EECS 16A Designing Information Devices and Systems I
Fall 2018 Discussion 5A

Consider a vector in the standard basis,

$$\vec{x} = a\vec{e}_1 + b\vec{e}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \mathbf{I}\vec{x} \tag{1}$$

where, a, b are $\vec{x}$'s coordinates in the standard basis.

Given a new set of basis vectors, $\mathscr{V} = \{\vec{v}_1, \vec{v}_2\}$, if $\vec{x} \in span\{\mathscr{V}\}$, then we can find new coordinates in terms of this new basis. The new coordinates are called a_v, b_v and are described,

$$\vec{x} = a_v\vec{v}_1 + b_v\vec{v}_2 = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} \begin{bmatrix} a_v \\ b_v \end{bmatrix} = \mathbf{V}\vec{x}_v \tag{2}$$

Now consider another set of basis vectors, $\mathscr{U} = \{\vec{u}_1, \vec{u}_2\}$, if $\vec{x} \in span\{\mathscr{U}\}$, then we can find the coordinates in terms of this basis. These coordinates are called a_u, b_u and are described,

$$\vec{x} = a_v\vec{u}_1 + b_v\vec{u}_2 = \begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{bmatrix} \begin{bmatrix} a_u \\ b_u \end{bmatrix} = \mathbf{U}\vec{x}_u \tag{3}$$

All of these bases are equivalent representations of any vector $\vec{x} \in \mathbb{R}^2$; each with their own set of coordinates.

$$\vec{x} = \begin{bmatrix} | & | \\ \vec{e}_1 & \vec{e}_2 \\ | & | \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{bmatrix} \begin{bmatrix} a_u \\ b_u \end{bmatrix} = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} \begin{bmatrix} a_v \\ b_v \end{bmatrix} \tag{4}$$

$$\vec{x} = I\vec{x} = \mathbf{V}\vec{x}_v = \mathbf{U}\vec{x}_u \tag{5}$$

1. Coordinate Change Examples

(a) **Transformation From Standard Basis To Another Basis in $\mathbb{R}^3$**

Calculate the coordinate transformation between the following bases

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

i.e. find a matrix $\mathbf{T}$, such that $\vec{x}_v = \mathbf{T}\vec{x}_u$ where $\vec{x}_u$ contains the coordinates of a vector in a basis of the columns of $\mathbf{U}$ and $\vec{x}_v$ is the coordinates of the same vector in the basis of the columns of $\mathbf{V}$.

Let $\vec{x}_u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and compute $\vec{x}_v$. Repeat this for $\vec{x}_u = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Now let $\vec{x}_u = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. What is $\vec{x}_v$?

Answer:

$$\vec{x} = \mathbf{U}\vec{x}_u = \mathbf{V}\vec{x}_v$$

$$\vec{x}_v = \mathbf{V}^{-1}\mathbf{U}\vec{x}_u = \mathbf{T}\vec{x}_u$$

$$\mathbf{T} = \mathbf{V}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\text{For } \vec{x}_u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{x}_v = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$$

$$\text{For } \vec{x}_u = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{x}_v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$\text{For } \vec{x}_u = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \vec{x}_v = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}.$$

(b) **Transformation Between Two Bases in $\mathbb{R}^3$**

Calculate the coordinate transformation between the following bases

$$\mathbf{U} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix},$$

i.e. find a matrix $\mathbf{T}$, such that $\vec{x}_v = \mathbf{T}\vec{x}_u$. Let $\vec{x}_u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and compute $\vec{x}_v$. Repeat this for $\vec{x}_u = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Now let $\vec{x}_u = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. What is $\vec{x}_v$?

Answer:

Again for any vector $\vec{x}$, we have that $\vec{x} = \mathbf{U}\vec{x}_u = \mathbf{V}\vec{x}_v$

$$\mathbf{T} = \mathbf{V}^{-1}\mathbf{U} = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$$

(c) What is the coordinate transformation from $\vec{x}_v$ to $\vec{x}_u$, i.e. find $\mathbf{W}$ such $\vec{x}_u = \mathbf{W}\vec{x}_v$?

Answer:

Given that $\mathbf{T}$ is the transformation from $\vec{x}_u$ to $\vec{x}_v$, $\mathbf{W} = \mathbf{T}^{-1}$.

(d) **Transformation Between General Bases in $\mathbb{R}^2$**

Calculate the coordinate transformation between the following bases

$$\mathbf{U} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix},$$

i.e. find a matrix $\mathbf{T}$, such that $\vec{x}_v = \mathbf{T}\vec{x}_u$. Let $\vec{x}_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and compute $\vec{x}_v$. Repeat this for $\vec{x}_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Now let $\vec{x}_u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. What is $\vec{x}_v$?

Answer:

$$\mathbf{T} = \mathbf{V}^{-1}\mathbf{U} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$$

2. Proofs

(a) Let $\mathbf{A}$ be an invertible matrix. Show that if λ is an eigenvalue of $\mathbf{A}$, then $\frac{1}{\lambda}$ is an eigenvalue of $\mathbf{A}^{-1}$.

Answer:

Let $\vec{v}$ be the eigenvector of $\mathbf{A}$ corresponding to λ.

$$\mathbf{A}\vec{v} = \lambda\vec{v}$$

Since we know that $\mathbf{A}$ is invertible, we can left-multiply both sides by $\mathbf{A}^{-1}$.

$$\mathbf{A}^{-1}\mathbf{A}\vec{v} = \lambda\mathbf{A}^{-1}\vec{v}$$

$$\vec{v} = \lambda\mathbf{A}^{-1}\vec{v}$$

$$\mathbf{A}^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}$$

3. Steady and Unsteady States

(a) You're given the matrix $\mathbf{M}$ (below) which describes some physical system (could describe either people or water):

$$\mathbf{M} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

Find the eigenspaces associated with the following eigenvalues:

i. span($\vec{v}_1$), associated with $\lambda_1 = 1$

ii. span($\vec{v}_2$), associated with $\lambda_2 = 2$

iii. span($\vec{v}_3$), associated with $\lambda_3 = \frac{1}{2}$

Answer: This is practice finding the null space.

i. $\lambda = 1$:

$$\left[\begin{array}{ccc|c} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{v}_1 = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \alpha \in \mathbb{R}$$

ii. $\lambda = 2$

$$\left[\begin{array}{ccc|c} \frac{-3}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} -3 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{v}_2 = \beta \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \beta \in \mathbb{R}$$

iii. $\lambda = \frac{1}{2}$

$$\left[\begin{array}{ccc|c} 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -2 & 0 \\ 0 & 0 & \frac{3}{2} & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\vec{v}_3 = \gamma \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \gamma \in \mathbb{R}$$

(b) Define $\vec{x} = \alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3$. The values $\alpha, \beta,$ and γ are the coordinates for the basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. For each of the cases in the table, determine if

$$\lim_{n\to\infty} \mathbf{M}^n \vec{x}$$

converges. If it does, what does it converge to?

α	β	γ	Converges?	$\lim_{n\to\infty} \mathbf{M}^n \vec{x}$
0	0	$\neq 0$		
0	$\neq 0$	0		
0	$\neq 0$	$\neq 0$		
$\neq 0$	0	0		
$\neq 0$	0	$\neq 0$		
$\neq 0$	$\neq 0$	0		
$\neq 0$	$\neq 0$	$\neq 0$		

Answer:

α	β	γ	Converges?	$\lim_{n\to\infty} \mathbf{M}^n \vec{x}$
0	0	$\neq 0$	Yes	$\vec{0}$
0	$\neq 0$	0	No	-
0	$\neq 0$	$\neq 0$	No	-
$\neq 0$	0	0	Yes	$\alpha\vec{v}_1$
$\neq 0$	0	$\neq 0$	Yes	$\alpha\vec{v}_1$
$\neq 0$	$\neq 0$	0	No	-
$\neq 0$	$\neq 0$	$\neq 0$	No	-

4. More Practice with Column Spaces and Null Spaces

- The **column space** is the possible outputs of a transformation/function/linear operation. It is also the **span** of the column vectors of the matrix.
- The **null space** is the set of input vectors that output the zero vector.

For the following matrices, answer the following questions:

i. What is the column space of $\mathbf{A}$? What is its dimension?

ii. What is the null space of $\mathbf{A}$? What is its dimension?

iii. Are the column spaces of the row reduced matrix $\mathbf{A}$ and the original matrix $\mathbf{A}$ the same?

iv. Do the columns of $\mathbf{A}$ form a basis for $\mathbb{R}^2$ (or $\mathbb{R}^3$ for part (b))? Why or why not?

(a) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

Answer:

Column space: span $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

Null space: span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

The two column spaces are not the same.
Not a basis for $\mathbb{R}^2$.

(b) $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

Answer:

Column space: $\mathbb{R}^2$

Null space: $\text{span}\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$

Yes, the two column spaces are the same as the column span $\mathbb{R}^2$.

This is a basis for $\mathbb{R}^2$.

(c) $\begin{bmatrix}-2 & 4\\3 & -6\end{bmatrix}$

Answer:

Column space: $\text{span}\left\{\begin{bmatrix}1\\-\frac{3}{2}\end{bmatrix}\right\}$

Null space: $\text{span}\left\{\begin{bmatrix}2\\1\end{bmatrix}\right\}$

The two column spaces are not the same.

Not a basis for $\mathbb{R}^2$.