EECS 16A   Designing Information Devices and Systems I
Fall 2018   Final Exam

Exam Location: Hearst Gym 220 (Last Digit of SID: 8)

PRINT your student ID: ___3033711308___

PRINT AND SIGN your name: ___He___ , ___Alan___   _(signature)_
                                      (last name)   (first name)   (signature)

PRINT the time of your Monday section and the GSI's name: ___4-5pm (Lydia) Etche. 3109___

PRINT the time of your Wednesday section and the GSI's name: ___4-5pm (Lydia). Etch. 3109___

Name and SID of the person to your left: ___Jack Michael Ronan 3033832598___

Name and SID of the person to your right: ___N/A___

Name and SID of the person in front of you: ___John McCormick Newsom 3033896038___

Name and SID of the person behind you: ___Neil Haeems 3031846208___

1. What is one of your favorite things to do in Berkeley? (1 Point)

Playing Avalon with my friends is one of my favorite things to do in Berkeley.

2. What are you looking forward to over the winter break? (1 Point)

I'm looking forward to meeting with my family and enjoying good food over the winter break.

| Do not turn this page until the proctor tells you to do so. You may work on the questions above. |
| --- |

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Extra page for scratchwork.
Work on this page will NOT be graded.

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3. Death Star Destruction (10 points)

(a) (2 points) Oh no! Darth Vader has just completed the Death Star for Emperor Palpatine, and they're getting ready to destroy Alderaan. The Death Star destroys planets by shooting a beam of charged particles at its target – this evil beam can be modeled as a stream of electrons fired from the reflector dish of the Death Star. You have been taken prisoner and forced to help the Empire design and operate the space station.

Darth Vader uses 3 identical capacitors to charge up his planet-destroying beam, but he wants to maximize the equivalent capacitance. How can you arrange three identical capacitors, each with capacitance C, to maximize the equivalent capacitance? **Draw the circuit, and write down the equivalent capacitance $C_{network}$ in terms of C.**
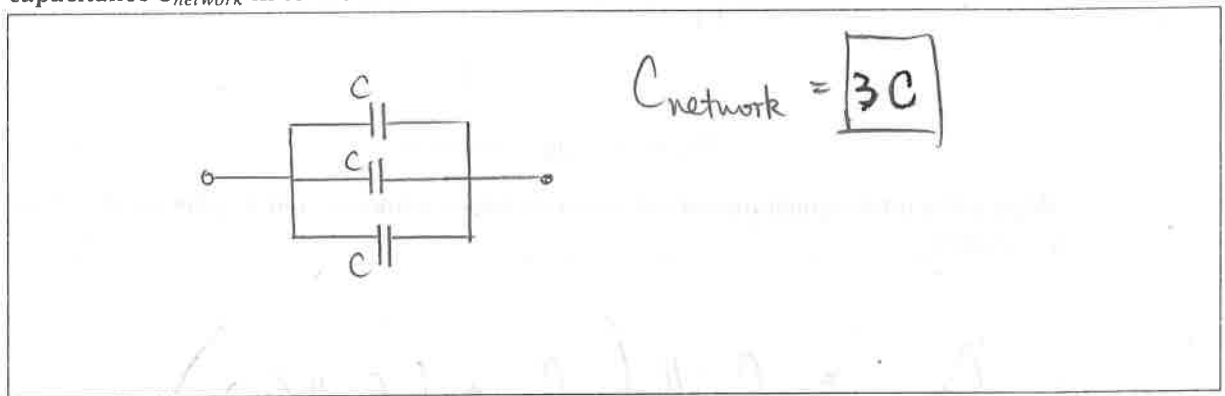
$$C_{network} = \boxed{3C}$$

(b) (2 points) Assume the equivalent capacitance of Darth Vader's arrangement is $C_{network} = 1\text{kF}$. (This is a lot – for reference, the capacitance of the Earth is just 700µF!) If $4.5 \times 10^{15}\text{J}$ are needed to destroy Alderaan, **to what voltage** will you have to charge your capacitor system ?

Since $E = \frac{1}{2}CV^2$, so $V^2 = \dfrac{E}{\frac{1}{2}C_{network}}$

$$= \frac{4.5 \times 10^{15}\text{J}}{\frac{1}{2} \cdot 1\text{kF}} = 9 \cdot 10^{12} \ (V^2)$$

Thus, $V = \sqrt{9 \cdot 10^{12} \ V^2} = \boxed{3 \cdot 10^6 \ V}$

$\frac{1}{2} \ 9 \cdot 10^{12} \ \text{kF}$
10^3
$= 9.5 \cdot 10^{15} \ \checkmark$

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(c) (2 points) Huzzah! You are rescued by the rebels, who need your help to decipher the plans of the Death Star that Princess Leia has helped recover.

With your help, the rebels realize there is one weakness in the design of the Death Star – if they can overload a port on the surface of the Death Star, the entire space station will explode! (Whoever drew up this Death Star design must have gone to Stanford...) Luke Skywalker's X-wing fires its charged beam from this capacitor network, where each capacitor has capacitance C_L:
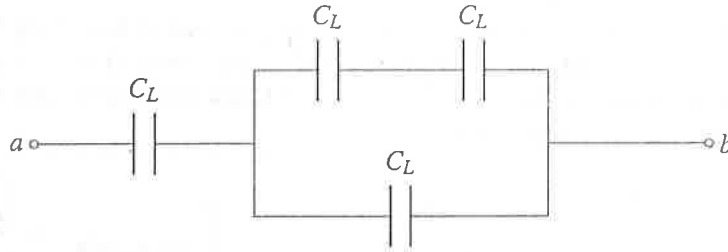

Figure 3.1: Figure for part (c)

What is the total capacitance of this network between nodes a and b, in terms of C_L? Simplify the expression.

$$C_{eq_{a,b}} = C_L \parallel \left(C_L + (C_L \parallel C_L) \right)$$

$$= C_L \parallel \left(C_L + 0.5 C_L \right)$$

$$= C_L \parallel 1.5 C_L$$

$$= \boxed{0.6 \, C_L}$$

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(d) (4 points) The port will overload if hit by more than 25kJ of energy. Assume that Luke's X-wing capacitor system from part (c) has 1kF of total capacitance, and that it is charged to 10V. If Luke's X-wing is hit during the raid, and his damaged X-wing can only discharge its capacitor system from 10V to 5V, **will Luke be able to overload the port if he hits it?**

Now, given that $V_{max} = 5V$, and $C_{eq} = 1kF = 1 \times 10^3 F$.

$$\text{So} \quad E_{max} = \frac{1}{2} C_{eq} \cdot V_{max}^2$$

$$= \frac{1}{2} \cdot 1 \cdot 10^3 F \cdot (5V)^2$$

$$= 25 \cdot 10^3 J = 25 \, kJ.$$

Since the maximum energy Luke can

produce $(25kJ)$ is not more than $25kJ$,
(strictly)

So $\boxed{No}$, he won't.

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4. (25 points) A Tool to Help Compute All the Fun You're Having

Starting in the 1950-1960's, the world began a series of missions to get to the moon. Back then though, computers took up entire rooms and could never fit on a spaceship! They needed a better way of computing values on the fly using what they did have: analog circuits.

In class you have seen circuits that can amplify, add, subtract and even integrate voltages, but we're missing a key ingredient to make computational circuits: **multiplication**. Although making a multiplier circuit is not as straightforward as we would like it to be, we can definitely use our now fully-developed EE16A skills to make this a reality.

You may find the following formulas useful throughout the problem:

$$\ln(e^a) = a \qquad\qquad \ln(a) + \ln(b) = \ln(ab)$$
$$e^{\ln(a)} = a \qquad\qquad \ln(a) - \ln(b) = \ln\left(\tfrac{a}{b}\right)$$

(a) (4 points) To start off, your TA Nick suggests that you first draw a block diagram that would do what you want, and then worry about how to implement it later. He starts you off with the following incomplete block diagram and blocks.

You are allowed to use any amount of the following logarithmic, exponential and summer blocks:
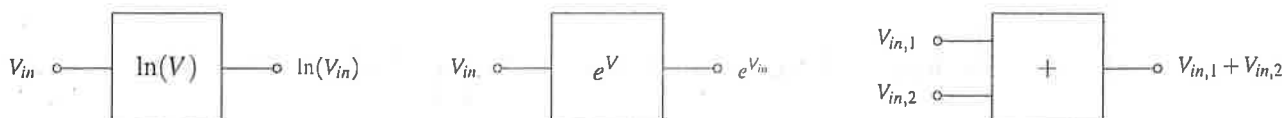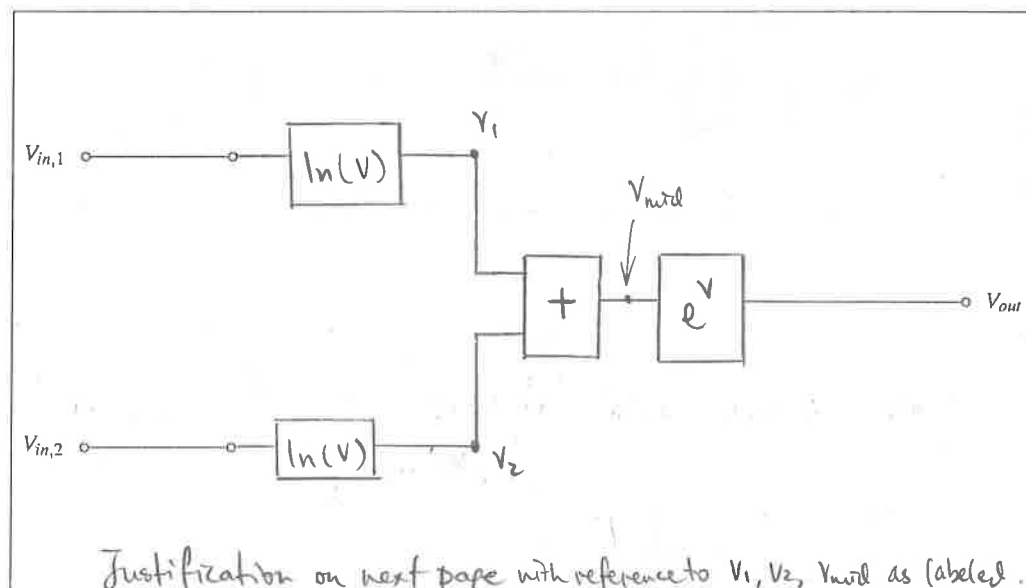

Figure 4.1: Function blocks you can use

In the box below, finish the **block diagram** so that the output is equal to $V_{out} = V_{in,1} V_{in,2}$. **Also provide a mathematical justification for why your block diagram works for full credit!**

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See previous page for labeled reference nodes.

Here, $V_1 = \ln(V_{in,1})$ and $V_2 = \ln(V_{in,2})$

So $V_{mid} = V_1 + V_2 = \ln(V_{in,1}) + \ln(V_{in,2}) = \ln(V_{in,1} \cdot V_{in,2})$

Thus, $V_{out} = e^{V_{mid}} = e^{\ln(V_{in,1} \cdot V_{in,2})} = V_{in,1} \cdot V_{in,2}$, as desired.

(One-liner: $V_{out} = e^{\ln(V_{in,1}) + \ln(V_{in,2})} = e^{\ln(V_{in,1} \cdot V_{in,2})} = V_{in,1} \cdot V_{in,2}$)   d.t.l).

While you were making your block diagram, your friend Vlad was watching over your shoulder. With his near-infinite knowledge of circuits, he predicts what you are trying to do and gives you a "magic device" that he claims will help you out.

This device is shown below: This device has the following IV characteristics:
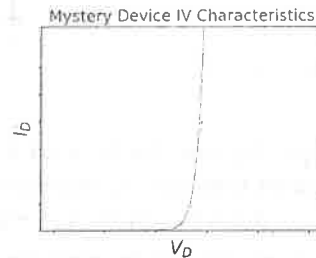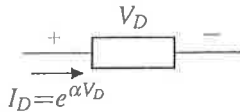
Numerically, this can be stated as:

$$I_D = e^{\alpha V_D}$$

where α is some constant and V_D is the voltage across the device as in the above figure.

IMPORTANT: Note the passive sign convention.

(b) (3 points) Your friend Gireeja sees you looking confused, and reads what you've done so far. She decides to help push you in the right direction. She reminds you that if you know how a device reacts when you put current through it or apply a voltage across it, then you are in a good spot to understand how it would behave in any circuit.

i. (2 points) **In the following circuit, express the voltage across the device, V_D, as a function of the source current, I_S.**
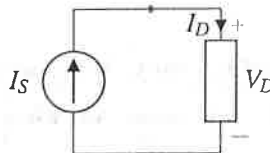
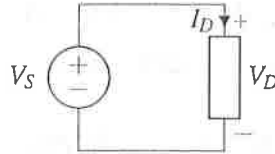Using simple nodal analysis (KCL), we could state that: $I_S = I_D = e^{\alpha V_D}$.

Thus, $\ln(I_S) = \ln(e^{\alpha \cdot V_D}) = \alpha \cdot V_D$, which gives us that:

$$V_D = \boxed{\frac{1}{\alpha} \ln(I_S)}$$

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ii. (1 point) **In the following circuit, express the current through the device, I_D, as a function of the source voltage, V_S.**


Again, using simple nodal analysis (i.e. KVL), we can state that $V_S = V_D$.

Since we have that $I_D = e^{\alpha \cdot V_D}$, so we can write.

$$I_D = \boxed{e^{\alpha \cdot V_S}}$$

(c) (6 point) By now we have all the tools necessary to make this a reality. Unfortunately, as you may remember from the touchscreen lab, ideal current sources don't exist. As such, we will use an op-amp to create the behavior we are interested in. Let's analyze the circuit below in steps.

Hint: *Your answers for some parts of this problem will not perfectly match with the ideal scenario in part (a). Don't be afraid of getting different answers! After all, a block diagram is just a starting point.*
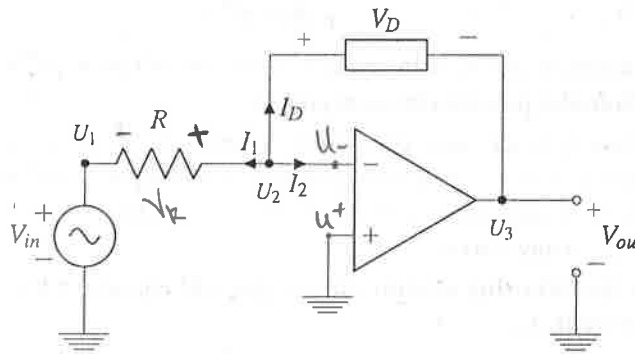
i. (1 point) **write V_D in terms of V_{out} only. You may assume the circuit is in negative feedback.**

Given that the circuit is in negative feedback, so $U_- = U_+ = 0V$ by the Golden Rules.

Since $U_2 - U_3 = V_D$ and $U_3 = V_{out}$, $U_2 = U_- = 0V$,

so $V_D = \boxed{-V_{out}}$

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ii. (1 point) **Now write out the KCL expression at the U_2 node in terms of the three currents I_D, I_1 and I_2.**

$$I_D + I_1 + I_2 = 0 \text{ A}$$

iii. (4 points) **Use the IV relations for each device and the KCL expression above to find V_{out} as a function of V_{in}.**

Firstly, $U_1 = V_{in}$ and $U_2 = 0V$, so labeling R with passive sign convention,

we have that $V_R = U_2 - U_1 = -V_{in}$.

Thus, $I_1 = \dfrac{V_R}{R} = \dfrac{-V_{in}}{R}$.

Then, using Golden Rules again, so $I_2 = 0A$.

Thus, since in part (c), we have $I_D + I_1 + I_2 = 0A$,

$$\text{So} \quad I_D = -I_1 - I_2 = \frac{V_{in}}{R}$$

Using a similar derivation as in part (b), since $I_D = e^{\alpha \cdot V_D}$,

$$\text{So} \quad V_D = \frac{1}{\alpha}(\alpha \cdot V_D) = \frac{1}{\alpha} \cdot \ln(e^{\alpha \cdot V_D}) = \frac{1}{\alpha} \cdot \ln(I_D) = \frac{V_{in}}{\alpha R}$$
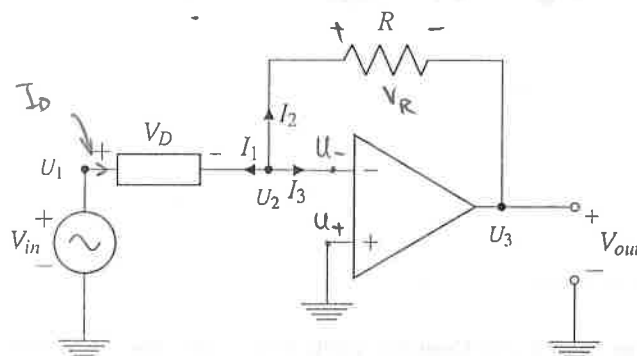
Thus, since in part (c)/(ii) we have $V_D = -V_{out}$,

$$\text{So} \quad V_{out} = -V_D = \boxed{-\frac{1}{\alpha R} V_{in}}$$

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(d) (6 points) Just one more piece left to analyze. You may assume the circuit is in negative feedback. **In the circuit below, write V_{out} as a function of V_{in}.**

Hint: *Note the direction of the voltage drop, V_D. Be careful when writing your KCL expressions to account for this!*

Since the circuit is in negative feedback, so the Golden Rules gives as.

$U_2 = U_- = U_+ = 0V$ and $I_3 = 0A$.

Since $U_1 = V_{in}$ and so, $V_D = U_1 - U_2 = V_{in} - 0V = V_{in}$.

The property of the "magic device" gives us that I_D, as labeled,

is $I_D = e^{\alpha \cdot V_D} = e^{\alpha \cdot V_{in}}$

So   a simple KCL gives $I_1 = -I_D = -e^{\alpha \cdot V_{in}}$ (A)

Now KCL at node U_2 gives $I_1 + I_2 + I_3 = 0 A$

So $I_2 = -I_1 - I_3 = e^{\alpha \cdot V_{in}}$ (A)

So $V_R = I_2 \cdot R = R \cdot e^{\alpha \cdot V_{in}}$ (V)

and since $V_{out} = U_3 - 0V = U_3$ and $U_2 - U_3 = V_R$,

So $V_{out} = U_3 = U_2 - V_R = 0V - R \cdot e^{\alpha \cdot V_{in}}$

So, $V_{out} = \boxed{- R \cdot e^{\alpha \cdot V_{in}}}$ (V)

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(e) (5 points) Given the polarity of circuits in parts (c) and (d), to put everything together, you need to implement **an inverting summer circuit, where** $V_{out} = -(V_{in_1} + V_{in_2})$.

You may use up to 1 op-amp (you do not need to label the power supplies) and as many resistors as you wish. **Any resistors used must be assigned a value. You must mathematically justify your circuit's behavior for full credit.**

As labeled, $R_1 = R_2 = R_3 = 1k\Omega$, $R_4 = 2R_3 = 2k\Omega$.

Since on the left side, I just built a summing circuit, so

we have $U_1 = V_{in_1} \cdot \dfrac{R_2}{R_1+R_2} + V_{in_2} \cdot \dfrac{R_1}{R_1+R_2} = \dfrac{1}{2}(V_{in_1} + V_{in_2})$

(N.B. This can also be analyzed via superposition, but detail is excluded).

Then, on the right half, I built an inverting op-amp, so

$U_2 = U_1 \cdot \left(-\dfrac{R_4}{R_3}\right) = U_1 \left(-\dfrac{2k\Omega}{1k\Omega}\right) = -2U_1 = -(V_{in_1} + V_{in_2})$

Finally, since $V_{out} = U_2 - 0V = U_2$, so we have the desired:

$V_{out} = -(V_{in_1} + V_{in_2})$. Q.E.D.

And that's it! Now that you have all the individual parts, you can string them together and start selling calculators. You may need to make a few small changes to your original design, but you can do it!

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(f) (1 points) Controversial question: Do you wet the toothbrush before or after applying toothpaste? *This problem is for fun. You can write whatever you want or nothing at all and you won't be penalized.*

I wet the toothbrush <u>after</u> applying toothpaste.
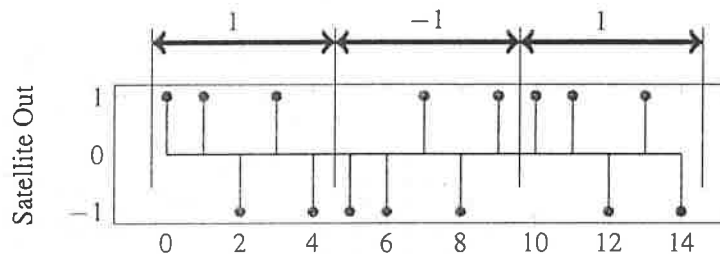
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5. *Gold*en Positioning System (20 points)

In this problem we will explore how real GPS systems work, and touch on a few aspects of implementing GPS receivers.

A Gold code is a sequence of 1's and -1's that has a high autocorrelation at a shift of 0, and small autocorrelations otherwise. Every GPS satellite has a unique Gold code assigned to it, and users are aware of the Gold code used by each satellite. The plot below shows a Gold code of length 5.
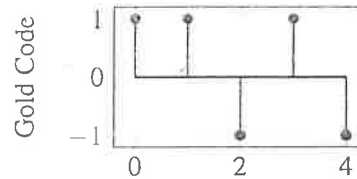
Each GPS satellite has a message that it transmits by modulating the Gold code. When the satellite is transmitting a 1, it sends just the Gold code sequence. When the satellite is transmitting a -1, it sends -1 times the Gold code. For example, if a satellite were transmitting the message [1, -1, 1], it would transmit the following (just as you have seen in the GPS homework problem):

(a) (2 Points) Suppose you receive the following from a GPS satellite that has the same Gold code as above. **What message is the satellite transmitting?**
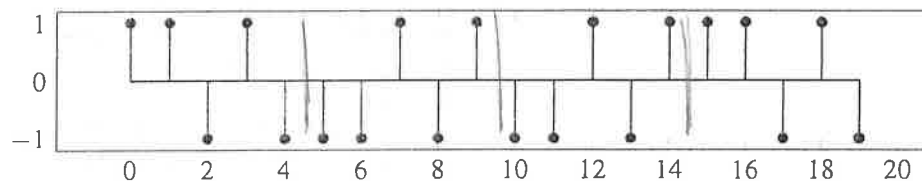
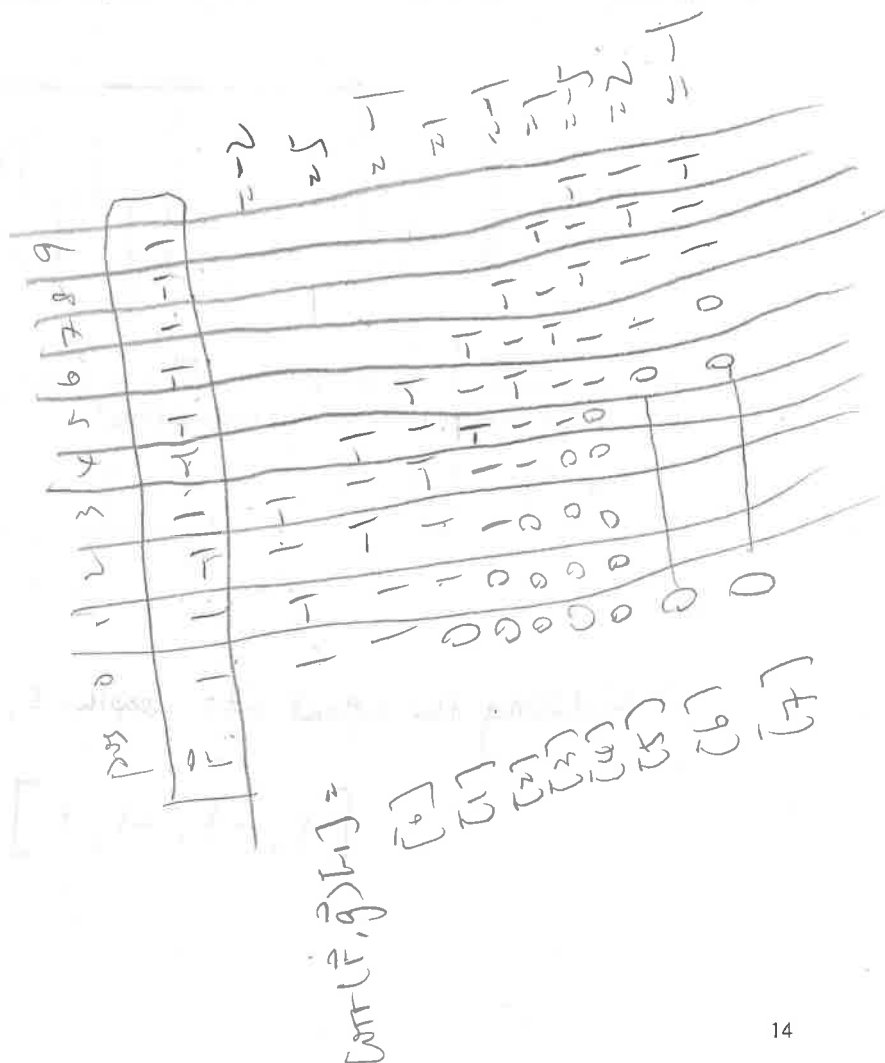Splitting the signal into length-5, we have that the message is:

$$[1, -1, -1, 1]$$

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Extra page for scratchwork.
**Work on this page will NOT be graded.**

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(b) (4 points) In order to find the message being sent by the satellite, the receiver will find the linear cross-correlation of the received signal with a replica of the satellite Gold code.

We need to find the **linear cross-correlation** of the signals shown below given by

$$\text{corr}(\vec{r}, \vec{g})[k] = \sum_{i=-\infty}^{\infty} r[i]g[i-k]$$

where $r[n]$ is the received signal and $g[n]$ is the Gold code sequence. *Note that neither of these signals is periodic in this part.*

Plot the values of $\text{corr}(\vec{r}, \vec{g})[k]$ for $-1 \leq k \leq 7$. What is the significance of the peaks in the linear cross-correlation?
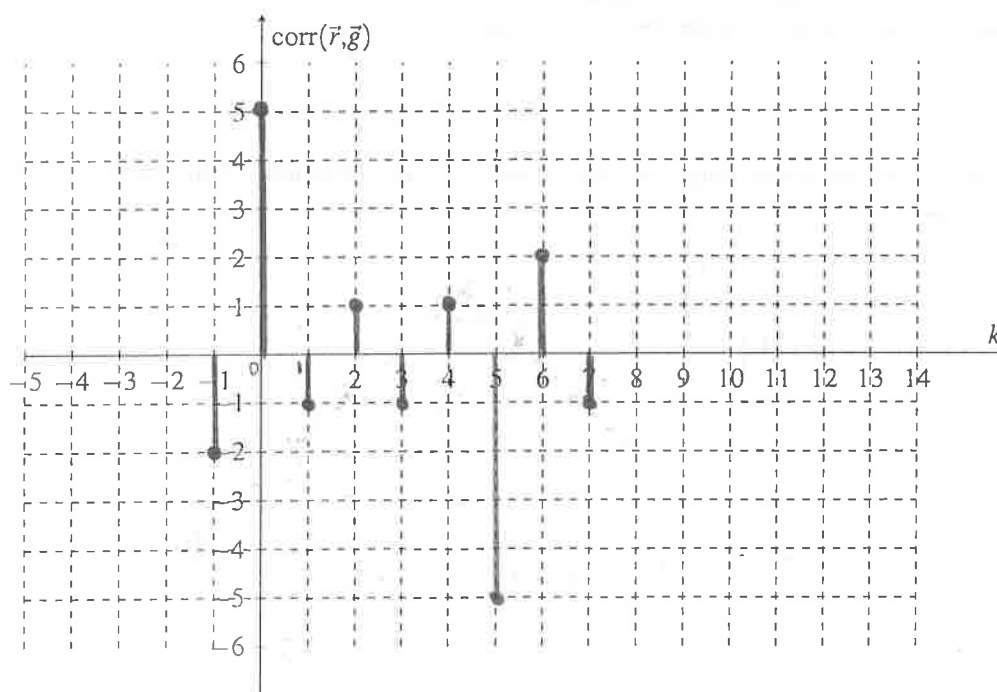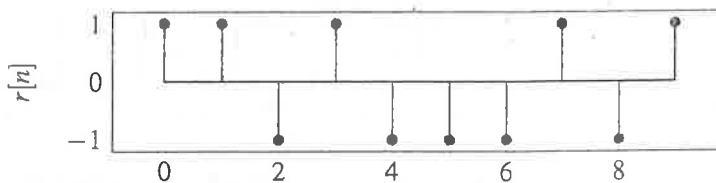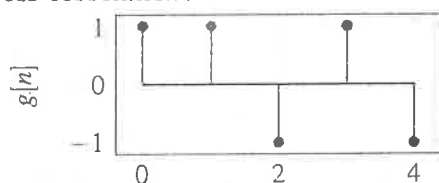
The peaks indicate the most correlated shift of the signal, i.e. the delay (shift) of the received message.

Figure 5.1: Your answer to part (b)

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(c) (3 points) Real GPS receivers have specialized hardware to perform cross-correlation using circuits. However, since these transmissions are continuous signals instead of discrete values, we will model the received signal $r(t)$ and the Gold code signal $g_2(t)$ as square waves, as shown in the plot below. Notice that $g_2(t)$ shows two periods of the Gold code.
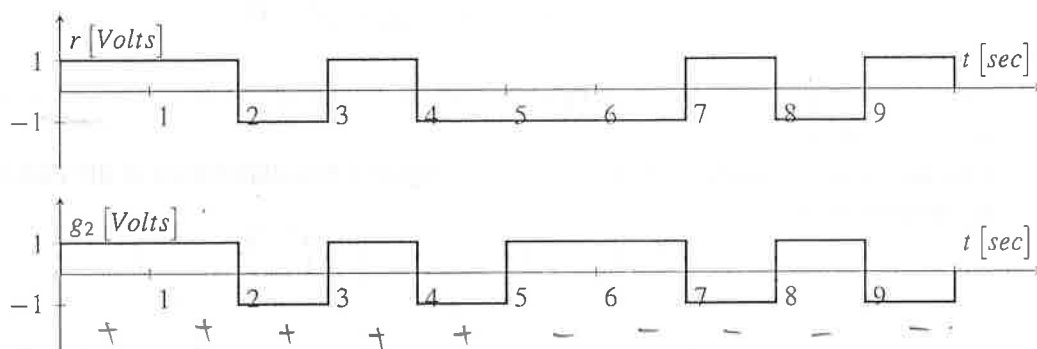
An essential hardware block to implementing a GPS correlator is *Multiply and Integrate*. This circuit multiplies its two inputs, then integrates the product over time. For example, the output of the *Multiply and Integrate* block given the above two inputs would be :

$$y(t) = \int_0^t r(\tau)g_2(\tau)d\tau$$

where $y(t)$ is the circuit output at time t. **Draw $y(t)$ as a function of time, for $t = 0$ to $t = 10$ sec.**
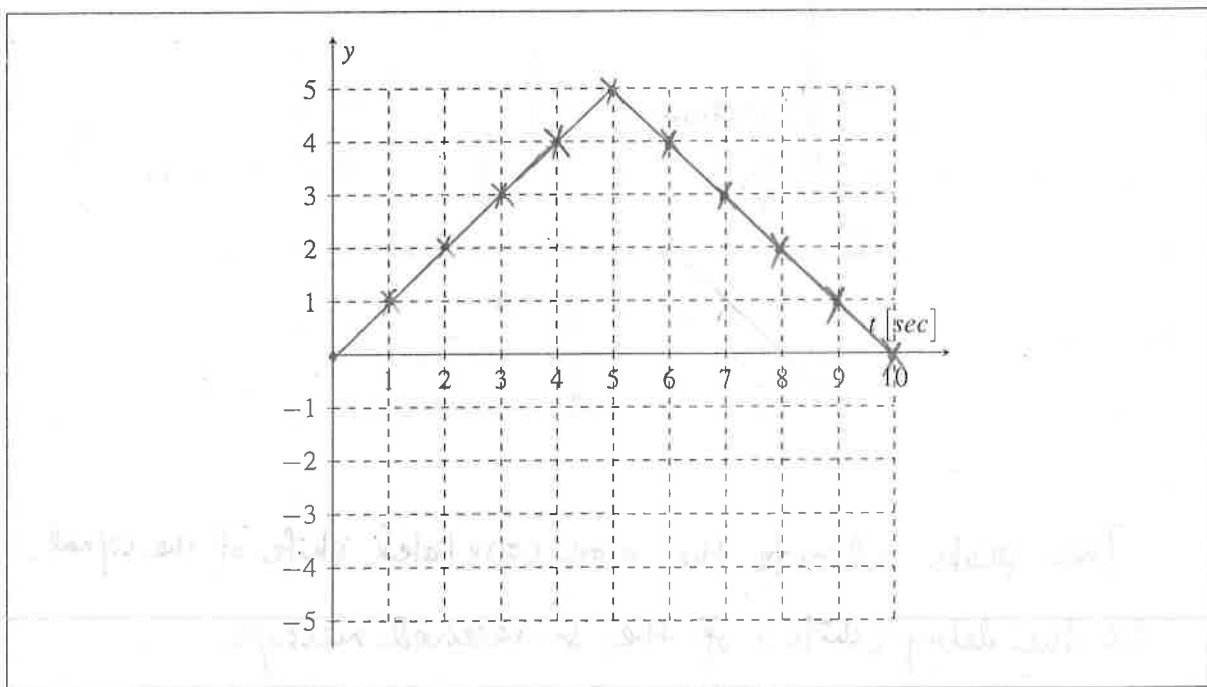
Figure 5.2: Your answer to part (c)

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(d) (7 points) The *Multiply and Integrate* circuit has to integrate the product of the two signals, $g_2(t)$ and $r(t)$. Your friend has already designed the multiplier circuit. A Thévenin model of her multiplier circuit is shown below:
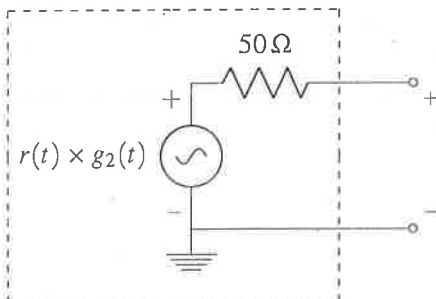
Design a circuit to complete the *Multiply and Integrate* block where the output of your circuit as a function of time, t, is:

$$v_{out}(t) = \int_0^t r(\tau) \times g_2(\tau)d\tau.$$

You may use **the multiplier circuit (Thevenin equivalent provided in the answer box), up to two op-amps, a capacitor, and two resistors** to implement your circuit. **Clearly define v_{out} and show numerical values of your circuit components.** As long as your circuits are in negative feedback, you may assume the voltage rails to your op-amps are large enough to not affect the output.

$C = 1mF, R_1 = R_2 = 1k\Omega.$

First, O_1 is just a unity pambufer, so $U_{mid}(t) = r(t) \times g_2(t)$.

Then, the integrator O_2 performs to integrate the voltage of U_{mid},

So $V_{out}(t) = \int_0^t U_{mid}(t)dt = \int_0^t r(t) \times g_2(t) d\tau$ as desired.

$(a-2.1?).$
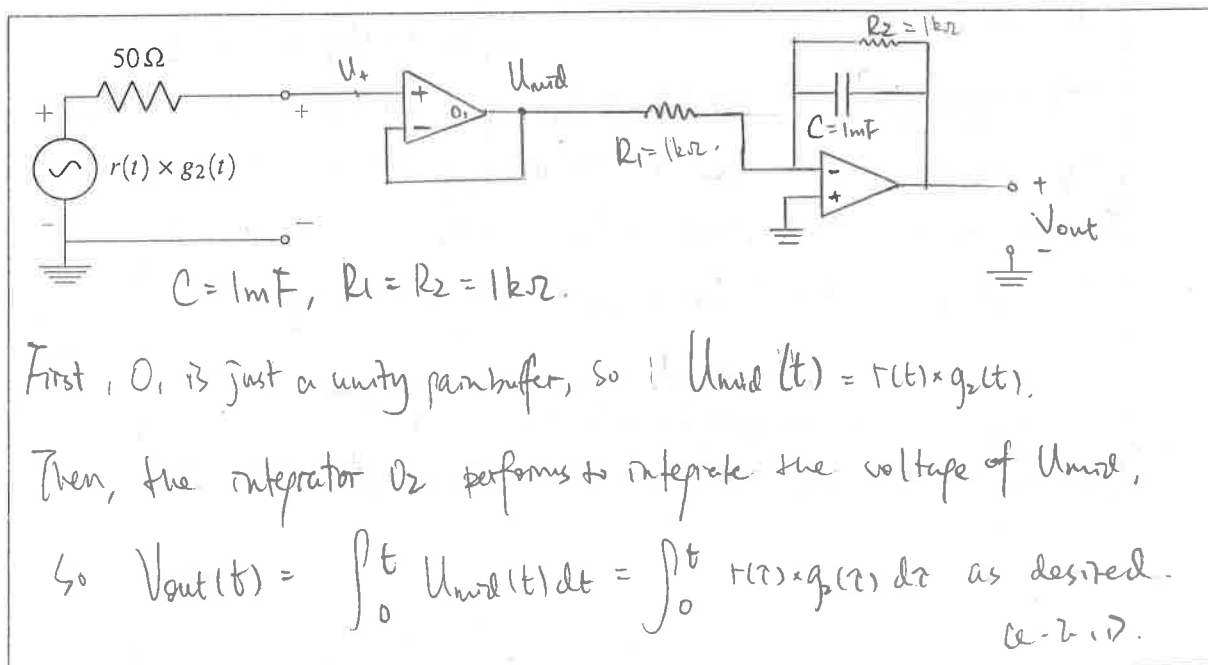
Figure 5.3: Your answer to part (d)

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(e) (2 points) Receivers also need to use the received data to calculate the position of the satellite. Each receiver will receive data from k satellites. Each satellite transmits the time, S_i, at which it started sending the message, where i is the index of the satellite, and $1 \leq i \leq k$. The receiver knows the time, T_i, at which each message arrives. You may assume the receiver and transmitter clocks are synchronized perfectly. Let c represent the speed of the signal.

Find an expression for d_i, the distance between the receiver and the ith satellite, in terms of S_i, T_i, and other relevant parameters.

$$\Delta T_i = T_i - S_i \quad \text{and} \quad v_i = c$$

$$\text{So} \quad d_i = v_i \cdot \Delta T_i = \boxed{c(T_i - S_i)}$$

(f) (2 points) Each satellite's position in 3D space is (u_i, v_i, w_i), where $1 \leq i \leq k$. The receiver position is given by (x, y, z). We need a linear system of equations the receiver can use to solve for its position, $\vec{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Due to limitations of the hardware, the receiver can only handle **linear** systems of equations.

How many satellites must the receiver get data from to solve for its position?

The equation for the ith satellite is: $d_i = \sqrt{(x-u_i)^2 + (y-v_i)^2 + (z-w_i)^2} = c(T_i - S_i)$

$\Rightarrow$ (squaring both sides) $(x-u_i)^2 + (y-v_i)^2 + (z-w_i)^2 = c^2(T_i - S_i)^2$.

Now, we need to "sacrifice" a satellite (say satellite 1) to linearize the equations (subtracting the equation for 1st satellite from the rest, here $2 \leq i \leq k$, $i \neq 1$).

$\Rightarrow 2(u_1 - u_i)x + 2(v_1 - v_i)y + 2(w_1 - w_i)z = c^2\left[(T_i - S_i)^2 - (T_1 - S_1)^2\right] + u_i^2 + v_i^2$

$\underbrace{}_{\text{Constants.} \Rightarrow} + w_i^2 - u_i^2 - v_i^2 - w_i^2$

which becomes linear, and we need 3 such equations to solve for 3 unknowns, i.e. $x, y,$ and z, so we need 3 more satellites' data (in addition to the sacrificed satellite 1) Thus, the receiver must get data from at least ④ satellites.

$x^2 - 2u_i x + u_i^2 + y^2 - 2v_i y + v_i^2 + z^2 - 2w_i z + w_i^2$

$-x^2 + 2u_1 x - u_1^2 - y^2 + 2v_1 y - v_1^2 - z^2 + 2w_1 z - w_1^2$

$= c^2(T_i - S_i)^2 - c^2(T_1 - S_1)^2$

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6. Besto Pesto (23 points)

Your TA Laura is struggling to keep her basil plant alive! She needs your help to determine how much water and sunlight her plant needs.

Let $x_h[k]$ be the plant's height on day k and $x_\ell[k]$ be the number of leaves on the plant on day k. The vector $\vec{x}[k] = \begin{bmatrix} x_h[k] \\ x_\ell[k] \end{bmatrix}$ defines the state of the plant. The evolution of the basil plant from one day to the next is defined by the **approximate** mathematical model:

$$\vec{x}[k+1] = A\vec{x}[k] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_h[k] \\ x_\ell[k] \end{bmatrix}. \tag{1}$$

$X_h[k+1] = a_{11} \cdot X_h[k] +$

(a) (3 points) Our first goal is to estimate the elements of state transition matrix, A: $a_{11}, a_{12}, a_{21}, a_{22}$. To do this we count the leaves and measure the height for the first N time steps, i.e. we know $\{\vec{x}[0], \vec{x}[1], \ldots, \vec{x}[N]\}$. Setup a least squares problem to estimate $\vec{a} = \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix}$:

$$\hat{\vec{a}} = \operatorname*{argmin}_{\vec{a}} \|M\vec{a} - \vec{b}\|^2. \tag{2}$$

Write the matrix, M, and vector, $\vec{b}$, that would be used in the above least squares problem for $N = 3$.

$$M = \begin{bmatrix} \vec{x}[0]^T & 0 & 0 \\ 0 & 0 & \vec{x}[0]^T \\ \vec{x}[1]^T & 0 & 0 \\ 0 & 0 & \vec{x}[1]^T \\ \vec{x}[2]^T & 0 & 0 \\ 0 & 0 & \vec{x}[2]^T \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} \vec{x}[1] \\ \vec{x}[2] \\ \vec{x}[3] \end{bmatrix} = \begin{bmatrix} x_h[1] \\ x_\ell[1] \\ x_h[2] \\ x_\ell[2] \\ x_h[3] \\ x_\ell[3] \end{bmatrix}$$

↑
Concatenate vertically.

Whichever version is preferred ...

$$\left(\text{Since } X_h[k+1] = a_{11} \cdot X_h[k] + a_{12} \cdot X_\ell[k], \right.$$
$$\left. X_\ell[k+1] = a_{21} \cdot X_h[k] + a_{22} \cdot X_\ell[k]. \right)$$

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(b) (4 points) Now we would like to use the observations of the current state of the plant to determine the initial state: $\vec{x}[0] = \begin{bmatrix} x_h[0] \\ x_\ell[0] \end{bmatrix}$. For convenience of measurement, we use a Launchpad with a camera attached that periodically takes pictures and gives us a measurement vector, $\vec{y}[k] = \begin{bmatrix} y_h[k] \\ y_\ell[k] \end{bmatrix} \in \mathbb{R}^2$, where $y_h[k]$ is a noisy measurement of the plant's height, $y_\ell[k]$ is a noisy count of leaves. Since the measurement might not be accurate, we will apply least squares to estimate $\vec{x}[0]$.

We have the dynamics and sensor models,

$$\vec{x}[k+1] = A\vec{x}[k], \tag{3}$$

$$\vec{y}[k] = C\vec{x}[k] + \vec{n}[k], \tag{4}$$

where $\vec{x}[k]$ is as before, $\vec{y}[k]$ is the sensor reading at time k, and $\vec{n}[k] \in \mathbb{R}^2$ is sensor noise/ error. The matrices C and A are given by

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{5}$$

$$A = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.5 \end{bmatrix}. \tag{6}$$

We took two actual readings from the Launchpad: $\vec{y}[0] = \begin{bmatrix} y_h[0] \\ y_\ell[0] \end{bmatrix}$ and $\vec{y}[1] = \begin{bmatrix} y_h[1] \\ y_\ell[1] \end{bmatrix}$. Now we need to use **both readings** to setup a least squares problem **to estimate the initial state of the plant, $\vec{x}[0]$**.

$$\hat{x}[0] = \underset{\vec{x}[0]}{\operatorname{argmin}} \|M\vec{x}[0] - \vec{b}\|^2. \tag{7}$$

Write the matrix, M, and vector, $\vec{b}$, that would be used in the above least squares problem to estimate $\vec{x}[0]$.

Given $\vec{y}[1] = C\vec{x}[1] + \vec{n}[1] = CA\vec{x}[0] + \vec{n}[1]$

and $\vec{y}[0] = C\vec{x}[0] + \vec{n}[0]$, and $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.5 \end{bmatrix}$

$\Rightarrow CA = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.5 \end{bmatrix}$

stacking $\vec{y}[0], \vec{y}[1]$ like part a

So

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.8 & 0 \\ 0 & 0.5 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} \vec{y}[0] \\ \vec{y}[1] \end{bmatrix} = \begin{bmatrix} y_h[0] \\ y_\ell[0] \\ y_h[1] \\ y_\ell[1] \end{bmatrix}$$

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(c) (2 points) The evolution of the basil plant from one day to the next is defined by:

$$\vec{x}[k+1] = A\vec{x}[k],$$

$$A = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.5 \end{bmatrix},$$

$$\vec{x}[0] \neq \vec{0}.$$

What will happen to the number of leaves and the height of the plant as $k \to \infty$?

Given $\vec{x}[k+1] = A\vec{x}[k] = A^2\vec{x}[k-1] = \cdots = A^{k+1}\vec{x}[0]$.

Since $A = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.5 \end{bmatrix}$, is a diagonal matrix, so $A^n = \begin{bmatrix} 0.8^n & 0 \\ 0 & 0.5^n \end{bmatrix}$.

$\Rightarrow$ $\vec{x}[k+1] = \begin{bmatrix} 0.8^{k+1} & 0 \\ 0 & 0.5^{k+1} \end{bmatrix} \cdot \vec{x}[0]$. $\Rightarrow \begin{cases} x_h[k+1] = 0.8^{k+1} x_h[0] \\ x_\ell[k+1] = 0.5^{k+1} x_\ell[0] \end{cases}$

and also $\lim\limits_{k\to\infty} 0.8^{k+1} = \lim\limits_{k\to\infty} 0.5^{k+1} = 0$, $\vec{x}[0] \neq \vec{0}$

So $\lim\limits_{k\to\infty} x_h[k+1] = \lim\limits_{k\to\infty} x_\ell[k+1] = 0$, i.e. Both numbers will become 0.

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(d) (6 points) Now suppose the system evolves according to a new state transition matrix A, where

$$\vec{x}[k+1] = A\vec{x}[k],$$

$$A = \begin{bmatrix} -1 & -3 \\ 4 & 6 \end{bmatrix}.$$

We want to diagonalize A to calculate the system evolution easily.

Find matrices V and Λ such that $A = V\Lambda V^{-1}$. Use this diagonalization to write out the value of $\vec{x}[N]$ in terms of $\vec{x}[0]$.

$(A - \lambda I)\vec{x} = \vec{0}$ $\Rightarrow$ $\det(A - \lambda I) = \det\left(\begin{bmatrix} -1-\lambda & -3 \\ 4 & 6-\lambda \end{bmatrix}\right) = 0$

So $-6+\lambda - 6\lambda + \lambda^2 + 12 = 0$ $\Rightarrow$ $\lambda^2 - 5\lambda + 6 = 0$

So $\lambda_1 = 2$ $\lambda_2 = 3$ $\Rightarrow$ $\Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

For $\lambda_1 = 2$ $A - \lambda_1 I = \begin{bmatrix} -3 & -3 \\ 4 & 4 \end{bmatrix}$ nullspace is span $\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$

So $\begin{cases} -3x_1 - 3x_2 = 0 \\ 4x_1 + 4x_2 = 0 \end{cases}$ $\Rightarrow$ $x_1 = -x_2$

So one of the eigenvectors is $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

For $\lambda_2 = 3$. So $A - \lambda_2 I = \begin{bmatrix} -4 & -3 \\ 4 & 3 \end{bmatrix}$ $\Rightarrow$ $x_1 = -\frac{3}{4}x_2$

So one of the eigenvectors (corresponding) is $\vec{v}_2 = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$

So $\boxed{V = \begin{bmatrix} 1 & 3 \\ -1 & -4 \end{bmatrix}}$ and $\Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

Now, we still have $\vec{x}[N] = A\vec{x}[N-1] = \cdots = A^N \vec{x}[0]$

Since $A^N = V\Lambda^N V^{-1}$, So $\vec{x}[N] = \begin{bmatrix} 1 & 3 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} 2^N & 0 \\ 0 & 3^N \end{bmatrix}\begin{bmatrix} 4 & 3 \\ -1 & -1 \end{bmatrix}\vec{x}[0]$

with $V^{-1} = -\begin{bmatrix} -4 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -1 & -1 \end{bmatrix}$

$\Downarrow$

$\begin{bmatrix} 2^{N+2} - 3^{N+1} & 3\cdot2^N - 3^{N+1} \\ -2^{N+2} + 4\cdot3^N & -3\cdot2^N + 4\cdot3^N \end{bmatrix}$

which gives us that $\boxed{\vec{x}[N] = \begin{bmatrix} 2^{N+2} - 3^{N+1} & 3\cdot2^N - 3^{N+1} \\ -2^{N+2} + 4\cdot3^N & -3\cdot2^N + 4\cdot3^N \end{bmatrix}\vec{x}[0]}$

$\begin{bmatrix} 1 & 3 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ -1 & -1 \end{bmatrix}$

$\begin{bmatrix} 2^N & 3^{N+1} \\ -2^N & -4\cdot3^N \end{bmatrix}\begin{bmatrix} 4 & 3 \\ -1 & -1 \end{bmatrix}$

$= \begin{bmatrix} 2^{N+2} - 3^{N+1} & 3\cdot2^N - 3^{N+1} \\ -2^{N+2} + 4\cdot3^N & -3\cdot2^N + 4\cdot3^N \end{bmatrix}$

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(e) (2 points) Assuming that

$$\vec{x}[k+1] = A\vec{x}[k],$$

$$A = \begin{bmatrix} -1 & -3 \\ 4 & 6 \end{bmatrix},$$

$$\vec{x}[0] \neq \vec{0},$$

what will happen to the number of leaves and the height of the plant as $k \to \infty$?

Using the result from the diagonalization, we can see the numbers would vary between +∞ and -∞ ... ???

height → -∞
leaves → +∞

Using result from part (d) and taking the limit $k \to \infty$

(f) (3 points) Laura decides to enlist the help of her friend Vijay to design an automatic watering and shading system. Let $u_w[k] \in \mathbb{R}$ be the amount of water added or removed from the soil each day and let $u_s[k] \in \mathbb{R}$ be the amount of time the plant spends in the sun each day. Vijay decides to design the automatic control system as a negative feedback controller (inspired by op-amps)!

The new evolution of the basil plant from one day to the next is defined by:

$$\begin{bmatrix} 2 & 9 \\ -2 & -12 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -3 \\ 4 & 6 \end{bmatrix}$$

$$\vec{x}[k+1] = A\vec{x}[k] + \vec{u}[k] = A\begin{bmatrix} x_h[k] \\ x_\ell[k] \end{bmatrix} + \begin{bmatrix} u_w[k] \\ u_s[k] \end{bmatrix},$$

$$A = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.5 \end{bmatrix},$$

$$\vec{u}[k] = -F\vec{x}[k] = -\begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix} \begin{bmatrix} x_h[k] \\ x_\ell[k] \end{bmatrix}.$$

We want to express the new system as $\vec{x}[k+1] = R\vec{x}[k]$, where R is the new state transition matrix. **Find R in terms of f_1, f_2, f_3, f_4, and numerical constants.**

$$\vec{x}[k+1] = A\vec{x}[k] + \vec{u}[k]$$

$$= A\vec{x}[k] - F\vec{x}[k]$$

$$= (A-F)\vec{x}[k]$$

So   R = A - F.

Since $A - F = \begin{bmatrix} 0.8 - f_1 & -f_2 \\ -f_3 & 0.5 - f_4 \end{bmatrix}$

(next page)

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$$\text{So } R = \begin{bmatrix} 0.8 - f_1 & -f_2 \\ -f_3 & 0.5 - f_4 \end{bmatrix}$$

(g) (3 points) Laura wants her basil plant to be at steady state; she wants the number of leaves and the height of the plant to stay the same over time. Vijay needs to choose values for the elements of matrix **F** from part (f) and asks for your help. Assume that $f_2 = 0$ and $f_3 = 0$. **Choose values for f_1 and f_4 that will keep the height of the plant and number of leaves constant over time.**

This means we want $\lambda = 1$ to be an eigenvalue of R with $f_2 = f_3 = 0$. So $R = \begin{bmatrix} 0.8 - f_1 & 0 \\ 0 & 0.5 - f_4 \end{bmatrix}$ we want it to be I_2 identity matrix.

and so $0.8 - f_1 = 1$ and $0.5 - f_4 = 1$

Null Space : $\boxed{f_1 = -0.2, \quad f_4 = -0.5}$

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7. **OMP for Matrices (27 points)**

In the real world, it is not computationally practical to directly solve for the eigenbasis for large matrices as you might do for small matrices on paper. You need to build an algorithm that sequentially computes the eigenbasis for a square symmetric matrix $Q = A^T A$ (Note: Any matrix that can be written with $A \in \mathbb{R}^{N \times N}$ in this form is symmetric).

To accomplish this we are given access to a function,

$$(\vec{v}_1, \lambda_1) = f(Q), \tag{8}$$

that returns the **largest** eigenvalue of the matrix Q and the corresponding eigenvector. We will use this function to build a sequential greedy algorithm (similar to Orthogonal Matching Pursuit) that computes the eigenvectors and eigenvalues in descending order.

IMPORTANT: Throughout the problem, you should assume that the magnitude of each eigenvector is 1 (i.e. $\|\vec{v}_i\| = 1$), that the eigenvalues are real, unique, and distinct, and that $\lambda_1 > \lambda_2 > \cdots > \lambda_N > 0$.

(a) (2 points) Show that the matrix $Q = A^T A$ is symmetric (i.e. $Q = Q^T$) when,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}. \tag{9}$$

Given $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, so $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

So $Q = A^T A = \begin{bmatrix} 7 & 14 \\ 14 & 20 \end{bmatrix}$

And so $Q^T = \begin{bmatrix} 7 & 14 \\ 14 & 20 \end{bmatrix} = Q$

So Q is symmetric. ✓ Q.E.D.

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(b) (4 points) Given two distinct eigenvalue/eigenvector pairs, $(\lambda_k, \vec{v}_k)$ and $(\lambda_\ell, \vec{v}_\ell)$, show that for the symmetric matrix $\mathbf{Q} = \mathbf{A}^T\mathbf{A}$, if $\lambda_k \neq \lambda_\ell$ then $\langle \vec{v}_k, \vec{v}_\ell \rangle = 0$ (i.e. any pair of eigenvectors with distinct eigenvalues is orthogonal). Hint: *Consider:*

$$\mathbf{Q}\vec{v}_k = \lambda_k \vec{v}_k, \tag{10}$$

$$\vec{v}_\ell^T \mathbf{Q} = \lambda_\ell \vec{v}_\ell^T. \tag{11}$$

From Eq (10), Multiply $\vec{v}_\ell^T$ to both sides $\Rightarrow \mathbf{Q}\vec{v}_k \vec{v}_\ell^T = \lambda_k \vec{v}_k \vec{v}_\ell^T$ (12)

From Eq (11), multiply both sides by $\vec{v}_k \Rightarrow \vec{v}_k \vec{v}_\ell^T \mathbf{Q} = \vec{v}_k \lambda_\ell \vec{v}_\ell^T$ (13)

Since $\mathbf{Q} = \mathbf{A}^T\mathbf{A}$ is a symmetric matrix,

So matrix multiplication is commutative in this case

So $\mathbf{Q}\vec{v}_k \vec{v}_\ell^T = \vec{v}_k \vec{v}_\ell^T \mathbf{Q}$

and thus $\lambda_k \vec{v}_k \vec{v}_\ell^T = \vec{v}_k \lambda_\ell \vec{v}_\ell^T$ by Eq (12), (13)

So $\lambda_k \cdot \langle \vec{v}_k, \vec{v}_\ell \rangle = \lambda_\ell \cdot \vec{v}_k \cdot \vec{v}_\ell^T = \lambda_\ell \langle \vec{v}_k, \vec{v}_\ell \rangle$

Since $\lambda_k \neq \lambda_\ell$, so we must have that

$\langle \vec{v}_k, \vec{v}_\ell \rangle = 0$ Q.E.D.

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(c) (3 points) Let us consider **V** to be an orthonormal matrix, where

$$\mathbf{V} = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_N \\ | & | & & | \end{bmatrix}.$$ (12)

Show that if **V** is an orthonormal matrix, then $\mathbf{V}^T \cdot \mathbf{V} = \mathbf{I}$.

Given $V = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_N \\ | & & | \end{bmatrix}$ and that V is an orthonormal matrix

So $V^T V = \begin{bmatrix} - \vec{v}_1^T - \\ \vdots \\ - \vec{v}_N^T - \end{bmatrix} \cdot \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_N \\ | & & | \end{bmatrix}$

$= \begin{bmatrix} \vec{v}_1^T \vec{v}_1 & \vec{v}_1^T \vec{v}_2 & \cdots & \vec{v}_1^T \vec{v}_N \\ \vec{v}_2^T \vec{v}_1 & \vec{v}_2^T \vec{v}_2 & \cdots & \vec{v}_2^T \vec{v}_N \\ \vdots & & \ddots & \vdots \\ \vec{v}_N^T \vec{v}_1 & & \cdots & \vec{v}_N^T \vec{v}_N \end{bmatrix}$

Since we proved in part (b) that for $i \neq j$, then $\langle \vec{v}_i, \vec{v}_j \rangle = \vec{v}_i^T \vec{v}_j = 0$
($i,j,k \in \{1,\ldots,N\}$)
and that for any k, $\vec{v}_k^T \vec{v}_k = \langle \vec{v}_k, \vec{v}_k \rangle = \|\vec{v}_k\|^2 = 1^2 = 1$
(length =1, unit vector)

So $V^T V = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} = I.$ (a,b,i).

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(d) (3 points) Recall that the columns of orthonormal matrix $\mathbf{V}$ form a basis for $\mathbb{R}^N$, as you proved in the discussion sections. Assume $\vec{\alpha} = [\alpha_1 \dots \alpha_n]^T$ represents $\vec{x}$ in the basis of $\mathbf{V}$. Find $\langle \vec{v}_i, \vec{x} \rangle$.

Hint: *Write $\vec{x}$ as a linear combination of the column vectors $\vec{v}_1, \cdots, \vec{v}_N$.*

Given information

$$\Rightarrow \vec{x} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_N \vec{v}_N$$

So for any i, we have, using linearity of inner product,

So $\langle \vec{v}_i, \vec{x} \rangle = \langle \vec{v}_i, \alpha_1 \vec{v}_1 \rangle + \cdots + \langle \vec{v}_i, \alpha_i \vec{v}_i \rangle$

$\qquad + \langle \vec{v}_i, \alpha_{i+1} \vec{v}_{i+1} \rangle + \cdots + \langle \vec{v}_i, \alpha_N \vec{v}_N \rangle$

$\qquad = \alpha_1 \langle \vec{v}_i, \vec{v}_1 \rangle + \cdots + \alpha_i \langle \vec{v}_i, \vec{v}_i \rangle$

$\qquad + \cdots + \alpha_N \langle \vec{v}_i, \vec{v}_N \rangle$

Again, using part (b) and length-1 property,

So $\langle \vec{v}_i, \vec{v}_j \rangle = 0$ for $i \neq j$ and $\langle \vec{v}_i, \vec{v}_i \rangle = 1$

So $\langle \vec{v}_i, \vec{x} \rangle = 0 + \cdots + \alpha_i \cdot 1 + 0 + \cdots + 0$

$\qquad = \boxed{\alpha_i}$

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(e) (4 points) Let us define,

$$\mathbf{V}^{(2)} = \begin{bmatrix} | & & | \\ \vec{v}_2 & \cdots & \vec{v}_N \\ | & & | \end{bmatrix}. \qquad N \cdot (N-1). \tag{13}$$

Assume $\vec{\alpha} = [\alpha_1 \ldots \alpha_N]^T$ represents $\vec{x}$ in the basis of $\mathbf{V}$. Find $\hat{\vec{x}}$, the projection of a vector $\vec{x} \in \mathbb{R}^N$ onto the columns of the orthonormal matrix $\mathbf{V}^{(2)}$:

$$\hat{\vec{x}} = \mathrm{proj}_{\mathrm{Col}(\mathbf{V}^{(2)})}(\vec{x}). \tag{14}$$

What is the magnitude of the error vector, $\vec{e} = \hat{\vec{x}} - \vec{x}$?

Using equations, so

$$\hat{\vec{x}} = \mathrm{proj}_{Col(V^{(2)})}(\vec{x}) = V^{(2)} \cdot \left(V^{(2)T} V^{(2)} \right)^{-1} \cdot V^{(2)T} \cdot \vec{x}$$

Since $V^{(2)}$ is still an orthonormal matrix with width ≤ height,

So $V^{(2)T} V^{(2)} = I$

and so $\hat{\vec{x}} = V^{(2)} \cdot I \cdot V^{(2)T} \vec{x}$

So $\hat{\vec{x}} = V^{(2)} V^{(2)T} \vec{x}$

So $V^{(2)T} \left(\hat{\vec{x}} - \vec{x} \right) = \underbrace{V^{(2)T} \cdot V^{(2)}}_{I} \cdot V^{(2)T} \vec{x} - V^{(2)T} \vec{x}$

$$= V^{(2)T} \vec{x} - V^{(2)T} \vec{x} = \vec{0}$$

Since $V^{(2)} \neq \vec{0}$, so $V^{(2)T} \neq \vec{0}$, and so it must be the case that

$\vec{e} = \hat{\vec{x}} - \vec{x} = \vec{0}$. and Thus, $\|\vec{e}\| = \boxed{0.}$

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(f) (4 points) We know that because $\mathbf{Q}$ is symmetric, $\mathbf{V}$ is an orthonormal matrix. We use the idea of diagonalization and part (c) of the problem to express $\mathbf{Q}$ as:

$$\mathbf{Q} = \mathbf{A}^T\mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^{-1} = \mathbf{V}\Lambda\mathbf{V}^T = \sum_{i=1}^{N} \lambda_i \vec{v}_i \vec{v}_i^T \tag{15}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_N \end{bmatrix} \tag{16}$$

Let $\mathbf{Q}^{(2)} = \mathbf{Q} - \lambda_1 \vec{v}_1 \vec{v}_1^T$. Thus, $\mathbf{Q}^{(2)}$ represents $\mathbf{Q}$ after the component associated with direction $\vec{v}_1$ is removed. Show that $\vec{v}_1$ is in the null space of $\mathbf{Q}^{(2)}$.

Hint: *Can you write $\mathbf{Q}^{(2)}$ using Eq. (15)?*

Given that Eq(15) $\gg$ $Q = \sum_{i=1}^{N} \lambda_i \vec{v}_i \vec{v}_i^T$

So $Q^{(2)} = Q - \lambda_1 \vec{v}_1 \vec{v}_1^T = \sum_{i=2}^{N} \lambda_i \vec{v}_i \vec{v}_i^T$ (simple subtraction)

Thus, $Q^{(2)} \vec{v}_1 = \left(\sum_{i=2}^{N} \lambda_i \vec{v}_i \vec{v}_i^T \right) \cdot \vec{v}_1$

$= \lambda_2 \vec{v}_2 \cdot \left(\vec{v}_2^T \cdot \vec{v}_1 \right) + \lambda_3 \cdot \vec{v}_3 \left(\vec{v}_3^T \cdot \vec{v}_1 \right) + \cdots$

$+ \lambda_N \vec{v}_N \cdot \left(\vec{v}_N^T \vec{v}_1 \right)$

Again, since we've proven in part (b) that for any $i \neq 1$,

$\vec{v}_i^T \vec{v}_1 = \langle \vec{v}_i , \vec{v}_1 \rangle = 0$,

So $Q^{(2)} \vec{v}_1 = \lambda_2 \vec{v}_2 \cdot 0 + \lambda_3 \vec{v}_3 \cdot 0 + \cdots + \lambda_N \vec{v}_N \cdot 0 = \vec{0}$.

Thus, by definition, $\vec{v}_1$ is in the null space of $Q^{(2)}$.   Q.E.D.

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(g) (7 points) Recall the function that returns the **largest** eigenvalue and corresponding eigenvector of a matrix,

$$(\vec{v}_1, \lambda_1) = f(\mathbf{Q}). \tag{17}$$

Design a sequential greedy algorithm that returns a list of eigenvalues of matrix $\mathbf{Q}$ in descending order of values. You may assume that all the eigenvalues of $\mathbf{Q}$ are positive (> 0). You are allowed to use the function defined in Eq. (17) that returns the largest eigenvalue and corresponding eigenvector and what you know from part (f).

① Initialize the following values: $E = Q$, $A = []$, $F = \{\emptyset\}$.

② While $\|\vec{e}\| \neq 0$:

    1) Find the largest eigenvalue and corresponding eigenvector with

$$(\vec{v}_i, \lambda_i) = f(E).$$

    2) Add λ_i to the set F, which represents the list of eigenvalues in DESC order.

    3) Column concatenate matrix A with $\vec{v}_i$, i.e. $A = [A | \vec{v}_i]$.

    4) Update the residual $\vec{e} = \vec{e} - \lambda_i \vec{v}_i$

return F

This procedure should eventually return a list of eigenvalues of matrix Q in descending order.

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Extra page for scratchwork.
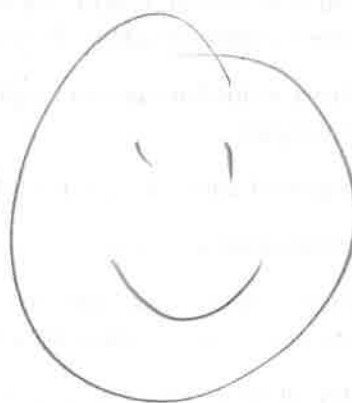**Work on this page will NOT be graded.**
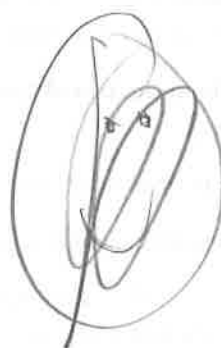
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Doodle page!
Draw us something if you want or give us suggestions, compliments, or complaints.
You can also use this page to report anything suspicious that you might have noticed.

Thanks for taking your final
for design & made the ~~exam~~ finals!

Enjoy your Xmas!

~~Xmas~~

(How I actually managed to write 3 hrs non-stop ————)
almost
(Didn't think it was possible ·———)

Read the following instructions before the exam.

There are 7 problems of varying numbers of points. You have 180 minutes for the exam. The problems are of varying difficulty, so pace yourself accordingly, do easier problems first, and avoid spending too much time on any one question until you have gotten all of the other points you can.

There are 34 pages on the exam, so there should be 17 sheets of paper in the exam. The exam is printed double-sided. Do not forget the problems on the back sides of the pages! Notify a proctor immediately if a page is missing. **Do not tear out or remove any of the pages. Do not remove the exam from the exam room.**

No collaboration is allowed, and do not attempt to cheat in any way. Cheating will not be tolerated.

Write your student ID on each page. If a page is found without a student ID, we are not responsible for identifying the student who wrote that page.

You may consult THREE handwritten 8.5" × 11" note sheets (front and back). No phones, calculators, tablets, computers, other electronic devices, or scratch paper are allowed.

Please write your answers legibly in the boxed spaces provided on the exam. The space provided should be adequate.

In general, show all of your work in order to receive full credit.

Partial credit will be given for substantial progress on each problem.

If you need to use the restrooms during the exam, bring your student ID card, your phone, and your exam to a proctor. You can collect them once you return from the restrooms.

Our advice to you: If you can't solve the problem, state and solve a simpler one that captures at least some of its essence. You might get some partial credit, and more importantly, you will perhaps find yourself on a path to the solution.

Good luck!

Do not turn this page until the proctor tells you to do so.