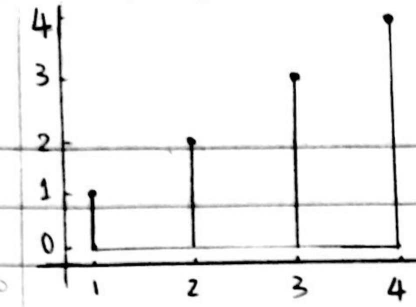
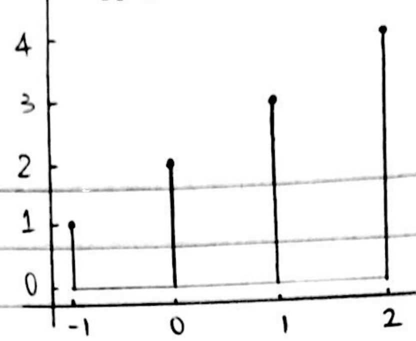
1. Mechanical Correlation

(a) $\vec{z}_1[n]$ ↙ 0 outside of the region drawn →

(both signals extends to $\pm\infty$)



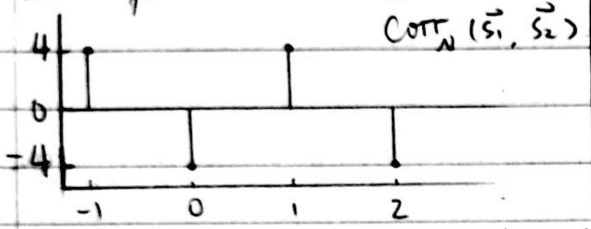
$\vec{z}_2[n]$



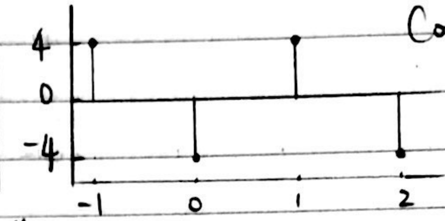
(b) Since each signal is periodic with period 4, so $\text{corr}(\vec{s}_1, \vec{s}_2)[0] = -4$. $\text{corr}(\vec{s}_1, \vec{s}_2)[1] = 4$. In general, $\text{corr}(\vec{s}_1, \vec{s}_2)[2k] = -4$ $\text{corr}(\vec{s}_1, \vec{s}_2)[2k+1] = 4$

Similarly, $\text{corr}(\vec{s}_2, \vec{s}_1)[2k] = -4$, $\text{corr}(\vec{s}_2, \vec{s}_1)[2k+1] = 4$ ← where $k \in \mathbb{Z}$

$\text{Corr}_N(\vec{s}_1, \vec{s}_2)$



$\text{Corr}_N(\vec{s}_2, \vec{s}_1)$.



They're the same in values, but different concept-wise.

They're the mirror image of each other, with symmetry axis at 0 (no shift)

(c) Here, period $N = 4$. Now, we can calculate:

$\text{circcorr}(\vec{s}_1, \vec{s}_2)[0] = \sum_{i=0}^{3} \vec{s}_1[i]\, \vec{s}_2[i_4] = 2\cdot1 + (-2)\cdot2 + 2\cdot3 + (-2)\cdot4 = -4.$

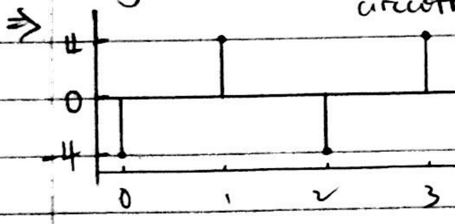
$\text{circcorr}(\vec{s}_1, \vec{s}_2)[1] = \sum_{i=0}^{3} \vec{s}_1[i]\, \vec{s}_2[(i-1)_4] = 2\cdot3 + (-2)\cdot1 + 2\cdot2 + (-2)\cdot3 = 4.$

Similarly, $\text{circcorr}(\vec{s}_1, \vec{s}_2)[2] = -4$, $\text{circcorr}(\vec{s}_1, \vec{s}_2)[3] = 4$.

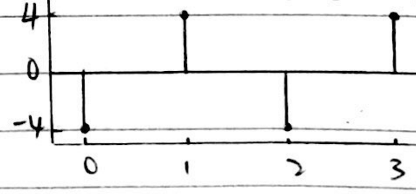
Thus. $\text{circcorr}(\vec{s}_1, \vec{s}_2) = [-4 \;\; 4 \;\; -4 \;\; 4]^T$.

Using a similar process, we can get $\text{circcorr}(\vec{s}_2, \vec{s}_1) = [-4 \;\; 4 \;\; -4 \;\; 4]^T$.

⇒ $\text{circcorr}(\vec{s}_1, \vec{s}_2)$



$\text{circcorr}(\vec{s}_2, \vec{s}_1)$



Again, they're the same in value, but different in concept (mirror with axis at 0)

They are related since they check for similarity between $\vec{s}_1, \vec{s}_2$ by shifting one of them.

(d) In this situation, we can calculate that (by definition):

$\text{corr}(\vec{s}_1, \vec{s}_2)[0] = \text{corr}(\vec{s}_2, \vec{s}_1)[0] = -4$, $\text{corr}(\vec{s}_1, \vec{s}_2)[1] = \text{corr}(\vec{s}_2, \vec{s}_1)[-1] = -4$.

$\text{corr}(\vec{s}_1, \vec{s}_2)[2] = \text{corr}(\vec{s}_2, \vec{s}_1)[-2] = -2$, $\text{corr}(\vec{s}_1, \vec{s}_2)[3] = \text{corr}(\vec{s}_2, \vec{s}_1)[-3] = -2$.

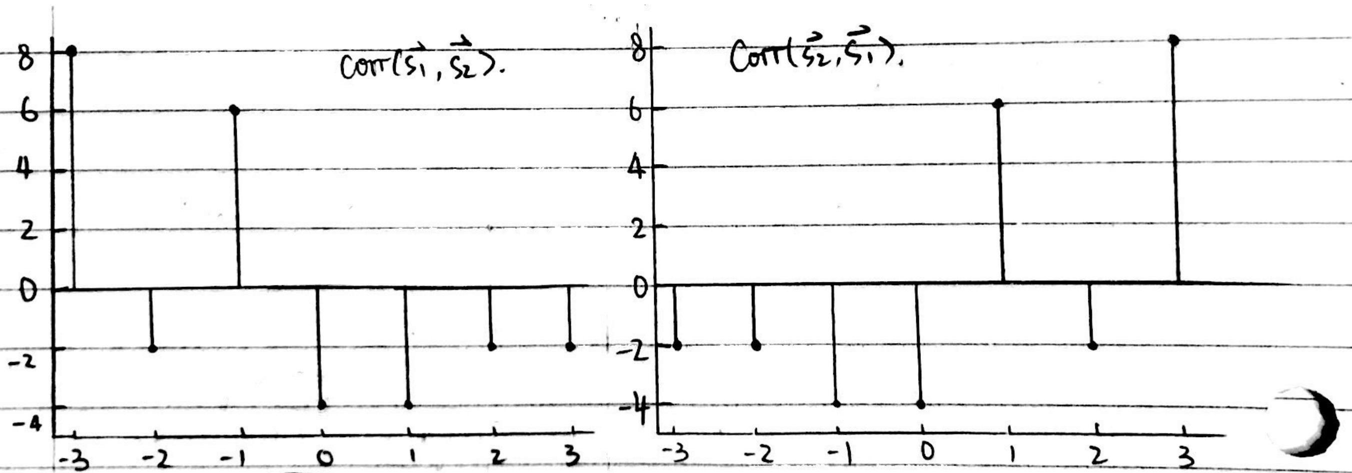
$\text{corr}(\vec{s}_1, \vec{s}_2)[-1] = \text{corr}(\vec{s}_2, \vec{s}_1)[1] = 6$, $\text{corr}(\vec{s}_1, \vec{s}_2)[-2] = \text{corr}(\vec{s}_2, \vec{s}_1)[2] = -2$

$\text{corr}(\vec{s}_1, \vec{s}_2)[-3] = \text{corr}(\vec{s}_2, \vec{s}_1)[3] = 8$

All other values of shifting (linear cross-correlation at other values) are 0 since they're sum of 0s

(d) Cont.

both signals extends to $\pm\infty$, with 0 outside the region shown.



They are $\boxed{\text{not}}$ the same, but they are mirror image of each other, with the symmetrical "axis" at 0 (no-shift).

(e) Checked. Should use mode $\boxed{\text{"full"}}$.

f. Since we've shown that the inner product is the cosine of the angle between any two unit vectors, and since $\mathrm{corr}_N(\vec{x}, \vec{x})[k]$ is just the inner product of two vectors, so $\frac{|\mathrm{corr}_N(\vec{x}, \vec{x})[k]|}{\|\vec{x}\|^2} \le \cos 0 = 1 \; \forall k \in \mathbb{Z}$. Then, $\mathrm{corr}_N(\vec{x}, \vec{x})[0] = \|\vec{x}\|^2$, with $|\mathrm{corr}_N(\vec{x}, \vec{x})[k]| \le \|\vec{x}\|^2$, so $\mathrm{corr}_N(\vec{x}, \vec{x})[0] \ge |\mathrm{corr}_N(\vec{x}, \vec{x})[m]|$ for all m. Q.E.D.

2. (a). I observe a graph with a bunch of small y-value indices from x=-1000 to 1000, and one very large y-value at x=0. (very high autocorrelation at 0 and low everywhere else)

(b). I see all y-values bounded by the range -80 to 80, which is relatively small compared to result in (a), In other, very low cross-correlation

(c). Again, the cross-correlation is very low (bounded by -100 to 75). This means that we have a strong ability to identify satellites.

(d). Again, very small y-values ⇒ the cross-correlation is small (-75 to 75)

(e). The satellites present are 4, 7, 13, 19

(f). Satellite 3, and message is [1 -1 -1 -1 1]

(g). Satellites 5 and 20. Delay is 506.

3. (a). No, sim_1 wouldn't be a good similarity measure.
because absolute value measures more of the distance (and thus the differences) between the two vectors, which is the opposite of what we want.

Yes, sim_2 would be good, because correlation measures the cosine of the angle between the vectors. The smaller the angle (i.e. the greater the similarity), the larger the score.
Thus, $\langle \vec{x_c}, \frac{\vec{S_A}}{\|S_A\|} \rangle$ is a good similarity measure.

(b). We can setup the system of linear equations with the information:

$$\begin{bmatrix} 40\% & 33\% & 22\% & 5\% \\ 70\% & 10\% & 10\% & 10\% \\ 20\% & 10\% & 15\% & 55\% \\ 5\% & 2\% & 20\% & 73\% \end{bmatrix} \cdot x_c = \begin{bmatrix} T_{food}\% \\ T_{movies}\% \\ T_{art}\% \\ T_{book}\% \end{bmatrix}$$

(c) Algorithm 1

1. procedure PROMOTION(M_{food}, M_{movies}, M_{art}, M_{books}, $\vec{S_{A_1}}$, ..., $\vec{S_{A_N}}$).

$\Rightarrow$
2. $T_{food} = M_{food} / (M_{food} + M_{movies} + M_{art} + M_{books})$
3. $T_{movies} = M_{movies} / (M_{food} + M_{movies} + M_{art} + M_{books})$
4. $T_{art} = M_{art} / (M_{food} + M_{movies} + M_{art} + M_{books})$.
5. $T_{books} = M_{books} / (M_{food} + M_{movies} + M_{art} + M_{books})$.
6. Setup and solve: $\begin{bmatrix} 0.4 & 0.33 & 0.22 & 0.05 \\ 0.7 & 0.1 & 0.1 & 0.1 \\ 0.2 & 0.1 & 0.15 & 0.55 \\ 0.05 & 0.02 & 0.2 & 0.73 \end{bmatrix} \cdot \vec{x_c} = \begin{bmatrix} T_{food} \\ T_{movies} \\ T_{art} \\ T_{books} \end{bmatrix}$

Solve for $\vec{x_c}$

7. ...
8. ... (algorithm on the hw pdf).
9. ...
10.

(d). First we calculate the spending percentage vector. $\vec{T_c} = [T_{food}\ T_{movies}\ T_{art}\ T_{books}]^T$.

where $T_{food} = 6/(6+4+1+5) = 0.375 = 37.5\%$

similarly $T_{movies} = 25\%$. $T_{art} = 6.25\%$. $T_{books} = 31.25\%$

Now, using IPython, we figured out that:

$$\vec{x_c} = \begin{bmatrix} 0.0782 \\ -2.1456 \\ 4.9815 \\ -0.8833 \end{bmatrix}$$

Then, using IPython, we figure out the best promotions with our similarity score in part (c), which is:

$$\Rightarrow \vec{S_{A_3}} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{5}{2} \\ -\frac{1}{2} \end{bmatrix}$$

e. $\boxed{\underline{No}}$ Using IPython, we found out that the 4×4 spending distribution matrix is full rank. In other words, it's invertible.

Thus, for any customer with percentage vector $\vec{T_c} = [T_{food}\ T_{movies}\ T_{art}\ T_{books}]^T$, we have $spending \cdot \vec{x_c} = \vec{T_c}$, so $\vec{x_c} = spending^{-1} \cdot \vec{T_c}$ is unique.

which implies that for all customers, the system has a unique solution.

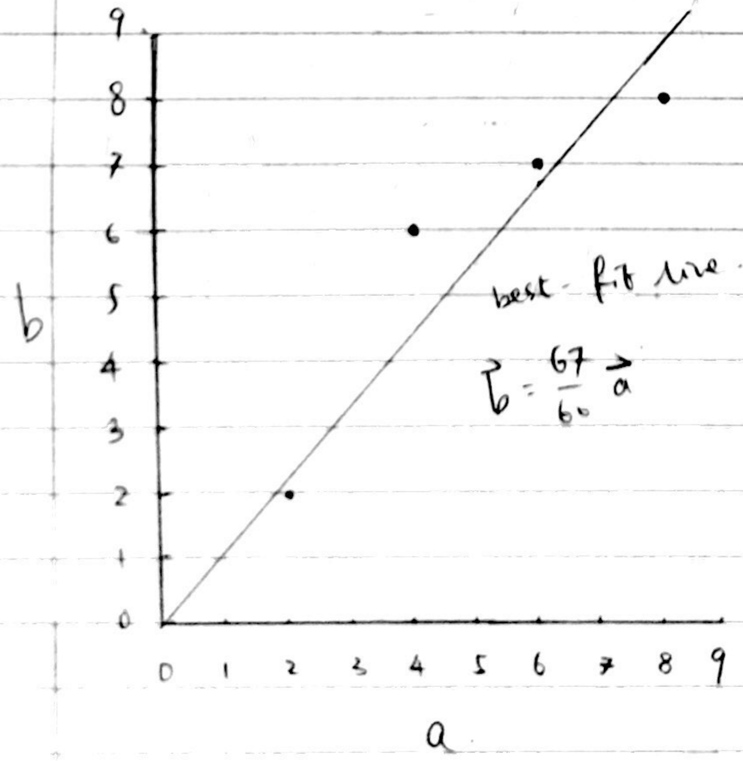
4. (a) Since $\vec{a} = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 2 \\ 6 \\ 7 \\ 8 \end{bmatrix}$, so $\vec{a}^T \vec{a} = 120$, $\vec{a}^T \vec{b} = 4+24+42+64$

$= 134$

So $x = (\vec{a}^T\vec{a})^{-1} \vec{a}^T \vec{b} = 120^{-1}\ 134 = \boxed{\frac{67}{60}}$.

Squared error $e_1 = \| \vec{b} - \vec{a}x \|^2 = \| [-\frac{7}{30} \quad \frac{23}{15} \quad \frac{3}{10} \quad -\frac{14}{15}] \|^2$

$= \left(\sqrt{\frac{101}{30}}\right)^2 = \frac{101}{30} \approx \boxed{3.37}$



(b) Since $A = \begin{bmatrix} 2 & 1 \\ 4 & 1 \\ 6 & 1 \\ 8 & 1 \end{bmatrix}$, so $A^T = \begin{bmatrix} 2 & 4 & 6 & 8 \\ 1 & 1 & 1 & 1 \end{bmatrix}$, and with $\vec{b} = \begin{bmatrix} 2 \\ 6 \\ 7 \\ 8 \end{bmatrix}$

So $A^T A = \begin{bmatrix} 120 & 20 \\ 20 & 4 \end{bmatrix}$, so $(A^T A)^{-1} = \frac{1}{120 \cdot 4 - 20^2} \cdot \begin{bmatrix} 4 & -20 \\ -20 & 120 \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{20} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{2} \end{bmatrix}$$

$$\Rightarrow (A^T A)^{-1} \cdot A^T = \begin{bmatrix} -\frac{3}{20} & -\frac{1}{20} & \frac{1}{20} & \frac{3}{20} \\ 1 & \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

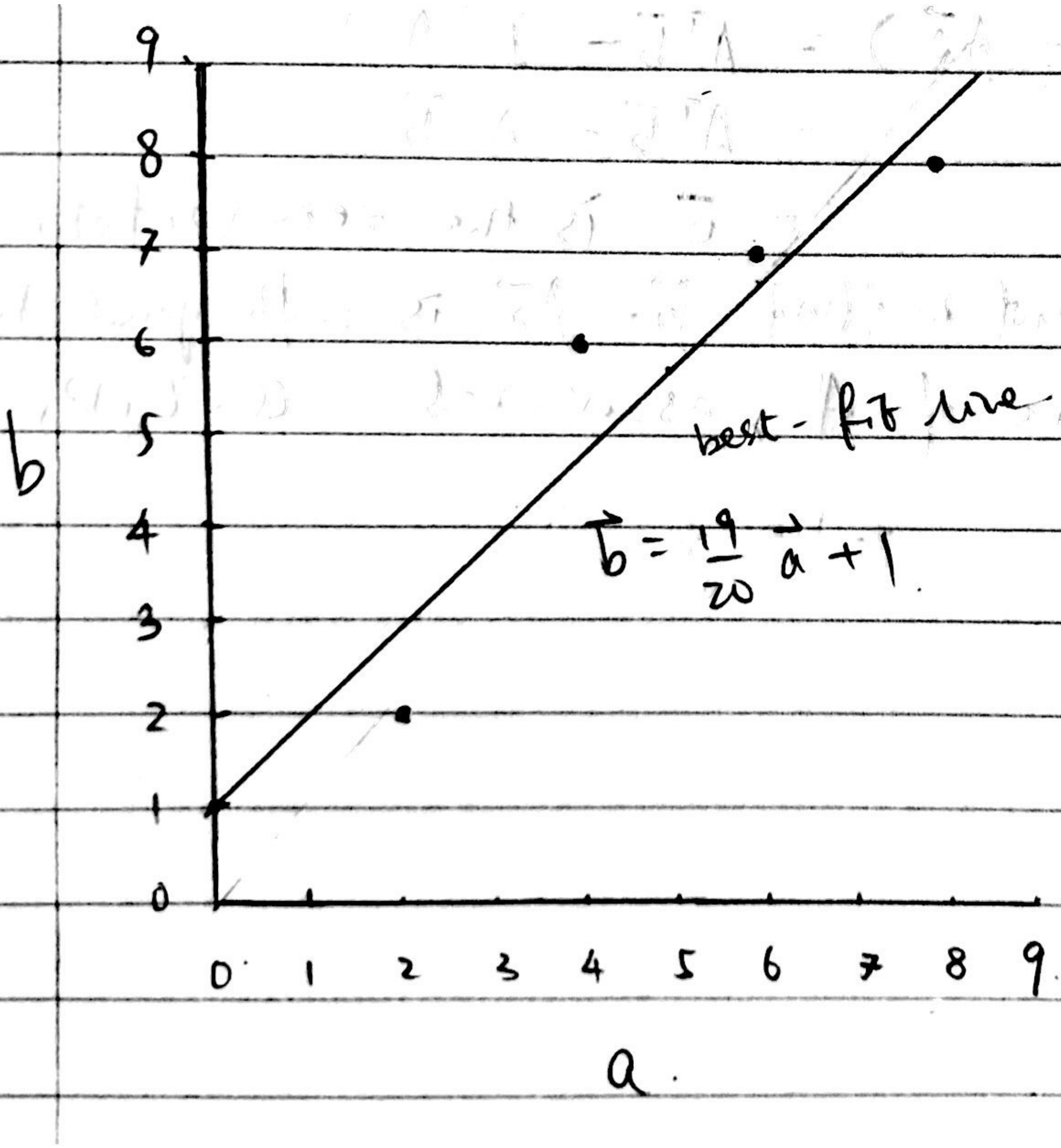
Thus, $\vec{x} = (A^T A)^{-1} A^T \vec{b} = \begin{bmatrix} \frac{19}{20} \\ 1 \end{bmatrix} \Rightarrow \boxed{\begin{matrix} x_1 = \frac{19}{20} \\ x_2 = 1 \end{matrix}}$

Here, Squared error $e_2 = \| \vec{b} - (x_1 \vec{a} + x_2) \|^2 = \| [2\ 6\ 7\ 8]^T - [\frac{29}{10}\ \frac{24}{5}\ \frac{67}{10}\ \frac{43}{5}]^T \|^2$

$$= \| [-\frac{9}{10}\ \frac{6}{5}\ \frac{3}{10}\ -\frac{3}{5}] \|^2$$

$$= \left(\sqrt{\frac{270}{100}}\right)^2 = \frac{27}{10} = \boxed{2.7}$$

Since $e_2 < e_1$, so $\boxed{\text{Yes}}$, it's a better fit. (also by the plot).



best-fit line
$\vec{b} = \frac{19}{20}\vec{a} + 1$
b
a
0 1 2 3 4 5 6 7 8 9

(c). From the notes we know that $\vec{\hat{x}} = (A^T A)^{-1} A^T \vec{b}$.

So $\vec{b} - A\vec{\hat{x}} = \vec{b} - A(A^T A)^{-1} A^T \vec{b}$.

Now, consider $A^T \cdot (\vec{b} - A\vec{\hat{x}}) = A^T \cdot (\vec{b} - A(A^T A)^{-1} A^T \vec{b})$

$= A^T \vec{b} - A^T A (A^T A)^{-1} A^T \vec{b}$.

Since $(A^T A) \cdot (A^T A)^{-1} = I$, the identity matrix,

so $A^T \cdot (\vec{b} - A\vec{\hat{x}}) = A^T \vec{b} - I \cdot A^T \vec{b}$

$= A^T \vec{b} - A^T \vec{b}$

$= \vec{0}$ is the zero vector.

which is equivalent to that $\vec{b} - A\vec{x}$ is orthogonal to the columns of A, as desired. Q.E.D.

5. Find an orthorgonal vector to $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ be orthorgonal to $\vec{v}$. so $\langle \vec{u}, \vec{v} \rangle = u_1 + 2u_2 + 3u_3 = 0$.

There are infinitely-many solutions since we have 3 variables and 1 equations. One solution is $u_1 = u_2 = 1$ $u_3 = -1$.

so $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ is orthorgonal to $\vec{v}$.

6. I worked alone without getting any help.