1. Counting Solutions.

(b). We first transfer this system of linear equations to an augmented matrix since it's equivalent to:

$$\begin{bmatrix} 0 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 0 & -1 & 2 & 1 \\ 2 & 0 & 1 & 2 \end{array}\right]$$

Switch the two rows, and since we have more columns than rows, so we add a row of 0.

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 0 & 1 & 2 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which means that I ended up with: (and since we have a row of 0s, so we have infinite solutions.)

$$\begin{cases} 2x+z=2 \\ -y+2z=1 \\ 0=0 \end{cases} \Rightarrow \begin{cases} x = 1-\frac{z}{2} \\ y=-1+2z \\ 0=0 \end{cases}$$

So, we have an **infinite number** of solutions, since I can't uniquely solve for x, y, z. The space of solutions:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ 2 \\ 1 \end{bmatrix} z.$$

(d). The system of linear equations is equivalent to:

$$\begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 1 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}$$

So, it's equivalent to the augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 1 & -3 & -5 \end{array}\right]$$

R_1: Multiply by 2 and Subtract R_2.
R_3: Multiply by 2 and Subtract R_2.

$$\Rightarrow \left[\begin{array}{cc|c} 0 & 5 & 5 \\ 2 & -1 & 1 \\ 0 & -5 & -11 \end{array}\right]$$

R_3: Add R_1

$$\Rightarrow \left[\begin{array}{cc|c} 0 & 5 & 5 \\ 2 & -1 & 1 \\ 0 & 0 & -6 \end{array}\right]$$

Since we end up with a row of 0s on the left and the right side is not 0, so (as discussed in class), there is no solution.

(e). The system of linear equations is equivalent to:

$$\begin{bmatrix} 1 & -1 \\ 5 & -5 \\ 3 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \\ 6 \end{bmatrix}$$

So, it's equivalent to this augmented matrix:

$$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 5 & -5 & 10 \\ 3 & -3 & 6 \end{array}\right]$$

R_2: Subtract $(5 \cdot R_1)$. and R_3: Subtract $(3 \cdot R_1)$ and then we have that.

$$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Since we have two rows of 0s, so only 1 linear equation with 2 variables, so we have infinite solutions. Alternatively, I end up with:

$$\begin{cases} x-y=2 \\ 0=0 \\ 0=0 \end{cases} \Rightarrow \begin{cases} x=2+y \\ 0=0 \\ 0=0 \end{cases}$$

which I can't uniquely solve for x, y with.

Thus, there is an infinite number of solutions. The space of solutions is:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} y.$$