PRINT your student ID: ________________________________________________

**3. System of Equations (4 points)**

Solve the following system of equations using Gaussian elimination. If there is no solution, explain why.

$$\begin{aligned} x+3y-z&=4 \\ 4x-y+2z&=8 \\ 2x-7y+4z&=-3 \end{aligned}$$

**4. Eig-dentical Eigenvalues (10 points)**

(a) (2 points) Consider the following matrix:

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ -1 & a \end{bmatrix}$$

Is $\vec{v} = \begin{bmatrix} 3 \\ -8 \end{bmatrix}$ in the column space of $\mathbf{A}$ when $a = 3$? Justify your answer in 1-2 sentences.

(b) (8 points) Solve for the value of $a$ that yields the **smallest** possible **identical** eigenvalues for the matrix $\mathbf{A}$.

## 5. Nullspace (8 points)

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 4 \\ 1 & 2 & 6 \\ 0 & 1 & x \end{bmatrix}$$

(a) (3 points) Find all the values for x such that **M** has a trivial nullspace (this means that the nullspace contains only the zero vector).

(b) (5 points) Find a value for $x$ such that it has a nontrivial nullspace and solve for the nullspace.

**6. Vectors and Bases and Spans, Oh My! (6 points)**

$$\mathbf{A} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ 5 \end{bmatrix} \right\}$$

$$\mathbf{B} = \left\{ \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix} \right\}$$

$$\mathbf{C} = \left\{ \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\mathbf{D} = \left\{ \begin{bmatrix} 2 \\ 4.5 \\ 5 \end{bmatrix}, \begin{bmatrix} 0.5 \\ 0 \\ 3.75 \end{bmatrix}, \begin{bmatrix} 1.5 \\ 4.5 \\ 1.25 \end{bmatrix} \right\}$$

(a) (2 points) Select from (A), (B), (C), (D) all sets of vectors above that *span* $\mathbb{R}^3$, and write their corresponding letter below. If none span $\mathbb{R}^3$, write "none".

(b) (2 points) Select from (A), (B), (C), (D) all sets of vectors above that *form a basis* for $\mathbb{R}^3$, and write their corresponding letter below. If none form a basis for $\mathbb{R}^3$, write "none".

(c) (2 points) Can a set that contains the zero vector ever be a basis for $\mathbb{R}^n$? Explain in one or two sentences why or why not.

**7. Proofs! Oh null! (10 points)**

(a) (3 points) Show that if $\vec{x} \in Null(\mathbf{A})$ then $\mathbf{A}\vec{x} + \mathbf{A}^2\vec{x} + \cdots + \mathbf{A}^n\vec{x} = \vec{0}$.

(b) (4 points) Suppose we have a square matrix $\mathbf{A}$ with full rank and a matrix $\mathbf{B}$. Show that $Null(\mathbf{AB}) = Null(\mathbf{B})$.

(c) (3 points) Conceptually, if a state-transition matrix has a non-trivial nullspace (i.e. non-zero) is the information about previous states preserved? Circle your answer, then give 1-2 sentences of justification.

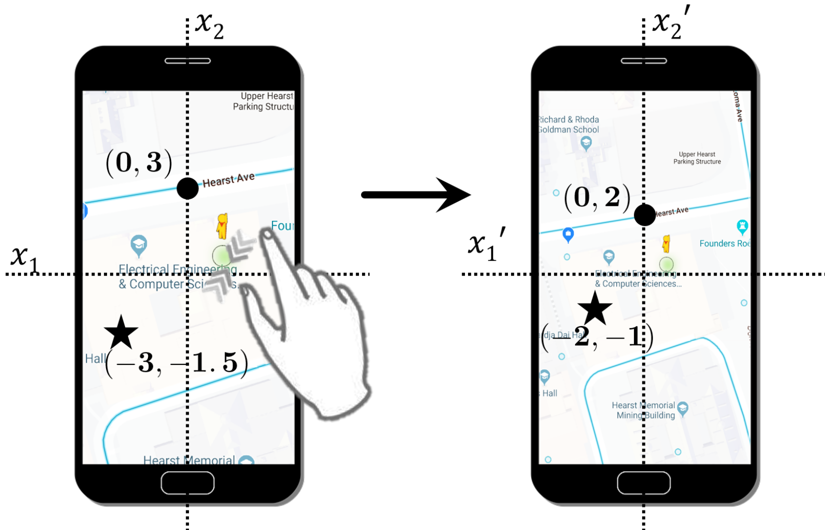
Yes No Sometimes

**8. Linear Algebra in a Pinch! (16 points)**

As you look at your smartphone to get Google Maps directions to TeaOne in Cory Hall, you realize that your smartphone is doing exactly what you do in class–linear algebra! As your fingers apply gestures to the screen, the phone is performing matrix transformations on the screen image.

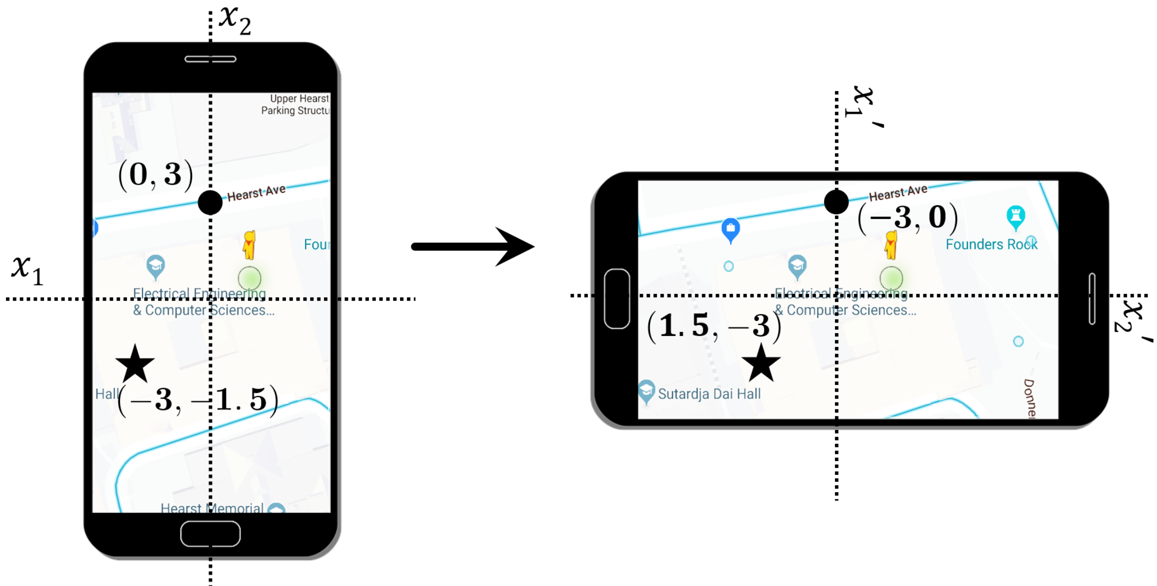
In the following parts, we represent coordinates before transformation as $\vec{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ and coordinates after transformation as $\vec{x'} = \begin{bmatrix} x_1' & x_2' \end{bmatrix}^T$. Although specific points are labeled, the transformation applies to all points in the $(x_1, x_2)$ plane.

(a) (5 points) You pinch the screen to zoom out, as shown in the figure below:



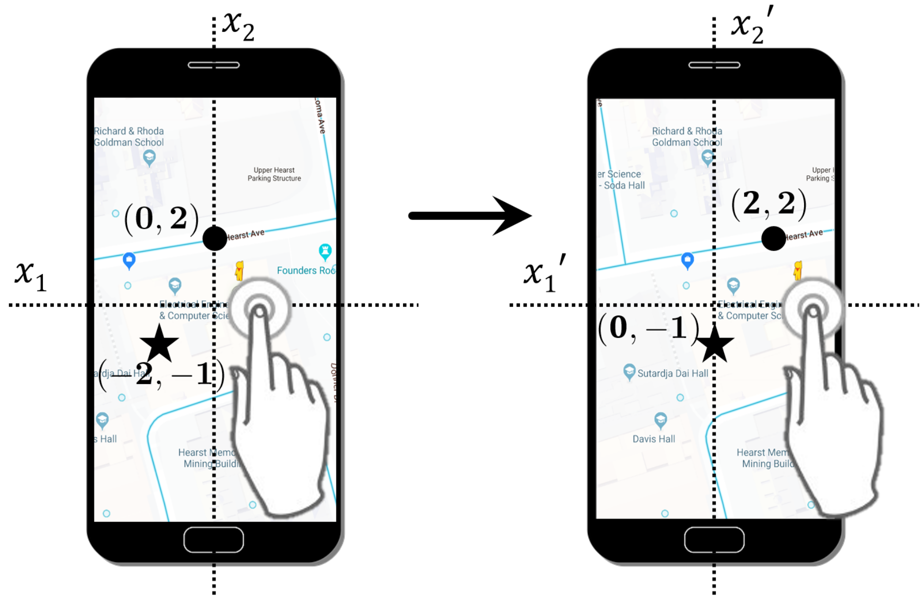

The point represented by a dot moves from $(0,3)$ to $(0,2)$, and the point represented by a star moves from $(-3,-1.5)$ to $(-2,-1)$. What is the transformation matrix, $\mathbf{A}$, such that $\vec{x'} = \mathbf{A}\vec{x}$ for this transformation?

(b) (5 points) Smartphones are smart, so if you rotate your phone, the map will reorient itself to make sure you can read it! That is, if you rotate your phone 90° clockwise, the map will rotate 90° counter-clockwise relative to the phone, as shown below.




The point represented by a dot moves from $(0,3)$ to $(-3,0)$, and the point represented by a star moves from $(-3,-1.5)$ to $(1.5,-3)$. What is the transformation matrix, $\mathbf{R}$, such that $\vec{x'} = \mathbf{R}\vec{x}$ for this transformation?

(c) (6 points) So far, we have only done transformations that involve rotation and scaling, but another key feature of Google Maps is that we can scroll laterally across a map, as shown below.




The point represented by a dot moves from $(0,2)$ to $(2,2)$, and the point represented by a star moves from $(-2,-1)$ to $(0,-1)$.

In the previous parts we were able to represent the map transformations in the form

$$\vec{x'} = \mathbf{A}\vec{x} + \vec{b}$$

(Previously $\vec{b} = \vec{0}$.) What are $\mathbf{A}$ and $\vec{b}$ for the scrolling operation above? Is this a linear transformation?

**9. Mining Population (20 points)**

There is a population of cryptocurrency miners. These miners are primarily interested in two coins: OskiCoin and BearCoin, and they switch between mining the two coins in a predictable way. Every week, 20% of OskiCoin miners switch to BearCoin and 30% of BearCoin miners switch to OskiCoin. The remaining miners keep mining the same coin for the following week.

Let $s_1[n]$ be the number of miners of OskiCoin on week $n$ and $s_2[n]$ be the number of miners of BearCoin on week $n$.

$$\vec{s}[n] = \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix}$$

(a) (2 points) Draw a well-labeled directed graph showing how the population of miners changes each week. Be sure to label each node and place appropriate weights on each edge.

(b) (2 points) Determine the state transition matrix $\mathbf{A}$.

(c) (5 points) Find the eigenvalues $(\lambda_1 \ldots \lambda_n)$ and eigenvectors $(\vec{v}_1 \ldots \vec{v}_n)$ of $\mathbf{A}$.

(d) (5 points) Express $\begin{bmatrix}5\\0\end{bmatrix}$ in the coordinate system of the eigenvectors, $\mathbf{V} = \{\vec{v}_1, \ldots, \vec{v}_n\}$.

(e) (6 points) If we start with 1000 miners mining OskiCoin and 0 miners mining BearCoin, then what will the steady state distribution of miners be?

**10. Bad Barometers (20 points)**

Two scientists, Alice and Bob, want to determine the air pressure at the bottom and the top of Mt. Diablo. They know that pressure changes linearly with altitude. We represent the air pressure at the top and bottom of the mountain as a vector:

$$\vec{p} = \begin{bmatrix} p_{bottom} \\ p_{top} \end{bmatrix}$$

Scientist Alice first takes a pressure measurement at the bottom of the mountain. Then she starts hiking up. 10% of the way up she she takes a second measurement. She knows she can invert her system of equations, so she turns around. We can represent Alice's measurements as

$$\vec{m}_a = \mathbf{A}_a\vec{p} \quad \text{where} \quad \mathbf{A}_a = \begin{bmatrix} 1 & 0 \\ 0.9 & 0.1 \end{bmatrix}$$

Scientist Bob also takes a pressure measurement at the bottom of the mountain. He starts hiking and is enjoying the view so he hikes 50% of the way to the top where he stops and takes a second measurement. We can represent Bob's measurements as

$$\vec{m}_b = \mathbf{A}_b\vec{p} \quad \text{where} \quad \mathbf{A}_b = \begin{bmatrix} 1 & 0 \\ 0.5 & 0.5 \end{bmatrix}$$

Unfortunately, both scientists have old barometers that don't work very well, so rather than measuring the true pressure, they measure the pressure plus some offset, $\vec{s}$. We define $\vec{q}$ to be the calculated pressure based on the inaccurate measurement:

$$\vec{m}_a + \vec{s}_a = \mathbf{A}_a\vec{q}_a \qquad\qquad \vec{m}_b + \vec{s}_b = \mathbf{A}_b\vec{q}_b$$

We are interested in the error, $\vec{e}$, which is the difference between the true pressure and the calculated pressure:

$$\vec{e}_a = \vec{q}_a - \vec{p} \qquad\qquad \vec{e}_b = \vec{q}_b - \vec{p}$$

(a) (4 points) Determine an expression for $\vec{e}_a$ as a function of $\mathbf{A}_a$ and $\vec{s}_a$.

(b) (6 points) Suppose that $\vec{s}_a = \vec{s}_b = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. This means that our scientists measure pressure that is 1 kPa too low in their first measurement and 1 kPa too high in their second measurement. (Both scientist have very similar old barometers, so they have the same offset for both measurements.)

Calculate Alice's error, $\vec{e}_a$, and Bob's error, $\vec{e}_b$.

(c) (6 points) Consider a general invertible, diagonalizable measurement matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ with eigenvalues $\lambda_1 \ldots \lambda_n$ and corresponding eigenvectors $\vec{v}_1 \ldots \vec{v}_n$, and a general offset vector $\vec{s} \in \mathbb{R}^n$. Show that

$$\vec{e} = \sum_{i=1}^{n} \frac{\alpha_i}{\lambda_i} \vec{v}_i$$

where $\alpha_1 \ldots \alpha_n$ are scalar values such that $\sum_{i=1}^{n} \alpha_i \vec{v}_i = \vec{s}$.

(d) (4 points) You calculate the eigenvalues and eigenvectors for $\mathbf{A}_a$ and $\mathbf{A}_b$ to be the following:

| $\mathbf{A}_a$ | $\mathbf{A}_b$ |
|---|---|
| $\lambda_1 = 1$ | $\lambda_1 = 1$ |
| $\lambda_2 = 0.1$ | $\lambda_2 = 0.5$ |
| $v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ | $v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ |

Based on the equation in Part (c) and the eigenvalues and eigenvectors above, which matrix ($\mathbf{A}_a$ or $\mathbf{A}_b$) is more sensitive to inaccurate measurements? In other words, which will have a larger error $\vec{e}$? Does this agree with the trend you saw in Part (b)?