EECS 16A Designing Information Devices and Systems I
Fall 2018 Homework 3

This homework is due September 14, 2018, at 23:59.
Self-grades are due September 18, 2018, at 23:59.

Submission Format

Your homework submission should consist of **two** files.

- `hw3.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.

  If you do not attach a PDF of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible.

- `hw3.ipynb`: A single IPython notebook with all of your code in it.

  In order to receive credit for your IPython notebook, you must submit both a "printout" and the code itself.

Submit each file to its respective assignment on Gradescope.

1. Counting Solutions

For each of the following systems of linear equations, determine if there is a unique solution, no solution, or an infinite number of solutions. **Show your work**. If there is a unique solution, find it. If there are an infinite number of solutions, describe the space of solutions.

(a) **(PRACTICE)**

$$\begin{aligned} x + y + z &= 3 \\ 2x + 2y + 2z &= 5 \end{aligned}$$

(b)

$$\begin{aligned} -y + 2z &= 1 \\ 2x + z &= 2 \end{aligned}$$

(c) **(PRACTICE)**

$$\begin{aligned} x + 2y &= 3 \\ 2x - y &= 1 \\ 3x + y &= 4 \end{aligned}$$

(d)

$$\begin{aligned} x + 2y &= 3 \\ 2x - y &= 1 \\ x - 3y &= -5 \end{aligned}$$

(e)

$$\begin{bmatrix} x - y = 2 \\ 5x - 5y = 10 \\ 3x - 3y = 6 \end{bmatrix}$$

2. Elementary Matrices

In lecture, we learned about an important technique for solving systems of linear equations called Gaussian elimination. It turns out that each row operation in Gaussian elimination can be performed by multiplying the augmented matrix on the left by a specific matrix called an *elementary matrix*. For example, suppose we want to row reduce the following augmented matrix:

$$\mathbf{A} = \left[\begin{array}{cccc|c} 1 & -2 & 0 & -5 & 15 \\ 0 & 1 & 0 & 3 & -7 \\ -2 & -3 & 1 & -6 & 9 \\ 0 & 1 & 0 & 2 & -5 \end{array}\right] \tag{1}$$

What matrix do you get when you subtract the 4th row from the 2nd row of $\mathbf{A}$ (putting the result in row 2)? (You don't have to include this in your solution.) Now, try multiplying the original $\mathbf{A}$ on the left by

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(You don't have to include this in your solutions either.) Notice that you get the same thing.

$$\mathbf{EA} = \left[\begin{array}{cccc|c} 1 & -2 & 0 & -5 & 15 \\ 0 & 0 & 0 & 1 & -2 \\ -2 & -3 & 1 & -6 & 9 \\ 0 & 1 & 0 & 2 & -5 \end{array}\right]$$

$\mathbf{E}$ is a special type of matrix called an *elementary matrix*. This means that we can obtain the matrix $\mathbf{E}$ from the identity matrix by applying an elementary row operation – in this case, subtracting the 4th row from the 2nd row.

In general, any elementary row operation can be performed by left multiplying by an appropriate elementary matrix.

In other words, you can perform a row operation on a matrix $\mathbf{A}$ by first performing that row operation on the identity matrix to get an elementary matrix (see below), and then left multiplying $\mathbf{A}$ by the elementary matrix (like we did above).

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 - R_4 \mapsto R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{E}$$

(a) Write down the elementary matrices required to perform the following row operations on a 4×5 augmented matrix.

i. $R_1 \mapsto R_3$
$R_3 \mapsto R_1$

ii. $-5R_3 \mapsto R_3$

iii. $3R_2 + R_4 \mapsto R_4$
$R_1 - R_2 \mapsto R_1$

Hint: For the last one, note that if you want to perform two row operations on the matrix $\mathbf{A}$*, you can perform them both on the identity matrix and then left multiply* $\mathbf{A}$ *by the resulting matrix.*

(b) In lecture we emphasized using Gaussian Elimination to reach an upper triangular form to determine the number of solutions for a given system of linear equations. When there is a unique solution, however, it is useful to determine exactly what that solution is by continuing Gaussian Elimination to reach a "fully reduced" form like that shown below. An example of this process can be found in Note 1, Example 1.7.

Compute a matrix $\mathbf{E}$ (by hand) that fully row reduces the augmented matrix $\mathbf{A}$ given in Equation (1)– that is, find $\mathbf{E}$ such that $\mathbf{EA}$ is a diagonal matrix with 1s along the diagonal. Show that this is true by multiplying out $\mathbf{EA}$. When an augmented matrix is in this final form, it will have the form

$$\mathbf{EA} = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & b_1 \\ 0 & 1 & 0 & 0 & b_2 \\ 0 & 0 & 1 & 0 & b_3 \\ 0 & 0 & 0 & 1 & b_4 \end{array}\right]$$

Once you have found the required elementary matrices,

i. use IPython to find the matrix $\mathbf{E}$
ii. verify by hand that multiplying $\mathbf{E}$ and $\mathbf{A}$ gives you the identity matrix augmented with constants (as $\mathbf{EA}$ shown above).

Hint: As before, note that you can either ***apply a set of row operations to the same identity matrix*** *or* ***apply them to separate identity matrices and then multiply the matrices together****. Make sure, though, that you apply the row operations and multiply the matrices in the correct order.*

3. Mechanical Inverses

For each of the following matrices, state whether the inverse exists. If so, find the inverse, $\mathbf{A}^{-1}$. If not, show why no inverse exists. **Solve the inverses by hand. You may use IPython for parts (a)-(d) to visualize how the matrix A changes a vector.**

(a) In addition to finding the inverse (if it exists), describe how the original matrix, $\mathbf{A}$, changes a vector it's applied to. For e.g., if $\mathbf{A}\vec{b} = \vec{c}$, then $\mathbf{A}$ could scale $\vec{b}$ by 2 to get $\vec{c}$, or $\mathbf{A}$ could reflect $\vec{b}$ across the x axis to get $\vec{c}$, etc. *Hint:* It may help to plot a few examples to recognize the pattern.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(b) **(PRACTICE)** In addition to finding the inverse (if it exists), describe how the original matrix, $\mathbf{A}$, changes a vector it's applied to. For e.g., if $\mathbf{A}\vec{b} = \vec{c}$, then $\mathbf{A}$ could scale $\vec{b}$ by 2 to get $\vec{c}$, or $\mathbf{A}$ could reflect $\vec{b}$ across the x axis to get $\vec{c}$, etc. *Hint:* It may help to plot a few examples to recognize the pattern.

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

(c) **(PRACTICE)** In addition to finding the inverse (if it exists), describe how the original matrix, $\mathbf{A}$, changes a vector it's applied to. For e.g., if $\mathbf{A}\vec{b} = \vec{c}$, then $\mathbf{A}$ could scale $\vec{b}$ by 2 to get $\vec{c}$, or $\mathbf{A}$ could reflect $\vec{b}$ across the x axis to get $\vec{c}$, etc. *Hint:* It may help to plot a few examples to recognize the pattern.

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

(d) In addition to finding the inverse (if it exists), describe how the original matrix, $\mathbf{A}$, changes a vector it's applied to. For e.g., if $\mathbf{A}\vec{b} = \vec{c}$, then $\mathbf{A}$ could scale $\vec{b}$ by 2 to get $\vec{c}$, or $\mathbf{A}$ could reflect $\vec{b}$ across the x axis to get $\vec{c}$, etc.

Hint: It may help to plot a few examples to recognize the pattern. What does the result look like when you apply this matrix to $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$? Draw it out. In the iPython notebook for this homework, we have provided you with a rotation function that may help to visualize what is happening.

$\mathbf{A} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$

(e) $\mathbf{A} = \begin{bmatrix}1 & 1\\ 2 & 0\end{bmatrix}$

(f) $\mathbf{A} = \begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 2\\ 1 & 4 & 4\end{bmatrix}$

(g) **PRACTICE:** $\mathbf{A} = \begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 1\\ 0 & 1 & -1\end{bmatrix}$

(h) **PRACTICE:** $\mathbf{A} = \begin{bmatrix}-1 & 1 & -\frac{1}{2}\\ 1 & 1 & -\frac{1}{2}\\ 0 & 1 & 1\end{bmatrix}$

(i) **(PRACTICE)**

$\mathbf{A} = \begin{bmatrix}3 & 0 & -2 & 1\\ 0 & 2 & 1 & 3\\ 3 & 1 & 0 & 4\\ 1 & 0 & 0 & 1\end{bmatrix}$

Hint 1: What do the linear (in)dependence of the rows and columns tell us about the invertibility of a matrix? Hint 2: We're reasonable people!

4. (PRACTICE) Powers Of Nilpotent Matrices

The following matrices are examples of a special type of matrix called a nilpotent matrix. What happens to each of these matrices when you multiply it by itself repeatedly? Multiply them to find out. Why do you think these are called "nilpotent" matrices? (Of course, there is nothing magical about 3×3 or 4×4 matrices. You can have nilpotent square matrices of any dimension greater than 1.)

(a) **(PRACTICE)** Calculate $\mathbf{C}^3$ by hand. Make sure you show what $\mathbf{C}^2$ is along the way.

$$\mathbf{C} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 2\\ 0 & 0 & 0\end{bmatrix}$$

(b) **(PRACTICE)** Calculate $\mathbf{A}^4$ by hand. Make sure you show what $\mathbf{A}^2$ and $\mathbf{A}^3$ are along the way.

$$\mathbf{A} = \begin{bmatrix}0 & 2 & 1 & 3\\ 0 & 0 & 1 & -2\\ 0 & 0 & 0 & 3\\ 0 & 0 & 0 & 0\end{bmatrix}$$

(c) **(PRACTICE)** Calculate $\mathbf{B}^4$. You are allowed to use iPython to find $\mathbf{B}^2$ and $\mathbf{B}^3$—write out in your homework what they are. Calculate the final multiplication by hand.

$$\mathbf{B} = \begin{bmatrix} 3 & 4 & 2 & 1 \\ -5 & -6 & -3 & -1 \\ 6 & 7 & 3 & 2 \\ 2 & 2 & 1 & 0 \end{bmatrix}$$

5. Properties of Pump Systems

Throughout this problem, we will consider a system of resevoirs connected to each other through pumps. An example system is shown below in Figure 1, represented as a graph. Each node in the graph is marked with a number and represents a resevoir. Each edge in the graph represents a pump which moves a fraction of the water from one resevoir to the next at every time step. The fraction of water is written on top of the edge.

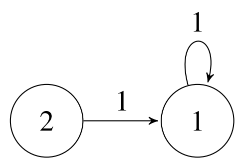


Figure 1: Pump system

(a) Consider the system of pumps shown above in Figure 1. Let $x_i[n]$ represent the amount of water in resevoir i at time step n. Find a system of equations that represents every $x_i[n+1]$ in terms of all the different $x_i[n]$.

(b) For the system shown in Figure 1, find the associated state transition matrix. That is find the matrix $\mathbf{A}$ such that:

$$\vec{x}[n+1] = \mathbf{A}\vec{x}[n], \text{ where } \vec{x}[n] = \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix}$$

(c) Suppose that the reservoirs are initialized to the following water levels: $x_1[0] = 0.5, x_2[0] = 0.5$. In a completely alternate universe, the reservoirs are initialized to the following water levels: $x_1[0] = 0.3, x_2[0] = 0.7$. For both initial states, what are the water levels at timestep 1 ($\vec{x}[1]$)? Use your answer from part (b) to compute your solution.

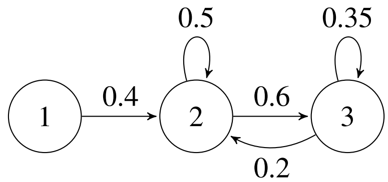
(d) If you observe the reservoirs at timestep 1, can you figure out what the initial ($\vec{x}[0]$) water levels were? Why or why not?

(e) Now generalize: if there exists a state transition matrix where two different initial state vectors lead to the same water levels/state vectors at a timestep in the future, can you recover the initial water levels? Prove your answer.
(*Hint: What does this say about the matrix* $\mathbf{A}$*?*)

(f) Suppose that we have 3 resevoirs and that there is a state transition matrix such that the entries of each column vector sum to one. Let the total amount of water in the system be s at timestep n. Show that the total amount of water at timestep $n+1$ will also be s. What is the physical interpretation about the total amount of water in the system? Prove this for 3 reservoirs first, then generalize to k reservoirs.
Hint: Consider the state vector at time n.

(g) **(PRACTICE)** Set up the state transition matrix $\mathbf{A}$ for the system of pumps shown below. Compute the sum of hte columns of the state transition matrix. Is it greater than/less than/equal to 1? Explain what this $\mathbf{A}$ matrix physically implies about the total amount of water in this system.

Note: If there is no "self-arrow/self-loop," then the water does not return.



(h) **(PRACTICE)** There is a state transition matrix where the entries of its rows sum to one. Prove that applying this system to a uniform vector will return the same uniform vector. A uniform vector is a vector whose elements are all the same.

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x \\ \vdots \\ x \end{bmatrix} = \begin{bmatrix} x \\ \vdots \\ x \end{bmatrix}$$

6. Audio File Matching

Each day a wide variety of quantities we interact with can be expressed as vectors. For example, an audio clip or a sound wave (continuous function in time) can be sampled at regular intervals to make a discrete sequence of values and represented in vector form.

This problem explores using inner products for measuring similarity between two sound signals represented as vectors. The same ideas here will be further developed in the third module of EE16A where we will learn about Locationing and GPS.

Let us consider a very simplified model for an audio signal; one that is composed of just two tones. One tone is represented by the value -1 and the other by the value $+1$. A vector of length n makes up the audio file.

(a) Say we want to compare two audio files of the same length n to decide how similar they are. First consider two vectors that are exactly identical $\vec{X}_1 = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T$ and $\vec{X}_2 = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T$. What is the inner product/dot product of these two vectors, i.e. $\vec{X}_1^T \vec{X}_2$? What if $\vec{X}_1 = \begin{bmatrix} 1 & 1 \cdots 1 \end{bmatrix}^T$ and $\vec{X}_2 = \begin{bmatrix} 1 & -1 & 1 & -1 \cdots 1 & -1 \end{bmatrix}^T$ (where the length of the vector is an even number)? For pairs of vectors of length n made of ± 1's, does a larger dot product imply that the vectors are more similar or less similar?

(b) Next suppose we want to search for a short audio clip in a longer one. We might want to do this for an application like *Shazam* to be able to identify a song from a signature tune. Consider the vector of length 8, $\vec{X} = \begin{bmatrix} 1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 \end{bmatrix}^T$. Let us label the elements of $\vec{X}$ so that $\vec{X} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \end{bmatrix}^T$. Our goal is to find the short segment $\vec{Y} = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}^T$ in the longer vector (i.e. we want to find i, such that the sequence represented by $\begin{bmatrix} x_i & x_{i+1} & x_{i+2} \end{bmatrix}^T$ is the closest to $\vec{Y}$). Come up with an approach to do this. Can your approach be written as a matrix vector multiplication $\mathbf{A}\vec{x}$ where $\mathbf{A}$ is 6×8 and $\vec{x}$ is 8×1? Applying your technique, which i gives the best match for $\vec{Y} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$?

(c) **(PRACTICE)** Now suppose our vector was represented using integers and not just by 1 and -1. We want to find the subsequence of the longer vector that is most similar to that of a sample vector. Say we wanted to locate the sequence closest in direction to $\vec{Y} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$ within $\vec{X} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{bmatrix}^T$. Can you explain why the approach in part (b) won't work? Consider using the norm/magnitude of the short segments (the norm/magnitude of a vector is defined as $\|\vec{z}\| = \sqrt{z_0^2 + z_1^2 + \ldots + z_n^2}$). How might you use this quantity to modify your approach for the new vectors to focus on the direction rather than the magnitude of the vectors?

(d) In the IPython notebook, `prob3.ipynb`, complete part 1.

(e) In the IPython notebook, `prob3.ipynb`, complete part 2.

7. Write Your Own Question And Provide a Thorough Solution.

Writing your own problems is a very important way to really learn material. The famous "Bloom's Taxonomy" that lists the levels of learning is: Remember, Understand, Apply, Analyze, Evaluate, and Create. Using what you know to create is the top level. We rarely ask you any homework questions about the lowest level of straight-up remembering, expecting you to be able to do that yourself (e.g. making flashcards). But we don't want the same to be true about the highest level. As a practical matter, having some practice at trying to create problems helps you study for exams much better than simply counting on solving existing practice problems. This is because thinking about how to create an interesting problem forces you to really look at the material from the perspective of those who are going to create the exams. Besides, this is fun. If you want to make a boring problem, go ahead. That is your prerogative. But it is more fun to really engage with the material, discover something interesting, and then come up with a problem that walks others down a journey that lets them share your discovery. You don't have to achieve this every week. But unless you try every week, it probably won't ever happen.

8. Homework Process and Study Group

Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.) How did you work on this homework?