3. Fun Times with Inverses (7 Points)

Consider the following matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

(a) (5 Points) *Calculate its inverse* $\mathbf{A}^{-1}$ if it exists.

(b) (2 Points) If an $n \times n$ matrix $\mathbf{A}$ does not have an inverse, is the dimension of $\mathbf{A}$'s column space (i.e., the span of the column vectors of $\mathbf{A}$) less than, greater than, or equal to n? *Circle your answer(s).*

LESS THAN **GREATER THAN** **EQUAL TO**

4. Mechanical Basis (4 Points)

Given $\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\1\end{bmatrix}$, and $\begin{bmatrix}1\\1\\x\end{bmatrix}$, *find a value of* x such that these three vectors do **not** form a basis for $\mathbb{R}^3$. Show why your choice of x makes it so these vectors do **not** form a basis for $\mathbb{R}^3$.

5. Mechanical Eigenvalues (5 Points)

Consider the matrix **A** below.

$$\mathbf{A} = \begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix}$$

Find the eigenvalues of **A** in terms of a and b.

6. Operation Crypto (17 Points)

Agent 16A, you have been tasked with an important mission. The Berkeley Intelligence Agency has been collecting important information on the other engineering schools (known as the adversary) using our field agents. Through their last reports, we have found out that our adversaries have been secretly sharing milk tea recipes through cryptographic ciphers in the form of Matrix Transformations, and we want to find out what these recipes are!

The adversary uses $n \times n$ preshared key matrices to encode plaintext vectors $\vec{p} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$.

$p_1,\ p_2,\ \ldots, p_n$ are real numbers from 0 to 25 that map to A through Z. That is, 0 maps to A, and 25 maps to Z.

To get the ciphertext vector, they multiply their key matrix

$$\mathbf{K} = \begin{bmatrix} k_{11} & \cdots & k_{1n} \\ \vdots & \ddots & \vdots \\ k_{n1} & \cdots & k_{nn} \end{bmatrix},$$

where all the elements in $\mathbf{K}$ are real numbers, with their plaintext vector $\vec{p}$ to get

$$\mathbf{K}\vec{p} = \vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix},$$

where the elements in $\vec{c}$ are real numbers that don't necessarily map to anything. To a passive observer, the entries in $\vec{c}$ look like nonsense.

(a) (4 Points) In a hypothetical scenario, MIT wants to send Stanford a milk tea recipe $\vec{p}$ by sending $\vec{c} = \mathbf{K}\vec{p}$. *What has to be true about* $\mathbf{K}$ *for Stanford be able to recover* $\vec{p}$*? In that case, how would Stanford recover the plaintext recipe* $\vec{p}$ *(decrypt the ciphertext vector)?*

(b) (4 Points) Agent 16A, your first objective is to find the key. Your mission is to perform what is called a *Known Plaintext Attack*, where the attacker (in this case, us) knows a plaintext vector $\vec{p}$ and its corresponding ciphertext vector $\vec{c}$, which we will call a plaintext/ciphertext pair. One field agent was able to uncover a plaintext/ciphertext pair:

$$\left(\vec{p}_1 = \begin{bmatrix} 1 \\ 14 \end{bmatrix}, \vec{c}_1 = \begin{bmatrix} 29 \\ 72 \end{bmatrix}\right)$$

For a 2×2 key matrix $\mathbf{K}$, *is it possible to fully determine* $\mathbf{K}$ *using just this pair? Briefly justify your answer.*

Hint: How many elements are there in $\mathbf{K}$?

(c) (4 Points) In general, for an $n \times n$ key matrix $\mathbf{K}$, *at least how many plaintext/ciphertext vector pairs do we need to be able to fully determine* $\mathbf{K}$*?*

(d) (5 Points) Through her dedication to the mission, one of the field agents was able to uncover a second plaintext/ciphertext pair. *Find the key matrix* $\mathbf{K}$*, such that* $\vec{c}_i = \mathbf{K}\vec{p}_i$. Don't let her efforts be in vain!
For your convenience, here are both plaintext/ciphertext pairs.

$$\left(\vec{p}_1 = \begin{bmatrix} 1 \\ 14 \end{bmatrix}, \vec{c}_1 = \begin{bmatrix} 29 \\ 72 \end{bmatrix}\right), \left(\vec{p}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{c}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$$

PRINT your name and student ID: ____

7. 3D Imaging (18 Points)

In lab, you built a 2D imaging system. In this problem, we will explore how to build a 3D tomography system. We will be trying to image voxels, which are a generalization of pixels in 3 dimensions. Each voxel has a value associated with it that represents how much light passes through it. This is equivalent to how dark or light the voxel appears.

We would like to image a $2 \times 2 \times 2$ structure. Each incident light ray passes all the way through the structure. Figure 7.1 demonstrates a few different example rays passing through our 3D structure. Note that lighter colored voxels represent voxels through which the light ray passes.

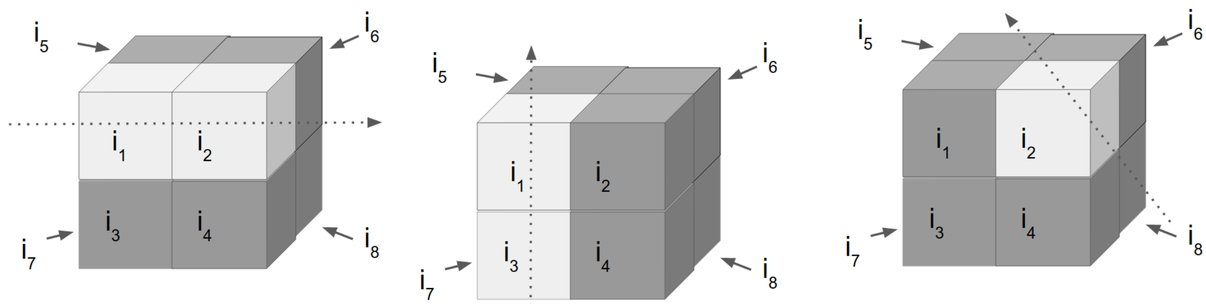

Figure 7.1: Three different incident rays.

As in the imaging lab, we will represent the three-dimensional $2 \times 2 \times 2$ structure as one column vector $\vec{i}$ with 8 elements. We will then apply rays that pass through the object represented by row vectors $\vec{h}_k^T$. Let the matrix $\mathbf{H}$ be the matrix whose row vectors are $\vec{h}_k^T$. Finally, we will measure the vector $\vec{s}$ where $\vec{s} = \mathbf{H}\vec{i}$.

(a) (5 Points) Suppose that we have the matrix $\mathbf{H}$ shown below. *Find the null space of this matrix* $\mathbf{H}$.

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) (5 Points) Find two objects that would give the same $\vec{s}$ when measured using the mask matrix $\mathbf{H}$ from part (a). *Write down the corresponding $\vec{i}$ vectors for the two objects.* Note that the two $\vec{i}$ vectors can contain any real number you would like them to, but the two vectors cannot be equal to each other. (For example $\vec{i}_1 = \vec{i}_2 = \vec{0}$ is not a valid solution.)

(c) (8 Points) Our matrix $\mathbf{H}$ only has four rows corresponding to four measurements. *Add another four rows to the original* $\mathbf{H}$ *matrix to make a new mask matrix* $\mathbf{H}'$ *of dimensions* 8×8 and select the entries of the additional four rows such that we can always uniquely reconstruct the imaged object. Each row of your new matrix can have at most two 1's in it, and the rest of the entries must be 0.

8. The Art of Proving (14 Points)

Dr. Alon and Dr. Sahai are two passionate scientists who are currently working on linear algebra. They have been working for a long time and have hit some major roadblocks. Help them out by proving some things that they are stuck on!

Let $\mathbf{A}$ and $\mathbf{B}$ be two matrices with dimensions, such that $\mathbf{AB}$ is valid.

(a) (4 Points) Show that *if $\vec{v}$ is in the null space of* $\mathbf{B}$*, then $\vec{v}$ is in the null space of* $\mathbf{AB}$.

(b) (4 Points) Show that *if λ is an eigenvalue of an $n \times n$ matrix* $\mathbf{A}$*, then for all $c \in \mathbb{R}$, $\lambda' = \lambda + c$ is an eigenvalue of the matrix* $\mathbf{A} + c\mathbf{I}$, where $\mathbf{I}$ is the $n \times n$ identity matrix.

(c) (6 Points) Suppose that **A** and **B** are $n \times n$ matrices that share the same n distinct eigenspaces. That is, there exist n non-zero vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, such that $\mathbf{A}\vec{v}_i = \lambda_{A,i}\vec{v}_i$ and $\mathbf{B}\vec{v}_i = \lambda_{B,i}\vec{v}_i$, where $\lambda_{A,i}$ is not necessarily equal to $\lambda_{B,i}$. Then, prove that *the matrices* **A** *and* **B** *commute*, that is, $\mathbf{AB} = \mathbf{BA}$.

Hint: Choose an appropriate basis.

9. Rabbits, Foxes, and the Circle of Life (21 Points)

If rabbits are such notoriously fast breeders, why haven't we all been crushed under a (warm, comfortable) mountain of rabbits by now? Well, consider the hungry foxes...

Let's examine the case of Tilden Park, circa 1000 CE. This vast beautiful space is initially filled with 200 foxes and 1000 rabbits. Since rabbits like to feast on the pleantiful greenery, the population of rabbits grows by 10% each month. Every month, 40% of the foxes either die or leave the park. The population of foxes increases by 20% of the population of rabbits each month. Similarly the population of rabbits decreases by 20% of the fox population each month. This can be summarized by the system shown below, where $f[t]$ and $r[t]$ represent the number of foxes and rabbits in the park each month t.

$$\begin{bmatrix} f[t+1] \\ r[t+1] \end{bmatrix} = \begin{bmatrix} 0.6 & 0.2 \\ -0.2 & 1.1 \end{bmatrix} \begin{bmatrix} f[t] \\ r[t] \end{bmatrix}$$

In this problem, we will use linear algebra to explore the predator-prey relationship to figure out if we should be submerged in rabbits, on the run from armies of foxes, or in some peaceful equilibrium state.

(a) (8 Points) We want to know what will happen to the populations of the two species as time goes on. Will the population numbers converge?

Note: You do not need to find what the populations converge to, just whether they converge or not. You may or may not find it useful to know that $1.7^2 = 2.89$.

(b) (5 Points) Assuming that there is some known total populaton of foxes and rabbits in the year 2000, *calculate what fraction of that total population is rabbits.* You can assume that 1000 years is a good approximation for an infinite amount of time.

(c) (8 Points) In the far future, a curious child digs up a strange fossil in the park. It seems like Ancient Dino-foxes once inhabited the park! Using advanced future Zoologic-Mathematics, graduate students from the University of MegaCalifornia, Berkeley derive the following eigenvalue/eigenvector pairs describing the species interactions from the fossils:

$$\left(\lambda_1 = 1, \vec{v}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}\right), \left(\lambda_2 = 0.5, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

Reconstruct the state transition matrix.