1. Mechanical Eigenvalues and Eigenvectors.

(a). We have $(A-\lambda I_2)\cdot\vec{x}=\vec{0}$, and $A-\lambda I_2=\begin{bmatrix}5-\lambda & 0\\ 0 & 2-\lambda\end{bmatrix}$

So we need $\det(A-\lambda I_2)=(5-\lambda)(2-\lambda)-0\cdot 0=0$. so eigenvalues $\lambda_1=2$, $\lambda_2=5$.
Each value will have its own corresponding eigenvector.

① $\lambda_1=2$, so $(A-2I_2)\vec{x}=\begin{bmatrix}3&0\\0&0\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\end{bmatrix}=\vec{0} \Rightarrow \left[\begin{array}{cc|c}3&0&0\\0&0&0\end{array}\right] \Rightarrow \left[\begin{array}{cc|c}1&0&0\\0&0&0\end{array}\right]$

So the eigenvectors are $\begin{bmatrix}0\\1\end{bmatrix}\alpha_1$ with $\alpha_1\in\mathbb{R}$.

② $\lambda_2=5$. so $(A-5I_2)\vec{x}=\begin{bmatrix}0&0\\0&-3\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\end{bmatrix}=\vec{0} \Rightarrow \left[\begin{array}{cc|c}0&0&0\\0&-3&0\end{array}\right] \Rightarrow \left[\begin{array}{cc|c}0&0&0\\0&1&0\end{array}\right]$

So the associated eigenvector are $\begin{bmatrix}1\\0\end{bmatrix}\alpha_2$, $\alpha_2\in\mathbb{R}$.

Thus. The eigenvalues are $\boxed{\lambda_1=2,\ \lambda_2=5}$; the eigenspace is $\boxed{\text{span}\left\{\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix}\right\}}$

(b). We have $(A-\lambda I_2)\cdot\vec{x}=\vec{0}$, and given $A=\begin{bmatrix}22&6\\6&13\end{bmatrix}$. So $A-\lambda I_2=\begin{bmatrix}22-\lambda&6\\6&13-\lambda\end{bmatrix}$

So, we need $\det(A-\lambda I_2)=(22-\lambda)(13-\lambda)-6\cdot 6=0 \Rightarrow \lambda^2-35\lambda+250=0$

so, $(\lambda-10)(\lambda-25)=0$, so we have eigenvalues $\lambda_1=10$, $\lambda_2=25$. and so:

① $\lambda_1=10$. so $(A-10I_2)\cdot\vec{x}=\begin{bmatrix}12&6\\6&3\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\end{bmatrix}=\vec{0} \Rightarrow \left[\begin{array}{cc|c}12&6&0\\6&3&0\end{array}\right]$ R_2: Subtract $\frac{1}{2}\cdot R_1$

So. $\left[\begin{array}{cc|c}12&6&0\\0&0&0\end{array}\right]$ which means that $12x_1+6x_2=0 \Rightarrow x_2=-2x_1$.

So the associated eigenvectors are. $\begin{bmatrix}1\\-2\end{bmatrix}\alpha_1$, with $\alpha_1\in\mathbb{R}$

② $\lambda_2=25$. so $(A-25I_2)\cdot\vec{x}=\begin{bmatrix}-3&6\\6&-12\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\end{bmatrix}=\vec{0} \Rightarrow \left[\begin{array}{cc|c}-3&6&0\\6&-12&0\end{array}\right]$. R_2, Add $2\cdot R_1$.

So, $\left[\begin{array}{cc|c}-3&6&0\\0&0&0\end{array}\right]$ which gives $-3x_1+6x_2=0 \Rightarrow x_1=2x_2$.

So the associated eigenvectors are $\begin{bmatrix}2\\1\end{bmatrix}\alpha_2$. $\alpha_2\in\mathbb{R}$.

Thus. the eigenvalues are: $\boxed{\lambda_1=10.\ \lambda_2=25}$; the eigenspace is $\boxed{\text{span}\left\{\begin{bmatrix}1\\-2\end{bmatrix},\begin{bmatrix}2\\1\end{bmatrix}\right\}}$.

(c) Since $A=\begin{bmatrix}\frac{\sqrt{3}}{2}&-\frac{1}{2}\\\frac{1}{2}&\frac{\sqrt{3}}{2}\end{bmatrix}=\begin{bmatrix}\cos\frac{\pi}{6}&-\sin\frac{\pi}{6}\\\sin\frac{\pi}{6}&\cos\frac{\pi}{6}\end{bmatrix}$, So this is a special matrix that would rotate any vector it applies to by $\frac{\pi}{6}$ $(=30^\circ)$ counterclockwise

Now, we have $(A-\lambda I_2)\cdot\vec{x}=\vec{0}$. so $\begin{bmatrix}\frac{\sqrt{3}}{2}-\lambda&-\frac{1}{2}\\\frac{1}{2}&\frac{\sqrt{3}}{2}-\lambda\end{bmatrix}=A-\lambda I_2$, so we have

$\det(A-\lambda I_2)=(\frac{\sqrt{3}}{2}-\lambda)\cdot(\frac{\sqrt{3}}{2}-\lambda)-\frac{1}{2}\cdot(-\frac{1}{2})=\lambda^2-\sqrt{3}\lambda+\frac{3}{4}+\frac{1}{4}=\lambda^2-\sqrt{3}\lambda+1=0$.

$\Rightarrow (\lambda-\frac{\sqrt{3}+i}{2})(\lambda-\frac{\sqrt{3}-i}{2})=0$, so eigenvalues $\lambda_1=\frac{\sqrt{3}+i}{2}$, $\lambda_2=\frac{\sqrt{3}-i}{2}$. So

① $\lambda_1=\frac{\sqrt{3}+i}{2}$, So $(A-\frac{\sqrt{3}+i}{2}I_2)\cdot\vec{x}=\begin{bmatrix}-\frac{i}{2}&-\frac{1}{2}\\\frac{1}{2}&-\frac{i}{2}\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\end{bmatrix}=\vec{0} \Rightarrow \left[\begin{array}{cc|c}-\frac{i}{2}&-\frac{1}{2}&0\\\frac{1}{2}&-\frac{i}{2}&0\end{array}\right]$

R_2: Subtract $i\cdot R_1 \Rightarrow \left[\begin{array}{cc|c}-\frac{i}{2}&-\frac{1}{2}&0\\0&0&0\end{array}\right]$ which gives $-\frac{i}{2}x_1-\frac{1}{2}x_2=0. \Rightarrow x_2=-ix_1$.

So the associated eigenvectors are $\begin{bmatrix}1\\-i\end{bmatrix}\alpha_1$ where $\alpha_1\in\mathbb{R}$.

② $\lambda_2=\frac{\sqrt{3}-i}{2}$. So $(A-\frac{\sqrt{3}-i}{2}I_2)\cdot\vec{x}=\begin{bmatrix}\frac{i}{2}&-\frac{1}{2}\\\frac{1}{2}&\frac{i}{2}\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\end{bmatrix}=\vec{0} \Rightarrow \left[\begin{array}{cc|c}\frac{i}{2}&-\frac{1}{2}&0\\\frac{1}{2}&\frac{i}{2}&0\end{array}\right]$

R_2. Add $iR_1 \Rightarrow \left[\begin{array}{cc|c}\frac{i}{2}&-\frac{1}{2}&0\\0&0&0\end{array}\right]$ which gives $\frac{i}{2}x_1-\frac{1}{2}x_2=0 \Rightarrow x_2=ix_1$.

So the associated eigenvectors are $\begin{bmatrix}1\\i\end{bmatrix}\alpha_2$ where $\alpha_2\in\mathbb{R}$.

Thus. the eigenvalues are $\boxed{\lambda_1=\frac{\sqrt{3}+i}{2}\quad \lambda_2=\frac{\sqrt{3}-i}{2}}$

and the eigenspace is $\boxed{\text{span}\left\{\begin{bmatrix}1\\-i\end{bmatrix},\begin{bmatrix}1\\i\end{bmatrix}\right\}}$.

(e). We have $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, and $(A - \lambda I_2) \cdot \vec{x} = 0$. with $A - \lambda I_2 = \begin{bmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}$

Since we need $\det(A - \lambda I_2) = (2-\lambda)(2-\lambda) - 0 \cdot 0 = 0$, so eigenvalues $\lambda_1 = \lambda_2 = 2$. which is a repeated eigenvalue of $\lambda = 2$.

Now, since $(A - 2I_2) \cdot \vec{x} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$, so the eigenspace is all of $\mathbb{R}^2$

This makes sense as $\forall \vec{v} \in \mathbb{R}^2$, with $\lambda = 2$, we have $A\vec{v} = \lambda \vec{v}$. A basis for $\mathbb{R}^2$ is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ by our definition (our regular view).

Thus, the eigenvalue is $\boxed{\lambda = 2}$; the eigenspace is $\boxed{\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}}$

2. Counting The Paths of a random Surfer.

(a). With $A=\begin{bmatrix}0&1\\1&0\end{bmatrix}$, since $a_{21}=1$, so there's $\boxed{1}$ one-hop path from webpage 1 to 2.

Then, $A^2=\begin{bmatrix}0&1\\1&0\end{bmatrix}\cdot\begin{bmatrix}0&1\\1&0\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, since for A^2, $a_{21}=0$, so there's $\boxed{\text{0 two-hop path}}$ from webpage 1 to webpage 2.

Then, since $A^3=A^2\cdot A=\begin{bmatrix}1&0\\0&1\end{bmatrix}\cdot\begin{bmatrix}0&1\\1&0\end{bmatrix}=\begin{bmatrix}0&1\\1&0\end{bmatrix}$,

Since for A^3, $a_{21}=1$ again, so there's $\boxed{\text{1 three-hop path}}$ from webpage 1 to webpage 2.

(b). By definition of transition matrix, so $T=\begin{bmatrix}0&1\\1&0\end{bmatrix}$. Consider when $\lambda=1$.

So $(T-\lambda I_2)\cdot\vec{x}=\begin{bmatrix}-1&1\\1&-1\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\end{bmatrix}=\vec{0} \Rightarrow \left[\begin{array}{cc|c}-1&1&0\\1&-1&0\end{array}\right]$ R_2: Add R_1, so:

$\Rightarrow \left[\begin{array}{cc|c}-1&1&0\\0&0&0\end{array}\right]$ which gives $-x_1+x_2=0$, so $x_1=x_2$

so the eigenvectors this $\lambda=1$ gives are: $\begin{bmatrix}1\\1\end{bmatrix}\alpha$, $\alpha\in\mathbb{R}$.

To have the values of this eigenvector sum to 1, so $1\cdot\alpha+1\cdot\alpha=1 \Rightarrow \alpha=0.5$, and so the specific eigenvector we're looking for is $\begin{bmatrix}0.5\\0.5\end{bmatrix}$.

Thus, the steady-state frequency for website 1 is $\boxed{0.5}$, for webpage 2 is $\boxed{0.5}$.

(c). From the graph, we can compute the adjacency matrix for graph B to be: $\boxed{B=\begin{bmatrix}0&1&1&1\\0&0&1&1\\0&0&0&1\\1&0&1&0\end{bmatrix}}$

(d). Since $B^2=B\cdot B=\begin{bmatrix}0&1&1&1\\0&0&1&1\\0&0&0&1\\1&0&1&0\end{bmatrix}\cdot\begin{bmatrix}0&1&1&1\\0&0&1&1\\0&0&0&1\\1&0&1&0\end{bmatrix}=\begin{bmatrix}1&0&2&2\\1&0&1&1\\1&0&1&0\\0&1&1&2\end{bmatrix}$ So, with $a_{31}=1$ in B^2, so there is $\boxed{\text{1 two-hop path}}$ from webpage 1 to webpage 3.

Then, $B^3=B^2\cdot B=\begin{bmatrix}1&0&2&2\\1&0&1&1\\1&0&1&0\\0&1&1&2\end{bmatrix}\cdot\begin{bmatrix}0&1&1&1\\0&0&1&1\\0&0&0&1\\1&0&1&0\end{bmatrix}=\begin{bmatrix}2&1&3&3\\1&1&2&2\\0&1&1&2\\2&0&3&2\end{bmatrix}$ With $a_{21}=1$ in B^3, so there is $\boxed{\text{1 three-hop path}}$ from webpage 1 to webpage 2.

(e). First, we calculate Graph B's transition matrix T, which by definition, gives: $T = \begin{bmatrix} 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} \\ 1 & 0 & \frac{1}{3} & 0 \end{bmatrix}$

Then using IPython, we discover that the normalized eigenvector (sum to one) corresponding to eigenvalue $\lambda = 1$ is: $\begin{bmatrix} 0.333 \\ 0.167 \\ 0.125 \\ 0.375 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{6} \\ \frac{1}{8} \\ \frac{3}{8} \end{bmatrix}$

Thus, the steady-state frequency for webpage 1 is $\boxed{\frac{1}{3}}$, webpage 2 is $\boxed{\frac{1}{6}}$, webpage 3 is $\boxed{\frac{1}{8}}$, webpage 4 is $\boxed{\frac{3}{8}}$

(f). The adjacency matrix for graph C is:

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

(g). There is $\boxed{\text{0 path}}$ from webpage 1 to webpage 3.

As we can see from Graph C, it is a disconnected graph, and so surfers from webpage 1 can only get to webpage 2 or back to webpage 1, which implies that there's no path between webpages 1 and 3, and thus, there's 0 path from webpage 1 to webpage 3.

Scanned with CamScanner

3. Noisy Images

(a). Since $\vec{y} = A\vec{x} + \vec{n}$, so $A\vec{x} = \vec{y} - \vec{n}$. Multiplying both sides by A^{-1}.

So: $\vec{x} = I_N\vec{x} = (A^{-1}A)\vec{x} = A^{-1}(A\vec{x}) = A^{-1}(\vec{y}-\vec{n}) \Rightarrow \boxed{\vec{x} = A^{-1}(\vec{y}-\vec{n})}$

(b). Since $\vec{w} = \alpha_1\lambda_1\vec{b}_1 + \cdots + \alpha_N\lambda_N\vec{b}_N$ where $\vec{w} = \alpha_1\vec{b}_1 + \cdots + \alpha_N\vec{b}_N$, so this means that:
for eigenvectors with large eigenvalues, the noise signals along will be $\boxed{\text{amplified}}$
vice versa, for eigenvectors with small eigenvalues, the noise will be: $\boxed{\text{attenuated}}$.

(c). A_1 is an $\boxed{\text{identity matrix } (100 \times 100)}$

$\boxed{\text{Yes,}}$ there are differences between A_2 and A_3 by inspection.

(d). By decreasing the absolute value of the eigenvalues, the pictures get much more vague, which means that the noises are amplified with small eigenvalues.

(e). Proof. Suppose λ is an eigenvalue of a matrix A where A is invertible.
So, there exists a corresponding eigenvector $\vec{v} \neq \vec{0}$ such that $A\vec{v} = \lambda\vec{v}$, $\lambda \neq 0$
Since A has an inverse A^{-1}, multiply both sides by A^{-1}, and we get:

$$A^{-1}(A\vec{v}) = A^{-1}(\lambda\vec{v}). \Rightarrow (A^{-1}A)\vec{v} = A^{-1}\lambda\vec{v}.$$

By definition of Inverses, so $A^{-1}A = I_N$, so $(A^{-1}A)\vec{v} = I_N \cdot \vec{v} = \vec{v}$
which gives: $\vec{v} = A^{-1}\lambda\vec{v}$. Then, since λ is a constant, $\lambda \neq 0$.
Divide both sides by λ and we have $\frac{1}{\lambda}\vec{v} = A^{-1}\vec{v} \Leftrightarrow A^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}$.
By definition, with $\vec{v} \neq \vec{0}$, so $\frac{1}{\lambda}$ is an eigenvalue of matrix A^{-1}.

Q.E.D.