4. The Dynamics of Romeo and Juliet's Love Affair.

(a). For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $a+b = c+d$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Consider $A\vec{v}_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix}$.

① Since $a+b = c+d$, so $\begin{bmatrix} a+b \\ c+d \end{bmatrix} = \begin{bmatrix} a+b \\ a+b \end{bmatrix} = (a+b) \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Since $\vec{v}_1 \neq 0$ and $(a+b)$ is a constant,

So, $\vec{v}_1$ by definition is an eigenvector of $\vec{A}$. Its corresponding eigenvalue $\boxed{\lambda_1 = a+b}$

② Similarly. for $\vec{v}_2 = \begin{bmatrix} b \\ -c \end{bmatrix}$, we have $A\vec{v}_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} b \\ -c \end{bmatrix} = \begin{bmatrix} ab-bc \\ bc-cd \end{bmatrix}$

Since $a+b = c+d$, so $a-c = d-b$, and so $(a-c) \cdot \vec{v}_2 = \begin{bmatrix} (a-c)b \\ (a-c)(-c) \end{bmatrix} = \begin{bmatrix} (a-c)b \\ (b-d)(-c) \end{bmatrix} =$

$= \begin{bmatrix} ab-bc \\ cd-bc \end{bmatrix}$, which implies that $A\vec{v}_2 = (a-c)\vec{v}_2$. Since $\vec{v}_2 \neq 0$, $(a-c)$ is constant,

so again, $\vec{v}_2$ by definition is an eigenvector of $\vec{A}$. Its corresponding eigenvalue is $\boxed{\lambda_2 = a-c}$

Thus, the eigenspace is $\boxed{\text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}b\\-c\end{bmatrix}\right\}}$ or equivalently, $\text{span}\left\{\begin{bmatrix}a\\a\end{bmatrix}, \begin{bmatrix}b\\-c\end{bmatrix}\right\}$

(b). Since $A = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$ is just a special case of the generalized part (a), with $a+b = c+d$. Specifically, $a = d = 0.75$, $b = c = 0.25$

So the first eigenpair is $\lambda_1 = a+b = 0.75+0.25 = 1$, and $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The second eigenpair is $\lambda_2 = a-c = 0.75-0.25 = 0.5$, and $\vec{v}_2 = \begin{bmatrix} b \\ -c \end{bmatrix} = \begin{bmatrix} 0.25 \\ -0.25 \end{bmatrix}$

Thus, the eigenpairs are: $\boxed{\left(1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \text{ and } \left(0.5, \begin{bmatrix} 0.25 \\ -0.25 \end{bmatrix}\right).}$

(c) To find the set of points $\{\vec{s}_* \mid A\vec{s}_* = \vec{s}_*\}$ is equivalent to finding the eigenvectors corresponding to $\lambda = 1$. so $(A - I_2) \cdot \vec{s}_* = \vec{0} \Rightarrow \begin{bmatrix} -0.25 & 0.25 \\ 0.25 & -0.25 \end{bmatrix} \cdot \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \vec{0}$.

$\Rightarrow \left[\begin{array}{cc|c} -0.25 & 0.25 & 0 \\ 0.25 & -0.25 & 0 \end{array}\right]$. R_2: Add R_1. $\Rightarrow \left[\begin{array}{cc|c} -0.25 & 0.25 & 0 \\ 0 & 0 & 0 \end{array}\right]$ which gives $-0.25 s_1 + 0.25 s_2 = 0$
$\Rightarrow s_1 = s_2$

So the steady states are $\boxed{\text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}}$.

(d). We've shown that A is a special case for our generalized situation in part (a), and so we proved that (with $b=c=0.25$), $\begin{bmatrix} b \\ -c \end{bmatrix} = \begin{bmatrix} 0.25 \\ -0.25 \end{bmatrix}$ is an eigenvector; which means that $\forall \alpha \in \mathbb{R}$, $\begin{bmatrix} 0.25 \\ -0.25 \end{bmatrix} \alpha$ is also an eigenvector. Take $\alpha = 4$, so $\begin{bmatrix} 0.25 \\ -0.25 \end{bmatrix} \cdot 4 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and so $\text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix} 0.25 \\ -0.25 \end{bmatrix}\right\}$ are eigenvectors with an eigenvalue $\lambda_2 = a-c = 0.5$.

Thus, $\forall \vec{s}[0] \in \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$, $\vec{s}[1] = A\vec{s}[0] = \lambda_2 \vec{s}[0] = 0.5\,\vec{s}[0] \in \text{span}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Thus, $\vec{s}[n] = 0.5^n\,\vec{s}[0] = \begin{bmatrix} 0.5^n \\ -0.5^n \end{bmatrix}$ which means that as $\boxed{n \to \infty,\ \vec{s}[n] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$.

which implies that Romeo and Juliet both have a neutral stance towards each other.

(e). Since here, $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. with $a=b=c=d$, so A is just another special case of part (a), with $a+b=c+d=2$.

So, the first eigenpair is: $\lambda_1 = a+b = 2$ and $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and

the second eigenpair is: $\lambda_2 = a-c = 1-1 = 0$. and $\vec{v}_2 = \begin{bmatrix} b \\ -c \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Thus, the eigenpairs are: $\boxed{\left(2, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \text{ and } \left(0, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)}$

f). Since $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector, so span$\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ are all eigenvectors.

which means that $\vec{S}[0] \in$ span$\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ is an eigenvector, so we have that $\vec{s}[1] = A\vec{s}[0] = \lambda_2 \vec{s}[0] = \vec{0}$, so $\forall n \geq 1$, $\vec{s}[n] = A\vec{s}[n-1] = \vec{0}$.

Thus, their relationship again falls into a $\boxed{\text{neutral stance towards each other}}$

Specifically, as $n \to \infty$, $\boxed{\vec{s}[n] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$

(g). Since $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector, so $\vec{s}[0] \in$ span$\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ is also an eigenvector.

So, $\vec{s}[1] = A\vec{S}[0] = \lambda_1 \vec{s}[0] = 2\vec{s}[0] = 2 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in$ span$\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

which we can then generalize that $\forall n \in \mathbb{N}$, $\vec{s}[n] \in$ span$\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

$\Rightarrow$ they are all eigenvectors.

which implies that $\vec{s}[n] = A\vec{s}[n-1] = \lambda_1 \vec{s}[n-1] = 2^n \vec{s}[0] = \begin{bmatrix} 2^n \\ 2^n \end{bmatrix}$.

Thus, as $n \to \infty$, $2^n \to \infty$, so $\boxed{\vec{s}[n] = \infty \cdot S[0]}$ if $R[0] = J[0] > 0$.

In other words, Romeo and Juliet would each have stronger love/like towards each other. and thus, developing a stronger relationship over time. and for $R[0] < 0$, vice versa. (stronger hatred)

(h) Again, since $A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$ with $a=d=1$, $b=c=-2$, this is yet another special case of part (a), with $a+b = c+d = -1$.

So ① the first eigenpair is $\lambda_1 = a+b = -1$. and $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

② the second eigenpair is $\lambda_2 = a-c = 1-(-2) = 3$, $\vec{v}_2 = \begin{bmatrix} b \\ -c \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$.

Thus, the eigenpairs are: $\boxed{\left(-1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \text{ and } \left(3, \begin{bmatrix} -2 \\ 2 \end{bmatrix}\right)}$

(i). Since $\begin{bmatrix} -2 \\ 2 \end{bmatrix} = -2 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ with $-2 \in \mathbb{R}$, so span$\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ = span$\left\{\begin{bmatrix} -2 \\ 2 \end{bmatrix}\right\}$ are all eigenvectors.

which implies $\vec{s}[0]$ is an eigenvector; so $\vec{s}[1] = A\vec{S}[0] = \lambda_2 \vec{s}[0] = 3\vec{s}[0] \in$ span$\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$.

In other words, $\forall n \in \mathbb{N}$, $\vec{s}[n] \in$ span$\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ is an eigenvector, with $\vec{s}[n] = 3^n \vec{s}[0]$.

$\to$ Case 1: if $R[0] > 0$, $J[0] < 0$, then Romeo will have growing love/like for Juliet, while Juliet will have growing hate towards Romeo

In other words, as $n \to \infty$, $3^n \to \infty$, so $\vec{s}[n] = \begin{bmatrix} \infty \\ -\infty \end{bmatrix}$.

$\to$ Case 2: if $R[0] < 0$, $J[0] > 0$, then vice versa, Romeo has growing hatred towards Juliet, and Juliet having growing love/like towards Romeo.

In other words, as $n \to \infty$, $3^n \to \infty$, so $\vec{s}[n] = \begin{bmatrix} -\infty \\ \infty \end{bmatrix}$.

(j) Similar to our argument in (g), so $\forall n \in \mathbb{N}$, $\vec{s}[n] \in$ span$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and so $\vec{s}[n] = (-1)^n \vec{s}[0]$.

Thus, as $n \to$ $\boxed{\text{we can't decide } \vec{s}[n]}$ and similarly we can't decide Romeo and Juliet's exact relationship — we only know they either maintained initial states or swapped entirely.