1. (a). $\det(A-\lambda I_2) = \det\left(\begin{bmatrix} 5-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}\right) = (5-\lambda)(2-\lambda) - 0 = 0. \Rightarrow \lambda = 5, 2.$

① $\boxed{\lambda_1 = 5}$ $\Rightarrow (A-5I_2)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & -3 & 0 \end{array}\right] \Rightarrow y = 0 \Rightarrow$ eigenspace: $\boxed{\text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}}.$

② $\boxed{\lambda_2 = 2.}$ $\Rightarrow (A-2I_2)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{cc|c} 3 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow x = 0 \Rightarrow$ eigenspace: $\boxed{\text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}}$

(b). $\det(A-\lambda I_2) = \det\left(\begin{bmatrix} 22-\lambda & 6 \\ 6 & 13-\lambda \end{bmatrix}\right) = (22-\lambda)(13-\lambda) - 36 = 0. \Rightarrow \lambda = 25, 10.$

① $\boxed{\lambda_1 = 25}$ $\Rightarrow (A-25I_2)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{cc|c} -3 & 6 & 0 \\ 6 & -12 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} -3 & 6 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow x = 2y$

so eigenspace correspondingly is: $\boxed{\text{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}}$

② $\boxed{\lambda_2 = 10}$ $\Rightarrow (A-10I_2)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{cc|c} 12 & 6 & 0 \\ 6 & 3 & 0 \end{array}\right]$

$\Rightarrow \left[\begin{array}{cc|c} 12 & 6 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow y = -2x \Rightarrow$ corresponding eigenspace is: $\boxed{\text{span}\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}}$

(d). $\det(A-\lambda I_2) = \det\left(\begin{bmatrix} \sqrt{3}/2-\lambda & -1/2 \\ 1/2 & \sqrt{3}/2-\lambda \end{bmatrix}\right) = \left(\frac{\sqrt{3}}{2}-\lambda\right)\left(\frac{\sqrt{3}}{2}-\lambda\right) - \left(\frac{1}{2}\right)\cdot\left(-\frac{1}{2}\right) = 0$

$\Rightarrow \lambda^2 - \sqrt{3}\lambda + 1 = 0. \Rightarrow \lambda = \frac{\sqrt{3}+i}{2}, \frac{\sqrt{3}-i}{2}$

① $\boxed{\lambda_1 = \frac{\sqrt{3}+i}{2}}$. So $\left(A - \frac{\sqrt{3}+i}{2} I_2\right)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{cc|c} -i/2 & -1/2 & 0 \\ 1/2 & -i/2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} -i/2 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\Rightarrow y = -ix \Rightarrow$ so corresponding eigenspace is: $\boxed{\text{span}\left\{\begin{bmatrix} 1 \\ -i \end{bmatrix}\right\}}$

② $\boxed{\lambda_2 = \frac{\sqrt{3}-i}{2}}$. So $\left(A - \frac{\sqrt{3}-i}{2} I_2\right)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{cc|c} i/2 & -1/2 & 0 \\ 1/2 & i/2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} i/2 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\Rightarrow y = ix \Rightarrow$ corresponding eigenvector: $\boxed{\text{span}\left\{\begin{bmatrix} 1 \\ i \end{bmatrix}\right\}}$

3. (c). See IPython Notebook. Matrix A_1 is the identity matrix.
And there are almost no visible differences between matrices A_2 and A_3.

(d). Notice that we're considering eigenvalue in absolute value.
As the absolute value of the smallest eigenvalue decreases, the noise $\boxed{\text{increases}}$.

4. (a). ① For $\vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so $A\vec{v_1} = \begin{bmatrix} a+b \\ c+d \end{bmatrix} = (a+b)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ since $a+b = c+d$.
Thus, $\lambda_1 = a+b$, its eigenspace is $\text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

② For $\vec{v_2} = \begin{bmatrix} b \\ -c \end{bmatrix}$, so $A\vec{v_2} = \begin{bmatrix} ab-bc \\ bc-cd \end{bmatrix} = (a-c)\begin{bmatrix} b \\ -c \end{bmatrix} = (d-b)\begin{bmatrix} b \\ -c \end{bmatrix}$ since $a+b = c+d$.
which gives $a-c = d-b$. So $\lambda_2 = a-c$, its eigenspace is $\text{span}\left\{\begin{bmatrix} b \\ -c \end{bmatrix}\right\}$.