Cryptography (CS-501)

Mathematics of Cryptography
Algebraic Structures

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Cryptography requires sets of integers and specific operations that are defined for those sets.

The combination of the set and the operations that are applied to the elements of the set is called an **algebraic structure***.*

Here, we will define three common algebraic structures: groups, rings, and fields.

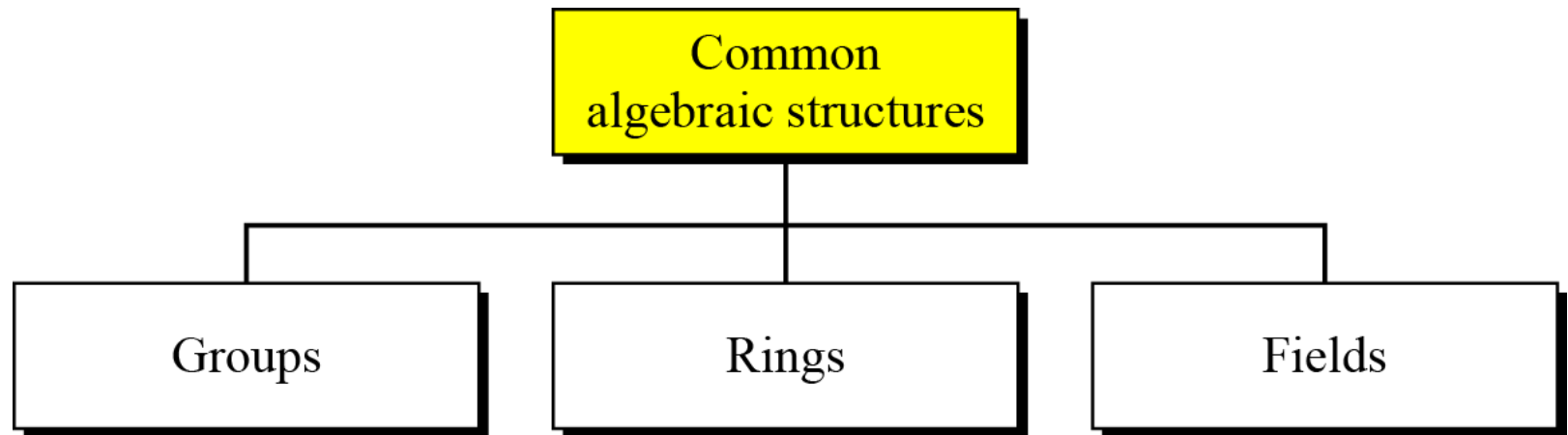**Figure 4.1** *Common algebraic structure*

A group (G) is a set of elements with a binary operation ($\bullet$) that satisfies four properties (or axioms).

A commutative group satisfies an extra property, commutativity:

- ❑ **Closure:**

- ❑ **Associativity:**

- ❑ **Commutativity:**

- ❑ **Existence of identity:**

- ❑ **Existence of inverse:**

Properties

1. Closure
2. Associativity
3. Commutativity (See note)
4. Existence of identity
5. Existence of inverse

Note:
The third property needs to be satisfied only for a commutative group.

{a, b, c, …}
Set

Operation

Group

Figure 4.2 *Group*

A group involves a single operation, the properties imposed on the operation allow the use of a pair of operations as long as they are inverses of each other.

Example 4.1

The set of residue integers with the addition operator,

$$G = <Z_n, +>,$$

is a commutative group. We can perform addition and subtraction on the elements of this set without moving out of the set.

Example 4.2

The set Z_n^* with the multiplication operator, $G = <Z_n^*, \times>$, is also an abelian group.

Example 4.3

Let us define a set $G = < \{a, b, c, d\}, \bullet>$ and the operation as shown in Table 4.1.

$\bullet$	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

Example 4.4

A very interesting group is the permutation group. The set is the set of all permutations, and the operation is composition: applying one permutation after another.
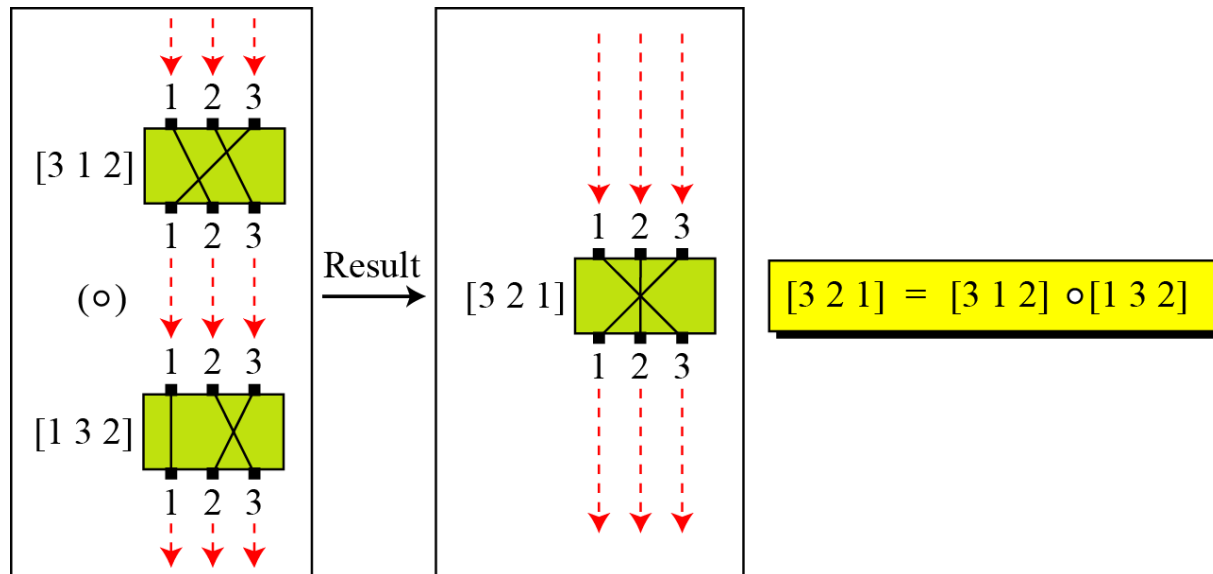

Figure 4.3 *Composition of permutation*

Example 4.4 *Continued*

◦	[1 2 3]	[1 3 2]	[2 1 3]	[2 3 1]	[3 1 2]	[3 2 1]
[1 2 3]	[1 2 3]	[1 3 2]	[2 1 3]	[2 3 1]	[3 1 2]	[3 2 1]
[1 3 2]	[1 3 2]	[1 2 3]	[2 3 1]	[2 1 3]	[3 2 1]	[3 1 2]
[2 1 3]	[2 1 3]	[3 1 2]	[1 2 3]	[3 2 1]	[1 3 2]	[2 3 1]
[2 3 1]	[2 3 1]	[3 2 1]	[1 3 2]	[3 1 2]	[1 2 3]	[2 1 3]
[3 1 2]	[3 1 2]	[2 1 3]	[3 2 1]	[1 2 3]	[2 3 1]	[1 3 2]
[3 2 1]	[3 2 1]	[2 3 1]	[3 1 2]	[1 3 2]	[2 1 3]	[1 2 3]

Table 4.2 *Operation table for permutation group*

Example 4.5

In the previous example, we showed that a set of permutations with the composition operation is a group.

This implies that using two permutations one after another cannot strengthen the security of a cipher, because we can always find a permutation that can do the same job because of the closure property.

❑ **Finite Group**

❑ **Order of a Group**

❑ **Subgroups**

Example 4.6

Is the group H = $<Z_{10}, +>$ a subgroup of the group G = $<Z_{12}, +>$?

Solution

The answer is no. Although H is a subset of G, the operations defined for these two groups are different. The operation in H is addition modulo 10; the operation in G is addition modulo 12.

Cyclic Subgroups

If a subgroup of a group can be generated using the power of an element, the subgroup is called the cyclic subgroup.

$$a^n \rightarrow a \bullet a \bullet \ldots \bullet a \quad (n \text{ times})$$

Example 4.7

Four cyclic subgroups can be made from the group G = <Z_6, +>. They are H_1 = <{0}, +>, H_2 = <{0, 2, 4}, +>, H_3 = <{0, 3}, +>, and H_4 = G.

$0^0 \bmod 6 = 0$

$1^0 \bmod 6 = 0$
$1^1 \bmod 6 = 1$
$1^2 \bmod 6 = (1 + 1) \bmod 6 = 2$
$1^3 \bmod 6 = (1 + 1 + 1) \bmod 6 = 3$
$1^4 \bmod 6 = (1 + 1 + 1 + 1) \bmod 6 = 4$
$1^5 \bmod 6 = (1 + 1 + 1 + 1 + 1) \bmod 6 = 5$

$2^0 \bmod 6 = 0$
$2^1 \bmod 6 = 2$
$2^2 \bmod 6 = (2 + 2) \bmod 6 = 4$

$3^0 \bmod 6 = 0$
$3^1 \bmod 6 = 3$

$4^0 \bmod 6 = 0$
$4^1 \bmod 6 = 4$
$4^2 \bmod 6 = (4 + 4) \bmod 6 = 2$

$5^0 \bmod 6 = 0$
$5^1 \bmod 6 = 5$
$5^2 \bmod 6 = 4$
$5^3 \bmod 6 = 3$
$5^4 \bmod 6 = 2$
$5^5 \bmod 6 = 1$

Example 4.8

Three cyclic subgroups can be made from the group $G = <Z_{10}*, \times>$. G has only four elements: 1, 3, 7, and 9. The cyclic subgroups are $H_1 = <\{1\}, \times>$, $H_2 = <\{1, 9\}, \times>$, and $H_3 = G$.

$1^0 \bmod 10 = 1$

$3^0 \bmod 10 = 1$
$3^1 \bmod 10 = 3$
$3^2 \bmod 10 = 9$
$3^3 \bmod 10 = 7$

$7^0 \bmod 10 = 1$
$7^1 \bmod 10 = 7$
$7^2 \bmod 10 = 9$
$7^3 \bmod 10 = 3$

$9^0 \bmod 10 = 1$
$9^1 \bmod 10 = 9$

Cyclic Groups

A cyclic group is a group that is its own cyclic subgroup.

$$\{e,\, g,\, g^2,\, \dots,\, g^{n-1}\},\ \text{where } g^n = e$$

Example 4.9

Three cyclic subgroups can be made from the group $G = \langle Z_{10}*, \times \rangle$. G has only four elements: 1, 3, 7, and 9. The cyclic subgroups are $H_1 = \langle\{1\}, \times\rangle$, $H_2 = \langle\{1, 9\}, \times\rangle$, and $H_3 = G$.

a. The group $G = \langle Z_6, + \rangle$ is a cyclic group with two generators, $g = 1$ and $g = 5$.

b. The group $G = \langle Z_{10}*, \times \rangle$ is a cyclic group with two generators, $g = 3$ and $g = 7$.

Lagrange's Theorem

Assume that G is a group, and H is a subgroup of G. If the order of G and H are |G| and |H|, respectively, then, based on this theorem, |H| divides |G|.

Order of an Element

The order of an element is the order of the cyclic group it generates.

Example 4.10

a. In the group $G = <Z_6, +>$, the orders of the elements are:
ord(0) = 1, ord(1) = 6, ord(2) = 3, ord(3) = 2, ord(4) = 3,
ord(5) = 6.

b. In the group $G = <Z_{10}^*, \times>$, the orders of the elements are:
ord(1) = 1, ord(3) = 4, ord(7) = 4, ord(9) = 2.

A ring, R = <{…}, •,   >, is an algebraic structure with two operations.

Distribution of ☐ over ●

1. Closure ●	1. Closure ☐
2. Associativity	2. Associativity
3. Commutativity	3. Commutativity
4. Existence of identity	
5. Existence of inverse	

Note:
The third property is only satisfied for a commutative ring.

{a, b, c, …}
Set

● ☐
Operations

Ring

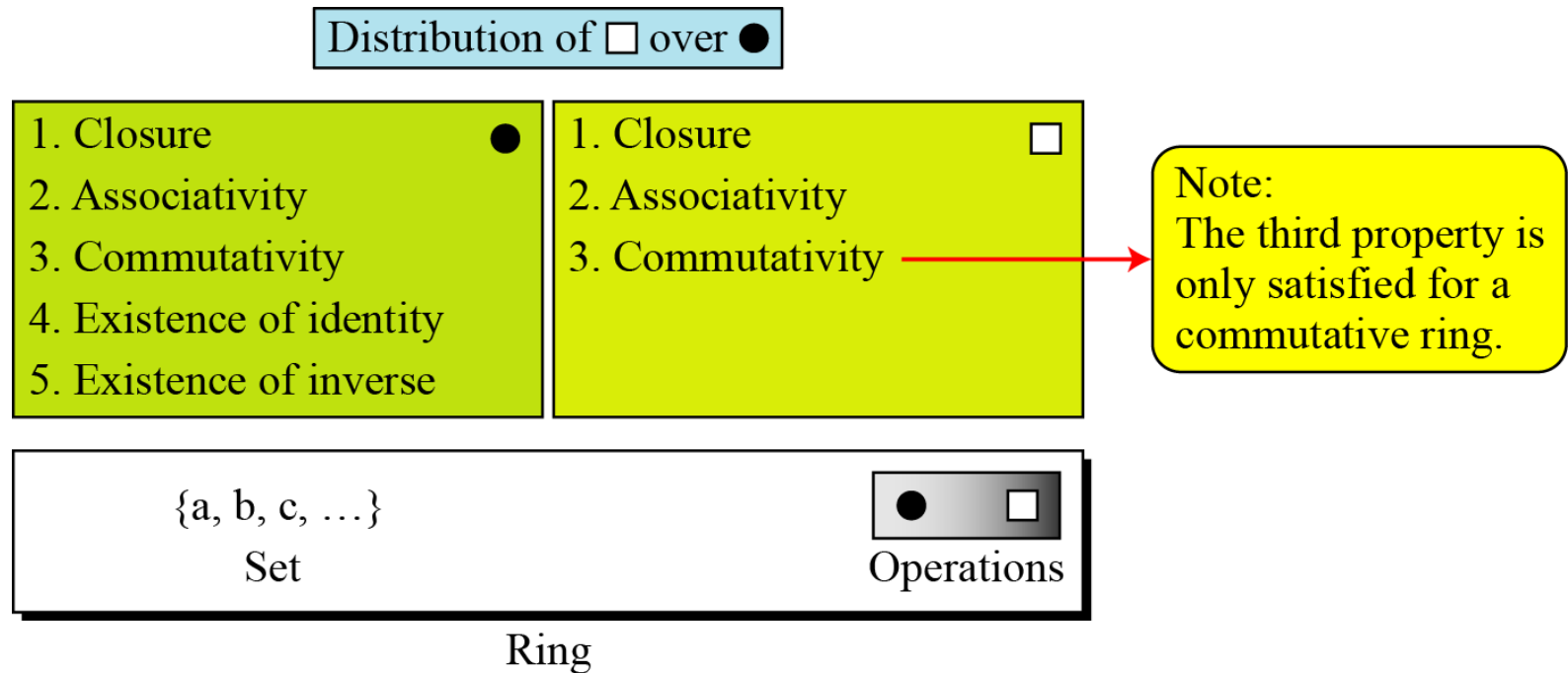**Figure 4.4** *Ring*

Example 4.11

The set Z with two operations, addition and multiplication, is a commutative ring. We show it by R = <Z, +, ×>. Addition satisfies all of the five properties; multiplication satisfies only three properties.

A field, denoted by F = <{…}, •,   > is a commutative ring in which the second operation satisfies all five properties defined for the first operation except that the identity of the first operation has no inverse.

Distribution of ☐ over ●

1. Closure ●
2. Associativity
3. Commutativity
4. Existence of identity
5. Existence of inverse

1. Closure ☐
2. Associativity
3. Commutativity
4. Existence of identity
5. Existence of inverse

Note:
The identity element of the first operation has no inverse with respect to the second operation.
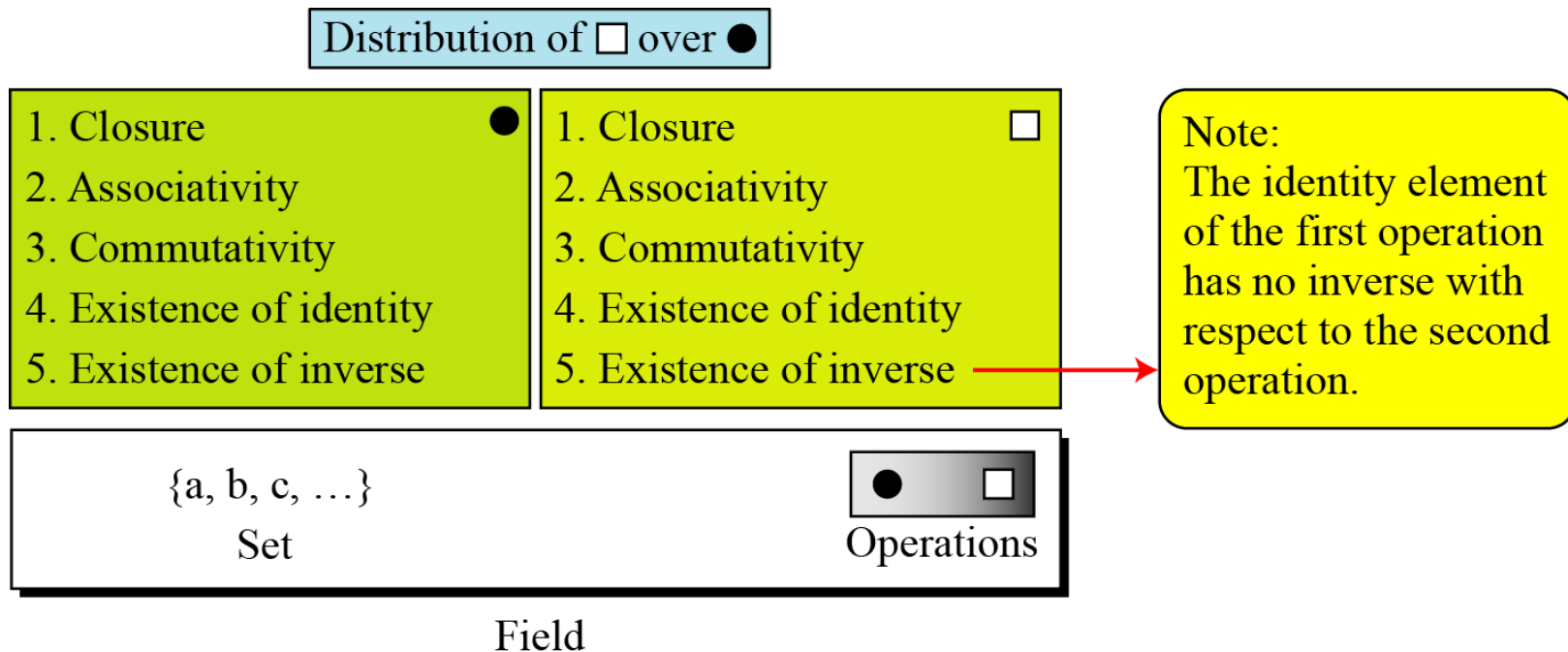
{a, b, c, …}
Set

● ☐
Operations

Field

Figure 4.5 *Field*

Finite Fields

Galois showed that for a field to be finite, the number of elements should be p^n, where p is a prime and n is a positive integer.

Note

A Galois field, GF(p^n), is a finite field with p^n elements.

GF(p) Fields

When $n = 1$, we have GF(p) field. This field can be the set Z_p, {0, 1, …, p − 1}, with two arithmetic operations.

Example 4.12

A very common field in this category is GF(2) with the set {0, 1} and two operations, addition and multiplication, as shown in Figure 4.6.

Figure 4.6 *GF(2) field*

Example 4.13

We can define GF(5) on the set Z_5 (5 is a prime) with addition and multiplication operators as shown in Figure 4.7.
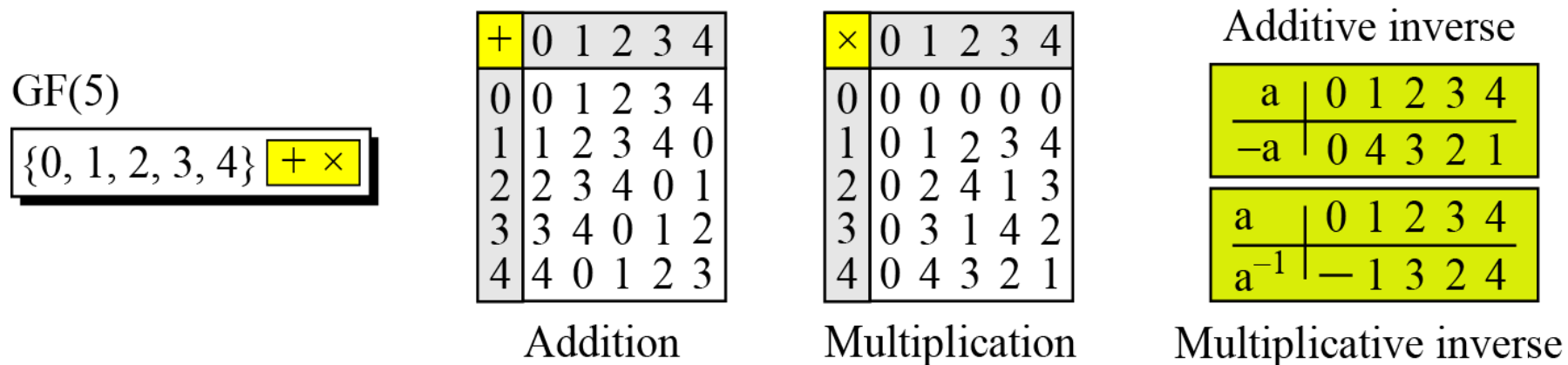
Figure 4.7 *GF(5) field*

Summary

Table 4.3

Algebraic Structure	Supported Typical Operations	Supported Typical Sets of Integers
Group	$(+ \ -)$ or $(\times \ \div)$	$\mathbf{Z}_n$ or $\mathbf{Z}_n^*$
Ring	$(+ \ -)$ and $(\times)$	$\mathbf{Z}$
Field	$(+ \ -)$ and $(\times \ \div)$	$\mathbf{Z}_p$

4-2   GF(2^n) FIELDS

In cryptography, we often need to use four operations (addition, subtraction, multiplication, and division). In other words, we need to use fields. We can work in GF(2^n) and uses a set of 2^n elements. The elements in this set are n-bit words.

Example 4.14

Let us define a GF(2^2) field in which the set has four 2-bit words: {00, 01, 10, 11}. We can redefine addition and multiplication for this field in such a way that all properties of these operations are satisfied, as shown in Figure 4.8.

Addition				
⊕	00	01	10	11
00	00	01	10	11
01	01	00	11	10
10	10	11	00	01
11	11	10	01	00

Identity: 00

Multiplication				
⊗	00	01	10	11
00	00	00	00	00
01	00	01	10	11
10	00	10	11	01
11	00	11	01	10

Identity: 01

Figure 4.8 *An example of GF(2^2) field*

A polynomial of degree $n - 1$ is an expression of the form

$$f(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x^1 + a_0x^0$$

where x^i is called the ith term and a_i is called coefficient of the ith term.

Example 4.15

Figure 4.9 show how we can represent the 8-bit word (10011001) using a polynomials.

n-bit word	1	0	0	1	1	0	0	1

Polynomial: $1x^7 + 0x^6 + 0x^5 + 1x^4 + 1x^3 + 0x^2 + 0x^1 + 1x^0$

First simplification: $1x^7 + 1x^4 + 1x^3 + 1x^0$

Second simplification: $x^7 + x^4 + x^3 + 1$
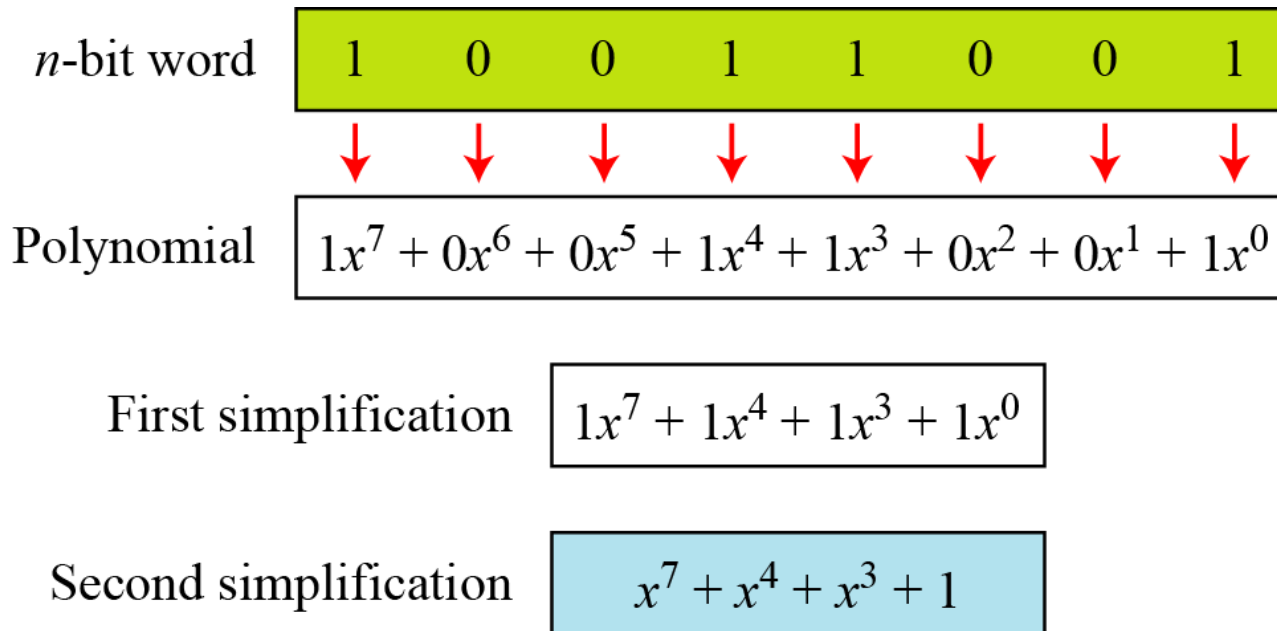
Figure 4.9 *Representation of an 8-bit word by a polynomial*

Example 4.16

To find the 8-bit word related to the polynomial $x^5 + x^2 + x$, we first supply the omitted terms. Since $n = 8$, it means the polynomial is of degree 7. The expanded polynomial is

$$0x^7 + 0x^6 + 1x^5 + 0x^4 + 0x^3 + 1x^2 + 1x^1 + 0x^0$$

This is related to the 8-bit word **00100110**.

GF(2^n) Fields

Note

Polynomials representing n-bit words use two fields: GF(2) and GF(2^n).

Modulus

For the sets of polynomials in GF(2^n), a group of polynomials of degree n is defined as the modulus. Such polynomials are referred to as **irreducible polynomials.**

Table 4.9 *List of irreducible polynomials*

Degree	Irreducible Polynomials
1	$(x + 1)$, (x)
2	$(x^2 + x + 1)$
3	$(x^3 + x^2 + 1)$, $(x^3 + x + 1)$
4	$(x^4 + x^3 + x^2 + x + 1)$, $(x^4 + x^3 + 1)$, $(x^4 + x + 1)$
5	$(x^5 + x^2 + 1)$, $(x^5 + x^3 + x^2 + x + 1)$, $(x^5 + x^4 + x^3 + x + 1)$, $(x^5 + x^4 + x^3 + x^2 + 1)$, $(x^5 + x^4 + x^2 + x + 1)$

Addition

Note

Addition and subtraction operations on polynomials are the same operation.

Example 4.17

Let us do $(x^5 + x^2 + x) \oplus (x^3 + x^2 + 1)$ in $GF(2^8)$. We use the symbol $\oplus$ to show that we mean polynomial addition. The following shows the procedure:

$$0x^7 + 0x^6 + 1x^5 + 0x^4 + 0x^3 + 1x^2 + 1x^1 + 0x^0 \quad \oplus$$
$$0x^7 + 0x^6 + 0x^5 + 0x^4 + 1x^3 + 1x^2 + 0x^1 + 1x^0$$
$$\text{--}$$
$$0x^7 + 0x^6 + 1x^5 + 0x^4 + 1x^3 + 0x^2 + 1x^1 + 1x^0 \quad \rightarrow \quad x^5 + x^3 + x + 1$$

Example 4.18

There is also another short cut. Because the addition in GF(2) means the exclusive-or (XOR) operation. So we can exclusive-or the two words, bits by bits, to get the result. In the previous example, $x^5 + x^2 + x$ is 00100110 and $x^3 + x^2 + 1$ is 00001101. The result is 00101011 or in polynomial notation $x^5 + x^3 + x + 1$.

Multliplication

1. **The coefficient multiplication is done in GF(2).**

2. **The multiplying x^i by x^j results in x^{i+j}.**

3. **The multiplication may create terms with degree more than $n - 1$, which means the result needs to be reduced using a modulus polynomial.**

Example 4.19

Find the result of $(x^5 + x^2 + x) \otimes (x^7 + x^4 + x^3 + x^2 + x)$ in GF(2^8) with irreducible polynomial $(x^8 + x^4 + x^3 + x + 1)$. Note that we use the symbol $\otimes$ to show the multiplication of two polynomials.

Solution

$$P_1 \otimes P_2 = x^5(x^7 + x^4 + x^3 + x^2 + x) + x^2(x^7 + x^4 + x^3 + x^2 + x) + x(x^7 + x^4 + x^3 + x^2 + x)$$

$$P_1 \otimes P_2 = x^{12} + x^9 + x^8 + x^7 + x^6 + x^9 + x^6 + x^5 + x^4 + x^3 + x^8 + x^5 + x^4 + x^3 + x^2$$

$$P_1 \otimes P_2 = (x^{12} + x^7 + x^2) \bmod (x^8 + x^4 + x^3 + x + 1) = x^5 + x^3 + x^2 + x + 1$$

To find the final result, divide the polynomial of degree 12 by the polynomial of degree 8 (the modulus) and keep only the remainder. Figure 4.10 shows the process of division.

$$x^4 + 1$$

$$x^8 + x^4 + x^3 + x + 1 \enclose{longdiv}{}$$

$$x^{12} + x^7 + x^2$$

$$x^{12} + x^8 + x^7 + x^5 + x^4$$

$$x^8 + x^5 + x^4 + x^2$$

$$x^8 + x^4 + x^3 + x + 1$$

Remainder $\boxed{x^5 + x^3 + x^2 + x + 1}$

Figure 4.10 *Polynomial division with coefficients in GF(2)*

Example 4.20

In GF (2^4), find the inverse of $(x^2 + 1)$ modulo $(x^4 + x + 1)$.

Solution

The answer is $(x^3 + x + 1)$ as shown in Table 4.5.

Table 4.5 *Euclidean algorithm for Exercise 4.20*

q	r_1	r_2	r	t_1	t_2	t
$(x^2 + 1)$	$(x^4 + x + 1)$	$(x^2 + 1)$	(x)	(0)	(1)	$(x^2 + 1)$
(x)	$(x^2 + 1)$	(x)	(1)	(1)	$(x^2 + 1)$	$(x^3 + x + 1)$
(x)	(x)	(1)	(0)	$(x^2 + 1)$	$(x^3 + x + 1)$	(0)
	(1)	(0)		$(x^3 + x + 1)$	(0)	

Example 4.21

In GF(2^8), find the inverse of (x^5) modulo ($x^8 + x^4 + x^3 + x + 1$).

Solution

The answer is ($x^5 + x^4 + x^3 + x$) as shown in Table 4.6.

Table 4.6 *Euclidean algorithm for Exercise 4.21*

q	r_1		r	t_1		t
(x^3)	($x^8 + x^4 + x^3 + x + 1$) (x^5)		($x^4 + x^3 + x + 1$)	(0) (1)		(x^3)
($x + 1$)	(x^5) ($x^4 + x^3 + x + 1$)		($x^3 + x^2 + 1$)	(1) (x^3)		($x^4 + x^3 + 1$)
(x)	($x^4 + x^3 + x + 1$) ($x^3 + x^2 + 1$)		(1)	(x^3) ($x^4 + x^3 + 1$)		($x^5 + x^4 + x^3 + x$)
($x^3 + x^2 + 1$)	($x^3 + x^2 + 1$) (1)		(0)	($x^4 + x^3 + 1$) ($x^5 + x^4 + x^3 + x$)		(0)
	(1) (0)			($x^5 + x^4 + x^3 + x$) (0)		

Multliplication Using Computer

The computer implementation uses a better algorithm, repeatedly multiplying a reduced polynomial by x.

Example 4.22

Find the result of multiplying $P_1 = (x^5 + x^2 + x)$ by $P_2 = (x^7 + x^4 + x^3 + x^2 + x)$ in GF(2^8) with irreducible polynomial ($x^8 + x^4 + x^3 + x + 1$) using the algorithm described above.

Solution

The process is shown in Table 4.7. We first find the partial result of multiplying x^0, x^1, x^2, x^3, x^4, and x^5 by P_2. Note that although only three terms are needed, the product of $x^m \otimes P_2$ for m from 0 to 5 because each calculation depends on the previous result.

Example 4.22 *Continued*

Table 4.7 *An efficient algorithm*

Powers	Operation	New Result	Reduction
$x^0 \otimes P_2$		$x^7 + x^4 + x^3 + x^2 + x$	No
$x^1 \otimes P_2$	$x \otimes (x^7 + x^4 + x^3 + x^2 + x)$	$x^5 + x^2 + x + 1$	**Yes**
$x^2 \otimes P_2$	$x \otimes (x^5 + x^2 + x + 1)$	$x^6 + x^3 + x^2 + x$	No
$x^3 \otimes P_2$	$x \otimes (x^6 + x^3 + x^2 + x)$	$x^7 + x^4 + x^3 + x^2$	No
$x^4 \otimes P_2$	$x \otimes (x^7 + x^4 + x^3 + x^2)$	$x^5 + x + 1$	**Yes**
$x^5 \otimes P_2$	$x \otimes (x^5 + x + 1)$	$x^6 + x^2 + x$	No
$\mathbf{P_1 \times P_2} = (x^6 + x^2 + x) + (x^6 + x^3 + x^2 + x) + (x^5 + x^2 + x + 1) = \mathbf{x^5 + x^3 + x^2 + x + 1}$			

Example 4.23

Repeat Example 4.22 using bit patterns of size 8.

Solution

We have P1 = 000100110, P2 = 10011110, modulus = 100011010 (nine bits). We show the exclusive or operation by ⊕.

Table 4.8 *An efficient algorithm for multiplication using n-bit words*

Powers	Shift-Left Operation	Exclusive-Or
$x^0 \otimes P_2$		10011110
$x^1 \otimes P_2$	00111100	(00111100) ⊕ (00011010) = **00100111**
$x^2 \otimes P_2$	01001110	**01001110**
$x^3 \otimes P_2$	10011100	10011100
$x^4 \otimes P_2$	00111000	(00111000) ⊕ (00011010) = 00100011
$x^5 \otimes P_2$	01000110	**01000110**
P$_1$ ⊗ P$_2$ = (00100111) ⊕ (01001110) ⊕ (01000110) = 00101111		

Example 4.24

The GF(2^3) field has 8 elements. We use the irreducible polynomial ($x^3 + x^2 + 1$) and show the addition and multiplication tables for this field. We show both 3-bit words and the polynomials. Note that there are two irreducible polynomials for degree 3. The other one, ($x^3 + x + 1$), yields a totally different table for multiplication.

Example 4.24 *Continued*

Table 4.9 *Addition table for GF(23)*

$\oplus$	000 (0)	001 (1)	010 (x)	011 (x + 1)	100 (x^2)	101 $x^2 + 1$	110 ($x^2 + x$)	111 ($x^2 + x + 1$)
000 (0)	000 (0)	001 (1)	010 (x)	011 (x + 1)	100 (x^2)	101 ($x^2 + 1$)	110 ($x^2 + x$)	111 ($x^2 + x + 1$)
001 (1)	001 (1)	000 (0)	011 (x + 1)	010 (x^2)	101 ($x^2 + 1$)	100 ($x^2 + x$)	111 ($x^2 + x + 1$)	110 ($x^2 + x$)
010 (x)	010 (x)	011 (x + 1)	000 (0)	001 (1)	110 ($x^2 + x$)	111 ($x^2 + x + 1$)	100 ($x^2 + x$)	101 ($x^2 + 1$)
011 (x + 1)	011 (x + 1)	010 (x)	001 (1)	000 (0)	111 ($x^2 + x + 1$)	110 ($x^2 + x$)	101 ($x^2 + 1$)	100 (x^2)
100 (x^2)	100 (x^2)	101 ($x^2 + 1$)	110 ($x^2 + x$)	111 ($x^2 + x + 1$)	000 (0)	001 (1)	010 (x)	011 (x + 1)
101 ($x^2 + 1$)	101 ($x^2 + 1$)	100 (x^2)	111 ($x^2 + x + 1$)	110 ($x^2 + x$)	001 (1)	000 (0)	011 (x + 1)	010 (x)
110 ($x^2 + x$)	110 ($x^2 + x$)	111 ($x^2 + x + 1$)	100 (x^2)	101 ($x^2 + 1$)	010 (x)	011 (x + 1)	000 (0)	001 (1)
111 ($x^2 + x + 1$)	111 ($x^2 + x + 1$)	110 ($x^2 + x$)	101 ($x^2 + 1$)	100 (x^2)	011 (x + 1)	010 (x)	001 (1)	000 (0)

Example 4.24 *Continued*

Table 4.10 *Multiplication table for GF(2^3)*

$\otimes$	000 (0)	001 (1)	010 (x)	011 ($x+1$)	100 (x^2)	101 (x^2+1)	110 (x^2+x)	111 (x^2+x+1)
000 (0)	000 (0)	000 (0)	000 (0)	000 (0)	000 (0)	000 (0)	000 (0)	000 (0)
001 (1)	000 (0)	001 (1)	010 (x)	011 ($x+1$)	100 (x^2)	101 (x^2+1)	110 (x^2+x)	111 (x^2+x+1)
010 (x)	000 (0)	010 (x)	100 (x)	110 (x^2+x)	101 (x^2+1)	111 (x^2+x+1)	001 (1)	011 ($x+1$)
011 ($x+1$)	000 (0)	011 ($x+1$)	110 (x^2+x)	101 (x^2+1)	001 (1)	010 (x)	111 (x^2+x+1)	100 (x)
100 (x^2)	000 (0)	100 (x^2)	101 (x^2+1)	001 (1)	111 (x^2+x+1)	011 ($x+1$)	010 (x)	110 (x^2+x)
101 (x^2+1)	000 (0)	101 (x^2+1)	111 (x^2+x+1)	010 (x)	011 ($x+1$)	110 (x^2+x)	100 (x^2)	001 (1)
110 (x^2+x)	000 (0)	110 (x^2+x)	001 (1)	111 (x^2+x+1)	010 (x)	100 (x^2)	011 ($x+1$)	101 (x^2+1)
111 (x^2+x+1)	000 (0)	111 (x^2+x+1)	011 ($x+1$)	100 (x^2)	110 (x^2+x)	001 (1)	101 (x^2+1)	010 (x)

Sometimes it is easier to define the elements of the $GF(2^n)$ field using a generator.

$$\{0, g, g, g^2, \ldots, g^N\}, \text{ where } N = 2^n - 2$$

Example 4.25

Generate the elements of the field $GF(2^4)$ using the irreducible polynomial $f(x) = x^4 + x + 1$.

Solution

The elements $0, g^0, g^1, g2,$ and g^3 can be easily generated, because they are the 4-bit representations of $0, 1, x^2,$ and x^3. Elements g^4 through g^{14}, which represent x^4 though x^{14} need to be divided by the irreducible polynomial. To avoid the polynomial division, the relation $f(g) = g^4 + g + 1 = 0$ can be used (See next slide).

Example 4.25   *Continued*

$$
\begin{aligned}
0 &= 0 &&= 0 &&= 0 &&\longrightarrow & 0 &= (0000) \\
g^0 &= g^0 &&= g^0 &&= g^0 &&\longrightarrow & g^0 &= (0001) \\
g^1 &= g^1 &&= g^1 &&= g^1 &&\longrightarrow & g^1 &= (0010) \\
g^2 &= g^2 &&= g^2 &&= g^2 &&\longrightarrow & g^2 &= (0100) \\
g^3 &= g^3 &&= g^3 &&= g^3 &&\longrightarrow & g^3 &= (1000) \\
g^4 &= g^4 &&= g^4 &&= g + 1 &&\longrightarrow & g^4 &= (0011) \\
g^5 &= g\,(g^4) &&= g\,(g + 1) &&= g^2 + g &&\longrightarrow & g^5 &= (0110) \\
g^6 &= g\,(g^5) &&= g\,(g^2 + g) &&= g^3 + g^2 &&\longrightarrow & g^6 &= (1100) \\
g^7 &= g\,(g^6) &&= g\,(g^3 + g) &&= g^3 + g + 1 &&\longrightarrow & g^7 &= (1011) \\
g^8 &= g\,(g^7) &&= g\,(g^3 + g + 1) &&= g^2 + 1 &&\longrightarrow & g^8 &= (0101) \\
g^9 &= g\,(g^8) &&= g\,(g^2 + 1) &&= g^3 + g &&\longrightarrow & g^9 &= (1010) \\
g^{10} &= g\,(g^9) &&= g\,(g^3 + g) &&= g^2 + g + 1 &&\longrightarrow & g^{10} &= (0111) \\
g^{11} &= g\,(g^{10}) &&= g\,(g^2 + g + 1) &&= g^3 + g^2 + g &&\longrightarrow & g^{11} &= (1110) \\
g^{12} &= g\,(g^{11}) &&= g\,(g^3 + g^2 + g) &&= g^3 + g^2 + g + 1 &&\longrightarrow & g^{12} &= (1111) \\
g^{13} &= g\,(g^{12}) &&= g\,(g^3 + g^2 + g + 1) &&= g^3 + g^2 + 1 &&\longrightarrow & g^{13} &= (1101) \\
g^{14} &= g\,(g^{13}) &&= g\,(g^3 + g^2 + 1) &&= g^3 + 1 &&\longrightarrow & g^{14} &= (1001)
\end{aligned}
$$

Example 4.26

The following show the results of addition and subtraction operations:

a. $g^3 + g^{12} + g^7 = g^3 + (g^3 + g^2 + g + 1) + (g^3 + g + 1) = g^3 + g^2 \rightarrow (1100)$

b. $g^3 - g^6 = g^3 + g^6 = g^3 + (g^3 + g^2) = g^2 \rightarrow (0100)$

Example 4.27

The following show the result of multiplication and division operations:.

a. $g^9 \times g^{11} = g^{20} = g^{20 \bmod 15} = g^5 = g^2 + g \rightarrow (0110)$

b. $g^3 / g^8 = g^3 \times g^7 = g^{10} = g^2 + g + 1 \rightarrow (0111)$

4.2.3   Summary

The finite field $GF(2^n)$ can be used to define four operations of addition, subtraction, multiplication and division over n-bit words. The only restriction is that division by zero is not defined.