

1. Gaussian Elimination

Use Gaussian elimination to solve the following systems. Does a solution exist? Is it unique?

(a)

$$\begin{bmatrix} 2 & 0 & 4 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix}$$

$$\begin{array}{ccc} 2 & 0 & 4 \mid 6 \\ 0 & 1 & 2 \mid -3 \\ 1 & 2 & 0 \mid 3 \end{array} \implies \begin{array}{ccc} 1 & 0 & 2 \mid 3 \\ 0 & 1 & 2 \mid -3 \\ 0 & 1 & -1 \mid 0 \end{array} \implies \begin{array}{ccc} 1 & 0 & 2 \mid 3 \\ 0 & 1 & 2 \mid -3 \\ 0 & 0 & -3 \mid 3 \end{array}$$

So
 $x_3 = -1$
 $x_2 = -1$
 $x_1 = 5$

(b)

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$$

R₃: Subtract 2*R₁, Add R₂

R₁: Subtract R₂

=>

$$\begin{array}{ccc} 1 & 0 & 1 \mid 2 \\ 0 & 1 & 1 \mid 0 \\ 0 & 0 & 0 \mid 0 \end{array}$$

$$\begin{array}{l} x + z = 2 \rightarrow x = 2 - z \\ y + z = 0 \rightarrow y = -z \end{array}$$

So, $[x_1 \ x_2 \ x_3] = [-1, -1, 1] * \alpha + [2 \ 0 \ 0]$

(c)

$$\begin{bmatrix} 1 & 4 & 2 \\ 1 & 2 & 8 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

R₂, R₃: each subtract R₁

$$\begin{array}{ccc} 1 & 4 & 2 \mid 2 \\ 0 & -2 & 6 \mid -2 \text{ then R}_2: \text{Subtract } 2 * \text{R}_3 \implies \\ 0 & -1 & 3 \mid 1 \end{array}$$

$$\begin{array}{ccc} 1 & 0 & 14 \mid 6 \\ 0 & -1 & 3 \mid 1 \implies x = 6 - 14z, y = 3z - 1 \\ 0 & 0 & 0 \mid 0 \end{array}$$

$$\begin{array}{ccc} 1 & 4 & 2 \mid 2 \\ 0 & 0 & 0 \mid -4 \text{ —No Solution} \\ 0 & -1 & 3 \mid 1 \end{array}$$

$[x_1 \ x_2 \ x_3] =$
 $[-14, 3, 1] * \alpha + [6, -1, 0]$

(d)
$$\begin{array}{ccc|c} 2 & 0 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & -2 & -2 & -6 \end{array} \implies$$

$$\begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$$

$$2x + z = 1$$

$$y + z = 3 \implies$$

$$x = 1/2 - 1/2 * z$$

$$y = 3 - z$$

$$[x_1 \ x_2 \ x_3] = [-1/2, -1, 1] * \text{alpha} + [1/2 \ 3 \ 0]$$

(e)

R_1: subtract R_2, R_1

$$\begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$$

$$0 \ 0 \ 0 \ | \ -1$$

xxxxxxx

→ No solution

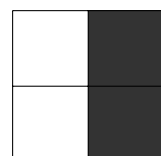
2. Finding The Bright Cave

Nara the one-handed druid and Kody the one-handed ranger find themselves in dire straits. Before them is a cliff with four cave entrances arranged in a square: two upper caves and two lower caves. Each entrance emits a certain amount of light, and the two wish to find exactly the amount of light coming from each cave. Here's the catch: after contracting a particularly potent strain of ghoulish fever, our intrepid heroes are only able to see the total intensity of light before them (so their eyes operate like a single-pixel camera). Kody and Nara are capable adventurers, but they don't know any linear algebra – and they need your help.

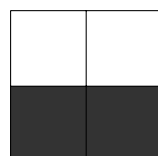
Kody proposes an imaging strategy where he uses his hand to completely block the light from two caves at a time. He is able to take measurements using the following four masks (black means the light is blocked from that cave):

Cave Labels

x_1	x_2
x_3	x_4



Measurement 1



Measurement 2



Measurement 3



Measurement 4

Figure 1: Four image masks.

1 0 1 0
1 1 0 0
0 1 0 1
0 0 1 1

- Let \vec{x} be the four-element vector that represents the magnitude of light emanating from the four cave entrances. Write a matrix \mathbf{K} that performs the masking process in Figure 1 on the vector \vec{x} , such that $\mathbf{K}\vec{x}$ is the result of the four measurements.
- Does Kody's set of masks give us a unique solution for all four caves' light intensities? Why or why not? **No, it doesn't. Using Gaussian Elimination, we'll reach a row of 0s.**
- Nara, in her infinite wisdom, places her one hand diagonally across the entrances, covering two of the cave entrances. However, her hand is not wide enough, letting in 50% of the light from the caves covered and 100% of the light from the caves not covered. The following diagram shows the percentage of light let through from each cave:

50%	100%
100%	50%

Does this additional measurement give them enough information to solve the problem? Why or why not?

Yes, it does.

Because we can use Gaussian Elimination to calculate out all the information necessary.