5. Properties of Pump Systems.

(a). Using the information from the graph, we have $\begin{cases} \vec{x_1}[n+1] = \vec{x_1}[n] + \vec{x_2}[n] \\ \vec{x_2}[n+1] = 0. \end{cases}$

(b). Thus, $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$\boxed{(c)}$ For both initial states, $\vec{x}[1] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

In Universe 1, where $x_1[0] = 0.5$, $x_2[0] = 0.5$,

so $\vec{x}[1] = A\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$;

In Universe 2. where $x_1[0] = 0.3$, $x_2[0] = 0.7$

So similarly, $\vec{x} = A\vec{x}[0] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Thus, no matter what, the water levels at timestep 1 is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(d) No, I can't. This is because that as we proved, both initial states ($x_1[0] = x_2[0] = 0.5$ and $x_1[0] = 0.3$, $x_2[0] = 0.7$) lead to the same result at timestep 1, so I can't figure out the initial water levels.

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(e) No, I can't. Proof by Contradiction.

Proof: Assume, for a contradiction, that there exists a state transition matrix A^* such that two different initial state vectors lead to the same water levels/ state vectors at timestep k, and that I can recover the unique initial water levels $\vec{x}[0]$. Let $A^* \cdot \vec{x}[i] = \vec{x}[i+1]$

Since we can recover an unique initial water levels, so it means that A^* is invertible, and then we can recover $\vec{x}[0]$ from $\vec{x}[k]$ by multiplying it to A^{*-1} for k times. Now, consider the state vectors at timestep $(k-1)$, $\vec{x}[k-1]$. Since we have proved in the lecture notes that if M is an invertible matrix, then its inverse must be unique. Thus, there is only one state vector possible, $\vec{x}[k-1]$. Similarly, we can deduce this for each timestep's state vector.

Therefore, it's not possible to have two different initial state vectors.

Q.E.D.

f. Consider, for k reservoirs, the state vector at time n,

$$\vec{x}[n] = \begin{bmatrix} x_1[n] \\ x_2[n] \\ \cdots \\ x_k[n] \end{bmatrix}$$

Since we know that the entries of each column vector of the state transition matrix A sum to one, so this implies that for any reservoir i, $1 \le i \le k$, all of its water goes to reservoirs 1 through k. In other words, the amount of water collectively, $x_1[n] + x_2[n] + \cdots + x_k[n] = s$, would be preserved for timestep $(n+1)$.

Thus, let $\vec{x}[n+1] = \begin{bmatrix} x_1[n+1] \\ x_2[n+1] \\ \vdots \\ x_k[n+1] \end{bmatrix}$, so $x_1[n+1] + \cdots + x_k[n+1] = x_1[n] + \cdots + x_k[n] = s$.

which means that the total amount of water at timestep $(n+1)$ is still s for k reservoirs.

(When $k=3$, this generalization provides the case for the first half of the problem).

Q.E.D.