EECS 16A Designing Information Devices and Systems I
Fall 2018 Discussion 5B

1. Diagonalization

One of the most powerful ways to think about matrices is to think of them in diagonal form [1].

(a) Consider a matrix $\mathbf{A}$, a matrix $\mathbf{V}$ whose columns are the eigenvectors of $\mathbf{A}$, and a diagonal matrix $\mathbf{\Lambda}$ with the eigenvalues of $\mathbf{A}$ on the diagonal (in the same order as the eigenvectors (or columns) of $\mathbf{V}$). From these definitions, show that

$$\mathbf{AV} = \mathbf{V\Lambda}$$

Answer:

$$\mathbf{AV} = \mathbf{A}\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \mathbf{A}\vec{v}_1 & \mathbf{A}\vec{v}_2 & \cdots & \mathbf{A}\vec{v}_k \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \cdots & \lambda_k\vec{v}_k \\ | & | & & | \end{bmatrix}$$

$$\mathbf{V\Lambda} = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \\ | & | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \cdots & \lambda_k\vec{v}_k \\ | & | & & | \end{bmatrix}$$

(b) We now multiply both sides on the right by $\mathbf{V}^{-1}$ and get $\mathbf{A} = \mathbf{V\Lambda V}^{-1}$, the diagonal form of $\mathbf{A}$. Consider the action of $\mathbf{A}$ on a coordinate vector $\vec{x}_u$ in the standard basis. Interpret each step of the following calculation in terms of coordinate transformations and scaling by eigenvalues.

$$\mathbf{A}\vec{x}_u = \mathbf{V\Lambda V}^{-1}\vec{x}_u$$

Answer:

Consider any vector $\vec{x}$ with coordinates $\vec{x}_u$ in the standard basis and $\vec{x}_v$ in a basis composed of the eigenvectors of $\mathbf{A}$. We can think of the action of $\mathbf{A}$ on $\vec{x}_u$ as

$$\mathbf{A}\vec{x}_u = \underbrace{\mathbf{V}\quad\underbrace{\mathbf{\Lambda}\quad\underbrace{\mathbf{V}^{-1}\vec{x}_u}_{\substack{\text{Coords of } \vec{x}_v \\ \text{in eg-vec basis}}}}_{\substack{\text{Coordinates for each eg-vec} \\ \text{scaled by appropriate eg-val}}}}_{\substack{\text{Result transformed} \\ \text{back into standard basis}}}$$

First, the coordinates in the standard basis are transformed into the eigenvector basis ($\mathbf{V}^{-1}\vec{x}_u$). Then the coordinates for each eigenvector are scaled by the appropriate eigenvalue ($\mathbf{\Lambda V}^{-1}\vec{x}_u$). This is the real transformation given by the matrix. Then the result is transformed back into the standard basis. ($\mathbf{V\Lambda V}^{-1}\vec{x}_u$).

[1] Not all matrices can be put in this form but most can. The ones that can't be diagonalized can be put in a similar form called the Jordan form.

2. Matrix Powers

One of the most powerful things about matrix diagonalization is that it gives us some insight into polynomial functions of matrices.

(a) Write $\mathbf{A}^N$ using the diagonalization of $\mathbf{A}$ and simplify your result as much as possible. What do you get?

Answer:

$$\begin{aligned}\mathbf{A}^N &= \underbrace{\mathbf{V}\mathbf{\Lambda}\underbrace{\mathbf{V}^{-1}\times\mathbf{V}}_{\mathbf{I}}\mathbf{\Lambda}\mathbf{V}^{-1}\times\cdots\times\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}}_{N\text{ times}}\\ &= \mathbf{V}\mathbf{\Lambda}^N\mathbf{V}^{-1}\end{aligned}$$

(b) How could you find $\mathbf{A}$ raised to any power while only doing three matrix multiplications?

Answer:

Write $\mathbf{A}^N = \mathbf{V}\mathbf{\Lambda}^N\mathbf{V}^{-1}$. $\mathbf{\Lambda}^N$ can be found by taking each diagonal entry to the Nth power.

3. Fibonacci Sequence

One of the most useful things about diagonalization is that it allows us to easily compute polynomial functions of matrices. This in turn lets us do far more, including solving many linear recurrence relations. This problem shows you how this can be done for the Fibonacci numbers, but you should notice that the same exact technique can be applied far more generally.

(a) The Fibonacci sequence can be constructed according to the following relation. The Nth number in the Fibonacci sequence, F_N, is computed by adding the previous two numbers in the sequence together:

$$F_N = F_{N-1} + F_{N-2}$$

We select the first two numbers in the sequence to be $F_1 = 0$ and $F_2 = 1$ and then we can compute the following numbers as

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

Write the operation of computing the next Fibonacci numbers from the previous two using matrix multiplication:

$$\begin{bmatrix} F_N \\ F_{N-1} \end{bmatrix} = \mathbf{A}\begin{bmatrix} F_{N-1} \\ F_{N-2} \end{bmatrix}$$

Answer:

$$\begin{bmatrix} F_N \\ F_{N-1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{\mathbf{A}}\begin{bmatrix} F_{N-1} \\ F_{N-2} \end{bmatrix}$$

(b) Diagonalize $\mathbf{A}$ to show that

$$F_N = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{N-1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{N-1}$$

is an analytical expression for the Nth Fibonacci number.

Answer:

Note that $\mathbf{A}$ has the following eigenvalues and corresponding eigenvectors:

$$\left\{\lambda_1 = \frac{1+\sqrt{5}}{2},\ \lambda_2 = \frac{1-\sqrt{5}}{2}\right\} \qquad \left\{\vec{p}_1 = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix},\ \vec{p}_2 = \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}\right\}$$

And recall the 2×2 inverse formula:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Using these facts, first we diagonalize $\mathbf{A}$

$$\mathbf{P} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \qquad \mathbf{P}^{-1} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix}$$

$$\begin{aligned} \mathbf{A} &= \mathbf{PDP}^{-1} \\ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} &= \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}\left(\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix}\right) \end{aligned}$$

Then, we have that F_N is equal to the first element of $\mathbf{A}^{N-2}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$$\begin{aligned} F_N &= \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{N-2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}^{N-2}\left(\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix}\right)\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{bmatrix}\begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{N-2} & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^{N-2} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{bmatrix} \\ &= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{N-1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{N-1} \end{aligned}$$