EECS 16A Designing Information Devices and Systems I
Fall 2018 Discussion 3B

1. Commutativity of Operations

You've learned about matrices as transformations, and so a question that we might have is: Does the *order* in which you apply operations matter? We'll be working with a unit square as an object we're going to transform. Follow your TA to obtain the answers to the following questions!

(a) Let's see what happens to the unit square when we rotate the square by $60°$ and then reflect it along the y-axis.

(b) Now, let's see what happens to the unit square when we first reflect the square along the y-axis and then rotate it by $60°$.

Answer: (For parts (a) and (b)): The two operations are not the same.

(c) Try to do steps (a) and (b) by multiplying the reflection and rotation matrices together (in the correct order for each case). What does this tell you?

Answer:

The resulting matrices that are obtained (by multiplying the two matrices) are different depending on the order of multiplication.

(d) If you reflected the unit square twice (along any pair of axes), do you think the order in which you applied the reflections would matter? Why/why not?

Answer:

It turns out that reflections are not commutative unless the two reflection axes are perpendicular to each other. For example, if you reflect about the x-axis and the y-axis, it is commutative. But if you reflect about the x-axis and $x = y$, it is not commutative.

2. Span Proofs

Given some set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, show the following:

(a)

$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \text{span}\{\alpha\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}, \text{ where } \alpha \text{ is a non-zero scalar}$$

In other words, we can scale our spanning vectors and not change their span.

(b)

$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \text{span}\{\vec{v}_2, \vec{v}_1, \ldots, \vec{v}_n\}$$

In other words, we can swap the order of our spanning vectors and not change their span.

(c) **[Practice Problem]:**

$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$$

In other words, we can replace one vector with the sum of itself and another vector and not change their span.

Answer:

(a) Suppose we have some arbitrary $\vec{q} \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. For some scalars a_i:

$$\vec{q} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = \left(\frac{a_1}{\alpha}\right)\alpha\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n.$$

Scalar multiplication cancels out. Thus, we have shown that $\vec{q} \in \text{span}\{\alpha\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Therefore, we have $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \subseteq \text{span}\{\alpha\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Now, we must show the other direction. Suppose we have some arbitrary $\vec{r} \in \text{span}\{\alpha\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. For some scalars b_i:

$$\vec{r} = b_1(\alpha\vec{v}_1) + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n = (b_1\alpha)\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n.$$

Thus, we have shown that $\vec{r} \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Therefore, we now have $\text{span}\{\alpha\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \subseteq \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Combining this with the earlier result, the spans are thus the same.

(b) Suppose $\vec{q} \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. For some scalars a_i:

$$\vec{q} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = a_2\vec{v}_2 + a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$$

Swapping the order in addition does not affect the sum, so $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \subseteq \text{span}\{\vec{v}_2, \vec{v}_1, \ldots, \vec{v}_n\}$. Similarly, starting with some $\vec{r} \in \text{span}\{\vec{v}_2, \vec{v}_1, \ldots, \vec{v}_n\}$, again swapping the order does not affect the sum, so putting both together, the spans are thus the same.

(c) Suppose $\vec{q} \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. For some scalars a_i:

$$\vec{q} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = a_1(\vec{v}_1 + \vec{v}_2) + (-a_1 + a_2)\vec{v}_2 + \cdots + a_n\vec{v}_n$$

We can change the scalar values to adjust for the combined vectors. Thus, we have shown that $\vec{q} \in \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$. Therefore, we have $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \subseteq \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$. Now, we must show the other direction. Suppose we have some arbitrary $\vec{r} \in \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$. For some scalars b_i:

$$\vec{r} = b_1(\vec{v}_1 + \vec{v}_2) + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n = b_1\vec{v}_1 + (b_1 + b_2)\vec{v}_2 + \cdots + b_n\vec{v}_n.$$

Thus, we have shown that $\vec{r} \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Therefore, we have $\text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\} \subseteq \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Combining this with the earlier result, the spans are thus the same.

3. Proofs

(a) **[Practice Problem]:** Suppose for some non-zero vector $\vec{x}$, $\mathbf{A}\vec{x} = \vec{0}$. Prove that the columns of $\mathbf{A}$ are linearly dependent.

Answer:

Begin by defining column vectors $\vec{a}_1 \ldots \vec{a}_n$.

$$\mathbf{A} = \begin{bmatrix} | & | & | & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & | & | \end{bmatrix}$$

Thus, we can represent the multiplication $\mathbf{A}\vec{x}$ as

$$\begin{bmatrix} | & | & | & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & | & | \end{bmatrix} \begin{bmatrix} | \\ \vec{x} \\ | \end{bmatrix} = \sum x_i\vec{a}_i = \vec{0}$$

Note that the equation above is the definition of linear dependence. That is, there exist coefficients, at least one which is non-zero, such that the sum of the vectors weighted by the coefficients is zero. These coefficients are the elements of the non-zero vector $\vec{x}$.

(b) **[Practice Problem]:** Suppose there exist two unique vectors $\vec{x}_1$ and $\vec{x}_2$ that both satisfy $\mathbf{A}\vec{x} = \vec{b}$, that is, $\mathbf{A}\vec{x}_1 = \vec{b}$ and $\mathbf{A}\vec{x}_2 = \vec{b}$. Prove that the columns of $\mathbf{A}$ are linearly dependent.

Answer:

Let us consider the difference of the two equations:

$$\mathbf{A}\vec{x}_1 - \mathbf{A}\vec{x}_2 = \mathbf{A}(\vec{x}_1 - \vec{x}_2) = \vec{b} - \vec{b} = \vec{0}$$

Once again, we've reached the definition of linear dependence since $\vec{x}_1 - \vec{x}_2 \neq \vec{0}$. We can apply the results from Part (a), setting $\vec{x} = \vec{x}_1 - \vec{x}_2$.

(c) Suppose there exists a matrix $\mathbf{A}$ whose columns are linearly dependent. Prove that if there exists a solution to $\mathbf{A}\vec{x} = \vec{b}$, then there are infinitely many solutions.

Answer:

Begin by defining column vectors $\vec{a}_1, \dots, \vec{a}_n$.

$$\mathbf{A} = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix}$$

Recall the definition of linear dependence:

$$\sum \alpha_i \vec{a}_i = \vec{0} \quad \exists i, \alpha_i \neq 0$$

Note the constraint that not all of the weights α_i can equal 0! (Equivalently, at least one of the weights must be non-zero.) This is extremely important–overlooking this detail will make the proof incorrect. What does this imply? It implies that there exists some $\vec{\alpha}$ such that $\mathbf{A}\vec{\alpha} = \vec{0}$, so that for any $\vec{x}$, where $\mathbf{A}\vec{x} = \vec{b}$, then $(\vec{x} + k\vec{\alpha}), \forall k \in \mathbb{R}$, is also a valid solution.

$$\begin{bmatrix} | & | & | & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & | & | \end{bmatrix} \begin{bmatrix} | \\ \vec{x} \\ | \end{bmatrix} = \begin{bmatrix} | \\ \vec{b} \\ | \end{bmatrix}$$

$$\begin{bmatrix} | & | & | & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & | & | \end{bmatrix} \left(\begin{bmatrix} | \\ \vec{x} \\ | \end{bmatrix} + \begin{bmatrix} | \\ \vec{\alpha} \\ | \end{bmatrix} \right) = \begin{bmatrix} | \\ \vec{b} \\ | \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Therefore, if a solution $\vec{x}$ exists, infinite solutions must exist: $\exists \vec{x}, \mathbf{A}\vec{x} = \vec{b} \iff \mathbf{A}(\vec{x} + k\vec{\alpha}) = \vec{b}, \forall k \in \mathbb{R}$. Physically, taking the example of $\vec{x} \in \mathbb{R}^3$, the set of solutions is not just a single point but a line or plane.

Reference Definitions

Vector spaces: A *vector space* V is a set of elements that is 'closed' under vector addition and scalar multiplication and contains a zero vector. What does closed mean?

That is, if you add two vectors in V, your resulting vector will still be in V. If you multiply a vector in V by a scalar, your resulting vector will still be in V.

More formally, a *vector space* (V, F) is a set of vectors V, a set of scalars F, and two operators that satisfy the following properties:

- Vector Addition
  - Associative: $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ for any $\vec{v}, \vec{u}, \vec{w} \in V$.
  - Commutative: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ for any $\vec{v}, \vec{u} \in V$.
  - Additive Identity: There exists an additive identity $\vec{0} \in V$ such that $\vec{v} + \vec{0} = \vec{v}$ for any $\vec{v} \in V$.
  - Additive Inverse: For any $\vec{v} \in V$, there exists $-\vec{v} \in V$ such that $\vec{v} + (-\vec{v}) = \vec{0}$. We call $-\vec{v}$ the additive inverse of $\vec{v}$.
- Scalar Multiplication
  - Associative: $\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}$ for any $\vec{v} \in V$, $\alpha, \beta \in F$.
  - Multiplicative Identity: There exists $1 \in F$ where $1 \cdot \vec{v} = \vec{v}$ for any $\vec{v} \in F$. We call 1 the multiplicative identity.
  - Distributive in vector addition: $\alpha(\vec{u} + \vec{v}) = \alpha\vec{u} + \alpha\vec{v}$ for any $\alpha \in F$ and $\vec{u}, \vec{v} \in V$.
  - Distributive in scalar addition: $(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$ for any $\alpha, \beta \in F$ and $\vec{v} \in V$.

Subspaces: A subset W of a *vector space* V is a *subspace* of V if the above conditions (closure under vector addition and scalar multiplication and existence of a zero vector) hold for the elements in the subspace W.

The vector spaces we will work with most commonly are $\mathbb{R}^n$ and $\mathbb{C}^n$ as well as their subspaces.

4. Identifying a Subspace: Proof

Is the set

$$V = \left\{ \vec{v} \;\middle|\; \vec{v} = c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \text{ where } c, d \in \mathbb{R} \right\}$$

a subspace of $\mathbb{R}^3$? Why/why not?

Answer:

Yes, V is a subspace of $\mathbb{R}^3$. We will *prove this* by using the definition of a subspace.

First of all, note that V is a subset of $\mathbb{R}^3$ – all elements in V are of the form $\begin{bmatrix} c+d \\ c \\ c+d \end{bmatrix}$, which is a 3-dimensional real vector.

Now, consider two elements $\vec{v}_1, \vec{v}_2 \in V$ and $\alpha \in \mathbb{R}$.

This means that there exists $c_1, d_1 \in \mathbb{R}$, such that $\vec{v}_1 = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Similarly, there exists $c_2, d_2 \in \mathbb{R}$, such that $\vec{v}_2 = c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Now, we can see that

$$\vec{v}_1 + \vec{v}_2 = (c_1 + c_2) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (d_1 + d_2) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

so $\vec{v}_1 + \vec{v}_2 \in V$.

Also,

$$\alpha\vec{v}_1 = (\alpha c_1)\begin{bmatrix}1\\1\\1\end{bmatrix} + (\alpha d_1)\begin{bmatrix}1\\0\\1\end{bmatrix},$$

so $\alpha\vec{v}_1 \in V$.

Furthermore, we observe that the zero vector is contained in V, when we set $c = 0$ and $d = 0$.

We have thus shown both of the no escape (closure) properties and the existence of a zero vector, so V is a subspace of $\mathbb{R}^3$.