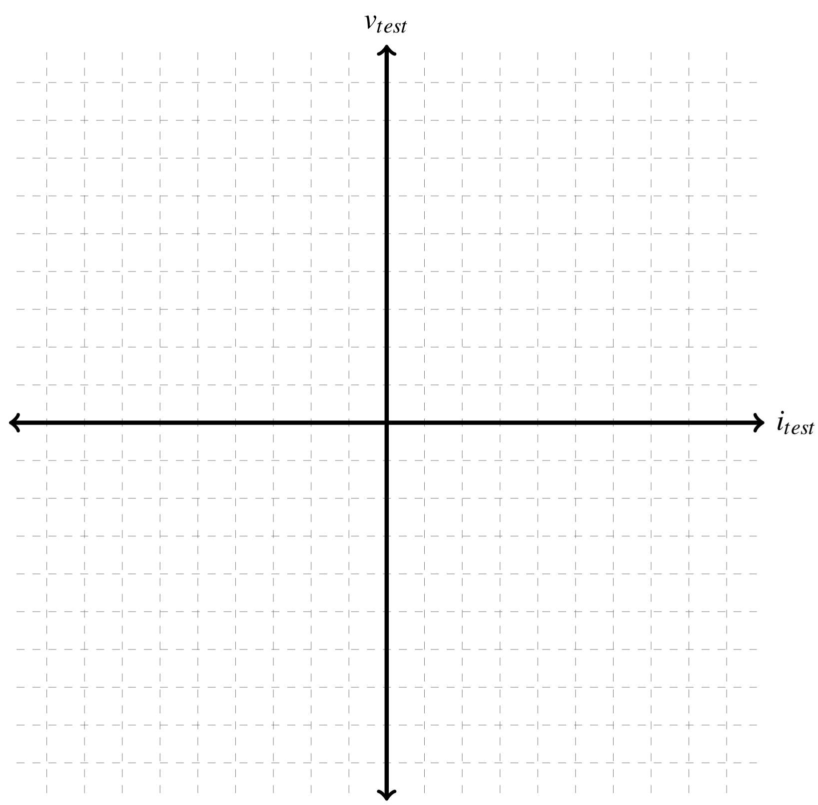
EECS 16A Designing Information Devices and Systems I
Fall 2018 Discussion 13A

1. Ohm's Law With Noise

We are trying to measure the resistance of a black box. We apply various i_{test} currents and measure the ouput voltage v_{test}. Sometimes, we are quite fortunate to get nice numbers. Oftentimes, our measurement tools are a little bit noisy, and the values we get out of them are not accurate. However, if the noise is completely random, then the effect of it can be averaged out over many samples. So we repeat our test many times:

Test	i_{test} (mA)	v_{test} (V)
1	10	21
2	3	7
3	-1	-2
4	5	8
5	-8	-15
6	-5	-11

(a) Plot the measured voltage as a function of the current.



Answer:
Notice that these points *do not* lie on a line!

(b) Suppose we stack the currents and voltages to get $\vec{I} = \begin{bmatrix} 10 \\ 3 \\ -1 \\ 5 \\ -8 \\ -5 \end{bmatrix}$ and $\vec{V} = \begin{bmatrix} 21 \\ 7 \\ -2 \\ 8 \\ -15 \\ -11 \end{bmatrix}$. Is there a unique solution for R? What conditions must $\vec{I}$ and $\vec{V}$ satisfy in order for us to solve for R uniquely?

Answer:

We cannot find the unique solution for R because $\vec{V}$ is not a scalar multiple of $\vec{I}$. In general, we need $\vec{V}$ to be a scalar multiple of $\vec{I}$ to be able to solve for R exactly (another linear algebraic way of saying this is that $\vec{V}$ is in the span of $\vec{I}$).

We know that the *physical* reason we are not able to solve for R is that we have imperfect observations of the voltage across the terminals, $\vec{V}$. Therefore, now that we know we cannot solve for R directly, a very pertinent goal would be to find a value of R that *approximates* the relationship between $\vec{I}$ and $\vec{V}$ as closely as possible.

Let's move on and see how we do this.

(c) Ideally, we would like to find R such that $\vec{V} = \vec{I}R$. If we cannot do this, we'd like to find a value of R that is the *best* solution possible, in the sense that $\vec{I}R$ is as "close" to $\vec{V}$ as possible. We are defining the sum of squared errors as a **cost function**. In this case the cost function for any value of R quantifies the difference between each component of $\vec{V}$ (i.e. v_j) and each component of $\vec{I}R$ (i.e. i_jR) and sum up the squares of these "differences" as follows:

$$\text{cost}(R) = \sum_{j=1}^{6} (v_j - i_jR)^2$$

Do you think this is a good cost function? Why or why not?

Answer:

For each point (i_j, v_j), we want $|v_j - i_jR|$ to be as small as possible. We can call this term the individual error term for this point.

One way of looking at the aggregate "error" in our fit is to add up the squares of the individual errors, so that all errors add up. This is precisely what we've done in the cost function. If we did not square the differences, then a positive difference and a negative difference would cancel each other out.

(d) Show that you can also express the above cost function in vector form, that is,

$$\text{cost}(R) = \left\langle (\vec{V} - \vec{I}R), (\vec{V} - \vec{I}R) \right\rangle$$

Hint: $\langle \vec{a}, \vec{b} \rangle = \vec{a}^T \vec{b} = \sum_i a_i b_i$

Answer:

Let's define the error vector as

$$\vec{e} = \vec{V} - \vec{I}R.$$

Then, we observe that $e_j = v_j - i_jR$.

Therefore,

$$\begin{aligned}
\text{cost}(R) &= \sum_{j=1}^{6} (v_j - i_j R)^2 \\
&= \sum_{j=1}^{6} e_j^2 \\
&= \|\vec{e}\|_2^2 \\
&= \langle \vec{e}, \vec{e} \rangle \\
&= \left\langle (\vec{V} - \vec{I}R), (\vec{V} - \vec{I}R) \right\rangle
\end{aligned}$$

(e) Find $\hat{R}$, which is defined as the optimal value of R that minimizes $\text{cost}(R)$.
Hint: Use calculus. The optimal $\hat{R}$ makes $\frac{d\text{cost}(\hat{R})}{dR} = 0$

Answer:

First, note that

$$\frac{d\text{cost}(R)}{dR} = -2 \sum_{j=1}^{6} i_j (v_j - i_j R)$$

For $R = \hat{R}$, we will have $\frac{d\text{cost}(R)}{dR} = 0$. This means that

$$-2 \sum_{j=1}^{6} i_j (v_j - i_j \hat{R}) = 0,$$

which will ultimately give us

$$\hat{R} = \frac{\sum_{j=1}^{6} i_j v_j}{\sum_{j=1}^{6} i_j^2} = \frac{\left\langle \vec{I}, \vec{V} \right\rangle}{\|\vec{I}\|^2}$$

In our particular example, $\left\langle \vec{I}, \vec{V} \right\rangle = 448$ and $\|\vec{I}\|^2 = 224$. Therefore, we will get $\hat{R} = 2\,\text{k}\Omega$.

Using the equation for least squares estimate with $A = \begin{bmatrix} 10 \\ 3 \\ -1 \\ 5 \\ -8 \\ -5 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 21 \\ 7 \\ -2 \\ 8 \\ -15 \\ -11 \end{bmatrix}$, we would have:

$$\begin{aligned}
\hat{R} &= (A^T A)^{-1} A^T \vec{b} \\
\hat{R} &= \frac{\left\langle \vec{I}, \vec{V} \right\rangle}{\left\langle \vec{I}, \vec{I} \right\rangle} \\
\hat{R} &= \frac{\left\langle \vec{I}, \vec{V} \right\rangle}{\|\vec{I}\|^2},
\end{aligned}$$

which gives us the same expression as before!

(f) On your original *IV* plot, also plot the line $v_{test} = \hat{R} i_{test}$. Can you visually see why this line "fits" the data well? How well would we have done if we had guessed $R = 3\,\mathrm{k}\Omega$? What about $R = 1\,\mathrm{k}\Omega$? Calculate the cost functions for each of these choices of R to validate your answer.

Answer:

When $\hat{R} = 2\,\mathrm{k}\Omega$, we have

$$\begin{aligned}\mathrm{cost}(2k) &= (21 - 2\cdot 10)^2 + (7 - 2\cdot 3)^2 + (-2 - 2\cdot(-1))^2 + \\ &\quad (8 - 2\cdot 5)^2 + (-15 - 2\cdot(-8))^2 + (-11 - 2\cdot(-5))^2 \\ &= 8.\end{aligned}$$

When $\hat{R} = 3\,\mathrm{k}\Omega$, we have

$$\begin{aligned}\mathrm{cost}(3k) &= (21 - 3\cdot 10)^2 + (7 - 3\cdot 3)^2 + (-2 - 3\cdot(-1))^2 + \\ &\quad (8 - 3\cdot 5)^2 + (-15 - 3\cdot(-8))^2 + (-11 - 3\cdot(-5))^2 \\ &= 232.\end{aligned}$$

When $\hat{R} = 1\,\mathrm{k}\Omega$, we have

$$\begin{aligned}\mathrm{cost}(1k) &= (21 - 1\cdot 10)^2 + (7 - 1\cdot 3)^2 + (-2 - 1\cdot(-1))^2 + \\ &\quad (8 - 1\cdot 5)^2 + (-15 - 1\cdot(-8))^2 + (-11 - 1\cdot(-5))^2 \\ &= 232.\end{aligned}$$

(g) Now, suppose that we add a new data point: $i_7 = 2\,\mathrm{mA}$, $v_7 = 4\,\mathrm{V}$. Will $\hat{R}$ increase, decrease, or remain the same? Why? What does that say about the line $v_{test} = \hat{R} i_{test}$?

Answer:

We can qualitatively see that $\hat{R}$ will remain 2. This is because we already obtained $\hat{R}$ to fit our previous data in the best way. Now, you should notice that this new piece of data (i_7, v_7) also lies exactly on the line $v_{test} = \hat{R} i_{test}$! Therefore, you have no reason to change $\hat{R}$. It is the best fit for the old data and will fit the new data anyway.

(h) Let's add another data point: $i_8 = 4\,\mathrm{mA}, v_8 = 11\,\mathrm{V}$. Will $\hat{R}$ increase, decrease, or remain the same? Why? What does that say about the line $v_{test} = \hat{R} i_{test}$?

Answer:

We can qualitatively see that $\hat{R}$ should be something greater than or equal to 2. This is because you have already obtained $\hat{R}$ to fit your previous data in the best way. Now, you notice that this new piece of data (i_8, v_8) also lies *above* the line $v_{test} = \hat{R} i_{test}$! Therefore, if you decreased $\hat{R}$, it would be a worse fit for the old data and the new data. You would increase $\hat{R}$ to find a better fit.

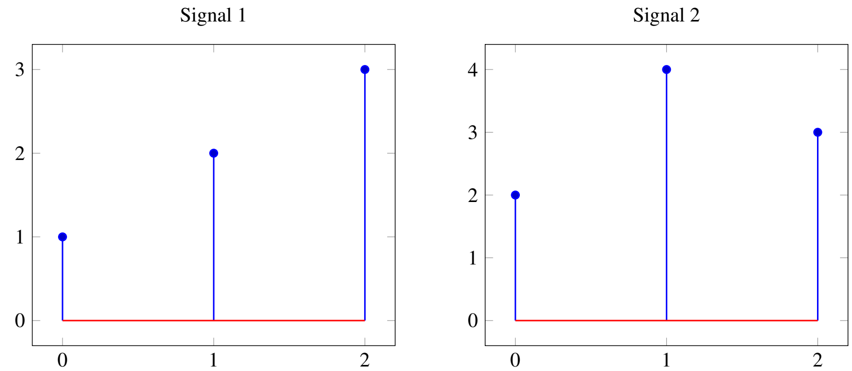
(i) Now your mischievous friend has hidden the black box. You want to predict what output voltage across the terminals if you applied $5.5\,\mathrm{mA}$ through the black box. What would your best guess be?

Answer:

Hopefully, by now, it makes sense to the class that you will estimate $\hat{V} = 5.5\,\mathrm{mA} \cdot \hat{R} = 5.5\,\mathrm{mA} \cdot 2\,\mathrm{k}\Omega = 11\,\mathrm{V}$. This is an example of estimation from machine learning! You have *learned* what is going on inside the black box, that is, $\hat{R}$, by making observations of $\vec{I}$ and $\vec{V}$. Now, you are using what you have learned, $\hat{R}$, to estimate $\hat{V}$ for new values of I.

2. Correlation Revisited

We are given the following two signals, $\vec{s}_1$ and $\vec{s}_2$ respectively:



We have find their cross-correlation for three different scenarios:

- Periodic linear cross-correlation
- Linear cross-correlation when the signals are not periodic
- Circular cross-correlation

Note that periodic linear cross correlation and circular cross correlation are essentially the same — while periodic linear cross correlation looks at linear shifts of an infinite periodic signal, circular correlation looks at circular shifts of one period of the signal.

(a) Find the periodic linear cross correlations, $\text{corr}_{N=3}(\vec{s}_1, \vec{s}_2)$ and $\text{corr}_{N=3}(\vec{s}_2, \vec{s}_1)$ for $\vec{s}_1$ and $\vec{s}_2$, assuming they are periodic with period $N = 3$, where periodic linear cross correlation is defined as:

$$\text{corr}_N(\vec{x}, \vec{y})[k] = \sum_{i=0}^{N-1} x[i]y[i-k]$$

The signals continue to $+\infty$ and $-\infty$, however, since they are periodic we can focus on just one period of the signal. Calculate the periodic linear cross-correlation assuming the signals are periodic as below (The first one is already done for you!):

$\text{corr}_{N=3}(\vec{s}_1, \vec{s}_2)$

$\vec{s}_1$	1		2		3	
$\vec{s}_2[n]$	2		4		3	
$\langle \vec{s}_1, \vec{s}_2[n] \rangle$	2	$+$	8	$+$	9	$= 19$

$\text{corr}_{N=3}(\vec{s}_2, \vec{s}_1)$

$\vec{s}_2$	2		4		3	
$\vec{s}_1[n]$						
$\langle \vec{s}_2, \vec{s}_1[n] \rangle$		$+$		$+$		$=$

$\vec{s}_1$	1		2		3	
$\vec{s}_2[n-1]$						
$\langle \vec{s}_1, \vec{s}_2[n-1] \rangle$		$+$		$+$		$=$

$\vec{s}_2$	2		4		3	
$\vec{s}_1[n-1]$						
$\langle \vec{s}_2, \vec{s}_1[n-1] \rangle$		$+$		$+$		$=$

$\vec{s}_1$	1		2		3	
$\vec{s}_2[n-2]$						
$\langle \vec{s}_1, \vec{s}_2[n-2] \rangle$		$+$		$+$		$=$

$\vec{s}_2$	2		4		3	
$\vec{s}_1[n-2]$						
$\langle \vec{s}_2, \vec{s}_1[n-2] \rangle$		$+$		$+$		$=$

Answer:

$\text{corr}_{N=3}(\vec{s}_1, \vec{s}_2)$

$\vec{s}_1$	1		2		3	
$\vec{s}_2[n]$	2		4		3	
$\langle \vec{s}_1, \vec{s}_2[n] \rangle$	2	+	8	+	9	$= 19$

$\vec{s}_1$	1		2		3	
$\vec{s}_2[n-1]$	3		2		4	
$\langle \vec{s}_1, \vec{s}_2[n-1] \rangle$	3	+	4	+	12	$= 19$

$\vec{s}_1$	1		2		3	
$\vec{s}_2[n-2]$	4		3		2	
$\langle \vec{s}_1, \vec{s}_2[n-2] \rangle$	4	+	6	+	6	$= 16$

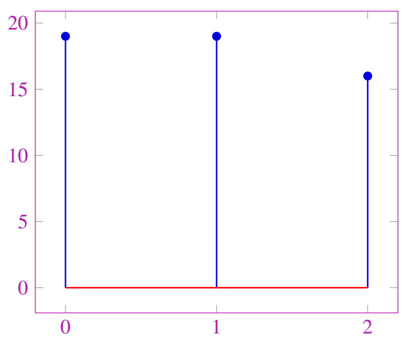
$\text{corr}_{N=3}(\vec{s}_2, \vec{s}_1)$

$\vec{s}_2$	2		4		3	
$\vec{s}_1[n]$	1		2		3	
$\langle \vec{s}_2, \vec{s}_1[n] \rangle$	2	+	8	+	9	$= 19$

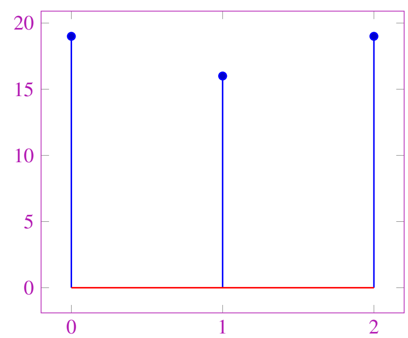
$\vec{s}_2$	2		4		3	
$\vec{s}_1[n-1]$	3		1		2	
$\langle \vec{s}_2, \vec{s}_1[n-1] \rangle$	6	+	4	+	6	$= 16$

$\vec{s}_2$	2		4		3	
$\vec{s}_1[n-2]$	2		3		1	
$\langle \vec{s}_2, \vec{s}_1[n-2] \rangle$	4	+	12	+	3	$= 19$

Periodic Cross-correlation of $\vec{s}_1$ and $\vec{s}_2$



Periodic Cross-correlation of $\vec{s}_2$ and $\vec{s}_1$



It is evident from the results that $\text{corr}_{N=3}(\vec{s}_1, \vec{s}_2)[k] = \text{corr}_{N=3}(\vec{s}_2, \vec{s}_1)[-k] = \text{corr}_{N=3}(\vec{s}_2, \vec{s}_1)[N-k]$.

(b) Find the two linear cross correlations, $\text{corr}(\vec{s}_1, \vec{s}_2)$ and $\text{corr}(\vec{s}_2, \vec{s}_1)$ for $\vec{s}_1$ and $\vec{s}_2$, assuming they are not periodic, where

$$\text{corr}(\vec{x}, \vec{y})[k] = \sum_{i=-\infty}^{\infty} x[i]y[i-k]$$

$\text{corr}(\vec{s}_1, \vec{s}_2)$

$\vec{s}_1$	0		0		1		2		3		0		0	
$\vec{s}_2[n+2]$														
$\langle \vec{s}_1, \vec{s}_2[n+2] \rangle$		+		+		+		+		+		+		=

$\vec{s}_1$	0		0		1		2		3		0		0	
$\vec{s}_2[n+1]$														
$\langle \vec{s}_1, \vec{s}_2[n+1] \rangle$		+		+		+		+		+		+		=

$\vec{s}_1$	0		0		1		2		3		0		0	
$\vec{s}_2[n]$														
$\langle \vec{s}_1, \vec{s}_2[n] \rangle$		+		+		+		+		+		+		=

$\vec{s}_1$	0		0		1		2		3		0		0	
$\vec{s}_2[n-1]$														
$\langle \vec{s}_1, \vec{s}_2[n-1] \rangle$		+		+		+		+		+		+		=

$\vec{s}_1$	0		0		1		2		3		0		0	
$\vec{s}_2[n-2]$														
$\langle \vec{s}_1, \vec{s}_2[n-2] \rangle$		+		+		+		+		+		+		=

$\text{corr}(\vec{s}_2, \vec{s}_1)$

$\vec{s}_2$	0		0		2		4		3		0		0	
$\vec{s}_1[n+2]$														
$\langle \vec{s}_2, \vec{s}_1[n+2] \rangle$		+		+		+		+		+		+		=

$\vec{s}_2$	0		0		2		4		3		0		0	
$\vec{s}_1[n+1]$														
$\langle \vec{s}_2, \vec{s}_1[n+1] \rangle$		+		+		+		+		+		+		=

$\vec{s}_2$	0		0		2		4		3		0		0	
$\vec{s}_1[n]$														
$\langle \vec{s}_2, \vec{s}_1[n] \rangle$		+		+		+		+		+		+		=

$\vec{s}_2$	0		0		2		4		3		0		0	
$\vec{s}_1[n-1]$														
$\langle \vec{s}_2, \vec{s}_1[n-1] \rangle$		+		+		+		+		+		+		=

$\vec{s}_2$	0		0		2		4		3		0		0	
$\vec{s}_1[n-2]$														
$\langle \vec{s}_2, \vec{s}_1[n-2] \rangle$		+		+		+		+		+		+		=

Answer: In this case, $\vec{s}_1[n]$ and $\vec{s}_2[n]$ are not periodic, so they contain only zeros outside the range: $0 \leq n \leq 2$. The linear cross-correlation is calculated by shifting the second signal both forward and backward until there is no overlap between the signals. When there is no overlap, the cross-correlation goes to zero. Both of these cross-correlations should have only zeros outside the range: $-2 \leq n \leq 2$.

$\text{corr}(\vec{s}_1, \vec{s}_2)$

$\vec{s}_1$	0		0		1		2		3		0		0	
$\vec{s}_2[n+2]$	2		4		3		0		0		0		0	
$\langle \vec{s}_1, \vec{s}_2[n+2] \rangle$	0	+	0	+	3	+	0	+	0	+	0	+	0	$= 3$

$\vec{s}_1$	0		0		1		2		3		0		0	
$\vec{s}_2[n+1]$	0		2		4		3		0		0		0	
$\langle \vec{s}_1, \vec{s}_2[n+1] \rangle$	0	+	0	+	4	+	6	+	0	+	0	+	0	$= 10$

$\vec{s}_1$	0		0		1		2		3		0		0	
$\vec{s}_2[n]$	0		0		2		4		3		0		0	
$\langle \vec{s}_1, \vec{s}_2[n] \rangle$	0	+	0	+	2	+	8	+	9	+	0	+	0	$= 19$

$\vec{s}_1$	0		0		1		2		3		0		0	
$\vec{s}_2[n-1]$	0		0		0		2		4		3		0	
$\langle \vec{s}_1, \vec{s}_2[n-1] \rangle$	0	+	0	+	0	+	4	+	12	+	0	+	0	$= 16$

$\vec{s}_1$	0		0		1		2		3		0		0	
$\vec{s}_2[n-2]$	0		0		0		0		2		4		3	
$\langle \vec{s}_1, \vec{s}_2[n-2] \rangle$	0	+	0	+	0	+	0	+	6	+	0	+	0	$= 6$

$\text{corr}(\vec{s}_2, \vec{s}_1)$

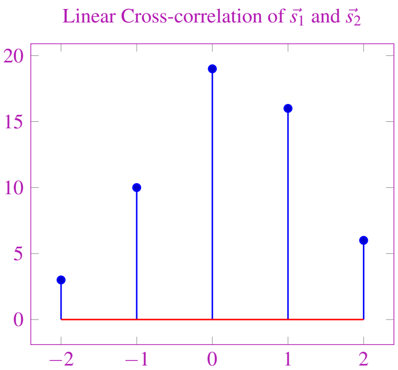
$\vec{s}_2[n]$	0		0		2		4		3		0		0	
$\vec{s}_1[n+2]$	1		2		3		0		0		0		0	
$\langle \vec{s}_2, \vec{s}_1[n+2] \rangle$	0	+	0	+	6	+	0	+	0	+	0	+	0	$= 6$

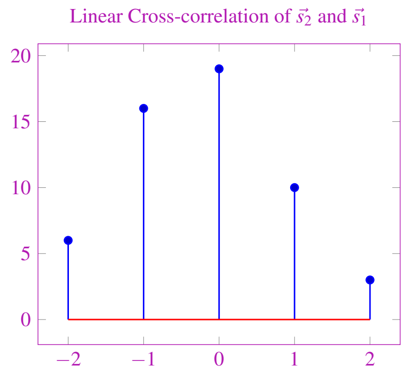
$\vec{s}_2[n]$	0		0		2		4		3		0		0	
$\vec{s}_1[n+1]$	0		1		2		3		0		0		0	
$\langle \vec{s}_2, \vec{s}_1[n+1] \rangle$	0	+	0	+	4	+	12	+	0 +		0	+	0	$= 16$

$\vec{s}_2[n]$	0		0		2		4		3		0		0	
$\vec{s}_1[n]$	0		0		1		2		3		0		0	
$\langle \vec{s}_2, \vec{s}_1[n] \rangle$	0	+	0	+	2	+	8	+	9	+	0	+	0	$= 19$

$\vec{s}_2[n]$	0		0		2		4		3		0		0	
$\vec{s}_2[n-1]$	0		0		0		1		2		3		0	
$\langle \vec{s}_2, \vec{s}_1[n-1] \rangle$	0	+	0	+	0	+	4	+	6	+	0	+	0	$= 10$

$\vec{s}_2[n]$	0		0		2		4		3		0		0	
$\vec{s}_2[n-2]$	0		0		0		0		1		2		3	
$\langle \vec{s}_2, \vec{s}_1[n-2] \rangle$	0	+	0	+	0	+	0	+	3	+	0	+	0	$= 3$





Notice that $\text{corr}(\vec{s}_1,\vec{s}_2)[k] = \text{corr}(\vec{s}_2,\vec{s}_1)[-k]$, i.e. changing the order of the signals reveres the cross-correlation sequence.

(c) Find the circular cross correlation, $\text{circcorr}(\vec{s}_1,\vec{s}_2)$ and $\text{circcorr}(\vec{s}_2,\vec{s}_1)$ for $\vec{s}_1$ and $\vec{s}_2$, where

$$\text{circcorr}(\vec{x},\vec{y})[k] = \sum_{i=0}^{N-1} x[i]y[i-k]_N$$

Answer:

$\text{circcorr}(\vec{s}_1,\vec{s}_2)$

$\vec{s}_1$	1		2		3	
$\vec{s}_2[n]$	2		4		3	
$\langle\vec{s}_1,\vec{s}_2[n]\rangle$	2	+	8	+	9	$=19$

$\vec{s}_1$	1		2		3	
$\vec{s}_2[n-1]$	3		2		4	
$\langle\vec{s}_1,\vec{s}_2[n-1]\rangle$	3	+	4	+	12	$=19$

$\vec{s}_1$	1		2		3	
$\vec{s}_2[n-2]$	4		3		2	
$\langle\vec{s}_1,\vec{s}_2[n-2]\rangle$	4	+	6	+	6	$=16$

$\text{circcorr}(\vec{s}_2,\vec{s}_1)$

$\vec{s}_2$	2		4		3	
$\vec{s}_1[n]$	1		2		3	
$\langle\vec{s}_2,\vec{s}_1[n]\rangle$	2	+	8	+	9	$=19$

$\vec{s}_2$	2		4		3	
$\vec{s}_1[n-1]$	3		1		2	
$\langle\vec{s}_2,\vec{s}_1[n-1]\rangle$	6	+	4	+	6	$=16$

$\vec{s}_2$	2		4		3	
$\vec{s}_1[n-2]$	2		3		1	
$\langle\vec{s}_2,\vec{s}_1[n-2]\rangle$	4	+	12	+	3	$=19$

Alternatively, we can utilize the circulant matrix to compute:

$$\text{circcorr}(\vec{s}_1,\vec{s}_2) = \begin{bmatrix} 2 & 4 & 3 \\ 3 & 2 & 4 \\ 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 19 \\ 19 \\ 16 \end{bmatrix}$$

$$\text{circcorr}(\vec{s}_2,\vec{s}_1) = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 19 \\ 16 \\ 19 \end{bmatrix}$$

You can easily verify that these results are the same as the periodic cross-correlation results.