6. OMP Exercise

(a) $x_1 = \frac{19}{3},\ x_2 = 0,\ x_3 = 0,\ x_4 = \frac{8}{3}$

Given that $\mathbf{M} = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

Iteration 1:

Calculating the four inner products ($\vec{b}$ with every column of $\mathbf{M}$) gives us $[10, 7, -3, -1]$, and the largest value is at column 1. Thus, we have $\mathbf{A_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{b_1} = \vec{b} = \begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix}$. So, we have the least squares solution $\vec{x_1} = (\mathbf{A_1}^T \mathbf{A_1})^{-1} \mathbf{A_1}^T \vec{b_1} = \frac{1}{2} \cdot 10 = 5$, which gives us that $x_1 = 5$, and so we have our estimate currently at $\vec{r_1} = 5x_1$ with the residue after first iteration being $\vec{y_1} = \vec{b_1} - \mathbf{A_1}\vec{x_1} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \neq \vec{0}$.

Iteration 2:

Again, calculate the four inner products ($\vec{y_1}$ with every column of $\mathbf{M}$) gives us $[0, 2, 2, 4]$, and the largest value is at column 4. Thus, we have $\mathbf{A_2} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\vec{b_2} = \vec{b} = \begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix}$. So, we can calculate the least squares solution $\vec{x_2} = \begin{bmatrix} 19/3 \\ 8/3 \end{bmatrix}$, which gives us updated value that $x_1 = \frac{19}{3}$ and $x_4 = \frac{8}{3}$, and so the second residue is $\vec{y_2} = \vec{b_2} - \mathbf{A_2}\vec{x_2} = \begin{bmatrix} 1/3 \\ -1/3 \\ 1/3 \end{bmatrix}$

Since we are given that $\vec{x}$ has only 2 non-zero entries, so we're done, with

$$x_1 = \frac{19}{3},\ x_2 = 0,\ x_3 = 0,\ x_4 = \frac{8}{3}$$

Therefore, we have that:

$$\vec{x} = \begin{bmatrix} 19/3 \\ 0 \\ 0 \\ 8/3 \end{bmatrix}$$

(b) $\vec{x} = \mathbf{A} \cdot \mathrm{OMP}(\mathbf{MA}, \vec{b})$

Given matrix $\mathbf{A}$ and that $\vec{x} = \mathbf{A}\vec{x_a}$, we first calculate the inverse of $\mathbf{A}$, $\mathbf{A}^{-1}$. Then, by multiplying both sides of the equation by $\mathbf{A}^{-1}$, we have that $\vec{x_a} = \mathbf{A}^{-1}\mathbf{A}\vec{x_a} = \mathbf{A}^{-1}\vec{x}$.

Since we have that $\mathbf{M}\vec{x} = \vec{b}$, and since by definition, $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$, so we have:

$$(\mathbf{MA})\vec{x_a} = \mathbf{MAA}^{-1}\vec{x} = \mathbf{M}\vec{x} = \vec{b}$$

Now, since we are given that $\vec{x_a}$ is sparse, so our OMP function would successfully compute $\vec{x_a}$ by plugging in parameters $\mathbf{MA}$ and $\vec{b}$, i.e. we can successfully determine $\vec{x_a} = \text{OMP}(\mathbf{MA}, \vec{b})$.

Finally, we can compute $\vec{x}$ back from $\vec{x_a}$ with the equation $\vec{x} = \mathbf{A}\vec{x_a}$. Therefore, in other words, we could compute $\vec{x}$ in this method:

$$\vec{x} = \mathbf{A} \cdot \text{OMP}(\mathbf{MA}, \vec{b})$$