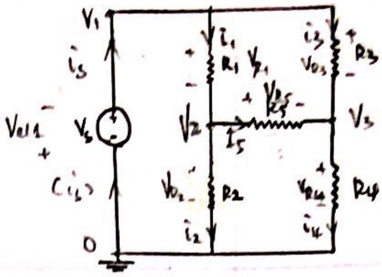
1. Circuit Analysis.

Labeling, see graph on right.



Then $A\vec{x} = \vec{b}$ where $\vec{x} = [\, i_1\ i_2\ i_3\ i_4\ i_5\ i_s\ V_1\ V_2\ V_3 \,]^T$.

Using KCL, so
$$\begin{cases} i_s = i_1 + i_3 \\ i_1 = i_2 + i_5 \\ i_3 + i_5 = i_4 \\ i_2 + i_4 = i_s \end{cases} \Rightarrow \begin{cases} i_1 + I_3 - i_s = 0 & ① \\ i_1 - i_2 - i_5 = 0 & ② \\ i_3 - i_4 + I_5 = 0 & ③ \\ i_2 + i_4 - i_s = 0. & ④ \end{cases}$$

(step 7) Voltage Source:
$$\begin{cases} -V_s = V_{el1}. \\ V_{el1} = 0 - V_1 = -V_1 \end{cases} \Rightarrow V_1 = V_s$$

Since $V_s = 5V$, so $V_1 = V_s = 5V$ ⑤

Then, $R_1 = 1\Omega$. $R_2 = 2\Omega$. $R_3 = 3\Omega$ $R_4 = 4\Omega$. $R_5 = 5\Omega$

Resisters:
$$\begin{cases} V_{R_1} = R_1 \cdot i_1 = 1_\Omega i_1 \\ V_{R_1} = V_1 - V_2. \\ V_{R_2} = R_2 \cdot i_2 = 2_\Omega i_2 \\ V_{R_2} = V_2 - 0. \\ V_{R_3} = R_3 \cdot i_3 = 3_\Omega i_3 \\ V_{R_3} = V_1 - V_3 \\ V_{R_4} = R_4 \cdot i_4 = 4_\Omega \cdot i_4 \\ V_{R_4} = V_3 - 0 \\ V_{R_5} = R_5 \cdot i_5 = 5_\Omega \cdot i_5 \\ V_{R_5} = V_2 - V_3. \end{cases} \Rightarrow \begin{cases} R_1 i_1 - V_1 + V_2 = 0 & ⑥ \\ R_2 i_2 - V_2 = 0 & ⑦ \\ R_3 i_3 - V_1 + V_3 = 0. & ⑧ \\ R_4 i_4 - V_3 = 0. & ⑨ \\ R_5 i_5 - V_2 + V_3 = 0. & ⑩ \end{cases}$$

With equations ② - ⑩, we can complete $A\vec{x} = \vec{b}$ as:

$$\begin{bmatrix} \cancel{1} & \cancel{0} & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{-1} & \cancel{0} & \cancel{0} & \cancel{0} \\ 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ R_1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & R_2 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & R_3 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & R_4 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & R_5 & 0 & 0 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_s \\ V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ V_s \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Using IPython Notebook.

So,
$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_s \\ V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 1.710 \\ 1.645 \\ 0.677 \\ 0.742 \\ 0.065 \\ 2.387 \\ 5 \\ 3.290 \\ 2.968 \end{bmatrix}$$

2. (MT Q3)

(a). 4

(b). $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 3 & 3 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_e \\ x_p \\ x_b \\ x_c \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 14 \\ 6 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 3 & 3 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}; \vec{y} = \begin{bmatrix} 4 \\ 2 \\ 14 \\ 6 \end{bmatrix}.$

(c). $\Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 4 \\ 0 & 2 & 0 & 0 & 2 \\ 2 & 4 & 1 & 1 & 14 \\ 0 & 0 & 1 & 1 & 6 \end{array}\right]$ Row-reduce it into: $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

So $\begin{cases} x_e = 2 \\ x_p = 1 \\ x_b + x_c = 6 \\ 0 = 0 \end{cases} \Rightarrow \vec{x} = \begin{bmatrix} 2 \\ 1 \\ 6-x_c \\ x_c \end{bmatrix}$ with x_c as free variable.

so $\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 6 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \alpha, \quad \alpha \in \mathbb{R}$

3. (MT Q4)

(a). $V_0 \vec{x} = \vec{0} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 2 & 3 & 10 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right]$ Row reduce it into $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\Rightarrow \begin{cases} x_0 + 2x_2 = 0 \\ 3x_1 + 6x_2 = 0 \\ 0 = 0 \end{cases}$ Let x_2 be a free variable, so $Null(V_0) = \begin{bmatrix} -2x_2 \\ -2x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} x_2$

so a basis for $Null(V_0)$ is $\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$

(b). No, it isn't. Since its nullspace is not trivial, (or in other words, its rows are linearly dependant as we've shown via Gaussian elimination,) so it's non-invertible. And it's not a good encoding matrix because we can't recover $\vec{x}$ from $\vec{y}$ without its inverse.

(c). $V_1 V_1^{-1} = I_3 \Rightarrow \left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$ Row-reduce it $\Rightarrow \left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & -1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right].$

$\Rightarrow \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 0 & 1/2 & -1/2 & 1/2 \\ 1 & 0 & 0 & -1/2 & 1/2 & 1/2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 1/2 & 1/2 \\ 0 & 1 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right]$

So $V_1^{-1} = \begin{bmatrix} -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix}$ Yes it's a good encoding matrix to use because given any $\vec{y}$, we can calculate $\vec{x}$ with $\vec{y} = V_1 \vec{x}$, so $\vec{x} = V_1^{-1} V_1 \vec{x} = V_1^{-1} \vec{y}$.

4. (a) (MT Q5) $\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \cdot \begin{bmatrix} 500 \\ 500 \end{bmatrix} = \begin{bmatrix} 500 \\ 1500 \end{bmatrix}$ and $\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \cdot \begin{bmatrix} 1000 \\ 500 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1500 \end{bmatrix}$

$$\Rightarrow \begin{cases} 500a_1 + 500a_2 = 500 \\ 500a_3 + 500a_4 = 1500 \\ 1000a_1 + 500a_2 = 1000 \\ 1000a_3 + 500a_4 = 1500 \end{cases} \Rightarrow \begin{cases} a_1 = 1 \\ a_2 = 0 \\ a_3 = 0 \\ a_4 = 3 \end{cases} \Rightarrow A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

(b) $R = R_2 R_1$ where R_1 is a rotation matrix ($30°$ CCW) and R_2 is a scaling matrix (factor $= 2$)

So $R_1 = \begin{bmatrix} \cos(30°) & -\sin(30°) \\ \sin(30°) & \cos(30°) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$

and $R_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$\therefore R = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} = \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$

(c) $A_p \begin{bmatrix} x_p \\ y_p \end{bmatrix} = \vec{0} \Rightarrow \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 6 & 0 \end{array}\right]$ Row-reduce to $\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

So $\begin{cases} x_p + 3y_p = 0 \\ 0 = 0 \end{cases} \Rightarrow x_p = -3y_p$

So possible positions are $(-3\alpha, \alpha)$ where $\alpha \in \mathbb{R}$

(d) Given information implies:

$A_p \begin{bmatrix} x_q \\ y_q \end{bmatrix} = \lambda \begin{bmatrix} x_q \\ y_q \end{bmatrix} = \begin{bmatrix} u_q \\ v_q \end{bmatrix}$. So we need to find eigenspace of A_p.

So $(A_p - \lambda I_2)\vec{x} = \vec{0} \Rightarrow \det(A_p - \lambda I_2) = \det\left(\begin{bmatrix} 1-\lambda & 3 \\ 2 & 6-\lambda \end{bmatrix}\right)$

$= (1-\lambda)(6-\lambda) - 3 \cdot 2 = 0 \Rightarrow \lambda_1 = 0, \ \lambda_2 = 7$

① $\lambda_1 = 0$, so $\begin{bmatrix} u_q \\ v_q \end{bmatrix} = \lambda_1 \begin{bmatrix} x_q \\ v_q \end{bmatrix} = \vec{0}$ but torpedo can't fire on its initial position $(0, 0)$.

Thus, this is impossible.

② $\lambda_2 = 7$. So $(A_p - \lambda_2 I_2)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{cc|c} -6 & 3 & 0 \\ 2 & -1 & 0 \end{array}\right]$

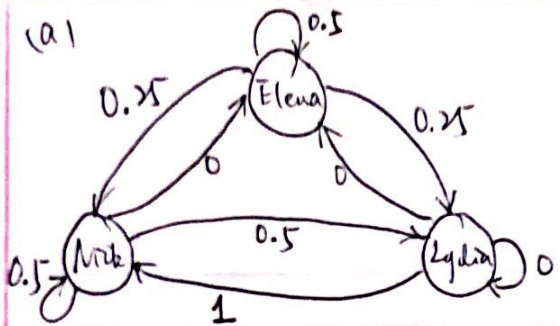
Row-reduce into $\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} 2x_q - y_q = 0 \\ 0 = 0 \end{cases} \Rightarrow y_q = 2x_q$.

So $\begin{bmatrix} x_q \\ y_q \end{bmatrix} = \begin{bmatrix} x_q \\ 2x_q \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} x_q$ with x_q as a free variable.

Thus, possible positions of the probe is

$(x_q, y_q) = (\alpha, 2\alpha)$ where $\alpha \in \mathbb{R}$.

5. (MT Q6) (a)



(b). Since P has a steady state. So $\lambda = 1$ is an eigenvalue and the $\vec{x}$ we wish to find is a corresponding eigenvector.

So, $(P - \lambda I_3)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{ccc|c} -0.8 & 0.5 & 0 & 0 \\ 0.5 & -0.5 & 0 & 0 \\ 0.3 & 0 & 0 & 0 \end{array}\right]$.

Row-reduce gives us: $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x_{elena} = 0 \\ x_{mark} = 0 \\ 0 = 0 \end{cases} \Rightarrow \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \alpha$ $(\alpha \in \mathbb{R})$

$\Rightarrow$ one of the steady state: $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(c). ● $\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ○ $\begin{bmatrix} 513 \\ 513 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix}$

○ $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ ● $\begin{bmatrix} 1026 \\ 0 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ ○ $\begin{bmatrix} 0 \\ 1026 \\ 0 \end{bmatrix}$.

(d). ○ $\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$ ● $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ○ $\begin{bmatrix} 513 \\ 513 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix}$

○ $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 1026 \\ 0 \\ 0 \end{bmatrix}$ ● $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ ○ $\begin{bmatrix} 0 \\ 1026 \\ 0 \end{bmatrix}$

(e) ○ $\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ○ $\begin{bmatrix} 513 \\ 513 \\ 0 \end{bmatrix}$ ● $\begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix}$

○ $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 1026 \\ 0 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ ● $\begin{bmatrix} 0 \\ 1026 \\ 0 \end{bmatrix}$.

(f). Consider $\vec{x}[0] = \vec{V_i}$. With $\vec{x}[n] = T\vec{x}[n-1] = \cdots = T^n \vec{x}[0]$

So, $\lim_{n\to\infty} \vec{x}[n] = \lim_{n\to\infty} T^n \vec{x}[0] = \lim_{n\to\infty} T^n \vec{V_i}$

Then since by the definition of eigenvector/eigenvalue, so $T\vec{V_i} = \lambda \vec{V_i}$

So, $\lim_{n\to\infty} \vec{x}[n] = \lim_{n\to\infty} T^n \vec{V_i} = \lim_{n\to\infty} \lambda_i^n \vec{V_i}$

With $|\lambda_i| > 1$, so $\lim_{n\to\infty} \lambda^n$ does not converge.

Given non-zero eigenvector $\vec{V_i}$, so $\lim_{n\to\infty} \vec{x}[n] = \lim_{n\to\infty} \lambda_i^n \vec{V_i}$ does not converge, either, as desired (Q.E.D).

(g). Of course it does!

6. (a). $\vec{y}[0] = C\vec{x}[0]$ using given information.

(MT Q7). (b). $\vec{x}[1] = A\vec{x}[0]$
$\vec{y}[1] = CA\vec{x}[0]$. again, use the given linear model

(c). Now, $Q\vec{x}[0] = \begin{bmatrix} \vec{y}[0] \\ \vec{y}[1] \end{bmatrix} = \begin{bmatrix} C\vec{x}[0] \\ CA\vec{x}[0] \end{bmatrix} = \begin{bmatrix} C \\ CA \end{bmatrix} \vec{x}[0]$

So, $Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0.5 \end{bmatrix}$. Let $QQ^{-1} = I_4$. if exists.

$$\Rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0.5 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0.5 & 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 & 0 & 2 \end{array}\right] \Rightarrow Q^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 2 \end{bmatrix}$$

So, Q is invertible. Since $Q\vec{x}[0] = \vec{z}$, so $\vec{x}[0] = I\vec{x}[0] = Q^{-1}Q\vec{x}[0] = Q^{-1}\vec{z}$.
which means that my friend can recover $\vec{x}[0]$ from any given $\vec{z}$.

Here, $\vec{z} = \begin{bmatrix} 5.0 \\ 0.1 \\ 6.0 \\ 0.2 \end{bmatrix}$, so $\vec{x}[0] = Q^{-1}\vec{z} = \begin{bmatrix} 5.0 \\ 2.0 \\ 0.1 \\ 0.2 \end{bmatrix}$.

(d). Again, here, $Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Since $R_2 = R_4$ for Q,

So the rows of Q are linearly dependent, so Q is not invertible.
which means that we can't calculate $\vec{x}[0] = Q^{-1}\vec{z}$ as Q^{-1} does not exist.
Thus, we can't recover $\vec{x}[0]$ from $\vec{z}$.

(e). Here, if we only take two measurements, then $Q_2 = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Again, since $R_2 = R_4$ in Q_2, so Q_2 is linearly dependent, and so non-invertible
Similar to part (d), so $\vec{x}[0]$ can't be recovered in this case.

Now, if we take 3 measurements, then
Since $\vec{y}[2] = C\vec{x}[2] = CA\vec{x}[1] = CA^2\vec{x}[0]$, so: $Q_3 = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

($A^2 = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$). Now, if we row-reduce Q_3, we'll get $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, which means

that 4 unique components of $\vec{x}[0]$ can be decided based on the given $\vec{z}$ since $\vec{x}[0] \in \mathbb{R}^4$. Thus, we can uniquely decide $\vec{x}[0]$ from any given $\vec{z}$, which is equivalent to recovering $\vec{x}[0]$.
Thus, the minimum number of measurements needed is 3.