EECS 16A Designing Information Devices and Systems I
Fall 2018 Discussion 4B

Reference Definitions: Orthogonality We've seen that the following statements are equivalent for an $n \times n$ matrix $\mathbf{A}$:

- $\mathbf{A}$ is invertible
- The equation $\mathbf{A}\vec{x} = \vec{0}$, has a unique solution, which is $\vec{x} = \vec{0}$
- The columns of $\mathbf{A}$ are linearly independent
- For each column vector $\vec{b} \in \mathbb{R}^n$, $\mathbf{A}\vec{x} = \vec{b}$ has a unique solution $\vec{x}$
- Null($\mathbf{A}$) $= \vec{0}$

Conversely, the opposites are equivalent statements:

- $\mathbf{A}$ is not invertible
- $\mathbf{A}\vec{x} = \vec{0}$ for some $\vec{x} \neq \vec{0}$
- The columns of $\mathbf{A}$ are linearly dependent
- There is not be a unique $\vec{x}$ for every $\vec{b}$ where $\mathbf{A}\vec{x} = \vec{b}$
- Null($\mathbf{A}$) contains more than just the zero vector $\vec{0}$

These are part of what is known as the Invertible Matrix Theorem.

1. Steady State Reservoir Levels

We have 3 reservoirs: A, B and C. The pumps system between the reservoirs is depicted in Figure 1.

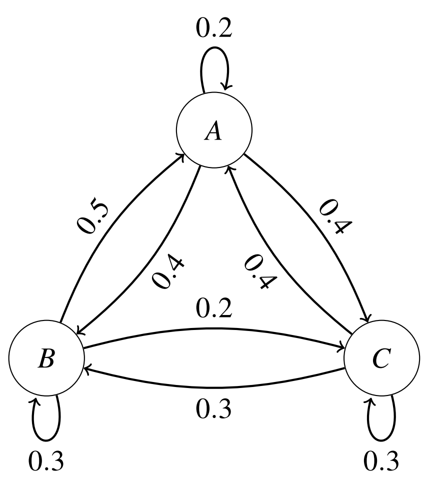

Figure 1: Reservoir pumps system.

(a) Write out the transition matrix representing the pumps system.

Answer:

$$\mathbf{T} = \begin{bmatrix} 0.2 & 0.5 & 0.4 \\ 0.4 & 0.3 & 0.3 \\ 0.4 & 0.2 & 0.3 \end{bmatrix}$$

(b) Assuming that you start the pumps with the water levels of the reservoirs at $A_0 = 129, B_0 = 109, C_0 = 0$ (in kiloliters), what would be the steady state water levels (in kiloliters) according to the pumps system described above?

Hint: If $\vec{x}_{ss} = \begin{bmatrix} A_{ss} \\ B_{ss} \\ C_{ss} \end{bmatrix}$ is a vector describing the steady state levels of water in the reservoirs (in kiloliters), what happens if you fill the reservoirs A, B and C with A_{ss}, B_{ss} and C_{ss} kiloliters of water, respectively, and apply the pumps once?

Hint II: Note that the pumps system preserves the total amount of water in the reservoirs. That is, no water is lost or gained by applying the pumps.

Answer:

If $\vec{x}_{ss} = \begin{bmatrix} A_{ss} \\ B_{ss} \\ C_{ss} \end{bmatrix}$ is a vector describing the steady state levels of water in the reservoirs, then we know that $\mathbf{T}\vec{x}_{ss} = 1 \cdot \vec{x}_{ss}$—that is, applying the pumps one more time wouldn't change the level of water in any of the reservoirs. This means that $\vec{x}_{ss}$ is an eigenvector of $\mathbf{T}$ associated with the eigenvalue $\lambda = 1$. Therefore,

$$\vec{x}_{ss} \in \text{Null}\,(\mathbf{T} - 1 \cdot \mathbf{I}) = \text{Null}\left(\begin{bmatrix} 0.2 & 0.5 & 0.4 \\ 0.4 & 0.3 & 0.3 \\ 0.4 & 0.2 & 0.3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = \text{Null}\left(\begin{bmatrix} -0.8 & 0.5 & 0.4 \\ 0.4 & -0.7 & 0.3 \\ 0.4 & 0.2 & -0.7 \end{bmatrix} \right)$$

We calculate the null space of the matrix $\begin{bmatrix} -0.8 & 0.5 & 0.4 \\ 0.4 & -0.7 & 0.3 \\ 0.4 & 0.2 & -0.7 \end{bmatrix}$, which is simply $\text{span}\left\{ \begin{bmatrix} 43 \\ 40 \\ 36 \end{bmatrix} \right\}$, which means that our steady state reservoirs levels vector is of the form $\begin{bmatrix} 43\alpha \\ 40\alpha \\ 36\alpha \end{bmatrix}, \alpha \in \mathbb{R}$.

Furthermore, we know that the pumps system conserves the water, i.e., no water is lost by running the pumps system. Therefore, we know that the total amount of water in the reservoirs at any point in time will be $129 + 109 + 0 = 238$ (equal to the original total amount of water in the system). Therefore, we are looking for an eigenvector whose components sum to 238. In other words, we are looking for α such that $43\alpha + 40\alpha + 36\alpha = 238$, which yields $\alpha = 2$. Therefore, the steady state levels of the water in the reservoirs will be $\begin{bmatrix} 86 \\ 80 \\ 72 \end{bmatrix}$.

2. Mechanical Eigenvalues and Eigenvectors

In each part, find the eigenvalues of the matrix $\mathbf{M}$ and the associated eigenvectors.

(a) $\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

Answer:

Let's begin by finding the eigenvalues:

$$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1-\lambda & 0 \\ 0 & 9-\lambda \end{bmatrix}\right) = 0$$

The determinant of a diagonal matrix is the product of the entries.

$$(1-\lambda)(9-\lambda) = 0$$

From the above equation, we know that the eigenvalues are $\lambda = 1$ and $\lambda = 9$.

For the eigenvalue $\lambda = 1$:

$$\begin{aligned} (\mathbf{M} - 1\mathbf{I})\vec{x} &= \vec{0} \\ \left(\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} - 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\vec{x} &= \vec{0} \\ \begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix}\vec{x} &= \vec{0} \end{aligned}$$

which is simply $x_2 = 0$ or equivalently $\begin{bmatrix} x_1 \\ 0 \end{bmatrix}$ or equivalently $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$.

For the eigenvalue $\lambda = 9$:

$$\begin{aligned} (\mathbf{M} - 9\mathbf{I})\vec{x} &= \vec{0} \\ \left(\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} - 9\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\vec{x} &= \vec{0} \\ \left(\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}\right)\vec{x} &= \vec{0} \\ \begin{bmatrix} -8 & 0 \\ 0 & 0 \end{bmatrix}\vec{x} &= \vec{0} \end{aligned}$$

which is simply $x_1 = 0$ or equivalently $\begin{bmatrix} 0 \\ x_2 \end{bmatrix}$ or equivalently $\operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$.

(b) $\mathbf{M} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

Answer:

Let's begin by finding the eigenvalues:

$$\det\left(\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1-\lambda & 1 \\ 2 & 2-\lambda \end{bmatrix}\right) = 0$$

$$(1-\lambda)(2-\lambda) - 2 = \lambda^2 - 3\lambda = \lambda(\lambda - 3) = 0$$

From the above equation, we know that the eigenvalues are $\lambda = 0$ and $\lambda = 3$.

For the eigenvalue $\lambda = 0$:

$$(\mathbf{M} - 0\mathbf{I})\vec{x} = \mathbf{M}\vec{x} = \vec{0}$$
$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\vec{x} = \vec{0}$$

which is simply x_1 is free and $x_2 = -x_1$ or equivalently $\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$ or equivalently span $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$.

For the eigenvalue $\lambda = 3$:

$$(\mathbf{M} - 3\mathbf{I})\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \right)\vec{x} = \vec{0}$$
$$\begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix}\vec{x} = \vec{0}$$

which is simply x_1 is free and $x_2 = 2x_1$ or equivalently $\begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix}$ or equivalently span $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

(c) **(PRACTICE)** $\mathbf{M} = \begin{bmatrix} 0 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix}$

Answer:

Let's begin by finding the eigenvalues:

$$\det\left(\begin{bmatrix} 0 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right) = \det\left(\begin{bmatrix} -\lambda & 0 & 0 \\ -3 & 4-\lambda & 9 \\ 0 & 0 & 3-\lambda \end{bmatrix} \right) = 0$$

Without changing the determinant, we can subtract $\frac{3}{\lambda}$ of row 1 from row 2.

$$\det\left(\begin{bmatrix} -\lambda & 0 & 0 \\ -3 & 4-\lambda & 9 \\ 0 & 0 & 3-\lambda \end{bmatrix} \right) = \det\left(\begin{bmatrix} -\lambda & 0 & 0 \\ 0 & 4-\lambda & 9 \\ 0 & 0 & 3-\lambda \end{bmatrix} \right) = 0$$
$$-\lambda(4-\lambda)(3-\lambda) = 0$$

From the above equation, we know that the eigenvalues are $\lambda = 0$, $\lambda = 3$, and $\lambda = 4$.

For the eigenvalue $\lambda = 0$:

$$(\mathbf{M} - 0\mathbf{I})\vec{x} = \mathbf{M}\vec{x} = \vec{0}$$
$$\begin{bmatrix} 0 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix}\vec{x} = \vec{0}$$

which is simply $x_3 = 0$, x_2 is free, and $x_1 = \frac{4}{3}x_2$ or equivalently $\begin{bmatrix} \frac{4}{3}x_2 \\ x_2 \\ 0 \end{bmatrix}$ or equivalently span $\left\{ \begin{bmatrix} \frac{4}{3} \\ 1 \\ 0 \end{bmatrix} \right\}$.

For the eigenvalue $\lambda = 3$:

$$(\mathbf{M} - 3\mathbf{I})\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix} 0 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix} 0 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right)\vec{x} = \vec{0}$$
$$\begin{bmatrix} -3 & 0 & 0 \\ -3 & 1 & 9 \\ 0 & 0 & 0 \end{bmatrix}\vec{x} = \vec{0}$$

which is simply $x_1 = 0$, x_3 is free, and $x_2 = -9x_3$ or equivalently $\begin{bmatrix} 0 \\ -9x_3 \\ x_3 \end{bmatrix}$ or equivalently span $\left\{\begin{bmatrix} 0 \\ -9 \\ 1 \end{bmatrix}\right\}$.

For the eigenvalue $\lambda = 4$:

$$(\mathbf{M} - 4\mathbf{I})\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix} 0 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix} 0 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}\right)\vec{x} = \vec{0}$$
$$\begin{bmatrix} -4 & 0 & 0 \\ -3 & 0 & 9 \\ 0 & 0 & -1 \end{bmatrix}\vec{x} = \vec{0}$$

which is simply $x_1 = x_3 = 0$ and $x_2 = x_2$ or equivalently $\begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix}$ or equivalently span $\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$.

(d) **(PRACTICE) M =** $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Answer:

Let's begin by finding the eigenvalues:

$$\det\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = 0$$

Without changing the determinant, we can add $\frac{1}{\lambda}$ of row 1 to row 2.

$$\det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} -\lambda & -1 \\ 0 & -\lambda - \frac{1}{\lambda} \end{bmatrix}\right) = 0$$

$$-\lambda\left(-\lambda-\frac{1}{\lambda}\right)=\lambda^2+1=0$$

From the above equation, we know that the eigenvalues are $\lambda=i$ and $\lambda=-i$.
For the eigenvalue $\lambda=i$:

$$\begin{aligned}(\mathbf{M}-i\mathbf{I})\vec{x}&=\vec{0}\\ \left(\begin{bmatrix}0&-1\\1&0\end{bmatrix}-i\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)\vec{x}&=\vec{0}\\ \left(\begin{bmatrix}0&-1\\1&0\end{bmatrix}-\begin{bmatrix}i&0\\0&i\end{bmatrix}\right)\vec{x}&=\vec{0}\\ \begin{bmatrix}-i&-1\\1&-i\end{bmatrix}\vec{x}&=\vec{0}\end{aligned}$$

which is simply $x_1=ix_2$ and x_2 is free or equivalently $\begin{bmatrix}ix_2\\x_2\end{bmatrix}$ or equivalently span $\left\{\begin{bmatrix}i\\1\end{bmatrix}\right\}$.
For the eigenvalue $\lambda=-i$:

$$\begin{aligned}(\mathbf{M}+i\mathbf{I})\vec{x}&=\vec{0}\\ \left(\begin{bmatrix}0&-1\\1&0\end{bmatrix}+i\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)\vec{x}&=\vec{0}\\ \left(\begin{bmatrix}0&-1\\1&0\end{bmatrix}+\begin{bmatrix}i&0\\0&i\end{bmatrix}\right)\vec{x}&=\vec{0}\\ \begin{bmatrix}i&-1\\1&i\end{bmatrix}\vec{x}&=\vec{0}\end{aligned}$$

which is simply $x_1=-ix_2$ and x_2 is free or equivalently $\begin{bmatrix}-ix_2\\x_2\end{bmatrix}$ or equivalently span $\left\{\begin{bmatrix}-i\\1\end{bmatrix}\right\}$.

3. Mechanical Determinants

(a) Compute the determinant of $\begin{bmatrix}2&0\\0&3\end{bmatrix}$.

Answer:

$$\det\left(\begin{bmatrix}2&0\\0&3\end{bmatrix}\right)=6$$

(b) Compute the determinant of $\begin{bmatrix}2&1\\0&3\end{bmatrix}$.

Answer:

$$\det\left(\begin{bmatrix}2&1\\0&3\end{bmatrix}\right)=6$$

(c) Compute the determinant of $\begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & 17 & 0 & 0 \\ 0 & 0 & -31 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

Answer:

$$\det\left(\begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & 17 & 0 & 0 \\ 0 & 0 & -31 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}\right) = 4216$$

4. Row Operations and Determinants

In this question we explore the effect of row operations on the determinant of a matrix. Note that scaling a row by a will increase the determinant by a factor of a, and adding a multiple of one row to another will not change the determinant. Swapping two rows of a matrix and computing the determinant is equivalent to multiplying the determinant of the original matrix by -1. The determinant of an identity matrix is 1. Feel free to prove these properties to convince yourself that they hold for general square matrices.

(a) An upper triangular matrix is a matrix with zeros below its diagonal. For example a 3×3 upper triangular matrix is:

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{bmatrix}$$

By considering row operations and what they do to the determinant, argue that the determinant of a general $n \times n$ upper triangular matrix is the product of its diagonal entries if they are non-zero. For example, the determinant of the 3×3 matrix above is $a_1 \cdot b_2 \cdot c_3$ if $a_1, b_2, c_3 \neq 0$.

Answer:

An $n \times n$ upper-triangular matrix looks like:

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ 0 & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{n,n} \end{bmatrix}$$

For every row i, divide it by $a_{i,i}$. Then we get 1s on the diagonal.

$$\mathbf{A}' = \begin{bmatrix} 1 & \frac{a_{1,2}}{a_{1,1}} & \cdots & \frac{a_{1,n}}{a_{1,1}} \\ 0 & 1 & \cdots & \frac{a_{2,n}}{a_{2,2}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

The determinant of this new matrix has been decreased by a factor of $\frac{1}{a_{1,1} \cdot a_{2,2} \cdot \cdots \cdot a_{n,n}}$:

$$\det(\mathbf{A}') = \frac{\det(\mathbf{A})}{a_{1,1} \cdot a_{2,2} \cdot \cdots \cdot a_{n,n}}$$

Finally, starting from the last row, subtract multiples of the row from the ones above it, so that we get the $n \times n$ identity matrix $\mathbf{I}_n$. This does not change the determinant since we are subtracting rows from each other. Thus:

$$1 = \det(\mathbf{I}_n) = \det\big(\mathbf{A}'\big) = \frac{\det(\mathbf{A})}{a_{1,1} \cdot a_{2,2} \cdot \dots \cdot a_{n,n}}$$
$$\det(\mathbf{A}) = a_{1,1} \cdot a_{2,2} \cdot \dots \cdot a_{n,n}$$

(b) If the diagonal of an upper-triangular matrix has a zero entry, argue that its determinant is still the product of its diagonal entries.

Answer:

If an upper-triangular matrix has a zero in its diagonal, it cannot be row reduced to the identity matrix, which means that that its rows are linearly dependent. Therefore its determinant is zero, which is the product of all diagonal entries (since one of them is zero).

5. (PRACTICE) StateRank Car Rentals

You are an analyst at StateRank Car Rentals, which operates in California, Oregon, and Nevada. You are hired to analyze the number of rental cars going into and out of each of the three states (CA, OR, and NV).

The number of cars in each state on day $n \in \{0, 1, \dots\}$ can be represented by the state vector $\vec{s}[n] = \begin{bmatrix} s_{\text{CA}}[n] \\ s_{\text{OR}}[n] \\ s_{\text{NV}}[n] \end{bmatrix}$.

The state vector follows the state evolution equation $\vec{s}[n+1] = \mathbf{A}\vec{s}[n], \forall n \in \{0, 1, \dots\}$, where the transition matrix, $\mathbf{A}$, of this linear dynamic system is

$$\mathbf{A} = \begin{bmatrix} 7/10 & 1/10 & 1/10 \\ 1/10 & 6/10 & 1/10 \\ 2/10 & 3/10 & 8/10 \end{bmatrix}$$

(a) Use the designated boxes in Figure 2 to **fill in the weights for the daily travel dynamics of rental cars between the three states**, as described by the state transition matrix $\mathbf{A}$.

Note the order of the elements in the state vector $\vec{s}[n]$.

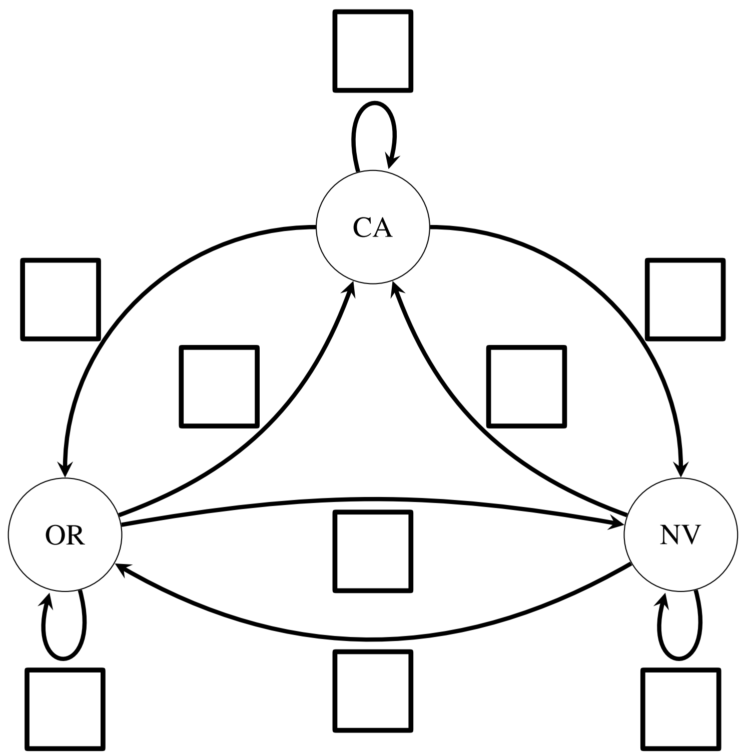

Figure 2: StateRank Rental Cars Daily Travel Dynamics.

Answer:

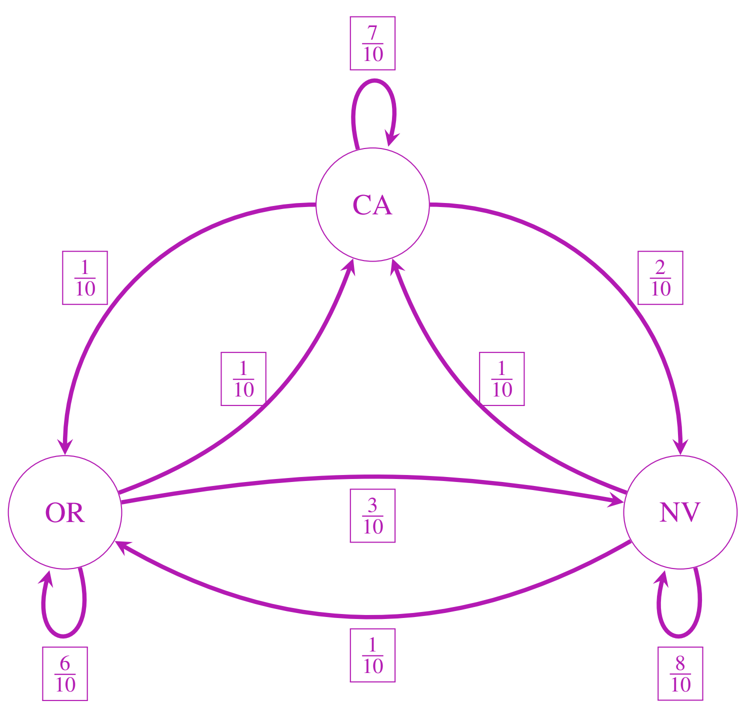

Figure 3: StateRank Rental Cars Daily Travel Dynamics – Solution.

A copy of the state transition matrix for reference: $\mathbf{A} = \begin{bmatrix} 7/10 & 1/10 & 1/10 \\ 1/10 & 6/10 & 1/10 \\ 2/10 & 3/10 & 8/10 \end{bmatrix}$

(b) Suppose the state vector on day $n = 4$ is $\vec{s}[4] = \begin{bmatrix} 100 \\ 200 \\ 100 \end{bmatrix}$. **Calculate the state vector on day 5, $\vec{s}[5]$.**

Answer:

$$\begin{bmatrix} 7/10 & 1/10 & 1/10 \\ 1/10 & 6/10 & 1/10 \\ 2/10 & 3/10 & 8/10 \end{bmatrix} \begin{bmatrix} 100 \\ 200 \\ 100 \end{bmatrix} = \begin{bmatrix} 70+20+10 \\ 10+120+10 \\ 20+60+80 \end{bmatrix} = \begin{bmatrix} 100 \\ 140 \\ 160 \end{bmatrix}$$

(c) We want to express the number of cars in each state on day n as a function of the initial number of cars in each state on day 0. That is, we write $\vec{s}[n]$ in terms of $\vec{s}[0]$ as follows:

$$\vec{s}[n] = \mathbf{B}\vec{s}[0]$$

Express the matrix B in terms of A and n.

Answer:

$$\vec{s}[n] = \mathbf{A}\vec{s}[n-1] = \mathbf{A}^2\vec{s}[n-2] = \mathbf{A}^3\vec{s}[n-3] = \cdots = \mathbf{A}^n\vec{s}[n-n] = \mathbf{A}^n\vec{s}[0]$$

$$\mathbf{B} = \mathbf{A}^n$$

A copy of the state transition matrix for reference: $\mathbf{A} = \begin{bmatrix} 7/10 & 1/10 & 1/10 \\ 1/10 & 6/10 & 1/10 \\ 2/10 & 3/10 & 8/10 \end{bmatrix}$

(d) We denote the eigenvalue/eigenvector pairs of the matrix $\mathbf{A}$ by

$$\left(\lambda_1 = 1, \vec{u}_1 = \begin{bmatrix} 50 \\ 40 \\ 110 \end{bmatrix}\right), \left(\lambda_2, \vec{u}_2 = \begin{bmatrix} 0 \\ -10 \\ 10 \end{bmatrix}\right), \text{ and } \left(\lambda_3, \vec{u}_3 = \begin{bmatrix} -10 \\ 0 \\ 10 \end{bmatrix}\right).$$

Find the eigenvalues λ_2 and λ_3 corresponding to the eigenvectors $\vec{u}_2$ and $\vec{u}_3$, respectively. Note that since $\lambda_1 = 1$ is given, you don't have to calculate it.

Answer:

Recall that if $\vec{u}, \lambda$ are an eigenpair of a matrix $\mathbf{A}$, then $\mathbf{A}\vec{u} = \lambda\vec{u}$. By left-multiplying the eigenvectors by $\mathbf{A}$, we get:

i. $\begin{bmatrix} 7/10 & 1/10 & 1/10 \\ 1/10 & 6/10 & 1/10 \\ 2/10 & 3/10 & 8/10 \end{bmatrix} \begin{bmatrix} 0 \\ -10 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 5 \end{bmatrix} = 0.5 \cdot \begin{bmatrix} 0 \\ -10 \\ 10 \end{bmatrix}$, which means that $\lambda_2 = 0.5 = \frac{1}{2}$.

ii. $\begin{bmatrix} 7/10 & 1/10 & 1/10 \\ 1/10 & 6/10 & 1/10 \\ 2/10 & 3/10 & 8/10 \end{bmatrix} \begin{bmatrix} -10 \\ 0 \\ 10 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \\ 6 \end{bmatrix} = 0.6 \cdot \begin{bmatrix} -10 \\ 0 \\ 10 \end{bmatrix}$, which means that $\lambda_3 = 0.6 = \frac{3}{5}$.

(e) For the given dynamics in this problem, **does a matrix $\mathbf{C}$ exists such that $\vec{s}[n-1] = \mathbf{C}\vec{s}[n]$, for $n \in \{1, 2, \ldots\}$? Justify your answer.**

Answer:

Yes. The matrix $\mathbf{A}$ is invertible because no eigenvalue is equal to 0.

A copy of the state transition matrix for reference: $\mathbf{A} = \begin{bmatrix} 7/10 & 1/10 & 1/10 \\ 1/10 & 6/10 & 1/10 \\ 2/10 & 3/10 & 8/10 \end{bmatrix}$

(f) Suppose that the initial number of rental cars in each state on day 0 is

$$\vec{s}[0] = \begin{bmatrix} 7000 \\ 5000 \\ 8000 \end{bmatrix} = 100\vec{u}_1 - 100\vec{u}_2 - 200\vec{u}_3,$$

where $\vec{u}_1, \vec{u}_2$ and $\vec{u}_3$ are the eigenvectors from part (d).
After a very large number of days n, how many rental cars will there be in each state?
That is, i) calculate

$$\vec{s}^* = \lim_{n\to\infty} \vec{s}[n]$$

<u>and</u> ii) show that the system will indeed converge to $\vec{s}^*$ as $n \to \infty$ if it starts from $\vec{s}[0]$.
Hint: If you didn't solve part (d), the eigenvalues satisfy $\lambda_1 = 1, |\lambda_2| < 1$ and $|\lambda_3| < 1$.

Answer:

We know that $\vec{s}[n] = \mathbf{A}^n\vec{s}[0]$. We also know that $\vec{s}[0] = 100\vec{u}_1 - 100\vec{u}_2 - 200\vec{u}_3$.
Therefore, we can write:

$$\begin{aligned}
\vec{s}[n] &= \mathbf{A}^n\vec{s}[0] \\
&= \mathbf{A}^n\left(100\vec{u}_1 - 100\vec{u}_2 - 200\vec{u}_3\right) \\
&= 100\mathbf{A}^n\vec{u}_1 - 100\mathbf{A}^n\vec{u}_2 - 200\mathbf{A}^n\vec{u}_3 \\
&= 100\lambda_1^n\vec{u}_1 - 100\lambda_2^n\vec{u}_2 - 200\lambda_3^n\vec{u}_3
\end{aligned}$$

From that, we can write:

$$\lim_{n\to\infty} \vec{s}[n] = \lim_{n\to\infty} 100\lambda_1^n\vec{u}_1 - 100\lambda_2^n\vec{u}_2 - 200\lambda_3^n\vec{u}_3$$

Since $|\lambda_2| < 1$ and $|\lambda_3| < 1$, we know that:

$$\lim_{n\to\infty} \left(-100\lambda_2^n\vec{u}_2 - 200\lambda_3^n\vec{u}_3\right) = \vec{0}$$

From that and the fact that $\lambda_1 = 1$, we are left with:

$$\begin{aligned}
\lim_{n\to\infty} \vec{s}[n] &= \lim_{n\to\infty} 100\lambda_1^n\vec{u}_1 \\
&= \lim_{n\to\infty} 100\cdot 1^n\vec{u}_1 \\
&= 100\vec{u}_1 \\
&= \begin{bmatrix} 5000 \\ 4000 \\ 11000 \end{bmatrix}
\end{aligned}$$