AI 비전공자를 위한 기초 수학 1: 선형 대수학

Math for AI Beginner Part 1: Linear Algebra

Week 2: Introduction of Linear Algebra

Linear Algebra

What we will do during this course…

Review of matrices

Systems of linear algebraic equations AX=B

Finding solution(s) by row operations

Inverse of a square matrix A

Finding inverse by row operations

Determinant of a square matrix A

Calculating determinant by row operations

Bigger pictures: how are these 3 topics connected together with AI?

In between, we will look at vector space and linearly independent vectors.

Matrix eigenproblem

Diagonalisation problem

Review on matrices

What is a matrix?

An $M \times N$ matrix is an array of MN numbers enclosed within a pair of brackets and arranged in M rows and N columns.

Examples:

(3) $\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & \pi \\ \sqrt{2} & \frac{1}{2} \\ 5 & 6 \end{pmatrix}$ $\begin{pmatrix} 3 & 6 & 9 \\ 2 & 3 & 9 \end{pmatrix}$

square matrix

The numbers making up a matrix are referred to as "elements" of the matrix.

In a square matrix, the number of rows equals the number of columns.

We use bold capital letters such as **A**, **B**, **P** and **Q** to denote matrices. E.g. we can write:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

Notation

Let **A** be an $M \times N$ matrix.

Denote the element in the *i*-th row and *j*-th column of **A** by a_{ij}. Then we can write:

$$\mathbf{A} = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{pmatrix}$$

Let **A** = (a_{ij}) and **B** = (b_{ij}) be both $M \times N$ (same order).

Equality of matrices

A and **B** are said to be equal, that is, **A** = **B**, if $a_{ij} = b_{ij}$.

E.g.

$$\begin{pmatrix} a+b & c \\ c & a-b \end{pmatrix} = \begin{pmatrix} c & 3 \\ d & b \end{pmatrix}$$

$$\Rightarrow a+b=c, \quad c=3, \quad c=d, \quad a-b=b$$

Addition of matrices

If **A**+**B** = **C** then **C**=(c_{ij}) is $M \times N$ and $c_{ij} = a_{ij} + b_{ij}$.

E.g.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix} = \begin{pmatrix} 1+7 & 2+8 & 3+9 \\ 4+10 & 5+11 & 6+12 \end{pmatrix}$$

Multiplication of a number to a matrix

If **A** = (a_{ij}) and c is a number then c**A** = (ca_{ij}) and c**A** has the same order as **A**.

E.g.

$$-2\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} -2 & -4 \\ -6 & -8 \end{pmatrix}$$

We write (–1)**A** as –**A**. So **B**–**A** = **B** + (–**A**).

E.g.

$$\begin{pmatrix} 2 & 3 \\ 4 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 2-1 & 3-2 \\ 4-3 & 4-2 \end{pmatrix}$$

Product of matrices

$*\ A \times B = C$

$(a \times b)\ (b \times c)\quad (a \times c)$

Let **A**=(a_{ij}) and **B**=(b_{ij}) be $M \times N$ and $P \times Q$ matrices respectively.

If $N=P$, we can form the product matrix **AB.**

If **AB** is denoted by **C**= (c_{ij}) then **C** is $M \times Q$ and the element c_{kp} is calculated using the k-th row of **A** and p-th column of **B** as follows.

$$c_{kp} = \begin{pmatrix} a_{k1} & a_{k2} & \cdots & a_{kN} \end{pmatrix} \begin{pmatrix} b_{1p} \\ b_{2p} \\ \vdots \\ b_{Np} \end{pmatrix} = a_{k1}b_{1p} + a_{k2}b_{2p} + \cdots + a_{kN}b_{Np}$$

$$= \sum_{n=1}^{N} a_{kn} b_{nj}$$

What is the condition for forming the product ***BA?***
What is the order of ***BA*** *if it can be formed**?***

Example:

$$\mathbf{P} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}_{3\times 2} \qquad \mathbf{Q} = \begin{pmatrix} 5 & 1 & 2 & 2 \\ 3 & 3 & 1 & 2 \end{pmatrix}_{2\times 4}$$

We can form **PQ** but not **QP**.

$$\mathbf{PQ} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}_{3\times 2} \begin{pmatrix} 5 & 1 & 2 & 2 \\ 3 & 3 & 1 & 2 \end{pmatrix}_{2\times 4} = \begin{pmatrix} 11 & 7 & 4 & 6 \\ 27 & 15 & 10 & 14 \\ 43 & 23 & 16 & 22 \end{pmatrix}_{3\times 4}$$

Multiplication of matrices is not commutative, that is, even if **AB** and **BA** can be formed, **AB** may or may not be equal to **BA**.

Identity matrices

An $N \times N$ matrix (c_{ij}) such that $c_{11} = c_{22} = c_{33} = \ldots = c_{NN} = 1$ and $c_{ij} = 0$ for *i* not equal to *j* is called an identity matrix. (I)

$\simeq 1$

Examples:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

What's special about identity matrices?

Let $\mathbf{I}$ be an identity matrix and **A** be any matrix.

If the product $\mathbf{IA}$ can be formed then $\mathbf{IA} = \mathbf{A}$.

Similarly, if the product $\mathbf{AI}$ can be formed then $\mathbf{AI} = \mathbf{A}$.

Transpose of a matrix

If **A** is an $M \times N$ matrix then the transpose of **A** is the $N \times M$ matrix obtained as follows:

"The *i*-th column of the transpose of A is the *i*-th row of A."

The transpose of A is denoted by A^T.

Examples:

$$A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad A^T = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix} \qquad B^T = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}^T = \begin{pmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{pmatrix}$$

Let's start on linear algebra now…

A good starting point is to look at a system of linear algebraic equations.

What's a linear algebraic equation?

Many simultaneous linear algebraic equations form a system.

How can we solve a system of linear algebraic equations?

What's a linear algebraic equation?

An example of a linear algebraic equation in one unknown x is: $2x+1=0$

The **solution of the above linear algebraic equation is:**

$$x=-\frac{1}{2}$$

A linear algebraic equation in two unknowns x and y is an equation of the form:

$$3x-y=9$$

If we let y =0 then x = 3. So (x,y)=(3,0) is **a** solution of the above equation.

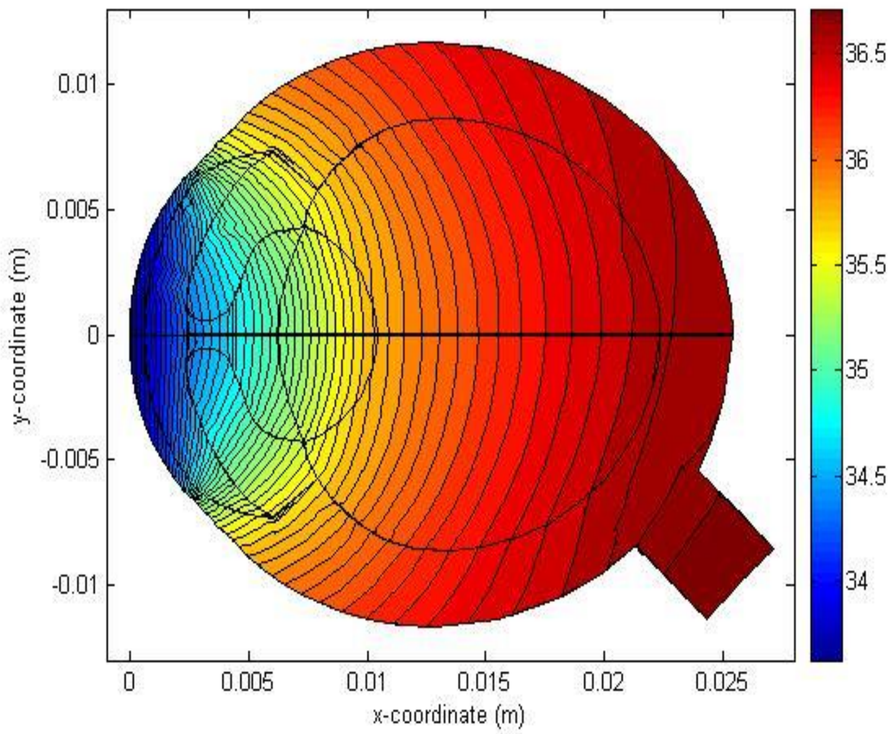
What’s a linear algebraic equation?

A linear algebraic equation in N unknowns x_1, x_2, …, $x_{N\text{-}1}$ and x_N is an equation of the form:

$$c_1 x_1 + c_2 x_2 + \ldots + c_N x_N = d_N$$

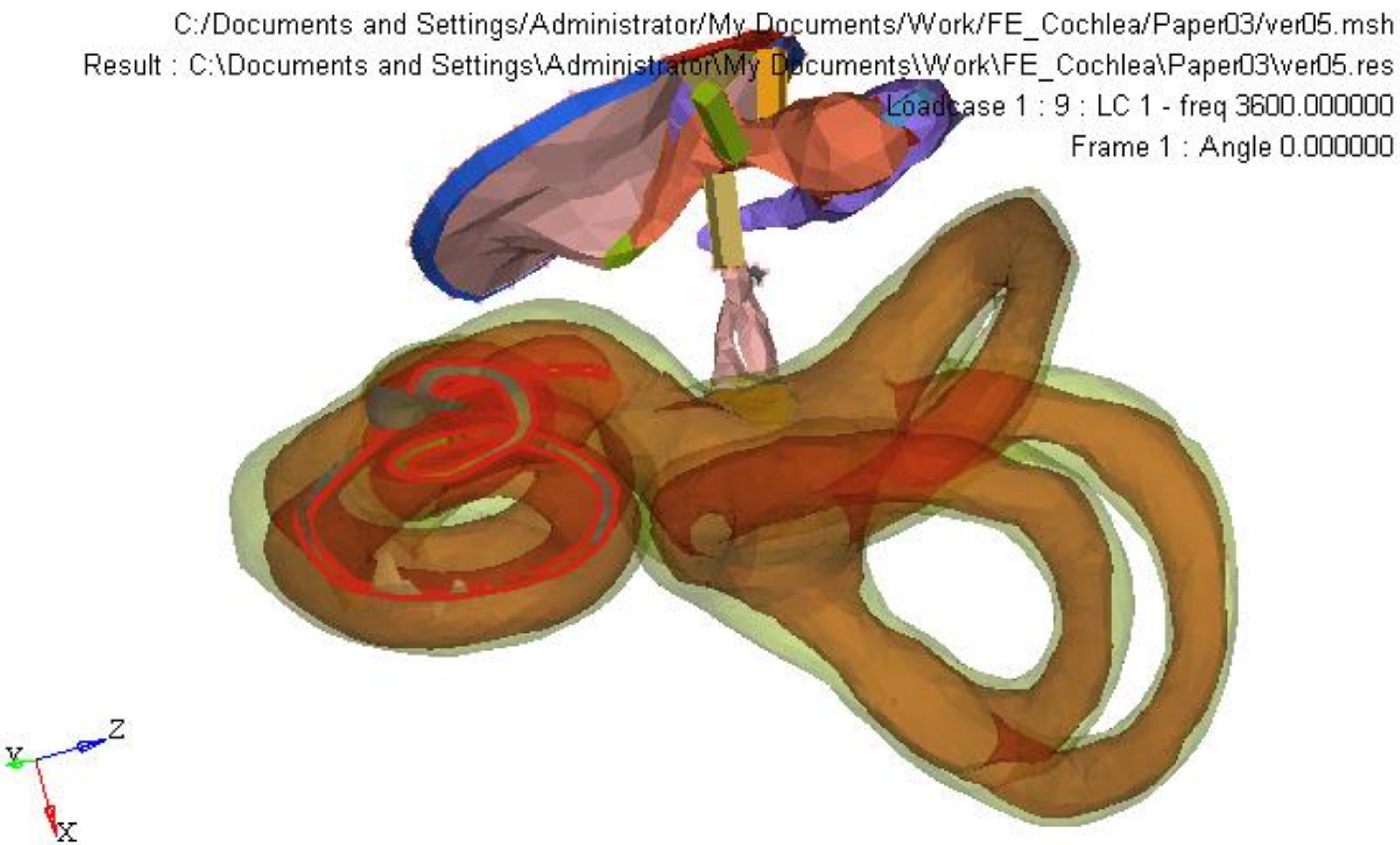
Why bother?

Many problems in engineering and physical sciences are formulated in terms of a *system* of linear algebraic equations.



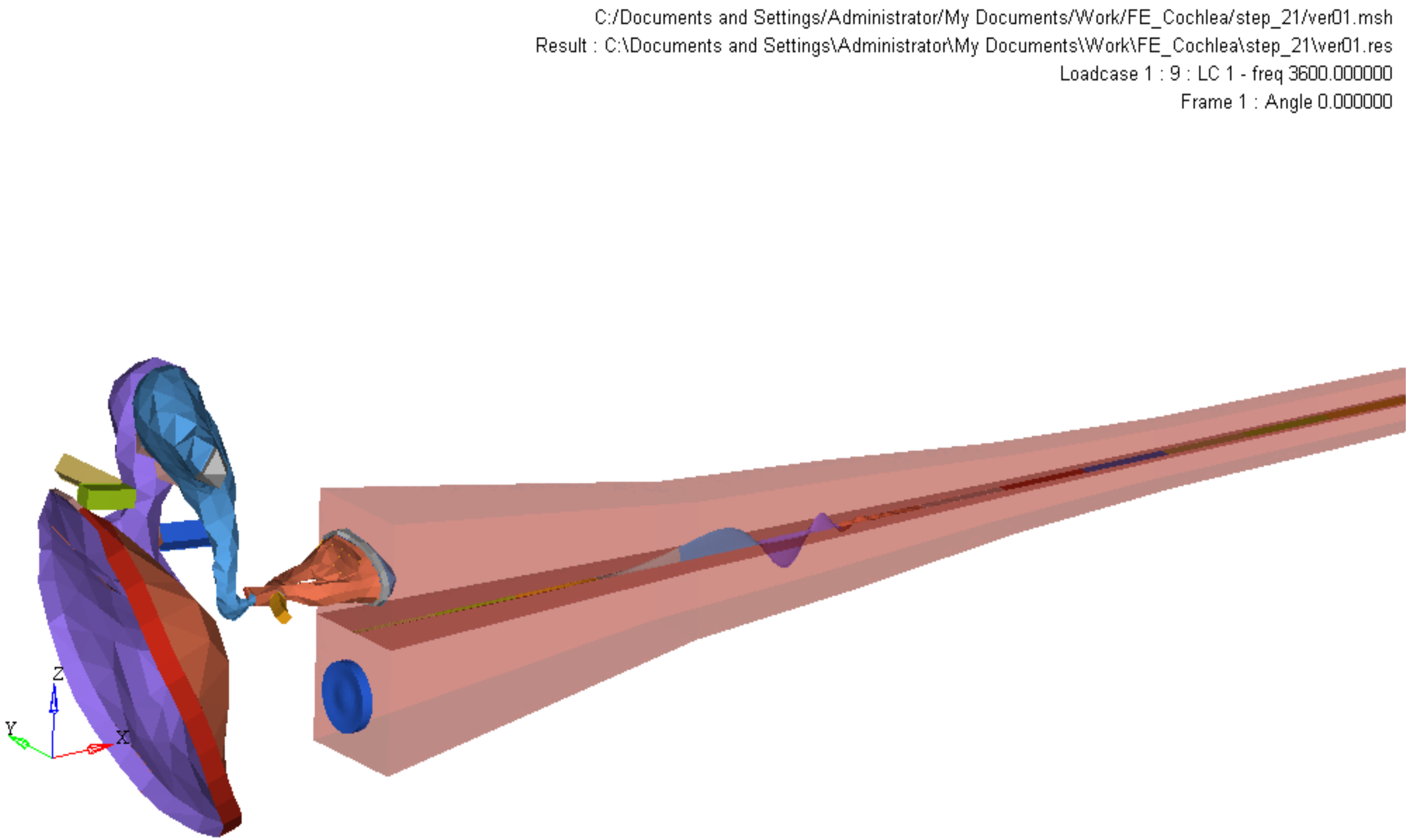
This temperature profile in the human eye was obtained from the boundary element method by solving a system of 470 linear algebraic equations in 470 unknowns.

Computational Human Ear – Real Model

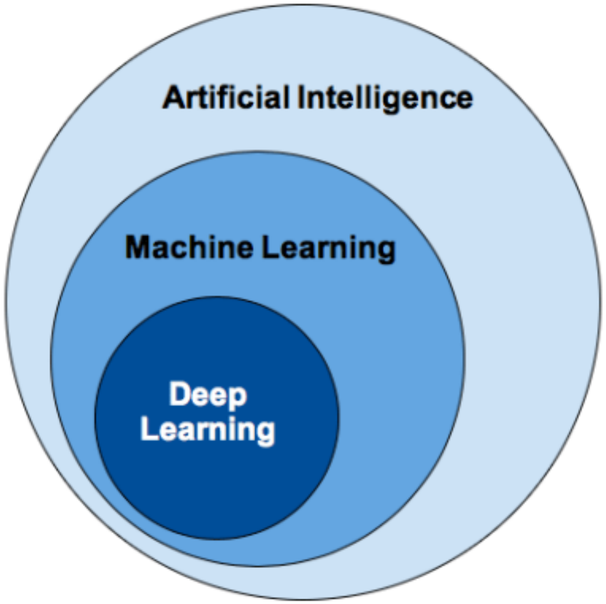


Computational Human Ear – Box Model

- ✓ **Acoustic FEM Simulation for 3.6 kHz**



Deep Learning, Machine Learning, Artificial Intelligence



Machine learning is an application of artificial intelligence (AI) that provides systems the ability to automatically learn and improve from experience without being explicitly programmed.

https://blog.knoldus.com/data-processing-and-using-ml-supervised-classification-algorithm-and-find-accuracy/

Here's a system of N linear algebraic equations in N unknowns.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1N}x_N = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2N}x_N = b_2 \\ \vdots \\ a_{N1}x_1 + a_{N2}x_2 + a_{N3}x_3 + \ldots + a_{NN}x_N = b_N \end{cases}$$

a_{ij} is the (constant) coefficient of the unknown x_j in the i-th equation.

b_i is the constant term in the i-th equation.

The system can be written in matrix form AX =B, where:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{12} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{pmatrix}$$

$$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix}$$

Example:

$$2x + 3y = 10$$
$$-x + y = 0$$

$$\begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2x + 3y \\ -x + y \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \end{pmatrix}$$

Solutions of a linear algebraic equation

Consider a single linear algebraic equation

$$c_1 x_1 + c_2 x_2 + \ldots + c_N x_N = d_N$$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{pmatrix}$$

is said to be a solution of the above linear algebraic equation if the LHS of the equation equals its RHS when we replace x_1, x_2, …, x_N by α_1, α_2, …, α_N respectively.

Example:

We can find many other solutions easily.

$$x + 2y - z = 0$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$ **is a solution but** $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$ **is not.**

Solution of a system of linear algebraic equations

If $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{pmatrix}$ is a solution of ***each and every*** linear algebraic equation in the system then it is said to be *a* **solution of the system**.

It is possible that a system of linear algebraic equations has no solution.

Example:

$$x + y = 10$$
$$x + y = 5$$

Such a system is said to be *inconsistent*.

It is possible that a consistent system has more than one solutions.

Example:

$$x + y = 10$$

$$2x + 2y = 20 \Rightarrow x + y = 10$$

The system really contains only one linear algebraic equation in 2 unknowns!

So we can find infinitely many solutions for the system.

If a consistent system of linear algebraic equations has only one solution, we say the system has a *unique solution.*

To summarise, a system of linear algebraic equations can either be consistent or inconsistent.

If it is consistent, it can have either a unique solution or infinitely many solutions.

Given a system of linear algebraic equations $\mathbf{AX} = \mathbf{B}$, how do we know whether it is consistent or not?

If it is consistent, how do we find all its solutions?

Reduce the system $AX = B$ to a simpler but equivalent system $UX = C$.

$AX = B$ and $UX = C$ are *equivalent* if they have exactly the same solution(s).

If we can work out the solution(s) of $UX = C$, we would have solved $AX = B$. 풀리면 풀린다

If the square matrix U is an upper triangular matrix, the system $UX = C$ would be simpler enough for us to work out its solution(s) (if any).

What is an upper triangular matrix?

E.g.

$$\begin{pmatrix} 1 & 4 & 5 & 6 \\ 0 & 1 & 6 & 8 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

To solve $\mathrm{AX} = \mathrm{B}$, reduce it to an equivalent system $\mathrm{UX} = \mathrm{C}$, where U is an upper triangular matrix.

How can this be done?

Write AX = B in tableau form A | B.

E.g.

$$
\begin{aligned}
2x+3y+4z&=6\\
3x+5y-2z&=7\\
x+10y+5z&=9
\end{aligned}
\quad \longrightarrow \quad
\left.\begin{array}{ccc|c}
2 & 3 & 4 & 6\\
3 & 5 & -2 & 7\\
1 & 10 & 5 & 9
\end{array}\right.
$$

Use <u>legitimate row operations</u> in a <u>systematic manner</u> to reduce the tableau to become U | C.

There are 2 types of legitimate row operations.

① $R_i \leftrightarrow R_j$ Interchange i-th and j-th rows.

② $R_i \to \alpha R_i + \beta R_j$ Use row j to change row i to become $\alpha R_i + \beta R_j$.

Important. The constant α is not allowed to be zero.

Why? Why $R_1 \to R_3$ (say) is not allowed?

A simple rule to observe

In changing a tableau by the second type of row operation, keep one row fixed. Use the fixed row to change other row(s).

$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 3 \\ -3 & -3 \end{bmatrix}$

Q1. find AB

$$AB = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$$

Q2. find BA

$$BA = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 8 & 8 \end{bmatrix}$$

Q3. find AC

$$AC = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 3 & 3 \\ -3 & -3 \end{bmatrix} = 3\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = 3\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Q4. find CA

$$CA = \begin{bmatrix} 3 & 3 \\ -3 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = 3\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = 3\begin{bmatrix} 3 & 3 \\ -3 & -3 \end{bmatrix} = \begin{bmatrix} 9 & 9 \\ -9 & -9 \end{bmatrix}$$