EECS 16A   Designing Information Devices and Systems I
Fall 2018
Homework 8

This homework is due October 19, 2018, at 23:59.
Self-grades are due October 23, 2018, at 23:59.

Submission Format

Your homework submission should consist of **one** file.

- `hw8.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned).

Submit the file to the appropriate assignment on Gradescope.

1. **(PRACTICE) Thévenin and Norton Equivalent Circuits**

   (a) Find the Thévenin and Norton equivalent circuits seen from outside of the box.
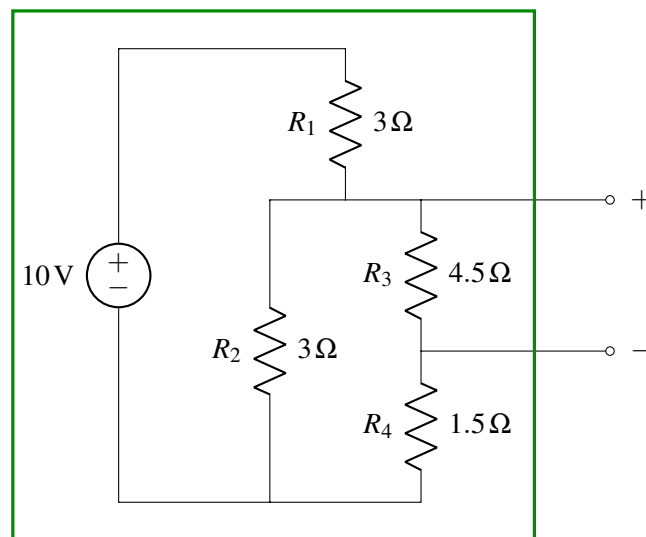

   Solution:

   To find the Thévenin and Norton equivalent circuits, we are going to find (1) the open circuit voltage between the output ports and (2) the current flowing through the output ports when the ports are shorted.

   For finding the open circuit voltage between the output ports, let us label the nodes as shown in the figure below.
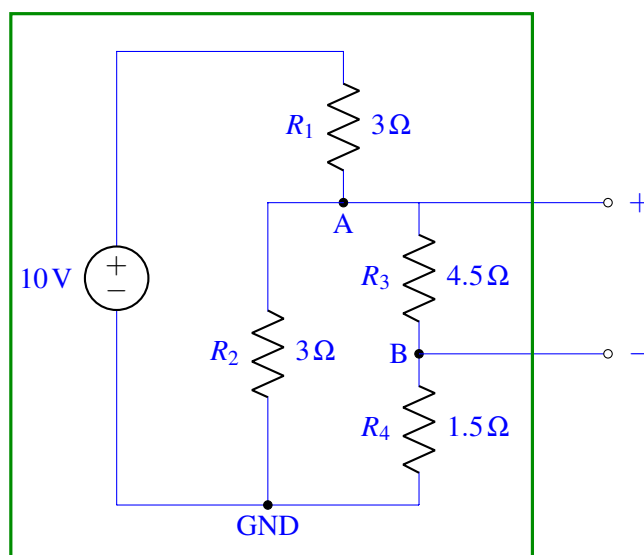
First, let us begin by calculating the effective resistance between nodes A and GND looking down from A. We have the $3\,\Omega$ resistor in parallel to the $4.5\,\Omega + 1.5\,\Omega$ resistance. This allows us to express the behavior of R_2, R_3, and R_4 together as an equivalent resistance of

$$R_{bot} = \frac{1}{\frac{1}{3\,\Omega} + \frac{1}{4.5\,\Omega + 1.5\,\Omega}} = 2\,\Omega.$$
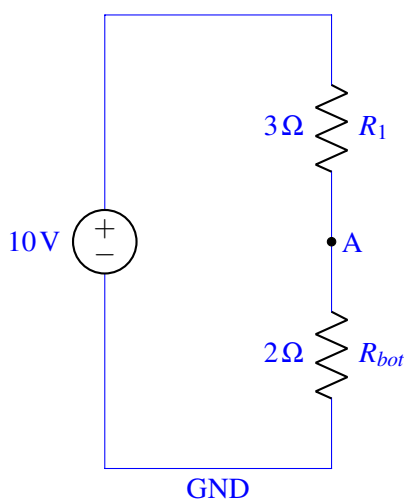
Notice how in this reduced diagram, node B isn't there!

Then we see that we have a voltage divider from the positive terminal of the $10\,V$ supply. The voltage divider is made up of two resistances in series, where the resistances are $3\,\Omega$ and $2\,\Omega$. This gives the voltage at node A equal to

$$V_A = 10\,V \frac{2\,\Omega}{3\,\Omega + 2\,\Omega} = 4\,V$$

To find V_B, we have to look back at the *original* diagram. To find the voltage at node B, note that we have another voltage divider between nodes A and GND. Hence, we can find the voltage at node B as

$$V_B = V_A \frac{1.5\,\Omega}{4.5\,\Omega + 1.5\,\Omega} = 4\,V \cdot \frac{1}{4} = 1\,V$$

Hence the open circuit voltage seen between the output ports is equal to

$$V_{\text{open}} = V_A - V_B$$
$$= 4\,\text{V} - 1\,\text{V}$$
$$= 3\,\text{V}$$

Now let us find the short circuit current flowing through the output ports. When doing this, we get the following circuit.
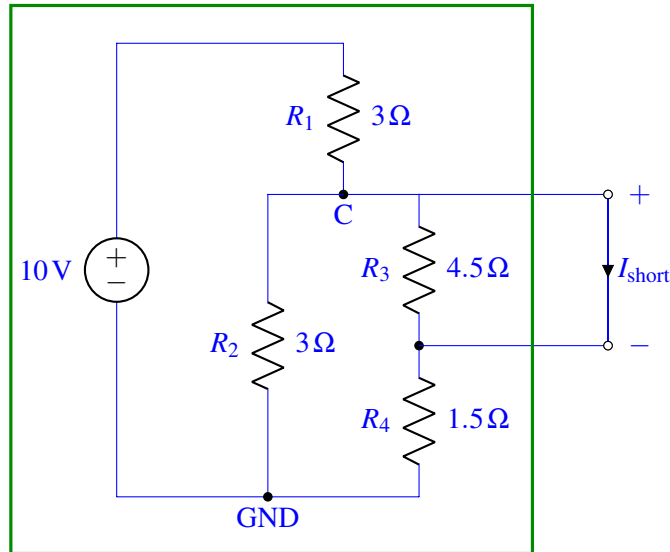
Note that when we short the output terminals, the voltages at the nodes change, this is why we changed the label of the node below the resistor R_1. Since there is a short circuit parallel to the resistor R_3, there will be no current flowing through it, hence we have

$$I_{\text{short}} = I_{R_4}$$

To find this current, let us find the equivalent resistance due to R_2 being connected in parallel to R_4 when we short the output ports. We have $3\,\Omega$ parellel to $1.5\,\Omega$, which gives an equivalent resistance

$$R_{bot} = \frac{1}{\frac{1}{3\,\Omega} + \frac{1}{1.5\,\Omega}} = 1\,\Omega$$

We again have a voltage divider between the positive side of the $10\,\text{V}$ supply and the ground. Using this voltage divider, we calculate the voltage at node C as

$$V_C = 10\,\text{V}\frac{1\,\Omega}{3\,\Omega + 1\,\Omega} = 2.5\,\text{V}$$

Hence, we see that the voltage across the resistor R_4 is equal to $2.5\,\text{V}$. Using Ohm's law, we get

$$I_{R_4} = \frac{2.5\,\text{V}}{1.5\,\Omega} = \frac{5}{3}\,\text{A}$$

Since we have $I_{\text{short}} = I_{R_4}$, we have

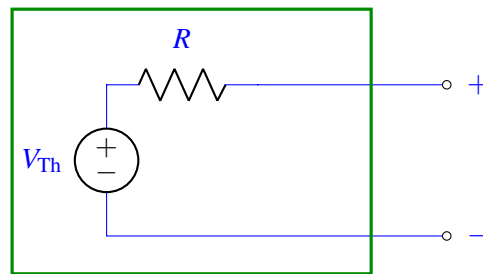$$I_{\text{short}} = I_{R_4}$$

Summarizing the results, we have

$$V_{\text{short}} = 3 \, \text{V}$$

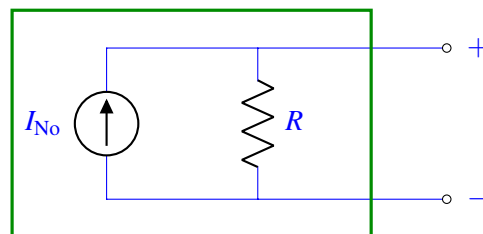$$I_{\text{short}} = \frac{5}{3} \, \text{A}$$

This gives

$$R_{\text{Th}} = \frac{V_{\text{open}}}{I_{\text{short}}} = \frac{9}{5} \, \Omega$$

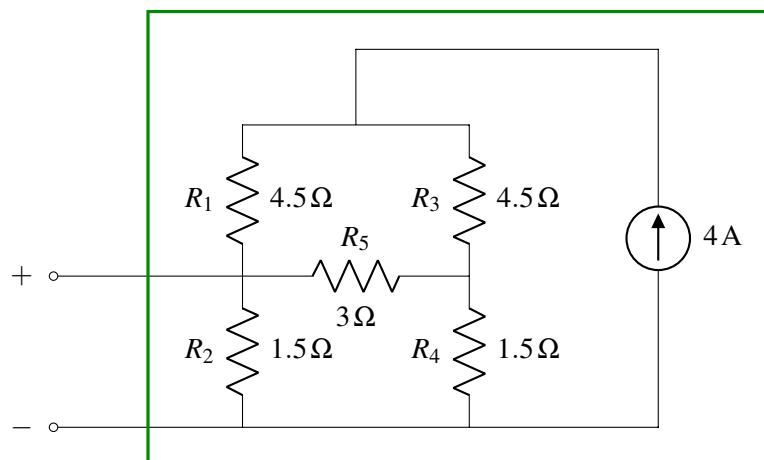Hence the Thévenin equivalent circuit is given by

where $R = R_{\text{Th}}$ and $V_{\text{Th}} = V_{\text{open}}$, and the Norton equivalent circuit is given by

where $R = R_{\text{Th}}$ and $I_{\text{No}} = I_{\text{short}}$.

(b) Find the Thévenin and Norton equivalent circuits seen from outside of the box.

Solution:

As with the previous part of this question, to find the Thévenin and Norton equivalent circuits we are going to find the open circuit voltage between the output ports and the current flowing through the output ports when the ports are shorted.

Let us first find the open circuit voltage between the output ports. We can use the symmetry in the circuit to see there will be no current flowing through R_5. Then, we have a current divider where each branch has the same resistance, hence the current $4\,\text{A}$ divides equally between the left and right branches. Hence we have

$$I_{R_1} = I_{R_2} = I_{R_3} = I_{R_4} = 2\,\text{A}$$

This gives us the voltage across R_2, equivalently the open circuit voltage between the output terminals, equal to

$$V_{\text{open}} = 2\,\text{A} \times 1.5\,\Omega = 3\,\text{V}$$

Now let us find the short circuit current across the output terminals. Let us find this using nodal analysis on the resulting circuit when we short the output ports. To help do the analysis, let us label the nodes as shown in the figure below.
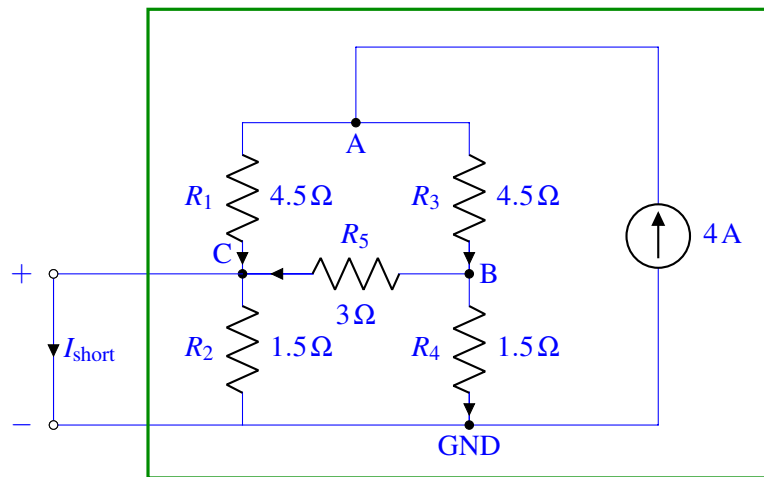
Now what are the unknown node voltages? We do not know the voltage at node A and B.

On the other hand, because node C is connected by a short circuit to GND, we know its voltage is equal to the GND which we set as 0; hence voltage at node C is known.

Next, because there is a short circuit across R_2, there will be no current flowing through it.

Let us write KCL at the nodes

$$4\,\text{A} = I_{R_1} + I_{R_3} \qquad\qquad \text{(Node A)}$$
$$I_{R_3} = I_{R_4} + I_{R_5} \qquad\qquad \text{(Node B)}$$
$$I_{\text{short}} = I_{R_1} + I_{R_5} \qquad\qquad \text{(Node C)}$$

Note that the A in this case is the symbol for amperes. Now let us relate the currents I_{R_1}, I_{R_2}, I_{R_3}, I_{R_4} and I_{R_5} to node voltages using Ohm's law. We have

$$I_{R_1} = \frac{V_A - V_C}{R_1} = \frac{V_A}{4.5\,\Omega}$$

since $V_C = 0\,\text{V}$ because it is connected to the ground by short circuit. Furthermore, we have

$$I_{R_2} = 0\,\text{A},$$
$$I_{R_3} = \frac{V_A - V_B}{R_3} = \frac{V_A - V_B}{4.5\,\Omega}$$
$$I_{R_4} = \frac{V_B}{R_4} = \frac{V_B}{1.5\,\Omega},$$
$$I_{R_5} = \frac{V_B - V_C}{R_5} = \frac{V_B}{3\,\Omega}$$

Plugging these into the first two KCL equations, we get

$$4\,\text{A} = \frac{V_A}{4.5\,\Omega} + \frac{V_A - V_B}{4.5\,\Omega}$$
$$\frac{V_A - V_B}{4.5\,\Omega} = \frac{V_B}{1.5\,\Omega} + \frac{V_B}{3\,\Omega}$$

These equations are solved by

$$V_A = 9.9\,\text{V},$$
$$V_B = 1.8\,\text{V}$$

Using the KCL at node C, we get

$$
\begin{aligned}
I_{\text{short}} &= I_{R_1} + I_{R_5} \\
&= \frac{V_A}{4.5\,\Omega} + \frac{V_B}{3\,\Omega} \\
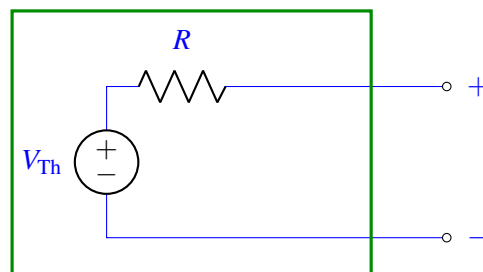&= \frac{9.9}{4.5\,\Omega} + \frac{1.8}{3\,\Omega} \\
&= 2.8\,\text{A}
\end{aligned}
$$

Summarizing the results, we have

$$V_{\text{open}} = 3\,\text{V}$$
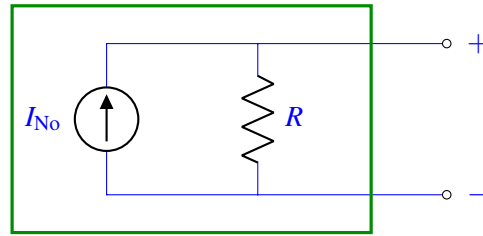$$I_{\text{short}} = 2.8\,\text{A}$$

This gives

$$
\begin{aligned}
R_{\text{Th}} &= \frac{V_{\text{open}}}{I_{\text{short}}} \\
&= \frac{15}{14}\,\Omega
\end{aligned}
$$

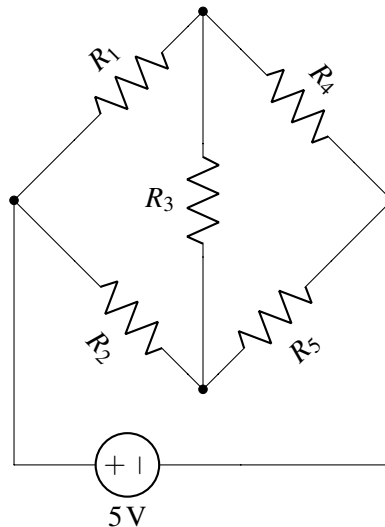Hence the Thévenin equivalent circuit is given by

where $R = R_{\text{Th}}$ and $V_{\text{Th}} = V_{\text{open}}$, and the Norton equivalent circuit is given by


where $R = R_{\text{Th}}$ and $I_{\text{No}} = I_{\text{short}}$.

2. Wheatstone Bridge

Thévenin equivalence is a powerful technique we can use to solve the Wheatstone bridge circuit shown below. This circuit is used in many sensor applications where a sensing element is the "bridge" resistor, R_3. It is often useful to find the current through the bridge resistor or the voltage across the bridge resistor. Intuitively, knowing I_{R_3} or V_{R_3} allows us to solve the rest of the circuit. In this problem, we want to find the current I_{R_3} flowing through the bridge resistor R_3.

(a) First, let's remove the bridge resistor R_3. Calculate the Thévenin equivalent voltage V_{th} between the two terminals a and b, for the circuit shown below, where the bridge resistor has been removed.
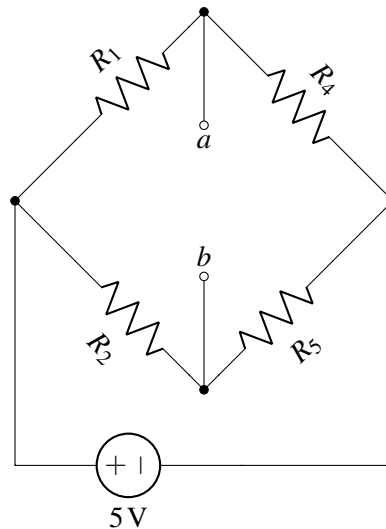
Solution:

Notice in the above circuit that there are two voltage dividers, so we can calculate v_a and v_b quickly.

$$v_a = \frac{R_4}{R_1 + R_4} \cdot 5\,\text{V}$$

$$v_b = \frac{R_5}{R_2 + R_5} \cdot 5\,\text{V}$$

Thus, the Thévenin voltage is the difference between the two voltages: $V_{\text{Th}} = v_a - v_b = \left(\frac{R_4}{R_1 + R_4} - \frac{R_5}{R_2 + R_5} \right) \cdot 5\,\text{V}$.

(b) Is the Thévenin voltage V_{Th} you found in part (a) equal to the actual voltage V_{R_3} across the bridge resistor? Why or why not?

Solution:

No, the Thévenin voltage we found in part (a) is the open-circuit voltage. If we add R_3 back into the original circuit, R_3 would load the other resistors (or, equivalently, the Thévenin resistance), so the Thévenin voltage is not equal to the actual voltage across the bridge resistor.

(c) Find the Thévenin resistance R_{Th} between the two terminals a and b for the above circuit. Draw the Thévenin equivalent between the terminals a and b for the circuit above.

Solution:

We find the Thévenin resistance by replacing the voltage source with a short and calculating the resistance between the two terminals a and b.

$$R_{\text{Th}} = (R_1 \parallel R_4) + (R_2 \parallel R_5), \text{ where } \parallel \text{ denotes the parallel operator.}$$
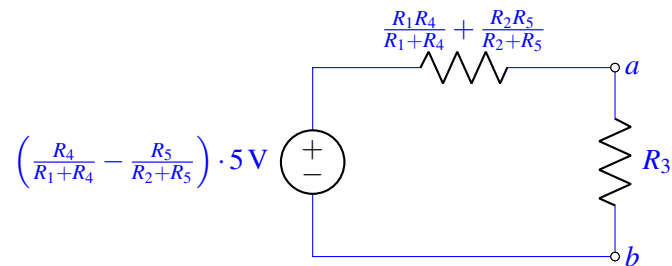$$= \frac{R_1 R_4}{R_1 + R_4} + \frac{R_2 R_5}{R_2 + R_5}$$

Using $V_{\text{Th}} = \left(\frac{R_4}{R_1 + R_4} - \frac{R_5}{R_2 + R_5} \right) \cdot 5\,\text{V}$ from part (a), we can construct the Thévenin equivalent circuit.
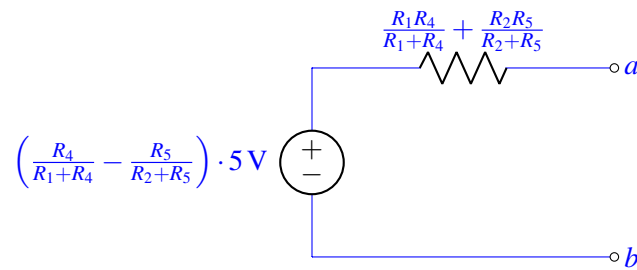
(d) With the Thévenin equivalent circuit, calculate the current I_{R_3} through the bridge resistor and the voltage V_{R_3} across the bridge resistor.

Solution:

Using the equivalent circuit, we now add R_3 back in.

$$I_{R_3} = \frac{\left(\frac{R_4}{R_1+R_4} - \frac{R_5}{R_2+R_5}\right) \cdot 5\,\text{V}}{R_3 + \frac{R_1 R_4}{R_1+R_4} + \frac{R_2 R_5}{R_2+R_5}}$$

$$V_{R_3} = I_{R_3} R_3 = \frac{\left(\frac{R_4}{R_1+R_4} - \frac{R_5}{R_2+R_5}\right) \cdot 5\,\text{V} \cdot R_3}{R_3 + \frac{R_1 R_4}{R_1+R_4} + \frac{R_2 R_5}{R_2+R_5}}$$

3. **Maximum Horsepower**

You are an engineer working on an electric car. Your job is to design a motor to be used on the car. Specifically, you are designing the resistance of this motor. The battery used by this car has some series resistance (R_s), as modeled by the circuit shown below:
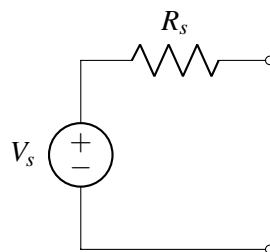
You attach your motor to the battery as shown below.
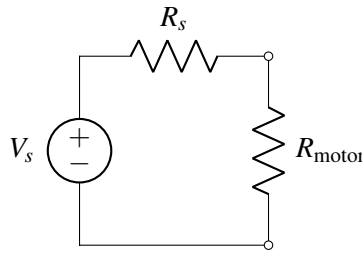
(a) Calculate the power P_s delivered by the voltage source in terms of V_s, R_s, and R_{motor}.

Solution:

$$I_s = \frac{V_s}{R_s + R_{\text{motor}}}$$

$$P_s = -I_s V_s = -\frac{V_s^2}{R_s + R_{\text{motor}}}$$

Using passive sign convention, we obtain the power dissipated by the source. This is a negative quantity, implying that the source is delivering power. The power delivered by the source is then $\frac{V_s^2}{R_s + R_{\text{motor}}}$.

(b) Now calculate the power P_{motor} dissipated by the load resistor in terms of V_s, R_s, and R_{motor}.

Solution:

$$V_{\text{motor}} = \frac{R_{\text{motor}}}{R_s + R_{\text{motor}}} \cdot V_s$$

$$P_{\text{motor}} = \frac{V_{\text{motor}}^2}{R_{\text{motor}}} = \frac{R_{\text{motor}}}{(R_s + R_{\text{motor}})^2} \cdot V_s^2$$

(c) Suppose we wanted to maximize the power dissipated across the load. Find the optimal value for R_{motor} in terms of R_s.

Hint: Use calculus.

Solution:

$$P_{\text{motor}} = \frac{R_{\text{motor}}}{(R_s + R_{\text{motor}})^2} \cdot V_s^2$$

$$\frac{dP_{\text{motor}}}{dR_{\text{motor}}} = \frac{(R_s + R_{\text{motor}})^2 - 2R_{\text{motor}}(R_s + R_{\text{motor}})}{(R_s + R_{\text{motor}})^4} \cdot V_s^2 = 0$$

$$R_s^2 - R_{\text{motor}}^2 = 0$$

$$R_{\text{motor}}^2 = R_s^2$$

$$R_{\text{motor}} = R_s$$

(d) Now you've swiched teams to designing the battery. Your job is now to pick the optimal R_s for maximizing the power delivered to the motor. What value of R_s should you pick?

Hint: Don't use calculus.

Solution:

To maximize the power delivered to the motor, we need to maximize the voltage across the motor. This is done when there is no R_s. Thus, the optimal value of R_s is $0\,\Omega$.

4. Digital to Analog Converter (DAC)

For some outputs, such as audio applications, we need to produce an analog output, or a continuous voltage from 0 to V_s. These analog voltages must be produced from digital voltages, that is sources, that can only be V_s or 0. A circuit that does this is known as a Digital to Analog Converter. It takes a binary representation of a number and turns it into an analog voltage.

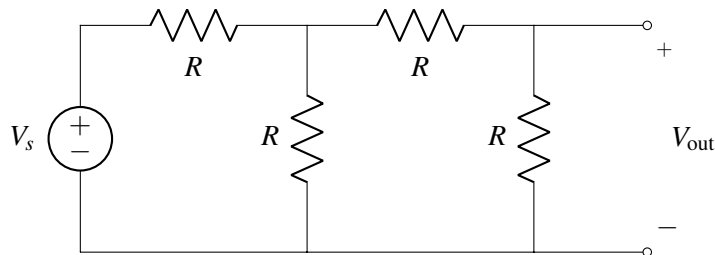The output of a DAC can be represented with the equation shown below:

$$V_{\text{out}} = V_s \sum_{n=0}^{N} \frac{1}{2^n} \cdot b_n$$

where each binary digit b_n is multiplied by $\frac{1}{2^n}$.

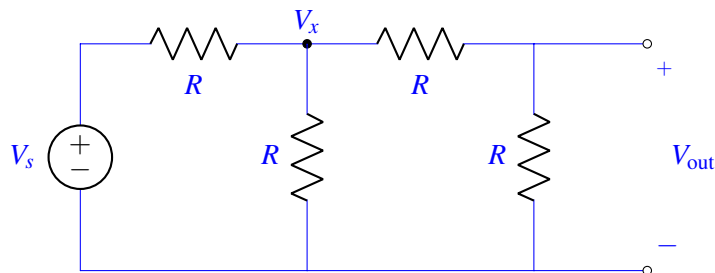(a) We know how to take an input voltage and divide it by 2:
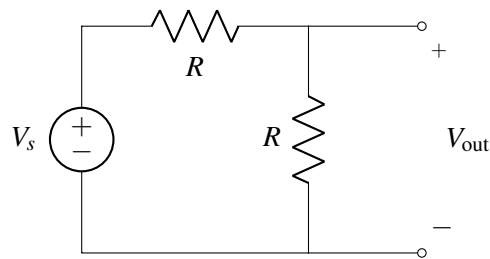
To divide by larger powers of two, we might hope to just "cascade" the above voltage divider. For example, consider:

Calculate V_{out} in the above circut. Is $V_{\text{out}} = \frac{1}{4}V_s$?
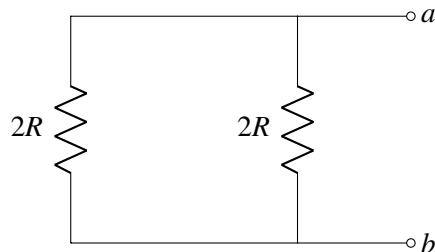
Solution:
We first find the potential V_x.

$$V_x = \frac{R \parallel 2R}{R + R \parallel 2R} V_s = \frac{\frac{2}{3}R}{R + \frac{2}{3}R} V_s = \frac{2}{5} V_s$$

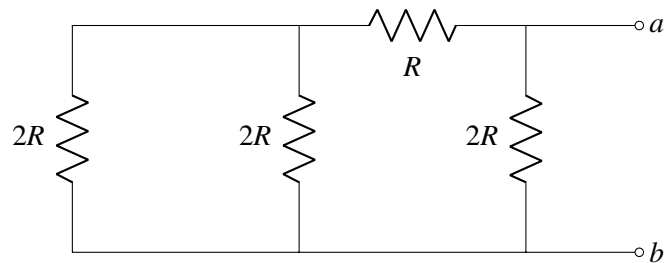$$V_{\text{out}} = \frac{R}{R+R}V_x = \frac{1}{2} \cdot \frac{2}{5}V_s = \frac{1}{5}V_s \neq \frac{1}{4}V_s$$

No, V_{out} does not equal $\frac{1}{4}V_s$.

(b) The R-$2R$ ladder, shown below, has a very nice property. For each of the circuits shown below, find the equivalent resistance looking in from points a and b. Do you see a pattern?
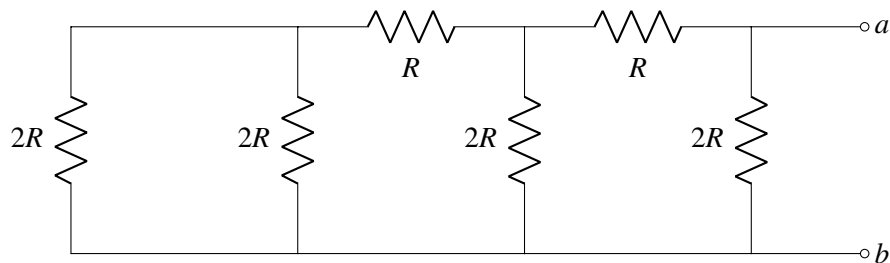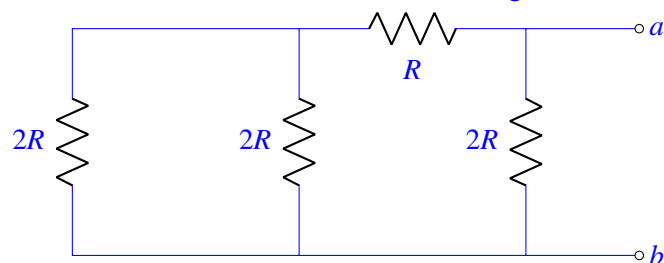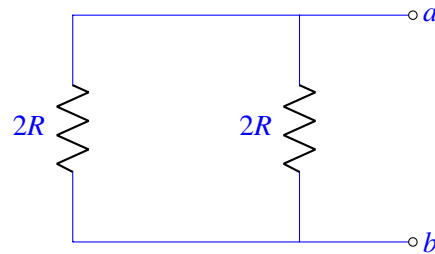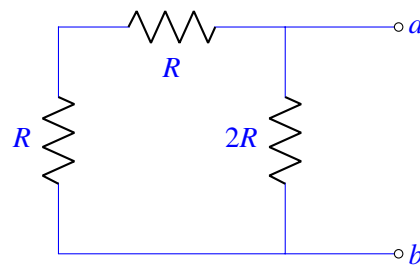
i.

ii.
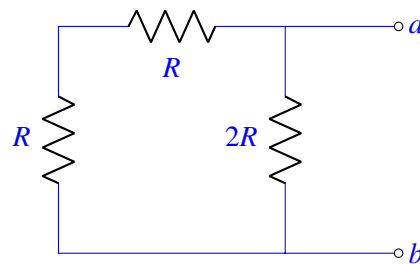

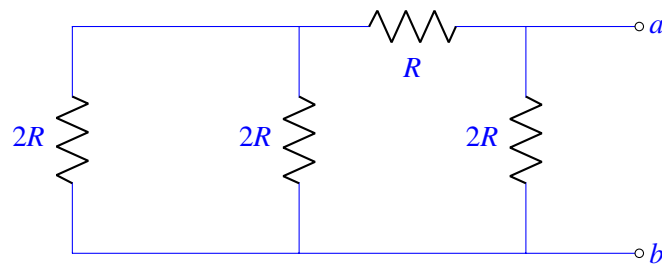iii.

Solution:

i.

$$R_{eq} = 2R \parallel 2R = R$$

ii. We find the equivalent resistance for the resistors from left to right.

$$R_{eq} = 2R \parallel 2R = R$$

iii. Again, we find the equivalent resistance for the resistors from left to right.
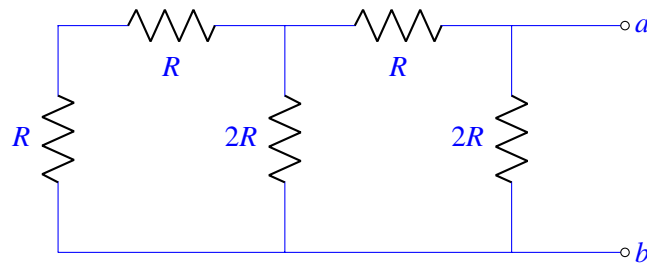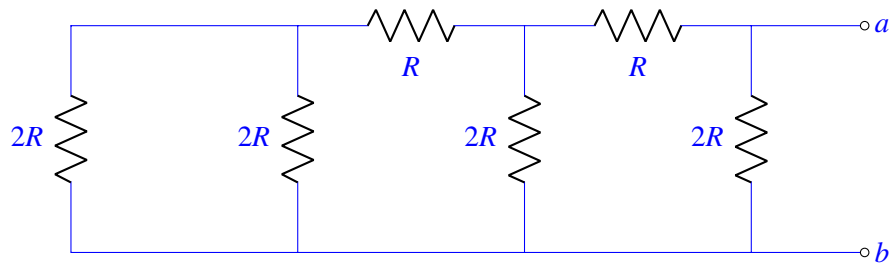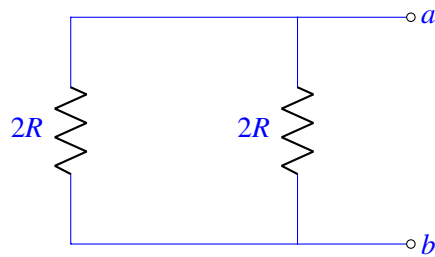
$$R_{eq} = 2R \parallel 2R = R$$
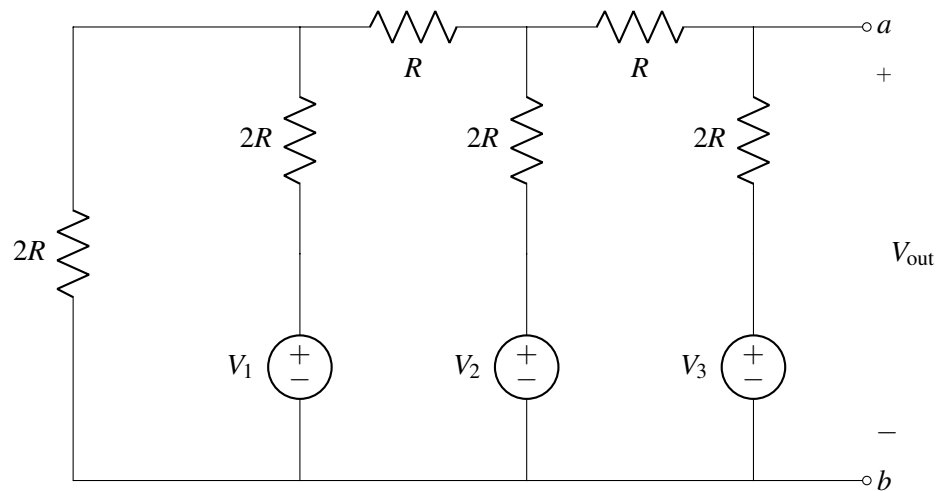
The equivalent resistance is always $R_{eq} = R$.

(c) The following circuit is an R-$2R$ DAC. To understand its functionality, use superposition to find V_{out} in terms of each V_k in the circuit.

Solution:

V_1:
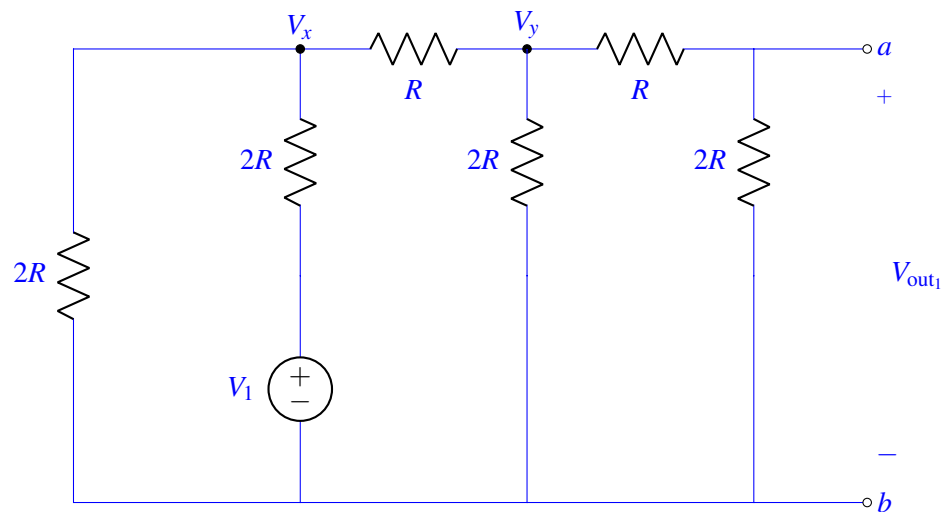
We first find the potential V_x. To do this, we can simplify the circuit.
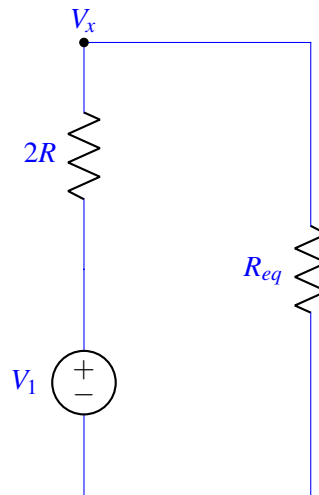
$$R_{eq} = 2R \parallel (R + (2R \parallel (R + 2R))) = \frac{22}{21}R$$

We can then find V_x using the voltage divider formula.

$$V_x = \frac{\frac{22}{21}R}{2R + \frac{22}{21}R}V_1 = \frac{11}{32}V_1$$

Similarly, we use the voltage divider formula to find V_y in terms of V_x.

$$V_y = \frac{2R \parallel (R + 2R)}{R + 2R \parallel (R + 2R)}V_x = \frac{\frac{6}{5}R}{R + \frac{6}{5}R}V_x = \frac{6}{11} \cdot \frac{11}{32}V_1 = \frac{3}{16}V_1$$

Applying the voltage divider formula again gives us V_{out_1}.

$$V_{out_1} = \frac{2R}{R + 2R}V_y = \frac{2}{3} \cdot \frac{3}{16}V_1 = \frac{1}{8}V_1$$

V_2:
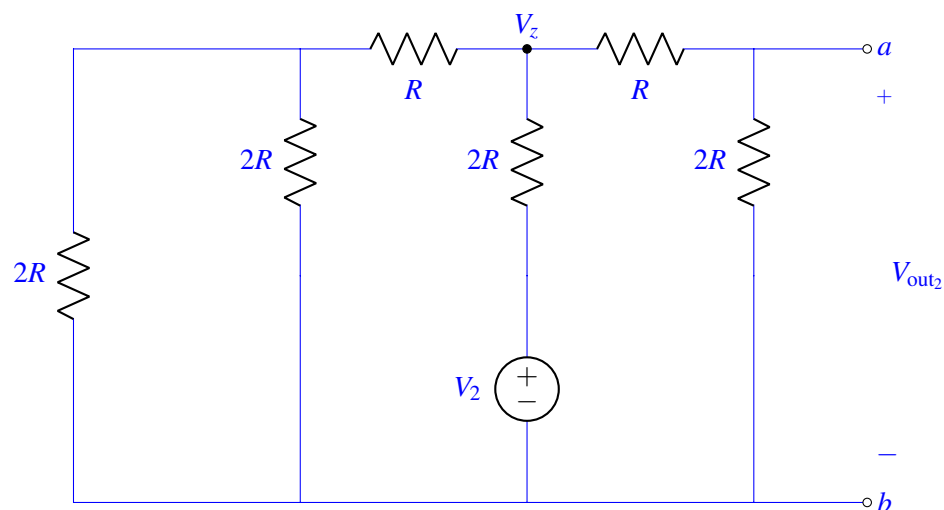
We first find the potential V_z. To do this, we can simplify the circuit.

$$R_{eq_1} = R + (2R \parallel 2R) = R + R = 2R$$

$$R_{eq_2} = R + 2R = 3R$$

We can then find V_z using the voltage divider formula.

$$V_z = \frac{2R \parallel 3R}{2R + (2R \parallel 3R)} V_2 = \frac{\frac{6}{5}R}{2R + \frac{6}{5}R} V_2 = \frac{3}{8} V_2$$
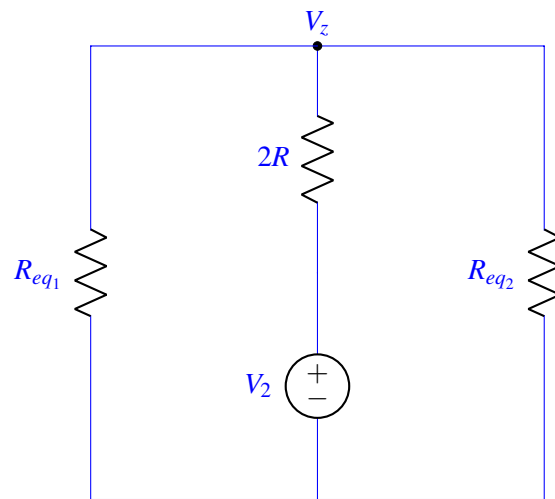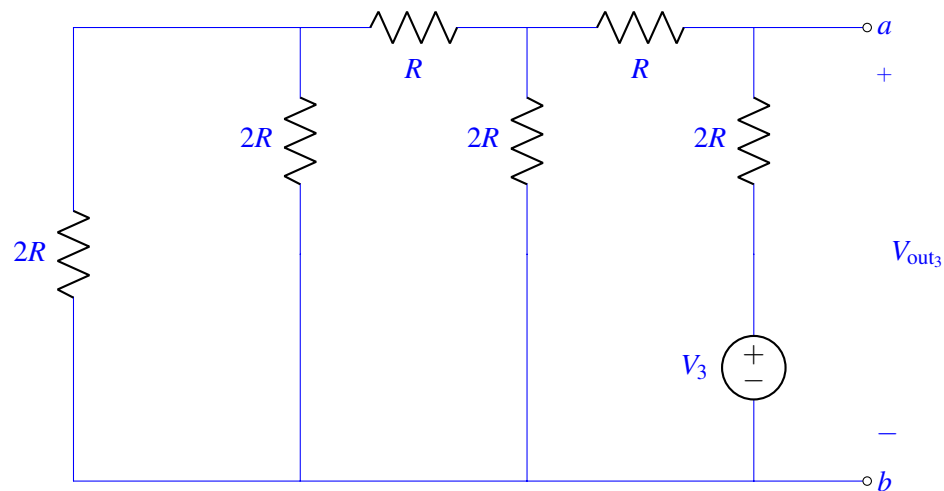
Applying the voltage divider formula again gives us V_{out_2}.

$$V_{\text{out}_2} = \frac{2R}{R + 2R} V_z = \frac{2}{3} \cdot \frac{3}{8} V_2 = \frac{1}{4} V_2$$

V_3:

We can simplify this circuit.
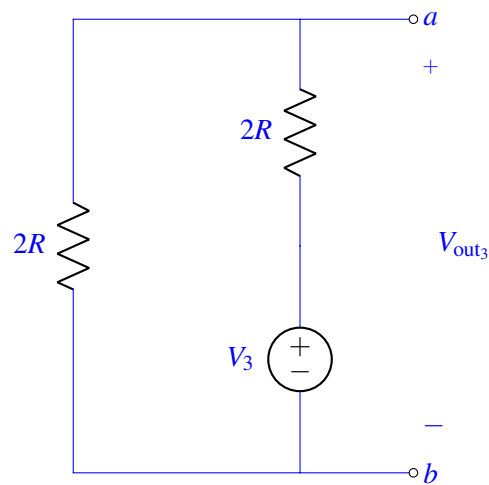
$$V_{\text{out}_3} = \frac{2R}{2R + 2R} V_3 = \frac{1}{2} V_3$$

$$V_{\text{out}} = V_{\text{out}_1} + V_{\text{out}_2} + V_{\text{out}_3} = \frac{1}{8} V_1 + \frac{1}{4} V_2 + \frac{1}{2} V_3$$

(d) We've now designed a 3-bit R-$2R$ DAC. What is the output voltage V_{out} if $V_2 = 1\,\text{V}$ and $V_1 = V_3 = 0\,\text{V}$?
Solution:

$$V_{\text{out}} = \frac{1}{8} \cdot 0\,\text{V} + \frac{1}{4} \cdot 1\,\text{V} + \frac{1}{2} \cdot 0\,\text{V} = \frac{1}{4}\,\text{V}$$

(e) Draw the Thévenin equivalent of the above circuit, looking in from the terminals a and b with $V_2 = 1\,\text{V}$ and $V_1 = V_3 = 0\,\text{V}$.
Solution:

$$V_{\text{Th}} = \frac{1}{4}\,\text{V}$$
$$R_{\text{Th}} = R$$
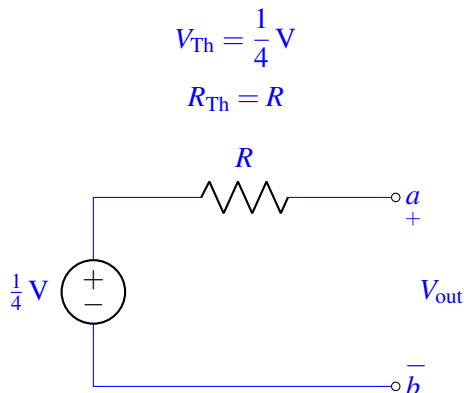
(f) Suppose that we now attach a speaker to the DAC with a resistance of $\frac{R}{3}$. Why is the voltage across the speaker lower than what we computed in part (d)? What is the actual output voltage?
Solution:

Attaching the speaker is equivalent to adding a load resistor to the output of the DAC. The speaker has some equivalent resistance and therefore draws some current from the DAC. As a result, the voltage across the output will be lower than expected. We can use the Thévenin equivalent circuit to calculate the actual output voltage across the speaker.
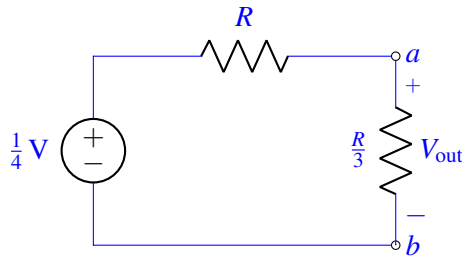
$$V_{\text{out}} = \frac{\frac{R}{3}}{R + \frac{R}{3}} \cdot \frac{1}{4}\,V = \frac{1}{4} \cdot \frac{1}{4}\,V = \frac{1}{16}\,V$$

5. Measuring Voltage and Current

In order to measure quantities such as voltage and current, engineers have designed circuits known as Analog to Digital Converters, or ADCs. While ADCs are very complicated circuits in reality, a simple model of an ADC can be represented as a resistor with a very high resistance, R_{ADC}. **The ADC measures the voltage across it's resistance** R_{ADC}. The measured voltage is then relayed to a microprocessor (such as the MSP430s used in Lab). We can use this ADC to create both a voltmeter and an ammeter (A voltmeter is an instrument that measures voltage, while an ammeter measures current). This model of an ADC is shown in Figure 1. Let us explore what happens when we connect this ADC to various circuits to measure voltages and currents. Throughout this problem assume $R_{\text{ADC}} = 1M\Omega$. Recall that the SI prefix M or Mega is 10^6.
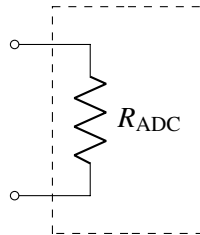
Figure 1: Our Model of an ADC, $R_{\text{ADC}} = 1M\Omega$

(a) Suppose we wanted to measure the voltage across R_2 (v_{out}) produced by the voltage divider circuit shown in Figure 2 on the left. The circuit on the right in Figure 2 shows how we would connect the ADC across R_2 to function as a voltmeter. Assume $R_1 = 100\Omega$ and $R_2 = 100\Omega$.

First calculate the value of v_{out}. Then calculate the voltage ADC would measure, i.e. v_{meas}.
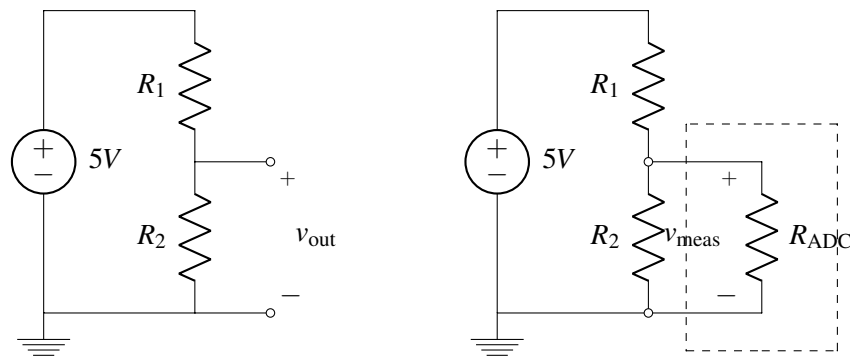
Figure 2: Left: Circuit without the ADC connected, Right: ADC measuring voltage across R_2

Solution: We start by finding v_{out} in the circuit on the left. Recognizing that this circuit is a voltage divider, we can directly find the following:

$$v_{\text{out}} = \frac{R_1}{R_1 + R_2} 5V = \frac{100\,\Omega}{200\,\Omega} 5V = 2.5V$$

Next we consider the circuit on the right. We start by combining the resistor R_2 and R_{ADC} since they are in parallel. Then we can apply the voltage divider formula to calculate the voltage across R_{ADC}.

$$R_2 || R_{\text{ADC}} = \frac{R_2 R_{\text{ADC}}}{R_2 + R_{\text{ADC}}} == \frac{1\,\text{M}\Omega \cdot 100\,\Omega}{1\,\text{M}\Omega + 100\,\Omega} = 99.99\,\Omega$$

$$v_{\text{out}} = \frac{R_2 || R_{\text{ADC}}}{R_1 + R_2 || R_{\text{ADC}}} = \frac{99.99\,\Omega}{99.99\,\Omega + 100\,\Omega} \cdot 5V = 2.49987\,V$$

(b) Repeat part a), but now $R_1 = 10M\Omega$ and $R_2 = 10M\Omega$. Is this ADC still a good tool to measure the output voltage?

Solution: We start by again finding v_{out} in the circuit on the left. Recognizing that this circuit is a voltage divider, we can directly find the following:

$$v_{\text{out}} = \frac{R_1}{R_1 + R_2} 5V = \frac{10\,\text{M}\Omega}{20\,\text{M}\Omega} 5V = 2.5V$$

Next we consider the circuit on the right. We start by combining the resistor R_2 and R_{ADC} since they are in parallel. Then we can apply the voltage divider formula to calculate the voltage across R_{ADC}.

$$R_2 || R_{\text{ADC}} = \frac{R_2 R_{\text{ADC}}}{R_2 + R_{\text{ADC}}} == \frac{10\,\text{M}\Omega \cdot 1\,\text{M}\Omega}{10\,\text{M}\Omega + 1\,\text{M}\Omega} = 0.909\,\text{M}\Omega$$

$$v_{\text{out}} = \frac{R_2 || R_{\text{ADC}}}{R_1 + R_2 || R_{\text{ADC}}} = \frac{0.909\,\text{M}\Omega}{0.909\,\text{M}\Omega + 10\,\text{M}\Omega} \cdot 5V = 0.4166\,V$$

Since the resistors R_1 and R_2 were larger than R_{ADC}, we see the addition of R_{ADC} changes what the calculated V_{out} is significantly. Thus we cannot use this ADC as a voltmeter here.

(c) Now suppose we are working with the same circuit as in Part (a), but we know that $R_2 = R_1$. What is the maximum value of R_1 that ensures that the difference between voltage measurement of the ADC (v_{meas}) and the actual value (v_{out}) remains within $\pm 10\%$ of v_{out}?

Solution: We will only have to consider the case where v_{meas} is less than v_{out}, because the parallel combination of R_2 and R_{ADC} can only make a resistor with total resistance smaller than R_2.

First let's symbolically represent what the outputs are in the two cases:

For the circuit without the voltmeter connected:

$$V_{\text{out}} = \frac{R_2}{R_1 + R_2} \cdot V_s$$

For the circuit with the voltmeter connected:

$$R_{\text{ADC}} \| R_2 = \frac{R_{\text{ADC}} R_2}{R_{\text{ADC}} + R_2}$$

$$V_{\text{meas}} = \frac{\frac{R_{\text{ADC}} R_2}{R_{\text{ADC}} + R_2}}{R_1 + \frac{R_{\text{ADC}} R_2}{R_{\text{ADC}} + R_2}} \cdot V_s = \frac{R_{\text{ADC}} R_2}{R_1(R_{\text{ADC}} + R_2) + R_{\text{ADC}} R_2} \cdot V_s$$

Now we need:

$$\frac{V_{\text{out}} - V_{\text{meas}}}{V_{\text{out}}} \leq \frac{1}{10}$$

$$\frac{\frac{R_2}{R_1 + R_2} \cdot V_s - \frac{R_{\text{ADC}} R_2}{R_1(R_{\text{ADC}} + R_2) + R_{\text{ADC}} R_2} \cdot V_s}{\frac{R_2}{R_1 + R_2} \cdot V_s} \leq \frac{1}{10}$$

$$\frac{\frac{R_2}{R_1 + R_2} - \frac{R_{\text{ADC}} R_2}{R_1(R_{\text{ADC}} + R_2) + R_{\text{ADC}} R_2}}{\frac{R_2}{R_1 + R_2}} \leq \frac{1}{10}$$

Since we know $R_1 = R_2$, we can simplify our final expression:

$$\frac{\frac{1}{2} - \frac{R_{\text{ADC}}}{R_{\text{ADC}} + R_2 + R_{\text{ADC}}}}{\frac{1}{2}} \leq \frac{1}{10}$$

$$1 - \frac{2R_{\text{ADC}}}{2R_{\text{ADC}} + R_2} \leq \frac{1}{10}$$

$$\frac{R_2}{2R_{\text{ADC}} + R_2} \leq \frac{1}{10}$$

$$\frac{9}{10}R_2 \leq \frac{2}{10}R_{\text{ADC}}$$

$$R_2 \leq \frac{2}{9}R_{\text{ADC}}$$

$$R_1 = R_2 \leq 0.22\,\text{M}\Omega$$

(d) As mentioned before, we can use an ADC to measure the current through an element as well, where the ADC functions as an ammeter. Your colleague suggests you use the circuit shown in Figure 3, where $R_x = 1\,\Omega$, then the measured current through R_x is $I_{\text{meas}} = \frac{V_{\text{ADC}}}{R_x}$ where V_{ADC} is the voltage across the ADC.

In Figure 4, the ADC is connected to measure the current through resistor $R_1 = 1k\Omega$. For the circuit

on the left, find the current through R_1 without the ADC connected (i.e. I_1). Then for the circuit on the right, find the current measured by the ADC when it is connected as an ammeter (i.e. I_{meas}).
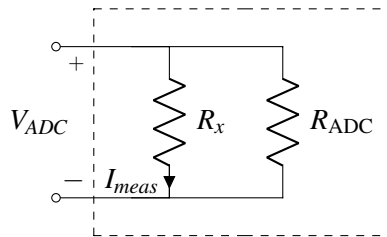

Figure 3: The ADC connected to measure current, $R_{\text{ADC}} = 1M\Omega$
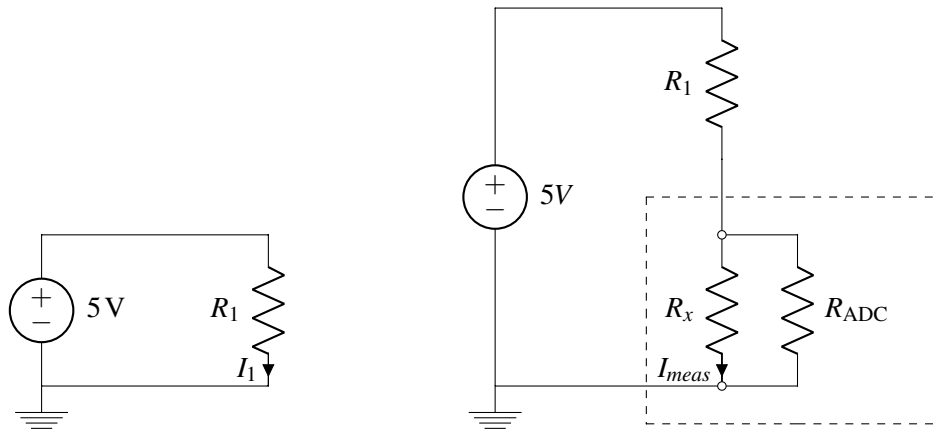

Figure 4: Circuits for Part (d) Left: Original circuit; Right: Circuit with the ADC measuring current

Solution:

We start with the circuit on the left

$$I_1 = \frac{5\,\text{V}}{1\,\text{k}\Omega} = 5\,\text{mA}$$

For the circuit on the right, we start by computing $R_x||R_{\text{ADC}}$.

$$R_x||R_{\text{ADC}} = \frac{R_x R_{\text{ADC}}}{R_x + R_{\text{ADC}}} = \frac{1\,\Omega \cdot 1\,\text{M}\Omega}{1\,\Omega + 1\,\text{M}\Omega} \approx 1\,\Omega$$

Next, we compute the voltage across the $R_x||R_{\text{ADC}}$ combination. Notice this circuit is again a voltage divider.

$$v_{R_{\text{ADC}}} = \frac{R_x||R_{\text{ADC}}}{R_1 + R_x||R_{\text{ADC}}} \cdot 5\,\text{V} = \frac{1\,\Omega}{1\,\text{k}\Omega + 1\,\Omega} \cdot 5\,\text{V} = 0.004\,995\,\text{V}$$

Now, the current is this voltage divided by the resistance $R_x||R_{\text{ADC}}$.

$$I_{\text{meas}} = \frac{v_{\text{ADC}}}{R_x||R_{\text{ADC}}} = \frac{0.004\,995\,\text{V}}{1\,\Omega} \approx 5\,\text{mA}$$

(e) What is the minimum value of R_1 that ensures the difference between current measurement (I_{meas}) and the the actual value (I_1) stays within $\pm 10\%$ of I_1?

Solution:

Once again, we will only consider the case where the measured current is smaler than the actual current, because the series combination of R_1 and $R_x || R_{ADC}$ can only create a resistor bigger than R_1. First let's symbolically represent what the outputs are in the two cases:

For the circuit without the ammeter connected:

$$I_1 = \frac{V_s}{R_1}$$

For the circuit with the ammeter connected:

$$R_{ADC} || R_x = \frac{R_{ADC} R_x}{R_{ADC} + R_x}$$

$$V_{ADC} = \frac{\frac{R_{ADC} R_x}{R_{ADC} + R_x}}{R_1 + \frac{R_{ADC} R_x}{R_{ADC} + R_x}} \cdot V_s$$

$$I_{meas} = \frac{V_{ADC}}{R_x || R_{ADC}} = \frac{V_s}{\frac{R_{ADC} R_x}{R_{ADC} + R_x} + R_1}$$

Now we need:

$$\frac{I_1 - I_{meas}}{I_1} \leq \frac{1}{10}$$

$$\frac{\frac{V_s}{R_1} - \frac{V_s}{\frac{R_{ADC} R_x}{R_{ADC} + R_x} + R_1}}{\frac{V_s}{R_1}} \leq \frac{1}{10}$$

$$1 - \frac{R_1}{\frac{R_{ADC} R_x}{R_{ADC} + R_x} + R_1} \leq \frac{1}{10}$$

$$\frac{\frac{R_{ADC} R_x}{R_{ADC} + R_x}}{\frac{R_{ADC} R_x}{R_{ADC} + R_x} + R_1} \leq \frac{1}{10}$$

We will approximate $\frac{R_{ADC} R_x}{R_{ADC} + R_x} = 1\,\Omega$.

$$\frac{1}{1 + R_1} \leq \frac{1}{10}$$

$$9 \leq R_1$$

6. Homework Process and Study Group

Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.) How did you work on this homework?

Solution:

I worked on this homework with...

I first worked by myself for 2 hours, but got stuck on problem 5, so I went to office hours on...

Then I went to homework party for a few hours, where I finished the homework.