
EECS 16A
Fall 2018

Designing Information Devices and Systems I

Discussion 4A

1. Identifying a Basis

Does each of these sets of vectors describe a basis for \mathbb{R}^3 ? What about for some subspace of \mathbb{R}^3 ?

$$V_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \quad V_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \quad V_3 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Answer:

- V_1 : The vectors are linearly independent, but they are not a basis for \mathbb{R}^3 . Instead, they are a basis for some 2-dimensional subspace of \mathbb{R}^3 .
- V_2 : Yes, the vectors are linearly independent and will form a basis for \mathbb{R}^3 .
- V_3 : No, $\vec{v}_2 + \vec{v}_3 = \vec{v}_1$, so the vectors are linearly dependent.

2. Constructing a Basis

Let's consider a subspace of \mathbb{R}^3 called V which has the following property: for every vector in V , the first entry is equal to two times the sum of the second and third entries. That is, if $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in V$, $a_1 = 2(a_2 + a_3)$.

Find a basis for V . What is the dimension of V ?

Answer:

Any vector \vec{v} in V is going to look as follows:

$$\vec{v} = \begin{bmatrix} 2(a_2 + a_3) \\ a_2 \\ a_3 \end{bmatrix} = a_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Now, we consider the set of vectors $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$. The vectors are linearly independent. Furthermore,

from the above equation, any vector $\vec{v} \in V$ can be expressed as a linear combination of the vectors in \mathcal{B} (the corresponding coefficients are a_2 and a_3). This means that $V = \text{span}\{\mathcal{B}\}$.

Therefore,

$$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

forms a basis for V .

$\dim(\mathcal{B}) = 2$ (there are two vectors in \mathcal{B}), so the dimension of V is 2.

3. Exploring Dimension, Linear Independence, and Basis

In this problem, we are going to talk about the connections between several concepts we have learned about in linear algebra – linear independence, dimension of a vector space/subspace, and basis.

Let's consider the vector space \mathbb{R}^m and a set of n vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ in \mathbb{R}^m .

- (a) For the first part of the problem, let $m > n$. Can $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ form a basis for \mathbb{R}^m ? Why/why not? What conditions would we need?

Answer:

No, $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ cannot form a basis for \mathbb{R}^m . The dimension of \mathbb{R}^m is m , so you would need m linearly independent vectors to describe the vector space. Since $n < m$, this is not possible.

- (b) Let $m = n$. Can $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ form a basis for \mathbb{R}^m ? Why/why not? What conditions would we need?

Answer:

Yes, this is possible. The only condition we need is that $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is linearly independent. If the vectors are linearly independent, since there are m of them, they will span \mathbb{R}^m .

- (c) Now, let $m < n$. Can $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ form a basis for \mathbb{R}^m ? What vector space could they form a basis for?

Hint: Think about whether the vectors can be linearly independent.

Answer:

No, $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ cannot form a basis for \mathbb{R}^m . \mathbb{R}^m will be spanned by m linearly independent vectors. Any additional vectors in \mathbb{R}^m must already exist in the span of the previous vectors, and are therefore linearly dependent. Since $n > m$, some of the vectors have to be linearly dependent, so they cannot form a basis.

The two regimes—one where $n > m$ and one where $n < m$ —give rise to two different classes of interesting problems. You might learn more about them in upper division courses!

4. Exploring Column Spaces and Null Spaces

- The **column space** is the possible outputs of a transformation/function/linear operation. It is also the **span** of the column vectors of the matrix.
- The **null space** is the set of input vectors that output the zero vector.

For the following matrices, answer the following questions:

- What is the column space of **A**? What is its dimension?
- What is the null space of **A**? What is its dimension?
- Are the column spaces of the row reduced matrix **A** and the original matrix **A** the same?
- Do the columns of **A** form a basis for \mathbb{R}^2 (or \mathbb{R}^3 for part (b))? Why or why not?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Answer:

Column space: $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

Null space: $\text{span} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

The matrix is already row reduced. The column spaces of the row reduced matrix and the original matrix are the same.
Not a basis for \mathbb{R}^2 .

5. Inverse Proof

Prove that a matrix \mathbf{A} is invertible if and only if its columns are linearly independent.

Answer:

The statement “if and only if” means that we need to prove two things:

- (a) If \mathbf{A} is invertible, then its columns are linearly independent.
- (b) If \mathbf{A} 's columns are linearly independent, then it is invertible.

For the first statement, we're trying to prove the following:

$$\mathbf{A}^{-1} \text{ exists} \implies \text{the columns of } \mathbf{A} \text{ are linearly independent}$$

We know that if the columns of \mathbf{A} are linearly independent, then $\mathbf{A}\vec{x} = \vec{0}$ only when $\vec{x} = \vec{0}$, so we can rephrase what we're trying to prove as

$$\mathbf{A}^{-1} \text{ exists} \implies (\mathbf{A}\vec{x} = \vec{0} \text{ only when } \vec{x} = \vec{0})$$

To prove this, assume that \mathbf{A} is invertible. Let \vec{v} be some vector such that $\mathbf{A}\vec{v} = \vec{0}$:

$$\begin{aligned} \mathbf{A}\vec{v} = \vec{0} &\longleftarrow \text{left-multiply by } \mathbf{A}^{-1} \\ \mathbf{A}^{-1}\mathbf{A}\vec{v} &= \mathbf{I}\vec{v} = \vec{0} \\ \vec{v} &= \vec{0} \end{aligned}$$

Boom! You've successfully proven the first statement.

The second statement is a little trickier to prove:

$$(\text{columns of } \mathbf{A} \text{ are linearly independent}) \implies \mathbf{A}^{-1} \text{ exists}$$

or, rewritten

$$(\mathbf{A}\vec{x} = \vec{b} \text{ has a unique solution } \vec{x}) \implies \mathbf{A}^{-1} \text{ exists}$$

Now suppose that $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ are the columns of the identity matrix. We want to see if there's a matrix \mathbf{M} which is \mathbf{A} 's inverse, i.e. it obeys the following properties:

$$\begin{aligned} \mathbf{AM} &= \mathbf{MA} = \mathbf{I} \\ &= \begin{bmatrix} | & | & \cdots & | \\ \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \\ | & | & \cdots & | \end{bmatrix} \end{aligned}$$

Using the definition of matrix-matrix multiplication as a series of stacked matrix-vector multiplications:

$$\mathbf{A} \begin{bmatrix} | & | & \cdots & | \\ \vec{m}_1 & \vec{m}_2 & \cdots & \vec{m}_n \\ | & | & \cdots & | \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{A}\vec{m}_1 & \mathbf{A}\vec{m}_2 & \cdots & \mathbf{A}\vec{m}_n \\ | & | & \cdots & | \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \\ | & | & \cdots & | \end{bmatrix}$$

What we need to prove now is that there are vectors $\{\vec{m}_1, \vec{m}_2, \dots, \vec{m}_n\}$ such that

$$\mathbf{A}\vec{m}_i = \vec{e}_i \forall i \in \{1, 2, \dots, n\}$$

but we already know this is true because this is a square matrix and because of the definition of linear independence above! We also already know from lecture that the left and right inverses are identical. Thus, we've proven the second statement because the m vectors exist, i.e. \mathbf{M} exists.

Note on Proofs and Implication:

When proving X is true if and only if Y , this requires a proof in both directions—that is, that X implies Y *and* that Y implies X .

The directionality of the proof can be quite subtle, so it helps to think of a situation where the implication does *not* go in both directions. For example, if shape S is a square, then it is a rectangle. However, if another shape R is a rectangle, it is not necessarily a square. In other words,

$$\begin{aligned} S \text{ is a square} &\implies S \text{ is a rectangle} \\ R \text{ is a rectangle} &\not\Rightarrow R \text{ is a square} \end{aligned}$$

When told to prove $X \implies Y$, it is the same as being told to prove $(\text{not } Y) \implies (\text{not } X)$.

Students aren't expected to know this yet, and it's something that they'll likely be exposed to for the first time in CS70—on problems, we will be explicit about the directionality of what we're asking to be proved.