EECS 16A Designing Information Devices and Systems I
Fall 2018 Homework 5

This homework is due September 28, 2018, at 23:59.
Self-grades are due October 2, 2018, at 23:59.

Submission Format

Your homework submission should consist of **two** files.

- `hw5.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.

  If you do not attach a PDF of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible.

- `hw5.ipynb`: A single IPython notebook with all of your code in it.

  In order to receive credit for your IPython notebook, you must submit both a "printout" and the code itself.

Submit each file to its respective assignment on Gradescope.

1. Mechanical Eigenvalues and Eigenvectors

Find the eignvalues and their eigenspaces — give a basis for the eigenspace when it is more than 1 dimensional

(a) $\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$

Solution:

There are two ways to do this.

First, we can do it by inspection. We can see that this matrix multiplies everything in the first coordinate by 5 and everything in the second by 2. Consequently, when given $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, it will return 2 times the input. And when given $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, it will return 5 times the input vector.

Alternatively, we can use determinants.

$$\begin{vmatrix} 5-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(2-\lambda) - 0 = 0$$

This is already factored for you! We see that, by definition, diagonal matrices have their eigenvalues on the diagonal.

$\lambda = 5$:

$$\mathbf{A}\vec{x} = 5\vec{x} \implies (\mathbf{A} - 5\mathbf{I}_2)\vec{x} = \vec{0}$$

$$\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix} \implies y = 0 \implies \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The eigenspace for $\lambda = 5$ is span$\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$.

$\lambda = 2$:

$$\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \implies x = 0 \implies \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The eigenspace for $\lambda = 2$ is span$\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$.

(b) $\begin{bmatrix} 22 & 6 \\ 6 & 13 \end{bmatrix}$

Solution:

Here, it is hard to guess the answers.

$$(22-\lambda)(13-\lambda) - 36 = 0 \implies \lambda = 10, 25$$

$\lambda = 10$:

$$\begin{bmatrix} 12 & 6 \\ 6 & 3 \end{bmatrix} \implies \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \implies 2x + y = 0 \implies \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

The eigenspace for $\lambda = 10$ is span$\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\}$.

$\lambda = 25$:

$$\begin{bmatrix} -3 & 6 \\ 6 & -12 \end{bmatrix} \implies \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} \implies 2y = x \implies \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The eigenspace for $\lambda = 25$ is span$\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\}$.

(c) **(PRACTICE)** $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

Solution:

This can also be seen by inspection. The matrix is clearly not invertible since the first two rows are linearly dependent. Therefore, there must be a 0 eigenvalue. This has the eigenvector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

The other eigenvector can be seen by noticing that the second row is twice the first. So $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a good guess to try and indeed it works with $\lambda = 5$.

Alternatively, we can brute force it.

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

$$\begin{aligned} (1-\lambda)(4-\lambda) - 4 &= 0 \\ \lambda^2 - 5\lambda = 0 &\implies \lambda(\lambda - 5) = 0 \\ \lambda &= 0, 5 \end{aligned}$$

$\lambda = 0$:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$x + 2y = 0 \implies y = -\frac{1}{2}x \implies \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

The eigenspace for $\lambda = 0$ is span$\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\}$.

$\lambda = 5$:

$$\begin{bmatrix} 1-5 & 2 \\ 2 & 4-5 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

$$2x - y = 0 \implies y = 2x \implies \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The eigenspace for $\lambda = 5$ is span$\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\}$.

(d) $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ (What special matrix is this?)

Solution:

This is a rotation matrix (counterclockwise by 30 degrees).

$$\begin{vmatrix} \frac{\sqrt{3}}{2} - \lambda & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} - \lambda \end{vmatrix} = 0$$

$$\left(\frac{\sqrt{3}}{2} - \lambda\right)^2 + \frac{1}{4} = 0 \implies \lambda = \frac{\sqrt{3} \pm i}{2}$$

$\lambda = \frac{\sqrt{3}+i}{2}$:

$$\begin{bmatrix} \frac{-i}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{-i}{2} \end{bmatrix} \implies \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \implies \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$

$$x - iy = 0 \implies \begin{bmatrix} i \\ 1 \end{bmatrix}$$

The eigenspace for $\lambda = \frac{\sqrt{3}+i}{2}$ is span$\{\begin{bmatrix} i \\ 1 \end{bmatrix}\}$.

$\lambda = \frac{\sqrt{3}-i}{2}$:

$$\begin{bmatrix} \frac{i}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{i}{2} \end{bmatrix} \implies \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \implies \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$$

$$x + iy = 0 \implies \begin{bmatrix} 1 \\ i \end{bmatrix}$$

The eigenspace for $\lambda = \frac{\sqrt{3}-i}{2}$ is span$\{\begin{bmatrix} 1 \\ i \end{bmatrix}\}$.

(e) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Solution:

$$\begin{vmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)^2 = 0 \implies \lambda = 2, 2$$

$\lambda = 2$:

$$\begin{bmatrix} 2-2 & 0 \\ 0 & 2-2 \end{bmatrix} \implies \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

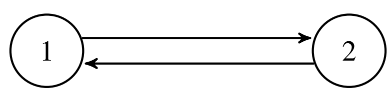
Now, both x, y are free variables. This means that our eigenspace for $\lambda = 2$ is span$\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$.

2. Counting The Paths of a Random Surfer

In homework and discussion, we have discussed the behavior of water flowing in reservoirs and the people flowing in social networks. We now consider the behavior of a random web-surfer who jumps from webpage to webpage. We would like to know how many possible paths there are for a random surfer to get from one webpage to another webpage. To do this, we represent the webpages as a graph.

If webpage 1 has a link to webpage 2, we have a directed edge from webpage 1 to webpage 2. This graph can further be represented by what is known as an "adjacency matrix", $\mathbf{A}$, with elements a_{ij}. We define $a_{ji} = 1$ if there is link from page i to page j. Note the ordering of the indices! Matrix operations on the adjacency matrix make it very easy to compute the number of paths to get from a particular webpage i to webpage j.

Consider the following graphs.



Graph A

(a) Based on this definition, the "adjacency matrix" for graph A, will be,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{1}$$

The element a_{ji} of $\mathbf{A}$ gives the number of one-hop paths from from website i to website j. Similarly, the elements of $\mathbf{A}^2$ give the number of two-hop paths from website i to website j. How many one-hop paths are there from webpage 1 to webpage 2? How many two-hop paths are there from webpage 1 to webpage 2? How about three-hop paths?

Solution:

We take the n^{th} power of the adjacency matrix to determine how many n-hops paths exist between the pages.

$$\mathbf{A}^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Therefore, there is 1 one-hop path between webpage 1 and webpage 2 (which can be checked trivially).

$$\mathbf{A}^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

There are no two-hop paths between webpage 1 and webpage 2! This matches the structure of the graph since two hops will always get the websurfer back to the page they started from.

Why does this work? Let's look at what the $\mathbf{A}^2$ matrix multiplication does:

The first element (describing the path to get to and from node 1 in two hops) is (number of paths from node 1 to node 1)^2+(number of paths from node 2 to node 1)(number of paths from node 1 to node 2). This is $0+(1)(1)=1$. The result is therefore the sum of any self-loops and the number of paths going to node 2 and back. A similar formula applies for the n^{th} powers.

This is because the concatenation of two paths is a valid path if one ends where the other begins. So the number of n-hop paths from i to j must in fact be the sum, over all intermediate pages k, of the number of ℓ-hop paths from i to k times the number of $(n-\ell)$-hop paths from k to j. This is precisely what matrix multiplication does.

$$\mathbf{A}^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

There is 1 three-hop path between webpage 1 and webpage 2. Note that $\mathbf{A}^3 = \mathbf{A}$.

(b) This path counting aspect is very related to the steady-state frequency for the fraction of people for each webpage. The steady-state frequency for a graph of websites is related to the eigenspace associated with eigenvalue 1 for the "transition matrix" of the graph.

The "transition matrix", $\mathbf{T}$, is slightly different from the "adjacency matrix". Its values, t_{ji}, are the *proportion* of the people who are at website i that click the link for website j. We assume people divide equally among the links on the website (e.g. if there are three links on a website, $\frac{1}{3}$ of the people will click each link).

Once computed, an eigenvector with eigenvalue 1 will have values which correspond to the steady-state frequency for the fraction of people for each webpage. When this eigenvector's values are made to sum to one (to conserve people), the i^{th} element of the eigenvector corresponds to the fraction of people on the i^{th} website.

For graph A, what are the steady-state frequencies for the two webpages?

Solution:

To determine the steady-state frequencies for the two pages, we need to find the appropriate eigenvector of the transition matrix. In this case, we are trying to determine the proportion of people who would be on a given page at steady state.

The transition matrix of graph A:

$$\mathbf{T} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{2}$$

To determine the eigenvalues of this matrix:

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\right) = \lambda^2 - 1 = 0 \tag{3}$$

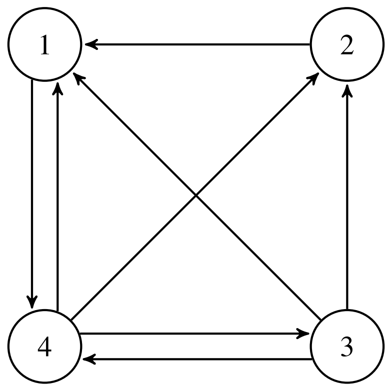
$\lambda = 1, -1$. The steady state vector is the eigenvector that corresponds to $\lambda = 1$. To find the eigenvector,

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{4}$$

The sum of the values of the vector should equal 1, so our conditions are:

$$v_1 + v_2 = 1$$
$$v_1 = v_2$$

The steady-state frequency eigenvector is $\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$ and each webpage has a steady-state frequency of 0.5.



Graph B

(c) Write out the adjacency matrix for graph B.

Solution:

Let $\mathbf{B}$ be the adjacency matrix for graph B. Then,

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

(d) For graph B, how many two-hop paths are there from webpage 1 to webpage 3? How many three-hop paths are there from webpage 1 to webpage 2? You may use your IPython notebook for this.

Solution:

Using the same procedure as part (b),

$$\mathbf{B}^2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

There is 1 two-hop path from webpage 1 to webpage 3. Let's look at what the $\mathbf{B}^2$ matrix multiplication does:

The B_{31} element (describing the path to get from webpage 1 to webpage 3) is

(paths from node 1 to node 3)(paths from node 1 to node 1)+(paths from node 2 to node 3)(paths from node 1 to node 2)+(paths from node 3 to node 3)(paths from node 1 to node 3)+(paths from node 4 to node 3)(paths from node 1 to node 4)+(paths from node 5 to node 3)(paths from node 1 to node 5).

This is $0+0+0+1 = 1$. A similar formula applies for the n^{th} powers.

For three hop paths,

$$\mathbf{B}^3 = \begin{bmatrix} 1 & 0 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 & 3 \\ 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 \\ 2 & 0 & 3 & 2 \end{bmatrix}$$

There is 1 three-hop path from webpage 1 to webpage 2.

(e) For graph B, what are the steady-state frequencies for the webpages? You may use your IPython notebook and the Numpy command numpy.linalg.eig for this.

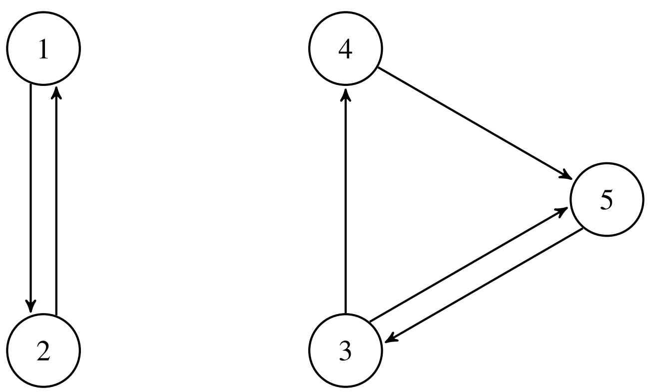
Solution:

To determine the steady-state frequencies, we need to create the transition matrix $\mathbf{T}$ first.

$$\mathbf{T} = \begin{bmatrix} 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} \\ 1 & 0 & \frac{1}{3} & 0 \end{bmatrix}$$

The eigenvector associated with eigenvalue 1 is $\begin{bmatrix} -0.61 & -0.31 & -0.23 & -0.69 \end{bmatrix}^T$ (found using IPython). Scaling it appropriately so the elements add to 1, we get $\begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{8} & \frac{3}{8} \end{bmatrix}^T$

These are the steady-state frequencies for the pages.



Graph C

(f) Write out the adjacency matrix for graph C.

Solution:

Let $\mathbf{C}$ be the adjacency matrix for graph C. Then,

$$\mathbf{C} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

(g) For graph C, how many paths are there from webpage 1 to webpage 3?

Solution:

There are no paths from webpage 1 to webpage 3 (and no n-hop paths either).

(h) **(PRACTICE)** Find the eigenspace that corresponds to the steady-state for graph C. How many independent systems (disjoint sets of webpages) are there in graph C versus in graph B? What is the dimension of the eigenspace corresponding to the steady-state for graph C?

Solution:

The transition matrix for graph C is

$$\mathbf{T} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & 0 \end{bmatrix}$$

The eigenvalues of this graph are $\lambda = 1, 1, -1, -\frac{1}{2} + -\frac{i}{2}, -\frac{1}{2} - -\frac{i}{2}$ (found using IPython). The eigenspace associated with $\lambda = 1$ is span by the vectors $\begin{bmatrix} 0 & 0 & 0.4 & 0.2 & 0.4 \end{bmatrix}^T$ and $\begin{bmatrix} 0.5 & 0.5 & 0 & 0 & 0 \end{bmatrix}^T$. While, any linear combination of these vectors is a eigenvector, these two particular vectors have a nice interpretation.

The first eigenvector describes the steady-state frequencies for the last three webpages, and the second vector describes the steady-state frequencies for the first two webpages. This makes sense since there are essentially “two internets", or two disjoint set of webpages. Surfers cannot transition between the two, so you cannot assign steady-state frequencies to webpage 1 and webpage 2 relative to the rest. This is why the eigenspace corresponding to the steady-state has dimension 2.

Assuming that each set of steady-state frequencies needs to add to 1, the first assigns steady-state frequencies of 0.4, 0.2, 0.4 to webpage 3, webpage 4, and webpage 5, respectively. The second assigns steady-state frequencies of 0.5 to both webpage 1 and webpage 2.

3. Noisy Images

In lab, we used a single pixel camera to capture many measurements of an image $\vec{x}$. A single measurement y_i is captured using a mask $\vec{a}_i$ such that $y_i = \vec{a}_i{}^T \vec{x}$. Many measurements can be expressed as a matrix-vector multiplication of the masks with the image, where the masks lie along the rows of the matrix.

$$\begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} \vec{a}_1 \\ \vdots \\ \vec{a}_N \end{bmatrix} \vec{x} \tag{5}$$

$$\vec{y} = \mathbf{A}\vec{x} \tag{6}$$

In the real world, noise, $\vec{n}$, creeps into our measurements, so instead,

$$\vec{y} = \mathbf{A}\vec{x} + \vec{n} \tag{7}$$

(a) Express $\vec{x}$ in terms of $\mathbf{A}$ (or its inverse), $\vec{y}$, and $\vec{n}$. (*Hint:* Think about what you did in the imaging lab.)

Solution:

$$\vec{y} = \mathbf{A}\vec{x} + \vec{n} \tag{8}$$
$$\mathbf{A}^{-1}\vec{y} = \mathbf{A}^{-1}(\mathbf{A}\vec{x} + \vec{n}) \tag{9}$$
$$\mathbf{A}^{-1}\vec{y} = \mathbf{A}^{-1}\mathbf{A}\vec{x} + \mathbf{A}^{-1}\vec{n} \tag{10}$$
$$\mathbf{A}^{-1}\vec{y} = \vec{x} + \mathbf{A}^{-1}\vec{n} \tag{11}$$
$$\vec{x} = \mathbf{A}^{-1}\vec{y} - \mathbf{A}^{-1}\vec{n} \tag{12}$$

(b) Now, because there is noise in our measurements, there will be noise in our recovered image, however, the noise is scaled. The noise in the recovered image, $\hat{\vec{w}}$, is related to $\vec{w}$, but it is transformed by $\mathbf{A}^{-1}$. Specifically,

$$\hat{\vec{w}} = \mathbf{A}^{-1}\vec{w} \tag{13}$$

To analyze how this transformation alters $\vec{w}$, consider representing $\vec{w}$ as a linear combination of the eigenvectors of $\mathbf{A}^{-1}$,

$$\vec{w} = \alpha_1\vec{b}_1 + \ldots + \alpha_N\vec{b}_N, \tag{14}$$

where, $\vec{b}_i$ is $\mathbf{A}^{-1}$'s eigenvector with eigenvalue λ_i. Now we can express the recovered image's noise as,

$$\hat{\vec{w}} = \mathbf{A}^{-1}\vec{w} = \mathbf{A}^{-1}(\alpha_1\vec{b}_1 + \ldots + \alpha_N\vec{b}_N) \tag{15}$$
$$= \alpha_1\mathbf{A}^{-1}\vec{b}_1 + \ldots + \alpha_N\mathbf{A}^{-1}\vec{b}_N \tag{16}$$
$$= \alpha_1\lambda_1\vec{b}_1 + \ldots + \alpha_N\lambda_N\vec{b}_N \tag{17}$$

Depending on the size of the eigenvalues, noise in the recovered image will be amplified or attenuated. For eigenvectors with large eigenvalues, will the noise signal along those eigenvectors be amplified of attenuated? For eigenvectors with small eigenvalues, will the noise signal along those eigenvectors be amplified of attenuated?

Solution: For eigenvectors with large eigenvalues, the noise signal will be amplified. This is bad and could corrupt the recovered image significantly.

For eigenvectors with small eigenvalues, the noise signal will be attenuated. This is better and will not corrupt the recovered image as much.

(c) We are going to try different $\mathbf{A}$ matrices in this problem and compare how they deal with noise. Run the associated cells in the attached IPython notebook. What special matrix is $\mathbf{A}_1$? Are there any differences between the matrices $\mathbf{A}_2$ and $\mathbf{A}_3$?

Solution:

See `sol5.ipynb`.

The matrix $\mathbf{A}_1$ is the identity matrix. Notice that there are almost no visibile differences between the matrices $\mathbf{A}_2$ and $\mathbf{A}_3$.

(d) Run the associated cells in the attached IPython notebook. Notice that each plot returns the result of trying to image a noisy image as well as the minimum absolute value of the eigenvalue of each matrix. Comment on the effect of small eigenvalues on the noise in the image.

Solution:

See `sol5.ipynb`.

Notice that we are printing the eigenvalue with the smallest absolute value. As the absolute value of the smallest eigenvalue decreases, the noise in the result increases.

(e) Depending on how large or small the eigenvalues of $\mathbf{A}^{-1}$ are, we will amplify or attenuate our measurement's noise. These eigenvalues are actually related to the eigenvalues of $\mathbf{A}$! Inverting the matrix $\mathbf{A}$ turns these small eigenvalues into large eigenvalues. Show that if λ is an eigenvalue of a matrix $\mathbf{A}$, then $\frac{1}{\lambda}$ is an eigenvalue of the matrix $\mathbf{A}^{-1}$.

Hint: Start with an eigenvalue λ and one corresponding eigenvector $\vec{v}$, such that they satisfy $\mathbf{A}\vec{v} = \lambda\vec{v}$.

Solution:

For some eigenvector $\vec{v}$ and associated eigenvalue λ, we know that:

$$\begin{aligned}
\mathbf{A}\vec{v} &= \lambda\vec{v} \\
\vec{v} &= \mathbf{A}^{-1}(\lambda\vec{v}) \\
\vec{v} &= \lambda\mathbf{A}^{-1}\vec{v} \\
\frac{1}{\lambda}\vec{v} &= \mathbf{A}^{-1}\vec{v} \\
\mathbf{A}^{-1}\vec{v} &= \frac{1}{\lambda}\vec{v}
\end{aligned}$$

Therefore, $\frac{1}{\lambda}$ is an eigenvalue of $\mathbf{A}^{-1}$ with the corresponding eigenvector $\vec{v}$.

4. The Dynamics of Romeo and Juliet's Love Affair

In this problem, we will study a discrete-time model of the dynamics of Romeo and Juliet's love affair—adapted from Steven H. Strogatz's original paper, *Love Affairs and Differential Equations*, Mathematics Magazine, 61(1), p.35, 1988, which describes a continuous-time model.

Let $R[n]$ denote Romeo's feelings about Juliet on day n, and let $J[n]$ denote Juliet's feelings about Romeo on day n. The sign of $R[n]$ (or $J[n]$) indicates like or dislike. For example, if $R[n] > 0$, it means Romeo likes Juliet. On the other hand, $R[n] < 0$ indicates that Romeo dislikes Juliet. $R[n] = 0$ indicates that Romeo has a neutral stance towards Juliet.

The magnitude (i.e. absolute value) of $R[n]$ (or $J[n]$) represents the intensity of that feeling. For example, a larger $|R[n]|$ means that Romeo has a stronger emotion towards Juliet (love if $R[n] > 0$ or hatred if $R[n] < 0$). Similar interpretations hold for $J[n]$.

We model the dynamics of Romeo and Juliet's relationship using the following linear system:

$$R[n+1] = aR[n] + bJ[n], \quad n = 0, 1, 2, \ldots$$

and

$$J[n+1] = cR[n] + dJ[n], \quad n = 0, 1, 2, \ldots,$$

which we can rewrite as

$$\vec{s}[n+1] = \mathbf{A}\,\vec{s}[n],$$

where $\vec{s}[n] = \begin{bmatrix} R[n] \\ J[n] \end{bmatrix}$ denotes the state vector and $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the state transition matrix for our dynamic system model.

The selection of the parameters a,b,c,d results in different dynamic scenarios. The fate of Romeo and Juliet's relationship depends on these model parameters (i.e. a,b,c,d) in the state transition matrix and the initial state ($\vec{s}[0]$). In this problem, we'll explore some of these possibilities.

(a) Consider the case where $a+b=c+d$ in the state-transition matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Show that

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is an eigenvector of $\mathbf{A}$, and determine its corresponding eigenvalue λ_1. Show that

$$\vec{v}_2 = \begin{bmatrix} b \\ -c \end{bmatrix}$$

is an eigenvector of $\mathbf{A}$, and determine its corresponding eigenvalue λ_2. Now, express the first and second eigenvalues and their eigenspaces in terms of the parameters $a,b,c,$ and d.

Hint: You could use the characteristic polynomial approach to find the eigenvalues and eigenvectors. You may find it easier to use the following approach instead:

- First find λ_1 by showing $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of $\mathbf{A}$.
- Then find λ_2 by showing $\vec{v}_2 = \begin{bmatrix} b \\ -c \end{bmatrix}$ is an eigenvector of $\mathbf{A}$.

Solution:

$$\begin{aligned} \mathbf{A} \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} a+b \\ c+d \end{bmatrix} \\ &= (a+b) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= (c+d) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

Let $\lambda_1 = a+b = c+d$. Then you can plug in to find that $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ_1. The first eigenpair $\mathbf{A}$ is,

$$\left(\lambda_1 = a+b = c+d, \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

To determine the other eigenpair $(\lambda_2, \vec{v}_2)$, we use the hint that $\vec{v}_2 = \begin{bmatrix} b \\ -c \end{bmatrix}$. Note that by modifying the constraint $a+b=c+d$, we can also get $a-c=d-b$, which helps simplify the following:

$$\mathbf{A}\begin{bmatrix} b \\ -c \end{bmatrix} = \begin{bmatrix} ab-bc \\ cb-dc \end{bmatrix} = \begin{bmatrix} b(a-c) \\ -c(d-b) \end{bmatrix} = (a-c)\begin{bmatrix} b \\ -c \end{bmatrix} = (d-b)\begin{bmatrix} b \\ -c \end{bmatrix}$$

Therefore, we have our second eigenpair:

$$\left(\lambda_2 = a-c = d-b, \vec{v}_2 = \begin{bmatrix} b \\ -c \end{bmatrix}\right).$$

For parts (b) - (d), consider the following state-transition matrix:

$$\mathbf{A} = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$$

(b) Determine the eigenpairs (i.e. $(\lambda_1, \vec{v}_1)$ and $(\lambda_2, \vec{v}_2)$) for this system. Note that this matrix is a special case of the matrix explored in part (a), so you can use results from that part to help you.

Solution:

From the results of part (a), we know that the eigenpairs of this matrix are

$$\left(\lambda_1 = a+b = 0.75+0.25 = 1, \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

and

$$\left(\lambda_2 = a-c = 0.75-0.25 = 0.5, \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right).$$

Note: If your choice of eigenvector $\vec{v}_1$ and $\vec{v}_2$ is a scaled version of the ones given in this solution, that is fine.

(c) Determine all of the *steady states* of the system. That is, find the set of points such that if Romeo and Juliet start at, or enter, any of those points, their states will stay in place forever: $\{\vec{s}_* \mid \mathbf{A}\vec{s}_* = \vec{s}_*\}$.

Solution: Any $\vec{s}_* \in \text{span}\{\vec{v}_1\}$, where $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, is the eigenvector which corresponds to the steady state, because $\vec{v}_1$ corresponds to the eigenvalue $\lambda_1 = 1$.

(d) Suppose Romeo and Juliet start from an initial state $\vec{s}[0] \in \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$. What happens to their relationship over time? Specifically, what is $\vec{s}[n]$ as $n \to \infty$?

Solution:

We note that $\vec{s}[0] \in \text{span}\{\vec{v}_2\}$. Therefore,

$$\begin{aligned}\vec{s}[1] &= \mathbf{A}\vec{s}[0] \\ &= \alpha\lambda_2\vec{v}_2\end{aligned}$$

where α is the scalar that expresses $\vec{s}[0]$ as a scaled version of $\vec{v}_2$.
If we continue to apply the state transition matrix, we will see that for this $\vec{s}[0]$,

$$\begin{aligned}\vec{s}[n] &= \mathbf{A}^n\vec{s}[0] \\ &= \alpha\lambda_2^n\vec{v}_2\end{aligned}$$

In this case $\lambda_2 = 0.5$. This means that as $n \to \infty$, $\lambda_2^n \to 0$.
Therefore,

$$\begin{aligned}\vec{s}[n] &= \alpha\lambda_2^n\vec{v}_2 \\ &= \alpha \cdot 0 \cdot \vec{v}_2 \\ &= \vec{0}\end{aligned}$$

which means that

$$\lim_{n\to\infty}(R[n], J[n]) = (0,0)$$

So, ultimately, Romeo and Juliet will become neutral to each other.

Now suppose we have the following state-transition matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Use this state-transition matrix for parts (e) - (g).

(e) Determine the eigenpairs (i.e. $(\lambda_1, \vec{v}_1)$ and $(\lambda_2, \vec{v}_2)$) for this system. Note that this matrix is a special case of the matrix explored in part (a), so you can use results from that part to help you.
Solution: From the results of part (a), we know that the eigenpairs of this matrix are

$$\left(\lambda_1 = a + b = 1 + 1 = 2, \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

and

$$\left(\lambda_2 = a - c = 1 - 1 = 0, \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right).$$

(f) Suppose Romeo and Juliet start from an initial state $\vec{s}[0] \in \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$. What happens to their relationship over time? Specifically, what is $\vec{s}[n]$ as $n \to \infty$?
Solution: The initial state $\vec{s}[0]$ lies in the span of the eigenvector $\vec{v}_2$, which has eigenvalue $\lambda_2 = 0$. Thus, $\vec{s}[1] = \vec{0}$. The state will remain at $\vec{0}$ for all subsequent time steps, i.e.

$$\vec{s}[n] = \vec{0}, n \geq 1$$

Therefore, Romeo and Juliet become neutral towards each other in the long run, i.e.

$$\lim_{n\to\infty} (R[n], J[n]) = (0,0)$$

(g) Now suppose that Romeo and Juliet start from an initial state $\vec{s}[0] \in \text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$. What happens to their relationship over time? Specifically, what is $\vec{s}[n]$ as $n \to \infty$?

Solution:

We note that $\vec{s}[0] \in \text{span}\{\vec{v}_1\}$. Therefore,

$$\begin{aligned}\vec{s}[1] &= \mathbf{A}\vec{s}[0]\\ &= \alpha\lambda_1\vec{v}_1\end{aligned}$$

where α is the scalar that expresses $\vec{s}[0]$ as a scaled version of $\vec{v}_1$.
If we continue to apply the state transition matrix, we will see that for this $\vec{s}[0]$,

$$\begin{aligned}\vec{s}[n] &= \mathbf{A}^n\vec{s}[0]\\ &= \alpha\lambda_1^n\vec{v}_1\end{aligned}$$

In this problem, $\lambda_1 = 2$. Therefore,

$$\vec{s}[n] = \alpha 2^n \vec{v}_1$$

This means that as $n \to \infty$, $\lambda_1^n \to \infty$. Essentially, the elements of the state vector continue to double at each time step and grow without bound to either $+\infty$ or $-\infty$.
Therefore, what happens to Romeo and Juliet depends on $\vec{s}[0]$. If $\vec{s}[0]$ is in the first quadrant, Romeo and Juliet will become "infinitely" in love with each other. On the other hand, if $\vec{s}[0]$ is in the third quadrant, then Romeo and Juliet will have "infinite" hatred for each other.

Finally, we consider the case where we have the following state-transition matrix:

$$\mathbf{A} = \begin{bmatrix}1 & -2\\ -2 & 1\end{bmatrix}$$

Use this state-transition matrix for parts (h) - (j).

(h) Determine the eigenpairs (i.e. $(\lambda_1, \vec{v}_1)$ and $(\lambda_2, \vec{v}_2)$) for this system. Note that this matrix is a special case of the matrix explored in part (a), so you can use results from that part to help you.

Solution: From the results of part (a), we know that the eigenpairs of this matrix are

$$\left(\lambda_1 = a + b = 1 - 2 = -1, \vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}\right)$$

and

$$\left(\lambda_2 = a - c = 1 - (-2) = 3, \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right).$$

(i) Suppose Romeo and Juliet start from an initial state $\vec{s}[0] \in \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$. What happens to their relationship over time if $R[0] > 0$ and $J[0] < 0$? What about if $R[0] < 0$ and $J[0] > 0$? Specifically, what is $\vec{s}[n]$ as $n \to \infty$?

Solution: The initial state $\vec{s}[0]$ lies in the span of the eigenvector $\vec{v}_2$, which has eigenvalue $\lambda_2 = 3$. Using similar methods to the solutions in part (d) and part (g), we can see that (for a given scalar α):

$$\begin{aligned}\vec{s}[n] &= \mathbf{A}^n \vec{s}[0] \\ &= \alpha \lambda_2^n \vec{v}_2 \\ &= \alpha 3^n \vec{v}_2\end{aligned}$$

There are two cases of long-term behavior.

Suppose, initially, that $R[0] > 0$ and $J[0] < 0$ (corresponding to $\alpha > 0$). Then as $n \to \infty$, $R[n] \to \infty$ and $J[n] \to -\infty$. Romeo will have "infinite" love for Juliet, while Juliet will have "infinite" hatred for Romeo.

Conversely, if initially $R[0] < 0$ and $J[0] > 0$ (corresponding to $\alpha < 0$), then as $n \to \infty$, $R[n] \to -\infty$ and $J[n] \to \infty$. Now Romeo would have "infinite" hatred for Juliet, while Juliet would have "infinite" love for Romeo.

(j) Now suppose that Romeo and Juliet start from an initial state $\vec{s}[0] \in \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$. What happens to their relationship over time? Specifically, what is $\vec{s}[n]$ as $n \to \infty$?

Solution: The initial state $\vec{s}[0]$ lies in the span of the eigenvector $\vec{v}_1$, which has eigenvalue $\lambda_1 = -1$. As with parts (d), (g), and (i), we can see that (for a given scalar α):

$$\begin{aligned}\vec{s}[n] &= \mathbf{A}^n \vec{s}[0] \\ &= \alpha \lambda_1^n \vec{v}_1 \\ &= \alpha (-1)^n \vec{v}_1\end{aligned}$$

The elements of the state vector continue to switch signs at each time step, while keeping the same magnitude.

Essentially, Romeo and Juliet maintain the same intensity (i.e. absolute value or magnitude) of feeling, but they keep changing their mind about whether that feeling is like or dislike at each time step. Note that $R[0]$ and $J[0]$ have the same sign, so they both either like each other or dislike each other at a given time step n.

5. (CHALLENGE PRACTICE) Is There A Steady State?

So far, we've seen that for a conservative state transition matrix $\mathbf{A}$, we can find the eigenvector, $\vec{v}$, corresponding to the eigenvalue $\lambda = 1$. This vector is the steady state since $\mathbf{A}\vec{v} = \vec{v}$. However, we've so far taken for granted that the state transition matrix even has the eigenvalue $\lambda = 1$. Let's try to prove this fact.

(a) Show that if λ is an eigenvalue of a matrix $\mathbf{A}$, then it is also an eigenvalue of the matrix $\mathbf{A}^T$.

Note: The transpose of the matrix is a new matrix where the columns of $\mathbf{A}$ are now along the rows of $\mathbf{A}^T$

Hint: The determinants of $\mathbf{A}$ and $\mathbf{A}^T$ are the same. This is because the volumes which these matrices represent are the same.

Solution:

Recall that we find the eigenvalues of a matrix $\mathbf{A}$ by setting the determinant of $\mathbf{A} - \lambda\mathbf{I}$ to 0.

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det\left((\mathbf{A} - \lambda\mathbf{I})^T\right) = \det\left(\mathbf{A}^T - \lambda\mathbf{I}\right) = 0$$

Since the two determinants are equal, the characteristic polynomicals of the two matrices must also be equal. Therefore, they must have the same eigenvalues.

(b) Let a square matrix $\mathbf{A}$ have rows that sum to one. Show that $\vec{1} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T$ is an eigenvector of $\mathbf{A}$. What is the corresponding eigenvalue?

Solution:

Recall that if the rows of $\mathbf{A}$ sum to one, then $\mathbf{A}\vec{1} = \vec{1}$. Therefore, the corresponding eigenvalue is $\lambda = 1$.

(c) Let's put it together now. From the previous two parts, show that any conservative state transition matrix will have the eigenvalue $\lambda = 1$. Recall that conservative state transition matrices are those that have columns that sum to 1.

Solution:

If we tranpose a conservative state transition matrix $\mathbf{A}$, then the rows of $\mathbf{A}^T$ (or the columns of $\mathbf{A}$) sum to one by definition of a conservative system. Then, from part (b), we know that $\mathbf{A}^T$ has the eigenvalue $\lambda = 1$. Furthermore, from part (a), we know that the $\mathbf{A}$ and $\mathbf{A}^T$ have the same eigenvalues, so $\mathbf{A}$ also has the eigenvalue $\lambda = 1$.

6. Homework Process and Study Group

Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.) How did you work on this homework?

Solution:

I worked on this homework with...

I first worked by myself for 2 hours, but got stuck on problem 5, so I went to office hours on...

Then I went to homework party for a few hours, where I finished the homework.