EECS 16A Designing Information Devices and Systems I
Fall 2017 Final

Final Solution

PRINT your student ID: ________________________________

PRINT AND SIGN your name: ____________________, ____________________ ____________________
(last name) (first name) (signature)

PRINT your discussion section and GSI(s) (the one you attend): ________________________________

Name and SID of the person to your left: ________________________________

Name and SID of the person to your right: ________________________________

Name and SID of the person in front of you: ________________________________

Name and SID of the person behind you: ________________________________

1. **Which lab was your favorite lab in EE16A? (1 Point)**

2. **What are your plans for winter break? (1 Point)**

Do not turn this page until the proctor tells you to do so. You may work on the questions above.

3. Mechanical Correlation (4 Points)

Given $\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, and that $\vec{a}$ and $\vec{b}$ are both circular (repeating) signals, **indicate (by circling it) which of the vectors below correspond to the cross correlation of $\vec{a}$ with $\vec{b}$.**

(a)

$$\begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

(d)

$$\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

Solution:

$$\mathbf{C}_{\vec{b}}^{T} \vec{a} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$$

This corresponds to vector (a).

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4. Mechanical Gram-Schmidt (5 Points)

Find a set of orthonormal vectors $\{\vec{u}_1, \vec{u}_2\}$ that spans the same subspace as the set of vectors $\{\vec{v}_1, \vec{v}_2\}$ given below. The vector $\vec{u}_1$ must be a multiple of $\vec{v}_1$, that is $\vec{u}_1 = \alpha\vec{v}_1$ for $\alpha \in \mathbb{R}$.

$$\left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

Solution:

We apply Gram-Schmidt on $\{\vec{v}_1, \vec{v}_2\}$.

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$\vec{u}_2 = \vec{v}_2 - \langle \vec{v}_2, \vec{q}_1 \rangle \vec{q}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \right\rangle \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$\vec{q}_2 = \frac{\vec{u}_2}{\|\vec{u}_2\|} = \sqrt{2} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$$

$$\left\{ \vec{q}_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \vec{q}_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} \right\}$$

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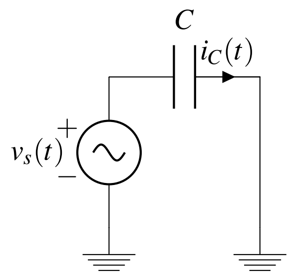
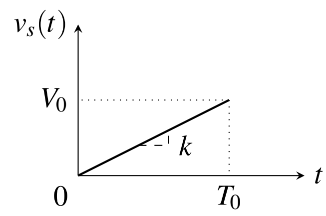
5. Capacitive Touch Revisited (18 Points)

In this problem, we will design a capacitive touch-sensing circuit without switches.

(a) (4 Points) A capacitor with capacitance C is driven by a voltage source $v_s(t)$ as shown below. **Derive an expression for the current through the capacitor** $i_C(t)$ when the voltage source has the time-dependent value shown in the plot below (i.e. $v_s(t) = k * t$ where k is the slope). Your expression for $i_c(t)$ should be in terms of C, V_0, and T_0. (Hint: Use nodal analysis and/or any other circuit analysis technique and the slope of $v_s(t)$.)



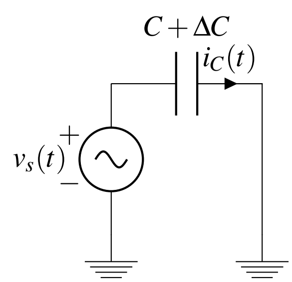


Solution:

$$v_s(t) = \frac{V_0}{T_0} t$$

$$i_C(t) = C\frac{dv_s(t)}{dt} = C\frac{V_0}{T_0}$$

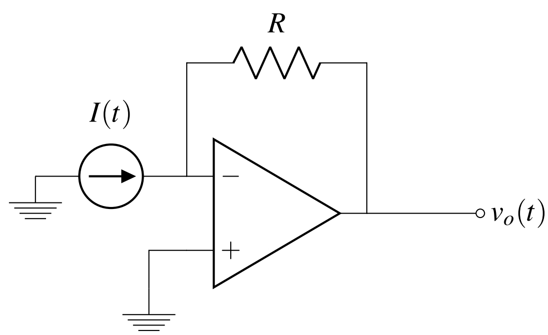
(b) (2 Points) For this problem, we will model the finger touch as having increased the capacitance from C to $C + \Delta C$, as shown below. **Find the current** $i_C(t)$ **for the new circuit shown below.**



Solution:

$$i_C(t) = (C + \Delta C)\frac{V_0}{T_0}$$

(c) (4 Points) **Derive an expression for** $v_o(t)$ for the circuit below in terms of R and $I(t)$.



Solution:

Applying the Golden Rules, we know that no current flows into the input terminals of the op-amp, so $I(t)$ must flow through R. Furthermore, $V^+ = V^- = 0$.

Applying Ohm' law, we get:

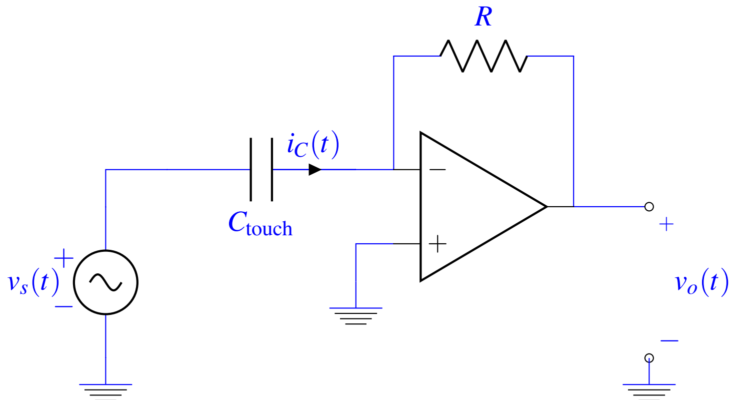
$$\frac{V^- - v_o(t)}{R} = I(t)$$

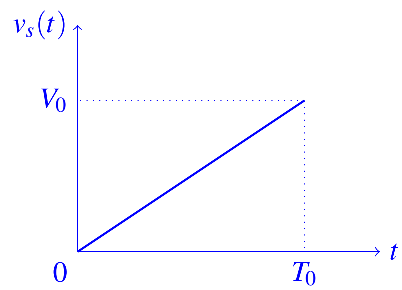
$$v_o(t) = -R \cdot I(t)$$

(d) (8 Points) Now let's **design a circuit whose output voltage is proportional to the capcitance of the touch screen capacitor (i.e., $V_{\text{out}} = \alpha C_{\text{touch}}$, where $\alpha \in \mathbb{R}$)**

You can only use the following components: the "ramp" voltage source $v_s(t)$ from part (a), constant (DC) voltage sources, op-amps, resistors, and the touch screen capacitor (C_{touch}). Clearly label all components that you use in your circuit. If you use a constant (DC) voltage source, clearly label its voltage. If you use the ramp voltage source, provide the value of its slope (k).

Solution:





The slope for $v_s(t)$ is $k = \frac{V_0}{T_0}$.

From part (a), we know that $i_C(t) = C_{\text{touch}}\frac{V_0}{T_0}$. From part (b), we know that $v_o(t) = -R \cdot I(t) = -RC_{\text{touch}}\frac{V_0}{T_0}$, so $\alpha = -Rk = -R\frac{V_0}{T_0}$.

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6. Eigenvectors (13 Points)

Consider a matrix $\mathbf{A} \in \mathbb{R}^{3\times 3}$ with eigenvalues $\lambda = 1,2,3$, and corresponding eigenvectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ respectively. Let the matrix $\mathbf{B} = \mathbf{A}^3 - 6\mathbf{A}^2 + 11\mathbf{A} - 6\mathbf{I}$.

(a) (5 Points) **Find $\mathbf{B}\vec{v}$**, where $\vec{v}$ is one of the eigenvectors of $\mathbf{A}$.

Hint:

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda - 1)(\lambda - 2)(\lambda - 3)$$

Solution:

$$\begin{aligned}\mathbf{B}\vec{v} &= \mathbf{A}^3\vec{v} - 6\mathbf{A}^2\vec{v} + 11\mathbf{A}\vec{v} - 6\mathbf{I}\vec{v} \\ &= \lambda^3\vec{v} - 6\lambda^2\vec{v} + 11\lambda\vec{v} - 6\vec{v} \\ &= (\lambda^3 - 6\lambda^2 + 11\lambda - 6)\vec{v} \\ &= (\lambda - 1)(\lambda - 2)(\lambda - 3)\vec{v}\end{aligned}$$

Note that the eigenvalues of $\mathbf{A}$ are $\lambda = 1,2,3$, which means that $\mathbf{B}\vec{v} = \vec{0}$.

(b) (4 Points) **Find all the eigenvalues of the matrix B.**

Solution:

For any eigenvalue λ of $\mathbf{A}$, where $\mathbf{A}\vec{v} = \lambda\vec{v}$ for some corresponding eigenvector $\vec{v}$,

$$\begin{aligned}\mathbf{B}\vec{v} &= \mathbf{A}^3\vec{v} - 6\mathbf{A}^2\vec{v} + 11\mathbf{A}\vec{v} - 6\mathbf{I}\vec{v} \\ &= \lambda^3\vec{v} - 6\lambda^2\vec{v} + 11\lambda\vec{v} - 6\vec{v} \\ &= (\lambda^3 - 6\lambda^2 + 11\lambda - 6)\vec{v} \\ &= (\lambda - 1)(\lambda - 2)(\lambda - 3)\vec{v}\end{aligned}$$

This implies that for every eigenvalue λ of $\mathbf{A}$, $(\lambda - 1)(\lambda - 2)(\lambda - 3)$ is an eigenvalue of $\mathbf{B}$ with the same eigenvector.

To find the eigenvalues of $\mathbf{B}$, we plug in every eigenvalue $\lambda = 1,2,3$ of $\mathbf{A}$ to get $\lambda = 0$ in all three cases. Since $\mathbf{B}$ has the same eigenvectors as $\mathbf{A}$, the dimension of the eigenspace of $\mathbf{B}$ corresponding to $\lambda = 0$ is 3, so $\mathbf{B}$ cannot have any other eigenvalues. Therefore, the only eigenvalue of $\mathbf{B}$ is $\lambda = 0$.

(c) (4 Points) **Write out the numerical values in the 3×3 matrix B and justify your answer.**

Solution:

Since $\mathbf{A}$ has 3 distinct eigenvalues in $\mathbb{R}^3$, it is diagonalizable, and its eigenvectors form an eigenbasis for $\mathbb{R}^3$. Thus, any $\vec{w} \in \mathbb{R}^3$ can be written as $\vec{w} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \alpha_3 \vec{v}_3$. Therefore, for any $\vec{w}$,

$$\begin{aligned}
\mathbf{B}\vec{w} &= \mathbf{B}\alpha_1 \vec{v}_1 + \mathbf{B}\alpha_2 \vec{v}_2 + \mathbf{B}\alpha_3 \vec{v}_3 \\
&= \alpha_1 \mathbf{B}\vec{v}_1 + \alpha_2 \mathbf{B}\vec{v}_2 + \alpha_3 \mathbf{B}\vec{v}_3 \\
&= \vec{0} + \vec{0} + \vec{0} \\
&= \vec{0}
\end{aligned}$$

Since $\mathbf{B}\vec{w} = \vec{0}$ for all $\vec{w}$, $\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ must be the zero matrix.

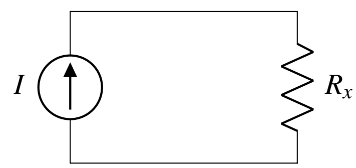
7. Measuring Currents (14 Points)

Your friends Elad and Anant, pleased with your help on Midterm 2, are working in lab and have another problem they need your help with. They found a current source in an unmarked box but have no idea what current it outputs.

Elad decided to attach 3 known resistors with varying values to the terminals of the current source and recorded the resistance and voltage across each resistor in the table below.

Resistor	Resistance	Voltage
R_1	$1\,\Omega$	$5\,\text{V}$
R_2	$3\,\Omega$	$15\,\text{V}$
R_3	$5\,\Omega$	$16\,\text{V}$

Below is the circuit he and Anant created.



(a) (6 Points) Anant thinks that there must have been some noise in the measurements since they do not get the same current for every data point. Because of this, he and Elad would like you to **use least squares to approximate the value of the current source I given their data. Setup this least squares problem in the format $A\vec{x} = \vec{b} + \vec{e}$, where $\vec{e}$ is the error vector, then find the vector $\vec{x}$ that minimizes the norm of $\vec{e}$.**

Solution:

We know that $RI = V$, so we can use least squares to find the best approximation for I.

$$\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \begin{bmatrix} I \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \\ 16 \end{bmatrix} + \vec{e}$$

To minimize $||\vec{e}||$, we apply the least squares formula.

$$I = \left(\begin{bmatrix} 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ 15 \\ 16 \end{bmatrix} = \frac{1}{35} \cdot 130 = \frac{130}{35} = \frac{26}{7}$$

Therefore, $I = \frac{26}{7}$ A.

(b) (8 Points) Anant and Elad repeat this procedure many times and find the variation in measured voltage for each resistor is different. This implies they should "trust" certain measurements more than others. Because of this, Anant constructs a diagonal matrix of weights $\mathbf{W}$, where all of the weights (diagonal entries in the matrix) have non-zero values. They would now like to find the vector $\vec{x}$ that minimizes the weighted error $\vec{e}_w = \mathbf{W}\vec{e}$. This amounts to solving the following optimization problem

$$\min_{\vec{x}} ||\mathbf{W}(\mathbf{A}\vec{x} - \vec{b})||$$

In terms of W, A, and $\vec{b}$ find the vector $\vec{x}$ that minimizes the norm of $\vec{e}_w$. Do not plug in any values for A, W, or $\vec{b}$ - i.e. your result should be symbolic. Do not use calculus in your solution.

Solution:

Since we trying to find $\min_{\vec{x}} ||\mathbf{W}(\mathbf{A}\vec{x} - \vec{b})|| = \min_{\vec{x}} ||\mathbf{WA}\vec{x} - \mathbf{W}\vec{b}||$, we can just apply least squares on $\mathbf{A}^{\dagger} = \mathbf{WA}$ and $\vec{b}^{\dagger} = \mathbf{W}\vec{b}$.

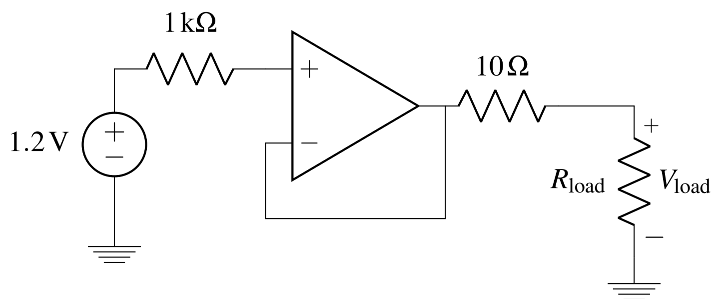
$$\vec{x} = (\mathbf{A}^{\dagger T}\mathbf{A}^{\dagger})^{-1}\mathbf{A}^{\dagger T}\vec{b}^{\dagger} = (\mathbf{A}^T\mathbf{W}^T\mathbf{WA})^{-1}\mathbf{A}^T\mathbf{W}^T\mathbf{W}\vec{b}$$

Since $\mathbf{W}$ is a diagonal matrix, $\mathbf{W}^T\mathbf{W} = \mathbf{W}^2$.

$$\vec{x} = (\mathbf{A}^T\mathbf{W}^2\mathbf{A})^{-1}\mathbf{A}^T\mathbf{W}^2\vec{b}$$

8. IoT4eva version ∞ (18 Points)

IoT4eva is working on their newest device, and in order to meet their form factor targets, they end up putting a cable in between the power module (that includes a simplified version of the votlage regulator you designed previously) and their electronics. This cable has substantial resistance associated with it, and the resulting circuit model is shown below. Throught this problem, you may assume the rails of the op amp are sufficiently large in magnitude to not affect the operation of the circuit.

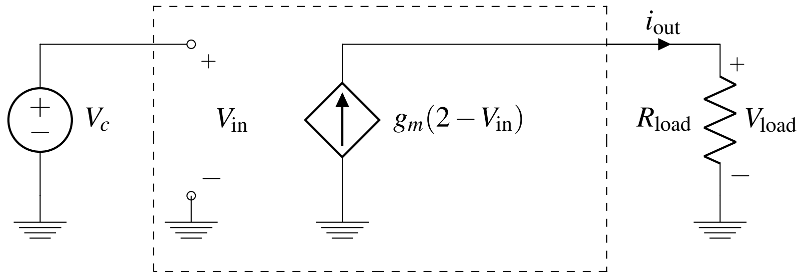


(a) (4 Points) The goal of the linear regulator is to provide a constant voltage across the R_{load} resistance (which models the electronics) independent of the value of R_{load}. **As a function R_{load}, for the circuit shown above, what will be the value of V_{load}?**

Solution:

This is a buffer, so we know by applying the Golden Rules that $V_{\text{out}} = V^- = V^+ = 1.2\,\text{V}$. The $10\,\Omega$ resistor and R_{load} form a voltage divider, so $V_{\text{load}} = \frac{R_{\text{load}}}{R_{\text{load}}+10\,\Omega} \cdot 1.2\,\text{V}$.

(b) (4 Points) One of your coleagues starts exploring the use of a "UCBMFET" device (which has three terminals) to try and improve the situation. The circuit model for a UCBMFET is shown below inside the dotted lines. **For the circuit shown below, what is the value of i_{out} as a function of R_{load}, V_c and g_m?**

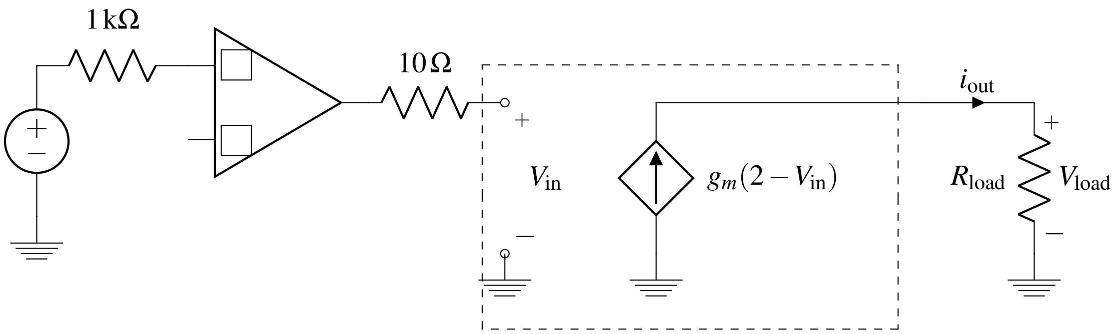


Solution:

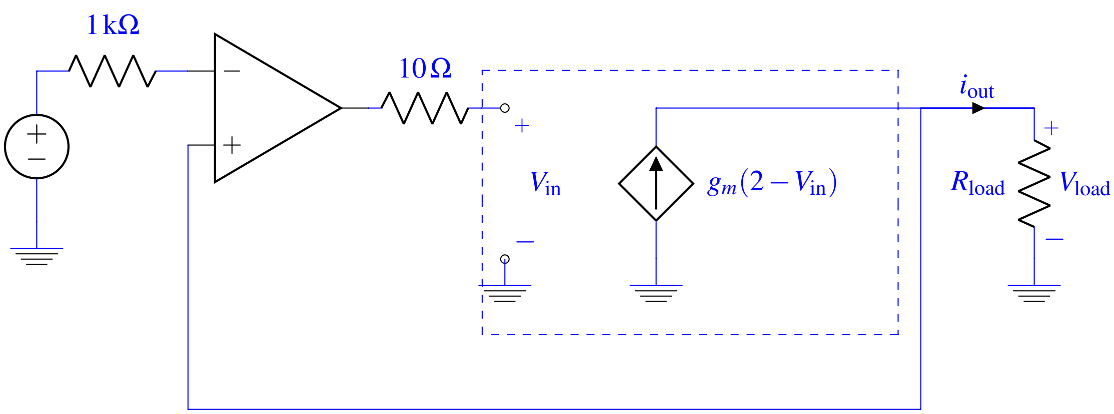
$$\begin{aligned} i_{\text{out}} &= g_m(2 - V_{\text{in}}) \\ &= g_m(2 - V_c) \end{aligned}$$

(c) (10 Points) Another one of your colleagues thinks they can achieve a constant voltage across R_{load} by combining the UCBMFET (placed on the same side of the cable as the electronics) with a modified version of the original regulator circuit. Your colleague started the design and came up with the incomplete circuit shown below.

Recalling that your goal is to design a circuit that will produce a constant voltage across R_{load} (where that voltage does not depend on the resistance of R_{load}), **complete the circuit shown below by adding in a single wire that connnects one of the nodes to the second input terminal of the op-amp, and fill in the boxes to indicate which of the op-amp inputs is positive (+) and which is negative (-). What is the voltage that your circuit would then produce across R_{load}?**



Solution:



This circuit is in negative feedback because if we "dink" V_{in} and increase V_{in}, the dependent current source will produce a smaller current. This current must flow through R_{load}, and by Ohm's law, V_{load} decreases. This causes V^+ of the op-amp to decrease, which then decreases V_{in}.
Since the circuit is in negative feedback, $V_{\text{load}} = V^+ = V^- = 1.2\,\text{V}$.

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9. STOMP Imaging (14 Points)

In the imaging lab, we projected masks onto objects and measured reflected light to create a single pixel camera. To image a 30×40 image, we created 1200 masks and took 1200 measurements.

We are going to image objects using a smaller set of masks, in this case, only 600 masks. Let the matrix $\mathbf{M}$ be the 600×1200 matrix of masks, where every mask is a row. Let the vector $\vec{i}$ of length 1200 represent the image and the vector $\vec{b}$ of length 600 represent what we measure after applying the masks. Concisely,

$$\vec{b} = \mathbf{M}\vec{i}$$

Assume that we have constructed the mask matrix $\mathbf{M}$ such that its columns are approximately orthogonal and have norm N. That is, if $\vec{v}_i$ and $\vec{v}_j$ are columns of $\mathbf{M}$, then $\langle \vec{v}_i, \vec{v}_j \rangle \leq \varepsilon$ for some small ε when $i \neq j$. Furthermore, for all i, $\langle \vec{v}_i, \vec{v}_i \rangle = N^2$.

(a) (3 Points) Because the number of pixels (1200) is greater than the number of masks (600), **is it possible that multiple distinct images, $\vec{i}$, give you the same measurements, $\vec{b}$? Justify your answer.**

Solution:

Yes. Since $\mathbf{M}$ is a 600×1200 matrix, $\mathbf{M}$ has a non-trivial null space. Let $\vec{i}_h$ be an image vector in the null space of $\mathbf{M}$, i.e., $\mathbf{M}\vec{i}_h = \vec{0}$. Then, for some image $\vec{i}_p$, where $\mathbf{M}\vec{i}_p = \vec{b}$, $\mathbf{M}(\vec{i}_p + \vec{i}_h) = \mathbf{M}\vec{i}_p + \mathbf{M}\vec{i}_h = \vec{b} + \vec{0} = \vec{b}$.

Therefore, $\vec{i}_p$ and $\vec{i}_p + \vec{i}_h$ would both give you the same measurements.

(b) (3 Points) Let the $\vec{m}_j$ be the j'th column of the matrix $\mathbf{M}$ and suppose the image contained only one pixel with magnitude α, i.e. the entries in $\vec{i}$ are 0 everywhere except for entry k which has value α. **For this $\vec{i}$, find the vector $\vec{b}$ in terms of the columns of the matrix M.**

Solution:

Matrix-vector multiplication is equivalent to producing a linear combination of the column vectors of $\mathbf{M}$.

$$\vec{b} = \mathbf{M}\vec{i} = \alpha\vec{m}_k$$

(c) (3 Points) If you know that the image contains only one pixel at entry k, **show that the inner product $\langle \vec{m}_i, \vec{b} \rangle$ is maximized at $i = k$**

Solution:

From the previous part, we know that $\vec{b} = \alpha \vec{m}_k$.

$$\langle \vec{m}_i, \vec{b} \rangle = \langle \vec{m}_i, \alpha \vec{m}_k \rangle = \alpha \langle \vec{m}_i, \vec{m}_k \rangle$$

There are now 2 cases for $\alpha \langle \vec{m}_i, \vec{m}_k \rangle$:

i. $i \neq k$.

$$\alpha \langle \vec{m}_i, \vec{m}_k \rangle \leq \alpha \varepsilon$$

ii. $i = k$:

$$\alpha \langle \vec{m}_k, \vec{m}_k \rangle = \alpha N^2$$

Since $N^2 \gg \varepsilon$, the inner product $\langle \vec{m}_i, \vec{b} \rangle$ is maximized at $i = k$.

(d) (5 Points) Now suppose there are mutliple pixels in the image, however the image is sparse. Using OMP, we could find which pixels were contained in the image. However, we would like to image faster, and so at every iteration of OMP, rather than picking the column of $\mathbf{M}$ with the largest inner product, we will pick all columns of $\mathbf{M}$ which have inner products greater than some threshold y_{magic}. This procedure is known as Stagewise OMP or STOMP.

Suppose that we run STOMP, selecting more than one pixel at each iteration. During the first 3 iterations, we find 13 pixels (1st iteration), 17 pixels (2nd iteration), and 4 pixels (3rd iteration). Recall that in the OMP procedure, we compute the projection $\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{b}$ at every iteration so that we can compute the residual $\vec{b}$. **At each of the first 3 iterations what is the size of the A matrix? What do the columns of A represent?**

Solution:

After the 1st iteration, $\mathbf{A}$ is a 600×13 matrix.

After the 2nd iteration, $\mathbf{A}$ is a 600×30 matrix.

After the 3rd iteration, $\mathbf{A}$ is a 600×34 matrix.

The columns of $\mathbf{A}$ represent the effect of each pixel in the image on the measurement $\vec{b}$.

10. Message Received! (14 Points)

Pat and Lucy are aspiring freshmen taking EE16A. They both live on opposite sides of Unit 2, but they want to communicate with each other. Worried about eavesdropping on WiFi or cellular data, they come up with a secure communcation scheme using Gold codes.

Lucy devises a Gold code $\vec{c}_p$ that has some unique properties and shares it with Pat. The Gold code is a sequence of $+1$ or -1 of length N. In order to send Pat a message, Lucy multiplies the Gold code $\vec{c}_p$ by a number x, and transmits $x\vec{c}_p$. Pat and Lucy want to communicate in binary, so x only takes the value $+1$ or -1.

Lucy designed the Gold codes such that the sum of the entries in $\vec{c}_p$ is equal to zero (i.e. the code has a mean of zero). Furthermore, the autocorrelation of the code $\vec{c}_p$ is 1 for no time shift and has magnitude less than or equal to 0.1 for all other time shifts. Recall that $\vec{c}_p^{(k)}$ denotes that the vector $\vec{c}_p$ is circularly shifted by k and that $\vec{c}_p[n]$ denotes the n'th entry in the vector $\vec{c}_p$. These properties are expressed mathematically below.

$$\frac{1}{N}\sum_{n=0}^{N-1}\vec{c}_p[n] = 0 \qquad \frac{1}{N}\left|\left\langle \vec{c}_p, \vec{c}_p^{(k)} \right\rangle\right| = 1 \text{ if } k = 0 \qquad \frac{1}{N}\left|\left\langle \vec{c}_p, \vec{c}_p^{(k)} \right\rangle\right| \leq 0.1 \text{ if } k \neq 0$$

Pat made a receiver whose output $\hat{x}$ should be equal to Lucy's messages. Throughout this problem, we will assume that Pat's receiver is synchronized with Lucy's transmission, that is when Lucy transmits $x * \vec{c}_p$, Pat also receives $\vec{r} = x * \vec{c}_p$ (i.e. no shift). Pat's receiver performs the inner product between the received signal, $\vec{r}$, and $\vec{c}_p$ to generate y. If y is greater than 0, $\hat{x} = 1$, and Pat knows Lucy was trying to send a $+1$. If y is less than 0, $\hat{x} = -1$, and Pat knows Lucy was trying to send a -1. The properties of Pat's receiver are summarized below.

Received Signal	Inner Product	Receiver Output
$\vec{r} = x * \vec{c}_p$	$y = \langle \vec{r}, \vec{c}_p \rangle$	$\hat{x} = \begin{cases} +1, & \text{if } y > 0 \\ -1, & \text{if } y < 0 \end{cases}$

(a) (6 Points) While Lucy and Pat are happily enjoying their secure chatting, Professor Miki Lustig starts operating his ham radio. Pat now starts picking up both Lucy's signal and Miki's signal. The input to Pat's receiver can be modeled as a sum of Lucy's signal $\vec{c}_p$ and Miki's interference signal $\vec{i}$, i.e., $\vec{r} = x * \vec{c}_p + \vec{i}$. We further assume that Miki's signal is constant during one period of the Gold code, so $\vec{i}[n] = T$ for all n and some real number T. **Will Pat still be able to recover Lucy's messages? Justify your answer**

Hint: Recall that Lucy's Gold codes have a mean of 0, that is

$$\frac{1}{N}\sum_{n=0}^{N-1}\vec{c}_p[n] = 0$$

Solution:

Lucy can still recover Pat's messages.

$$y = \langle \vec{r}, \vec{c}_p \rangle = \langle x * \vec{c}_p + \vec{i}, \vec{c}_p \rangle = \langle x * \vec{c}_p, \vec{c}_p \rangle + \langle \vec{i}, \vec{c}_p \rangle = x * \langle \vec{c}_p, \vec{c}_p \rangle + \langle \vec{i}, \vec{c}_p \rangle$$

Since the mean of the components in $\vec{c}_p$ is 0,

$$\langle \vec{i}, \vec{c}_p \rangle = \sum_{n=0}^{N-1}\vec{i}[n]\vec{c}_p[n] = \sum_{n=0}^{N-1} T\vec{c}_p[n] = T\sum_{n=0}^{N-1}\vec{c}_p[n] = 0$$

Therefore,

$$y = \langle \vec{r}, \vec{c}_p \rangle = x * \langle \vec{c}_p, \vec{c}_p \rangle + \langle \vec{i}, \vec{c}_p \rangle = x * N$$

(b) (8 Points) An Alien spaceship lands in Oakland and Lucy and Pat's transmissions are reflecting off this spaceship. This results in two paths for the signal from Lucy to Pat, as shown in Figure 10.1.

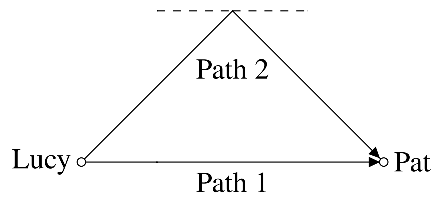

Figure 10.1: Two paths from Lucy to Pat

Path 1 is the same as in part (a). Path 2 is a reflected path that causes another copy of the signal to arrive at a delay ℓ with strength β. With these two paths, the input to Pat's receiver $\vec{r}$ is given below. We assume here that the delay l is less than one period of the Gold codes, i.e. $0 \le \ell < N$.

$$\vec{r} = x * \vec{c}_p + \beta * x * \vec{c}_p^{(l)}$$

What is the largest value of β for which Pat's receiver can still correctly recover Lucy's message (i.e., $\hat{x} = x$)? Hint: Recall that the inner product of Lucy's Gold Code with a shifted version of itself has an absolute value less that or equal to 0.1. That is:

$$\frac{1}{N}\left|\left\langle \vec{c}_p, \vec{c}_p^{(k)} \right\rangle\right| = 1 \text{ if } k = 0$$

$$\frac{1}{N}\left|\left\langle \vec{c}_p, \vec{c}_p^{(k)} \right\rangle\right| \le 0.1 \text{ if } k \neq 0$$

Solution:

$$y = \langle \vec{r}, \vec{c}_p \rangle = \left\langle x * \vec{c}_p + \beta * x * \vec{c}_p^{(l)}, \vec{c}_p \right\rangle = x * \langle \vec{c}_p, \vec{c}_p \rangle + \beta * x * \left\langle \vec{c}_p^{(l)}, \vec{c}_p \right\rangle$$

Since $x * \langle \vec{c}_p, \vec{c}_p \rangle = x * N$, we need $\left|\beta * x * \left\langle \vec{c}_p^{(l)}, \vec{c}_p \right\rangle\right| < x * N$. Therefore, $\left|\beta \left\langle \vec{c}_p^{(l)}, \vec{c}_p \right\rangle\right| < N$.

$$\left|\beta * \left\langle \vec{c}_p^{(l)}, \vec{c}_p \right\rangle\right| \le |\beta * 0.1N| < N$$

Therefore, $\beta < 10$.