + \cdots + Y_n) = var(Y_1) + \cdots + var(Y_n)$$ since $Y_1, \ldots, Y_n$ are independent,

$$= p(1-p) + \cdots + p(1-p) = np(1-p).$$

(c) Let X be a continuous random variable and prove that

$$P(a < X \le b) = P(a \le X \le b).$$

$$P(a \le X \le b) = P(a < X \le b \text{ or } X = a)$$ (union of disjoint events)

$$= P(a < X \le b) + P(X = a)$$

$$= P(a < X \le b) + 0$$ (since X is continuous)

$$= P(a < X \le b).$$

5. For a final exam, your professor gives you a list of 15 items to study. She indicates that she will choose 8 items for the actual exam. You must answer 5 of the 8 correctly to pass the exam. You decide to study 10 of the 15 items. What is the probability that you will pass the exam?

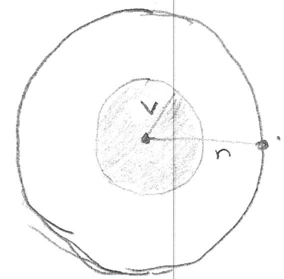
There are $R=10$ items that you study, $W=5$ items that you do not study, and $n=8$ items that are to be chosen. If X is the number of items you studied out of the 8 items chosen, then the probability you pass is

$$P(X\geq 5) = \frac{\binom{10}{5}\binom{5}{3} + \binom{10}{6}\binom{5}{2} + \binom{10}{7}\binom{5}{1} + \binom{10}{8}\binom{5}{0}}{\binom{15}{8}}$$

$\boxed{X \text{ has a Hypergeometric distribution with } R=10, W=5, n=8.}$

6. A point is chosen at random from inside a circle of radius r. Let the random variable V be the distance from the point to the center of the circle.

(a) Find the cumulative distribution function for V.



Let $v \in (0,r)$, then

$$P(V\leq v) = \frac{\pi v^2}{\pi r^2} = \frac{v^2}{r^2}.$$

$$F_V(v) = \begin{cases} 1, & \text{if } v\geq r \\ \frac{v^2}{r^2}, & \text{if } 0<v<r \\ 0, & \text{if } v\leq 0 \end{cases}$$

(b) Find the probability density function for V.

$$f_V(v) = F_V'(v) = \begin{cases} \frac{2v}{r^2}, & \text{if } 0<v<r \\ 0, & \text{otherwise} \end{cases}$$

(c) Find the expected value of V.

$$E(V) = \int_{-\infty}^{\infty} v f_V(v)\,dv = \int_0^r \frac{2}{r^2} v^2\,dv = \frac{2}{3r^2} v^3 \Big|_0^r = \frac{2r^3}{3r^2} = \frac{2}{3}r$$

$$E(V) = \frac{2}{3}r$$