EECS 16A Designing Information Devices and Systems I
Fall 2018 Homework 9

This homework is due October 26, 2018, at 23:59.
Self-grades are due October 30, 2018, at 23:59.

Submission Format
Your homework submission should consist of **one** file.

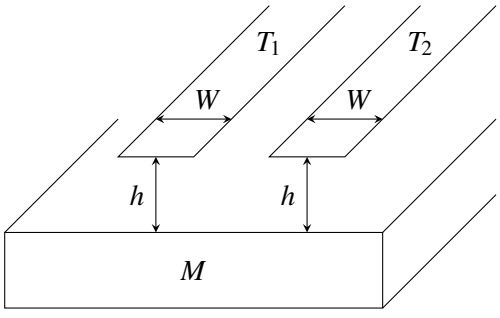
- `hw9.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned).

Submit the file to the appropriate assignment on Gradescope.

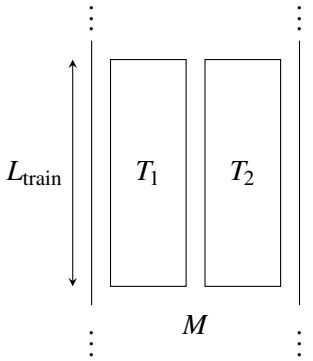
1. Maglev Train Height Control System

One of the fastest forms of land transportation are trains that actually travel slightly elevated from the ground using magnetic levitation (or "maglev" for short). Ensuring that the train stays at a relatively constant height above its "tracks" (the tracks in this case are what provide the force to levitate the train and propel it forward) is critical to both the safety and fuel efficiency of the train. In this problem, we'll explore how maglev trains use capacitors to stay elevated. (Note that real maglev trains may use completely different and much more sophisticated techniques to perform this function, so if you get a contract to build such a train, you'll probably want to do more research on the subject.)

(a) As shown below, we put two parallel strips of metal (T_1, T_2) along the bottom of the train and we have one solid piece of metal (M) on the ground below the train (perhaps as part of the track).



Side View



Top View

Assuming that the entire train is at a uniform height above the track and ignoring any fringing fields (i.e., we can use the simple equations developed in lecture to model the capacitance), as a function of L_{train} (the length of the train), W (the width of T_1 and T_2), and h (the height of the train off of the track), what is the capacitance between T_1 and M? What is the capacitance between T_2 and M?

Solution:

The distance between the plates (T_1 & M or T_2 & M) is h. The area of the parallel plate capacitor is $A = WL_{\text{train}}$. Using the formula for capacitance of a parallel plate capacitor, we get:

$$C = \frac{\varepsilon A}{d}$$

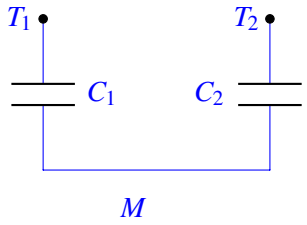
$$C_1 = \frac{\varepsilon W L_{\text{train}}}{h} \text{ (Capacitance between } T_1 \text{ and M)}$$

$$C_2 = \frac{\varepsilon W L_{\text{train}}}{h} \text{ (Capacitance between } T_2 \text{ and M)}$$

(b) Any circuit on the train can only make direct contact at T_1 and T_2. Thus, you can only measure the equivalent capacitance between T_1 and T_2. Draw a circuit model showing how the capacitors between T_1 and M and between T_2 and M are connected to each other.

Solution:

The capacitors C_1 and C_2 are in series. To realize this, let's consider the train circuit that is in contact with T_1 and T_2. If there is current entering plate T_1, the same current has to exit plate T_2. Thus, the circuit can be modeled as follows:



(c) Using the same parameters as in part (a), provide an expression for the equivalent capacitance between T_1 and T_2.

Solution:

Since the two capacitors are in series, the equivalent capacitance between T_1 and T_2 is given by:

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2}$$

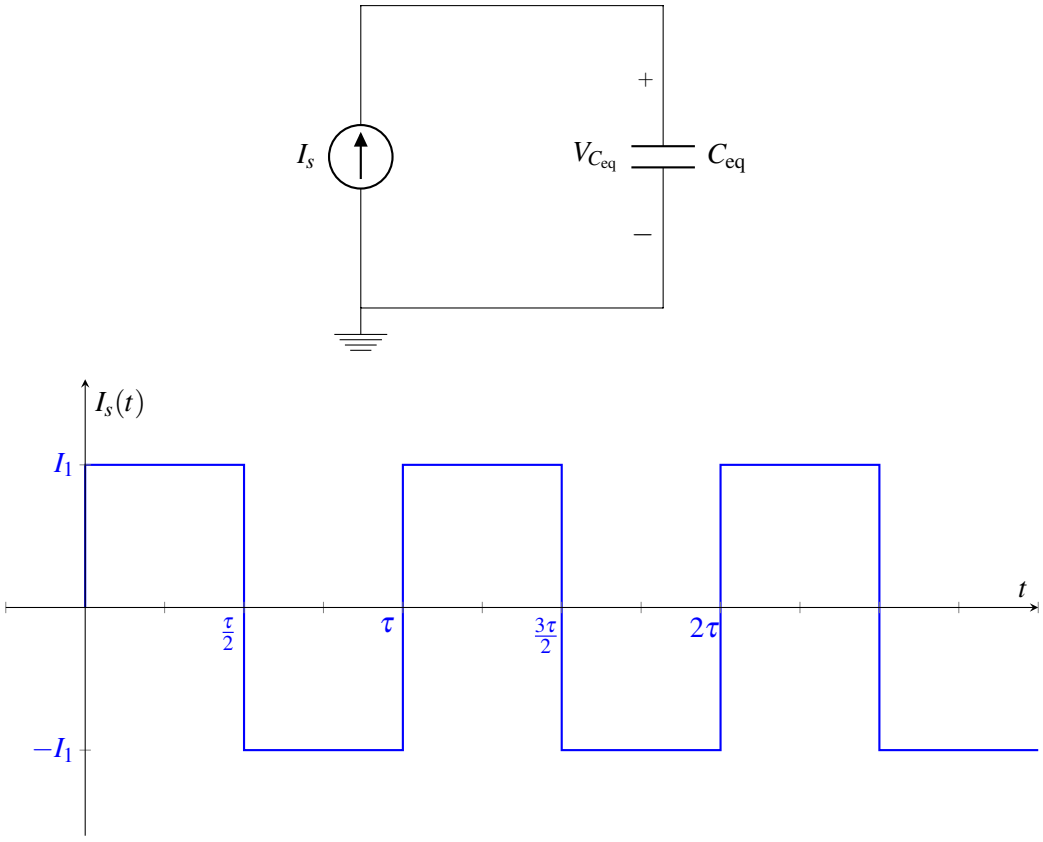
Thus, we get

$$\frac{1}{C_{\text{eq}}} = \frac{h}{\varepsilon W L_{\text{train}}} + \frac{h}{\varepsilon W L_{\text{train}}}$$

$$C_{\text{eq}} = \frac{\varepsilon W L_{\text{train}}}{2h}$$

(d) We want to build a circuit that creates a voltage waveform with an amplitude that changes based on the height of the train. Your colleague recommends you start with the circuit as shown below, where I_s is a periodic current source, and C_{eq} is the equivalent capacitance between T_1 and T_2. The graph below shows I_s, a square wave with period τ and amplitude I_1, as a function of time.

Find an equation for and draw the voltage $V_{C_{\text{eq}}}(t)$ as a function of time. Assume the capacitor C_{eq} is discharged at time $t = 0$, so $V_{C_{\text{eq}}}(0) = 0\,\text{V}$.



Solution: We know the rate of change of voltage across a capacitor is related to the the current into the capacitor. That is:

$$I_{C_{eq}} = C_{eq}\frac{dV_{C_{eq}}}{dt}$$

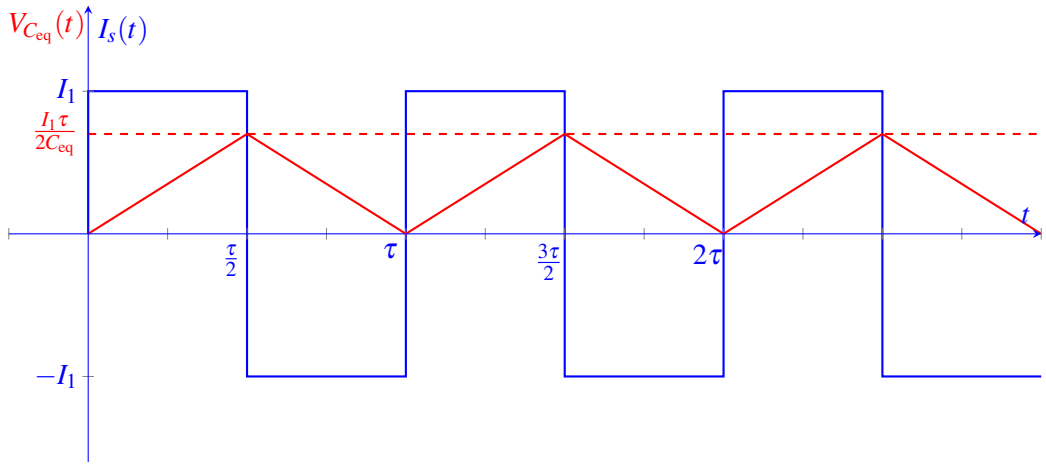
From KCL, we know $I_{C_{eq}} = I_s$. Then:

$$I_{C_{eq}} = I_s = C_{eq}\frac{dV_{C_{eq}}}{dt} \implies \frac{dV_{C_{eq}}}{dt} = \frac{I_s}{C_{eq}}$$

Since I_s is periodic, we can apply the procedure detailed in Note 17, Section 17.2.1 to get the following equation for $V_{C_{eq}}(t)$ for the first period, which repeats for subsequent periods. We recall that the capacitor is uncharged at $t = 0$ so that $V_{C_{eq}}(0) = 0\,\text{V}$.

$$V_{C_{eq}}(t) = \begin{cases} \frac{I_1}{C_{eq}}t & \text{when } 0 \le t \le \frac{\tau}{2} \\ \frac{-I_1}{C_{eq}}\left(t - \frac{\tau}{2}\right) + \frac{I_1\tau}{2C_{eq}} & \text{when } \frac{\tau}{2} < t \le \tau \end{cases}$$

Given this equation for the output voltage, $V_{C_{eq}}(t)$, as a function of the current, I_s, we can draw what the output waveform should look like.



(e) We now want to develop an indicator that alerts us when the train is too high above the tracks. We want to output a series of 5 V pulses that can be used to drive a horn when the train is above 1 cm, and not output anything when the train is below 1 cm.

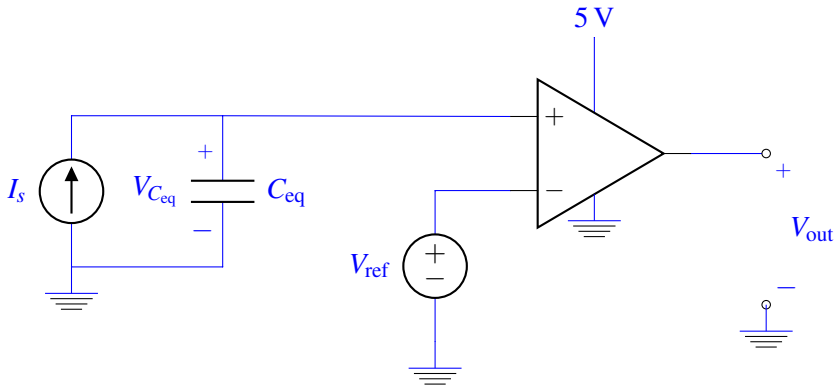
We will assume the train has length $L_{\text{train}} = 100\,\text{m}$ and that the metals, T_1 and T_2, have width $W = 1\,\text{cm}$ and permitivity $\varepsilon = 8.85 \times 10^{-12}\,\frac{\text{F}}{\text{m}}$.

Design a circuit using a square wave current source (i.e. I_s in part (d)) with period $\tau = 1\,\mu\text{s}$ and pulses of amplitude $I_1 = 1\,\text{mA}$, a comparator, and any number of voltage sources to implement this function.

Hint: you should use the circuit you analyzed in part (d).

Solution:

The circuit is shown below:



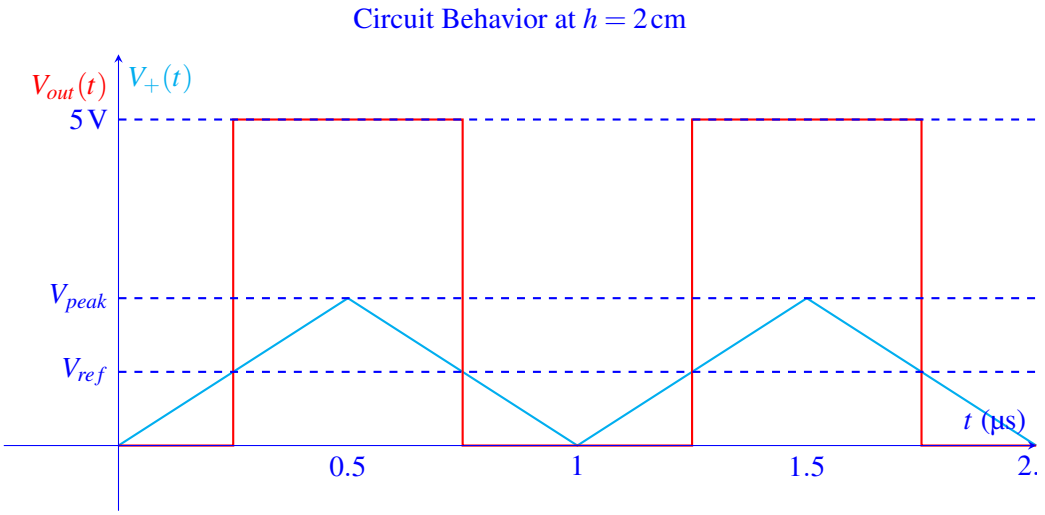
From the choice of supply voltages, we see that $V_{out} = 5\,\text{V}$ when $V_+ > V_-$.

We know the amplitude of $V_{C_{eq}}(t)$ when $h = 1\,\text{cm}$ is:

$$\frac{I_1\tau}{2C_{eq}} = \frac{1\,\text{mA}\cdot 1\,\mu\text{s}}{2\cdot C_{eq}} = \frac{1\,\text{mA}\cdot 1\,\mu\text{s}}{2\cdot\frac{\varepsilon W L_{\text{train}}}{2h}} = \frac{1\,\text{mA}\cdot 1\,\mu\text{s}\cdot h}{\varepsilon W L_{\text{train}}} = \frac{1\,\text{mA}\cdot 1\,\mu\text{s}\cdot 1\,\text{cm}}{8.85\times 10^{-12}\,\frac{\text{F}}{\text{m}}\cdot 1\,\text{cm}\cdot 100\,\text{m}} = 1.13\,\text{V}$$

Thus we can set V_{ref} to this peak value assuming the train is 1 cm above the ground. If the train's height is larger than 1 cm, the peak voltage rises, and we continue to get pulses. If the train's height is below 1 cm, the peak value is less than 1.13 V preventing any pulses from the output of the circuit.

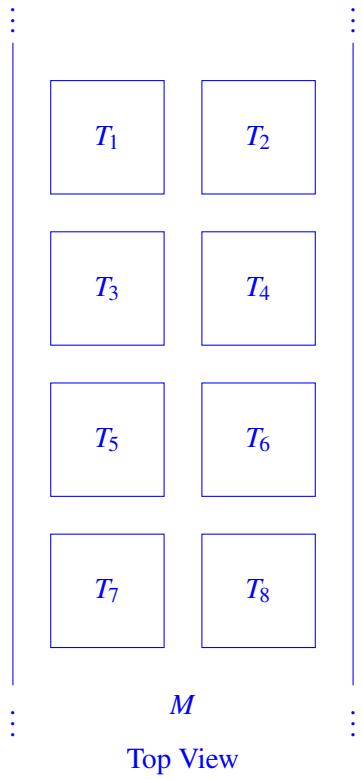
As an example, let's suppose the train's height is 2 cm. Then we would observe the following output for V_{out}. Note that the x-axis is in µs, that $V_{ref} = 1.13\,\text{V}$ as we found before, and that $V_{peak} = 2\cdot V_{ref} = 2.26\,\text{V}$. The cyan waveform is what we measure at V_+, and the red waveform is the 5 V pulse generated at V_{out}.



(f) So far we've assumed that the height of the train off of the track is uniform along its entire length, but in practice, this may not be the case. Suggest and sketch a modification to the basic sensor design (i.e., the two strips of metal T_1 and T_2 along the entire bottom of the train) that would allow you to measure the height at the train at 4 different locations.

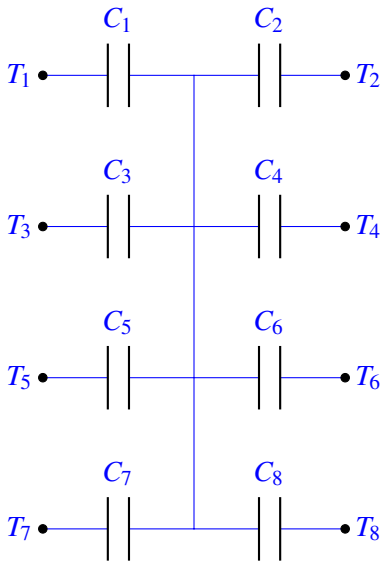
Solution:

One possible solution is shown below. Here we divide the T_1 and T_2 strips into many shorter strips.

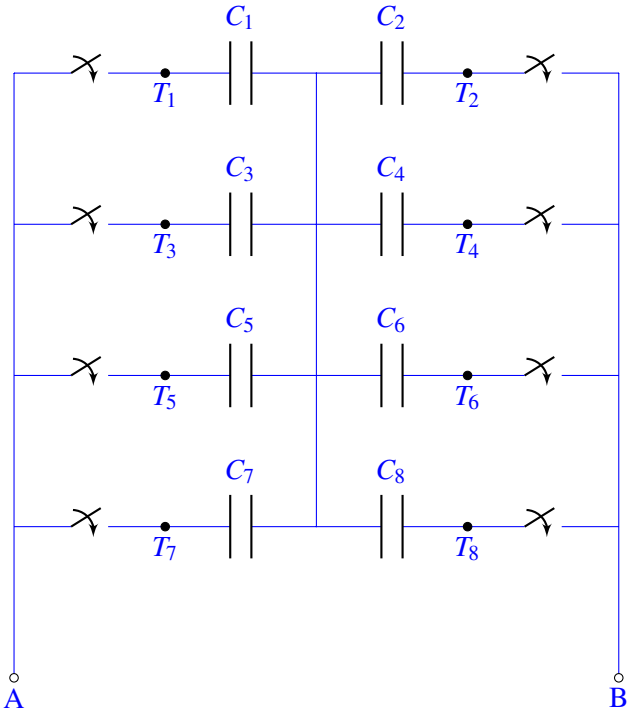


Top View

One important thing to note about this circuit is that it works only if extra care is taken during the capacitance measurement circuit. The equivalent model for this is:



Because all of the caps are connected together by the rail under the train, we need to have a way to select only some of the caps. We can accomplish this by having seperate switches on each T, so that you can measure the capacitance between only two terminals (like T_1 & T_2) and so that the effect of other capacitors is nullified. This is shown below, where the points A and B are where you connect your previous circuit.

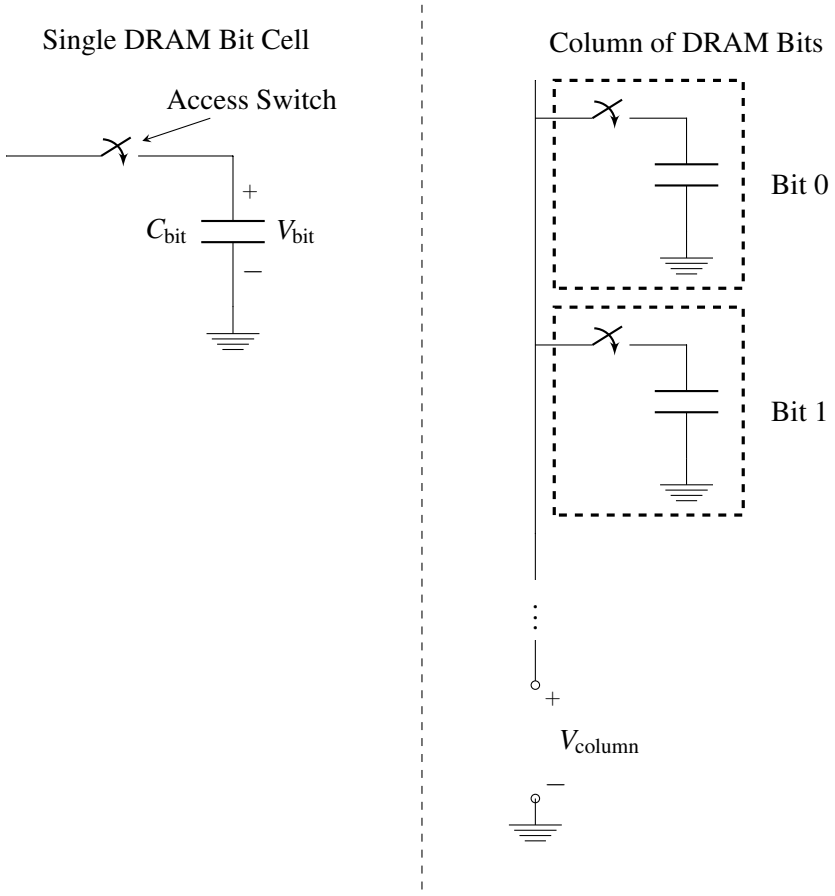


2. Dynamic Random Access Memory (DRAM)

Nearly all devices that include some form of computational capability (phones, tablets, gaming consoles, laptops, ...) use a type of memory known as Dynamic Random Access Memory (DRAM). DRAM is where the "working set" of instructions and data for a processor is typically stored, and the ability to pack an

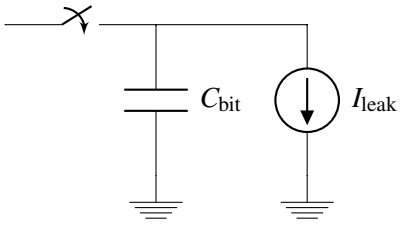
ever increasing number of bits on to a DRAM chip at low cost has been critical to the continued growth in computational capability of our systems. For example, a single DRAM chip today can store $>$ 8 billion bits and is sold for $\approx$ \$3-\$5.

At the most basic level and as shown below, every bit of information that a DRAM can store is associated with a capacitor. The amount of charge stored on that capacitor (and correspondingly, the voltage across the capacitor) sets whether a "1" or a "0" is stored in that location. As shown below, in order to pack as many bits together as possible on to a single chip, rather than running a massive number of wires to access every single bit of the DRAM individually, the bits are arranged into a set of columns, where each column uses a single wire to access information from one of the bits. By turning on the access switch within the particular bit cell via the single column wire, the corresponding bit is accessed (while leaving all of the switches in the rest of the cells off).



Building even on only what we've learned about capacitors so far, we can understand a lot about how DRAMs work and are designed. Thus, in this problem we will examine some of the issues and tradeoffs that actual DRAM designers deal with when engineering their products.

In any real capacitor, there is always a path for charge to "leak" off the capacitor and cause it to eventually discharge. In DRAMs, the dominant path for this leakage to happen is through the access switch, but let's ignore this for now and assume that this leakage can be modeled as shown below:



This leakage is actually responsible for the "D" in "DRAM" – the memory is "dynamic" because after a cell is written by storing some charge onto its capacitor, if you leave the cell alone for too long, the value you wrote in will disappear because the charge on the capacitor leaked away.

Let's now try to use some representative numbers to compute how long a DRAM cell can hold its value before the information leaks away. Let $C_{\text{bit}} = 18\,\text{fF}$ (note that $1\,\text{fF} = 1 \times 10^{-15}\,\text{F}$) and the capacitor be initially charged to $1.2\,\text{V}$ to store a "1." V_{bit} must be $> 0.8\,\text{V}$ in order for the circuits outside of the column to properly read the bit stored in the cell as a "1."

What is the maximum value of I_{leak} that would allow the DRAM cell retain its value for $> 1\,\text{ms}$?

Solution:

We want the time that a cell can read a '1' to be $t_{\text{store}} = 1\,\text{ms}$. We are given that $V_{\text{init}} = 1.2\,\text{V}$ and that $V_{\text{min}} = V_{\text{bit}} = 0.8\,\text{V}$. To get an expression for the leakage rate, we differentiate

$$Q_{\text{bit}} = V_{\text{bit}} C_{\text{bit}},$$

giving

$$I_{\text{leak}} = \frac{dV_{\text{bit}}}{dt} C_{\text{bit}}.$$

Assuming a constant leakage rate, we have

$$\begin{aligned} I_{\text{leak}} &= \frac{\Delta V_{\text{bit}}}{\Delta t} C_{\text{bit}} \\ &= \frac{V_{\text{init}} - V_{\text{min}}}{t_{\text{store}}} C_{\text{bit}}. \end{aligned}$$

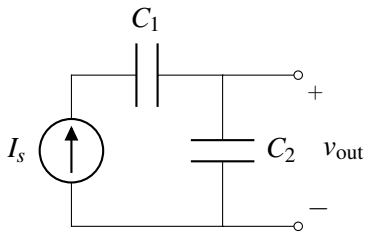
Plugging in the values from above, we get

$$\begin{aligned} I_{\text{leak}} &= \frac{(1.2\,\text{V} - 0.8\,\text{V}) \cdot 18\,\text{fF}}{1 \times 10^{-3}\,\text{s}} \\ &= 7.2\,\text{pA}. \end{aligned}$$

3. Current Sources And Capacitors

For the circuits given below, give an expression for $v_{\text{out}}(t)$ in terms of I_s, C_1, C_2, C_3, and t. Assume that all capacitors are initially uncharged, i.e. the initial voltage across each capacitor is $0\,\text{V}$.

(a)



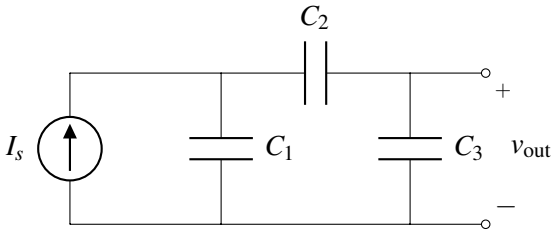
Solution:

By KCL, the current I_s flowing through C_1 must be the current flowing through C_2. $v_{\text{out}}(0) = 0$ because all capacitors are initially uncharged.

$$I_s = C_2 \frac{dv_{\text{out}}(t)}{dt}$$

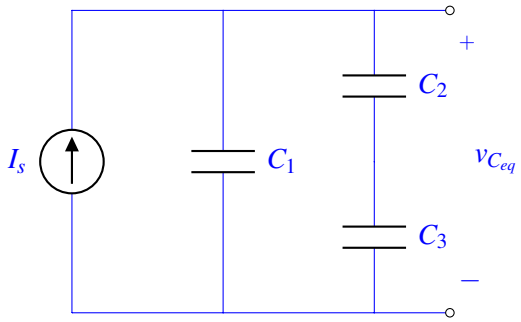
$$v_{\text{out}}(t) = \int \frac{I_s}{C_2} dt = \frac{I_s t}{C_2} + v_{\text{out}}(0) = \frac{I_s t}{C_2}$$

(b)

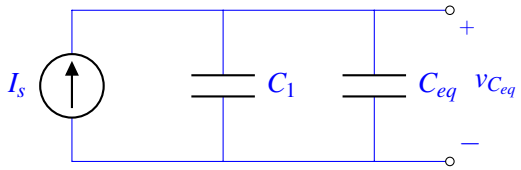


Solution:

Instead of finding v_{out} directly, let's first find the voltage $v_{C_{eq}}$ across C_2 and C_3.



To do this, we replace C_2 and C_3 with their equivalent capacitance $C_{eq} = C_2 \parallel C_3 = \frac{C_2 C_3}{C_2 + C_3}$.



We know that to solve for $v_{C_{eq}}$, we can find the equivalent capacitance of C_1 and C_{eq} first, which is $C_1 + C_{eq}$. Since the capacitors are initially uncharged, $v_{C_{eq}}(0) = 0$.

$$v_{C_{eq}}(t) = \int \frac{I_s}{C_1 + C_{eq}} dt = \frac{I_s t}{C_1 + C_{eq}} + v_{C_{eq}}(0) = \frac{I_s t}{C_1 + C_{eq}}$$

Now that we know that voltage across the equivalent capacitor C_{eq}, we can find the current flowing through the equivalent capacitor C_{eq}.

$$i_{C_{eq}}(t) = C_{eq} \frac{dv_{C_{eq}}(t)}{dt} = \frac{C_{eq} I_s}{C_1 + C_{eq}}$$

Note that the current $i_{C_{eq}}$ is equal to the current flowing through C_3 since C_2 and C_3 were originally connected in series.

$$i_{C_3}(t) = i_{C_{eq}}(t) = \frac{C_{eq} I_s}{C_1 + C_{eq}}$$

Since v_{out} is the voltage across the capacitor C_3, we integrate to find v_{out}. Again, since all capacitors are initially uncharged, $v_{\text{out}}(0) = 0$.

$$i_{C_3}(t) = C_3 \frac{dv_{\text{out}}(t)}{dt}$$

$$v_{\text{out}}(t) = \int \frac{C_{eq} I_s}{C_3(C_1 + C_{eq})} dt = \frac{C_{eq} I_s t}{C_3(C_1 + C_{eq})} + v_{\text{out}}(0) = \frac{\frac{C_2 C_3}{C_2 + C_3} I_s t}{C_3 \left(C_1 + \frac{C_2 C_3}{C_2 + C_3} \right)} = \frac{C_2 I_s t}{C_1 C_2 + C_1 C_3 + C_2 C_3}$$

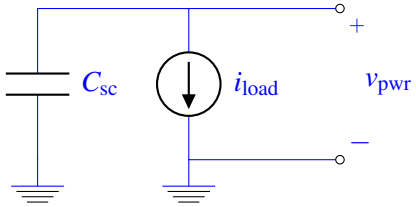
4. Super-Capacitors

In order to enable small devices for the "Internet of Things" (IoT), many companies and researchers are currently exploring alternative means of storing and delivering electrical power to the electronics within these devices. One example of these are "super-capacitors" - the devices generally behave just like a "normal" capacitor but have been engineered to have extremely high values of capacitance relative to other devices that fit in to the same physical volume.

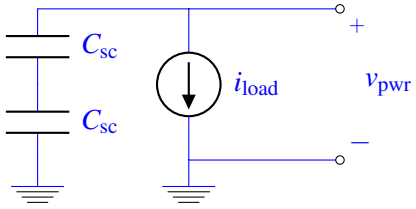
Your startup named **IoT4eva** is designing a new device that will revolutionize the process of making pizza, and you've been put in charge of selecting an energy source for it. You can't find a battery that quite suits your needs, so you decide to try out some super capacitors in various configurations. The super capacitors will be charged up to a certain voltage in the factory and will then act as the power supply (source of voltage) for the electronics in your device.

(a) Assuming that your electronic device can be modeled as drawing a constant current source with a value of i_{load}, draw circuit models for your device using super-capacitors as the power supply with the following configurations:

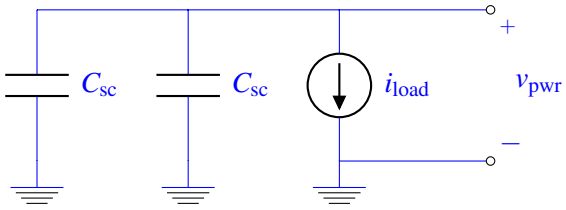
- Config 1: a single super-capacitor as the power supply
  Solution:



- Config 2: two super-capacitors stacked in series as the power supply
  Solution:



- Config 3: two super-capacitors connected in parallel as the power supply
  Solution:



(b) If each super-capacitor is charged to an initial voltage v_{init} and has a capacitance of C_{sc}, for each of the three configurations above, write an expression for the voltage supplied to your electronic device as a function of time after the device has been activated (i.e. connected to the super-capacitor(s)).

Solution:

Let the initial voltage each super capacitor is charged to be v_{init}. We'll now consider the three situations:

- Config 1: **Single super capacitor**
  In this case, the initial voltage that the super capacitor provides is v_{init}, and the intial charge stored in it is then given by $Q_{init} = v_{init}C_{sc}$. Let the voltage at any time, t be defined by $v_{pwr}(t)$. The charge drained by the constant current source, i_{load} in time t is given by $i_{load}t$. The effective charge stored in the capacitor after time t is given by $Q(t) = Q_{init} - i_{load}t$. Therefore,

$$\begin{aligned} v_{pwr}(t) &= \frac{Q(t)}{C_{sc}} \\ &= \frac{Q_{init} - i_{load}t}{C_{sc}} \\ &= \frac{v_{init}C_{sc} - i_{load}t}{C_{sc}} \\ &= v_{init} - \frac{i_{load}t}{C_{sc}} \end{aligned}$$

- Config 2: **Two super capacitors in series**
  In this case, the initial voltage that the effective super capacitor provides is $2v_{init}$, and the effective capacitance is $C_{eq} = \frac{C_{sc}}{2}$. Then, the initial effective charge stored in them is then given by $Q_{init} = 2v_{init}C_{eq} = 2v_{init}\frac{C_{sc}}{2} = v_{init}C_{sc}$. Let the voltage at any time, t be defined by $v_{pwr}(t)$. The charge drained by the constant current source in time t is given by $i_{load}t$. The effective charge stored in the combination after time t is given by $Q(t) = Q_{init} - i_{load}t$. Therefore,

$$\begin{aligned} v_{pwr}(t) &= \frac{Q(t)}{C_{eq}} \\ &= \frac{Q_{init} - i_{load}t}{C_{eq}} \\ &= \frac{v_{init}C_{sc} - i_{load}t}{C_{eq}} \\ &= 2v_{init} - \frac{2i_{load}t}{C_{sc}} \end{aligned}$$

- Config 3: **Two super capacitors in parallel**
  In this case, the initial voltage that the effective super capacitor provides is v_{init}, and the effective capacitance is $C_{eq} = 2C_{sc}$. Then, the initial effective charge stored in them is then given by $Q_{init} = v_{init}C_{eq} = 2v_{init}C_{sc}$. Let the voltage at any time, t be defined by $v_{pwr}(t)$. The charge drained by the constant current source in time t is given by $i_{load}t$. The effective charge stored in the combination

after time t is given by $Q(t) = Q_{\text{init}} - i_{\text{load}}t$. Therefore,

$$\begin{aligned} v_{\text{pwr}}(t) &= \frac{Q(t)}{C_{\text{eq}}} \\ &= \frac{Q_{\text{init}} - i_{\text{load}}t}{C_{\text{eq}}} \\ &= \frac{2v_{\text{init}}C_{\text{sc}} - i_{\text{load}}t}{C_{\text{eq}}} \\ &= v_{\text{init}} - \frac{i_{\text{load}}t}{2C_{\text{sc}}} \end{aligned}$$

(c) Now let's assume that your electronic device requires some minimum voltage v_{min} in order to function properly. For each of the three super-capacitor configurations, write an expression of the lifetime of the device.

Solution:

The lifetime of a device is the time it takes for the $v_{\text{pwr}}(t)$ to hit the threshold v_{min}. For each of the three configurations, let's find out their lifetime (denoted by t_0):

- Config 1: **Single super capacitor**
  Let us calculate at what time $t = t_0$, $v_{\text{pwr}}(t)$ equals v_{min}. We know from the previous part that

  $$v_{\text{pwr}}(t) = v_{\text{init}} - \frac{i_{\text{load}}t}{C_{\text{sc}}}.$$

  Substituting $v_{\text{pwr}}(t) = v_{\text{min}}$, we get

  $$t_0 = \frac{(v_{\text{init}} - v_{\text{min}})C_{\text{sc}}}{i_{\text{load}}}.$$

- Config 2: **Two super capacitors in series**
  Let us calculate at what time $t = t_0$, $v_{\text{pwr}}(t)$ equals v_{min}. We know from the previous part that

  $$v_{\text{pwr}}(t) = 2v_{\text{init}} - \frac{2i_{\text{load}}t}{C_{\text{sc}}}.$$

  Substituting $v_{\text{pwr}}(t) = v_{\text{min}}$, we get

  $$t_0 = \frac{(2v_{\text{init}} - v_{\text{min}})C_{\text{sc}}}{2i_{\text{load}}}.$$

- Config 3: **Two super capacitors in parallel**
  Let us calculate at what time $t = t_0$, $v_{\text{pwr}}(t)$ equals v_{min}. We know from the previous part that

  $$v_{\text{pwr}}(t) = v_{\text{init}} - \frac{i_{\text{load}}t}{2C_{\text{sc}}}.$$

  Substituting $v_{\text{pwr}}(t) = v_{\text{min}}$, we get

  $$t_0 = \frac{(v_{\text{init}} - v_{\text{min}})2C_{\text{sc}}}{i_{\text{load}}}.$$

Note: We could have also figured it out by finding out how much charge needs to be removed to cause the voltage at the effective capacitance to drop to v_{min}. Thus, we have

$$\Delta Q = (v_{\text{init}} - v_{\text{min}})C_{\text{eq}},$$

which gives us

$$t_0 = \frac{\Delta Q}{i_{\text{load}}}.$$

(d) Assume that a single super-capacitor doesn't provide you sufficient lifetime and so you have to spend the extra money (and device volume) for another super-capacitor. You consider the two following configurations:

- Config 2: two super-capacitors stacked in series
- Config 3: two super-capacitors connected in parallel

When is Config 3 (parallel) better than Config 2 (series)? Your answer should involve conditions on v_{init} and v_{min}.

Solution:

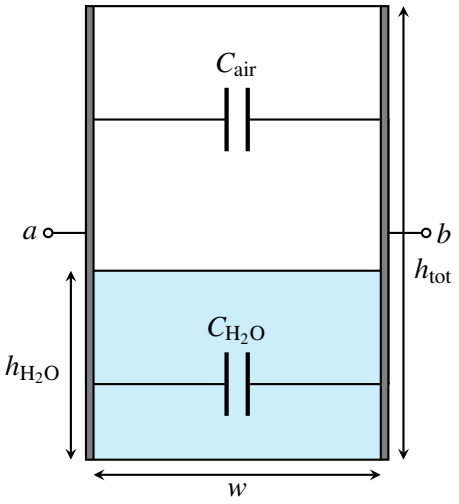
It is not obvious from the previous part whether configuration 2 or 3 will provide a longer lifetime. In fact, it depends on what v_{init} is with respect to v_{min}. Let us now see what conditions we need on v_{init}, such that the parallel configuration provides a longer lifetime, i.e., $t_{0,\text{ parallel}} > t_{0,\text{ series}}$. From the previous part, we get

$$\begin{aligned}
\frac{(v_{\text{init}} - v_{\text{min}})2C_{\text{sc}}}{i_{\text{load}}} &\geq \frac{(2v_{\text{init}} - v_{\text{min}})C_{\text{sc}}}{2i_{\text{load}}} \\
2(v_{\text{init}} - v_{\text{min}}) &\geq v_{\text{init}} - \frac{v_{\text{min}}}{2} \\
v_{\text{init}} &\geq \frac{3}{2}v_{\text{min}}
\end{aligned}$$

Thus, we see that when $v_{\text{init}} \geq \frac{3}{2}v_{\text{min}}$, the parallel configuration is better; otherwise the series configuration is better.

5. It's finally raining!

A lettuce farmer in Salinas Valley has grown tired of weather.com's imprecise rain measurements. Therefore, they decided to take matters into their own hands by building a rain sensor. They placed a rectangular tank outside and attached two metal plates to two opposite sides in an effort to make a capacitor whose capacitance varies with the amount of water inside.



The width and length of the tank are both w (i.e., the base is square) and the height of the tank is h_{tot}.

(a) What is the capacitance between terminals a and b when the tank is full? What about when it is empty? *Note:* the permittivity of air is ε, and the permittivity of rainwater is 81ε.

Solution:

Capacitance of parallel plates is governed by the equation:

$$C = \frac{\varepsilon A}{d},$$

where ε is the *permittivity* of the dielectric material, A is the area of the plates, and d is the distance between the plates. If we apply this to our physical structure, we find that the area of the plates are $h_{\text{tot}} \cdot w$, and the distance between the plates is w. The only difference here between a full and empty tank is the permittivity of the material between the two plates.

$$C_{\text{empty}} = \frac{\varepsilon_{\text{air}} h_{\text{tot}} w}{w} = \varepsilon h_{\text{tot}}$$

$$C_{\text{full}} = \frac{\varepsilon_{H_2O} h_{\text{tot}} w}{w} = 81\varepsilon h_{\text{tot}}$$

(b) Suppose the height of the water in the tank is h_{H_2O}. Modeling the tank as a pair of capacitors in parallel, find the total capacitance between the two plates. Call this capacitance C_{tank}.

Solution:

We can break the total capacitance into two parts. First, let's calculate the capacitance of the two plates separated by water:

$$C_{\text{water}} = \frac{\varepsilon_{H_2O} h_{H_2O} w}{w} = 81\varepsilon h_{H_2O}$$

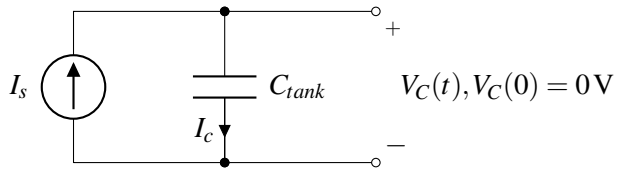
And now we can calculate the capacitance of the two plates separated by air:

$$C_{\text{air}} = \frac{\varepsilon_{\text{air}} (h_{\text{tot}} - h_{H_2O}) w}{w} = \varepsilon (h_{\text{tot}} - h_{H_2O})$$

Because these two capacitors appear in parallel, we can simply add our two previous results to find the total equivalent capacitance:

$$C_{\text{tank}} = C_{\text{water}} + C_{\text{air}} = \varepsilon (h_{\text{tot}} + 80 h_{H_2O})$$

(c) After building this capacitor, the farmer consults the internet to assist them with a capacitance-measuring circuit. A fellow internet user recommends the following:



In this circuit, C_{tank} is the total tank capacitance that you calculated earlier. I_s is a known current supplied by a current source.

The suggestion is to measure V_C for a brief interval of time, and then use the difference to determine C_{tank}.

Determine $V_C(t)$, where t is the number of seconds elapsed since the start of the measurement. You should assume that before any measurements are taken, the voltage across C_{tank}, i.e. V_C, is initialized to 0 V, i.e. $V_C(0) = 0$.

Solution: The element equation for the capacitor is:

$$I_C = C_{tank}\frac{dV_C}{dt}$$

We also know from KCL that:

$$I_C = I_s$$

Thus, we get the following differential equation for V_C:

$$\frac{dV_C}{dt} = \frac{I_s}{C_{tank}}$$

We recall that I_s and C_{tank} are constant values and the initial value of V_C is zero ($V_C(0) = 0$). Applying these facts and integrating the differential equation, we get the following equation for V_C:

$$V_C(t) = \frac{I_s}{C_{tank}}t$$

(d) Using the equation you derived for $V_C(t)$, describe how you can use this circuit to determine C_{tank} and $h_{\text{H}_2\text{O}}$.

Solution: We connect the current source providing I_s A to the capacitor C_{tank}. At the same time, we can measure $V_C(t)$. After some time passes, we measure $V_C(t)$ and plug it into the following equation (assuming, as before, that $V_C(0) = 0$):

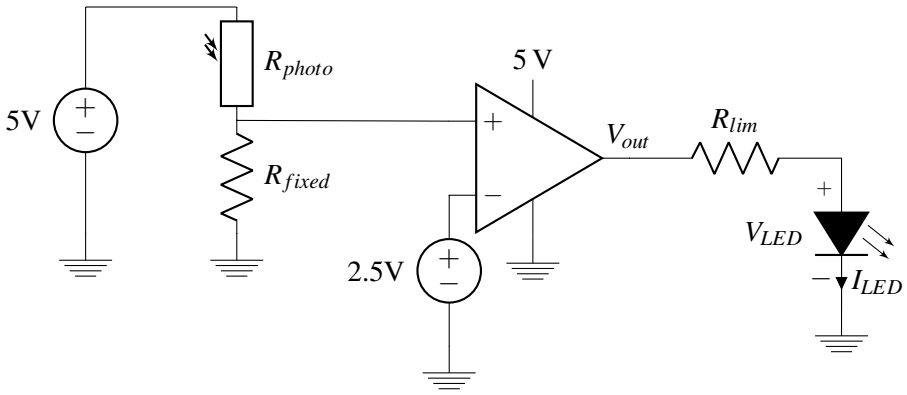
$$C_{tank} = \frac{I_s}{V_C(t)}t$$

If we know C_{tank}, we can determine $h_{\text{H}_2\text{O}}$. Using the equation derived in part (b), we see that

$$h_{\text{H}_2\text{O}} = \frac{C_{tank} - h_{tot}\varepsilon}{80\varepsilon}$$

6. LED Alarm Circuit

One day, you come back to your dorm to find that your favorite candy has been stolen. Determined to catch the perpetrator red-handed, you decide to put the candy inside a kitchen drawer. Using the following circuit design, you would like to turn on a light-emitting diode (LED) "alarm" if the kitchen drawer is opened.



Note R_{photo} is a photoresistor, which acts like a typical resistor but changes resistance based on the amount of light it is exposed to. This photoresistor is located inside the kitchen drawer, so we can tell when the drawer is opened or closed.

V_{LED} indicates the voltage across the LED; we will guide you through the IV behavior of this element later in the problem. The LED is located in your room (and connected to a long wire going to the kitchen), so that you can remotely tell when the kitchen drawer has been opened.

(a) We see that the op-amp is in a comparator configuration. What is V_+, the voltage at the positive voltage input? Your answer should be written in terms of R_{photo} and R_{fixed}.

Solution: V_+ is the output of a voltage divider:

$$V_+ = \frac{R_{fixed}}{R_{fixed} + R_{photo}} \cdot 5\,\text{V}$$

(b) We now want to choose a value for R_{fixed}. From the photoresistor's datasheet, we see the resistance in "light" conditions (i.e. drawer open) is $1\,\text{k}\Omega$. In "dark" conditions (i.e. drawer closed), the resistance is $10\,\text{k}\Omega$.

To ensure the comparator detects the light condition with more tolerance, we decide to design R_{fixed} so that V_+ is 3 V under the "light" condition. Solve for the value of R_{fixed} to meet this specification.

Solution: We start from the voltage divider equation we derived in the previous part:

$$V_+ = \frac{R_{fixed}}{R_{fixed} + R_{photo}} \cdot 5\,\text{V}$$

Now we plug in the known values, $V_+ = 3\,\text{V}$ and $R_{photo} = 1\,\text{k}\Omega$.

$$3\,\text{V} = \frac{R_{fixed}}{R_{fixed} + 1000\Omega} \cdot 5\,\text{V}$$

Solving this equation, we get $R_{fixed} = 1.5\,\text{k}\Omega$.

(c) Write down V_{out} with any conditions in terms of V_+. For simplicity, consider the case when $V_+ \neq V_-$ and assume the op-amp is ideal.

Solution:

Since the op-amp is ideal and in a comparator configuration, we know that V_{out} will be the voltage at either the positive rail (5 V) or at the negative rail (0 V) when $V_+ \neq V_-$. Which voltage depends on if V_+ is greater than V_- or not. Since V_- is 2.5 V, we get the following piecewise equation for V_{out}:

$$V_{out} = \begin{cases} 5\,\text{V}, & V_+ > 2.5\,\text{V} \\ 0\,\text{V}, & V_+ < 2.5\,\text{V} \end{cases}$$

(d) Using your answers to the previous parts, write down V_{out} with the conditions on its output in terms of R_{photo}. You can substitute the value of R_{fixed} you found in part (b). As before, you can assume that $V_+ \neq V_-$ and the op-amp is ideal.

Solution:

We substitute the equations for V_+ into the equation for V_{out}:

$$V_{out} = \begin{cases} 5\,\text{V}, & \frac{R_{fixed}}{R_{fixed}+R_{photo}} \cdot 5\,\text{V} > 2.5\,\text{V} \\ 0\,\text{V}, & \frac{R_{fixed}}{R_{fixed}+R_{photo}} \cdot 5\,\text{V} < 2.5\,\text{V} \end{cases}$$

Plugging in $R_{fixed} = 1.5\,\text{k}\Omega$ from part (b), we can get the following in terms of R_{photo}:

$$V_{out} = \begin{cases} 5\,\text{V}, & R_{photo} < 1.5\,\text{k}\Omega \\ 0\,\text{V}, & R_{photo} > 1.5\,\text{k}\Omega \end{cases}$$

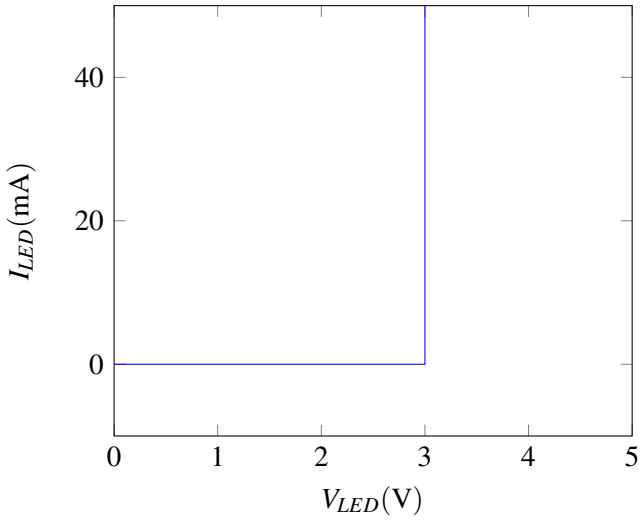
(e) From the design steps in the previous parts, we have designed a circuit that outputs non-zero voltage when the photoresistor is exposed to light (i.e. kitchen drawer open). We now want to design the LED portion of the circuit, so we get a visual alarm when the drawer is open.

From the LED's datasheet, the forward voltage, V_F is 3 V. Essentially, if V_{LED} is less than this voltage, the LED won't light up and I_{LED} will be 0 A.

Here is an idealized IV curve of this LED. The LED behaves in one of the following two modes:

i. If the voltage across the LED is less than $V_F = 3\,\text{V}$ or if $I_{LED} < 0\,\text{A}$, then the LED acts like an open circuit.

ii. If the voltage across the LED is $V_F = 3\,\text{V}$, then the LED acts like a voltage source, except that it only allows positive current flow (i.e. only in the direction of current marked on the circuit diagram).



To avoid exceeding the power rating of the LED (and having it burn out), the recommended value for I_{LED} is 20 mA.

Find the value of the current-limiting resistor, R_{lim}, such that when the photoresistor is in the "light" condition, $I_{LED} = 20\,\text{mA}$.

Solution: When the photoresistor is in the "light" condition, $R_{photo} = 1\,\text{k}\Omega$ so that $V_{out} = 5\,\text{V}$ per the analysis in the previous part. This implies that $V_{LED} = V_F$ and the LED acts like a power supply with positive current flow when in the "light" condition.

Using Ohm's Law and noting that the same current passes through R_{lim} and the LED itself,

$$V_{out} - V_F = I_{LED} R_{lim}$$

Rearranging and plugging in values when in the "light" condition:

$$R_{lim} = \frac{V_{out} - V_F}{I_{LED}}$$

$$R_{lim} = \frac{5 - 3\ \text{V}}{0.02\ \text{A}}$$

$$R_{lim} = 100\,\Omega$$

Note that when $V_{out} < 3\,\text{V}$, the LED will not light up and I_{LED} will be 0 mA. Thus by our design of the voltage divider, we were able to ensure the LED lights up only if the drawer is opened.

7. Homework Process and Study Group

Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.) How did you work on this homework?

Solution:

I worked on this homework with...

I first worked by myself for 2 hours, but got stuck on problem 5, so I went to office hours on...

Then I went to homework party for a few hours, where I finished the homework.