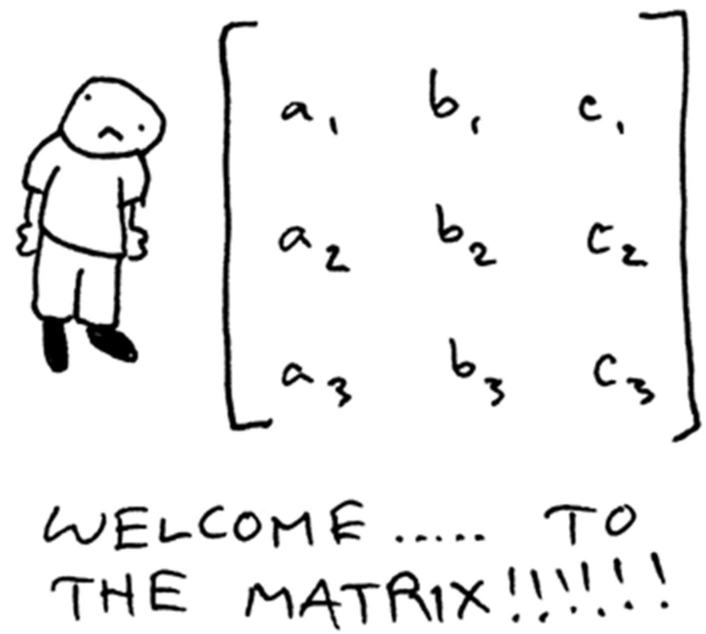
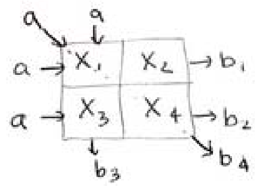


EE16A

Designing Information Devices and Systems I

Last time: Tomography

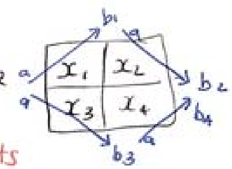


$$
\begin{aligned}
a x_1 + a x_2 \qquad\qquad &= b_1 \\
a x_3 + a x_4 &= b_2 \\
a x_1 \qquad + a x_3 \qquad &= b_3 \\
a x_1 \qquad\qquad + a x_4 &= b_4
\end{aligned}
$$

Linear System of Equations.

<u>Some good questions:</u>

1) Why not take diagonal measurements like



?

Answer: Too easy! So it's cheating.
Also, won't work for bigger objects
(like 3x3 milk jugs problem in notes.

If we did that, what would system of equations look like?

$a x_1 = b_1 \rightarrow x_1 = b_1/a$
$a x_2 = b_2$ ⋮
$a x_3 = b_3$ ⋮
$a x_4 = b_4$ ⋮

Essentially, these are direct measurements. Good if you can do it!

2) Why is $a \rightarrow \boxed{x_1 | x_2} \rightarrow b = \underbrace{a x_1 + a x_2}_{\text{adding}}$, instead of $b = a \underbrace{x_1 x_2}_{\text{multiplying}}$?

Answer: This is our model – we decided to use additive model.

3) Why does $\rightarrow \boxed{x} \rightarrow$ give same measurement as $\searrow \boxed{x} \searrow$ and not scaled by $\sqrt{2}$ because it goes along diagonal?

Answer: This is just our model!
* See "Beer's Law" for more details on physics of tomography to understand why the model is physically correct.

Vectors are arrays of numbers

$$\vec{\mathsf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

column vector

$$\mathsf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_N \end{bmatrix}$$

row vector

What are the dimensions?

N-dimensional vector

A matrix is a rectangular array of numbers

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & \boxed{A_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix}$$

This is element (component) 2n of the matrix

What are the dimensions of A?

m rows and **n columns** means it is a **m x n** matrix

Some special types of matrices

zero matrix

$$\vec{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

diagonal

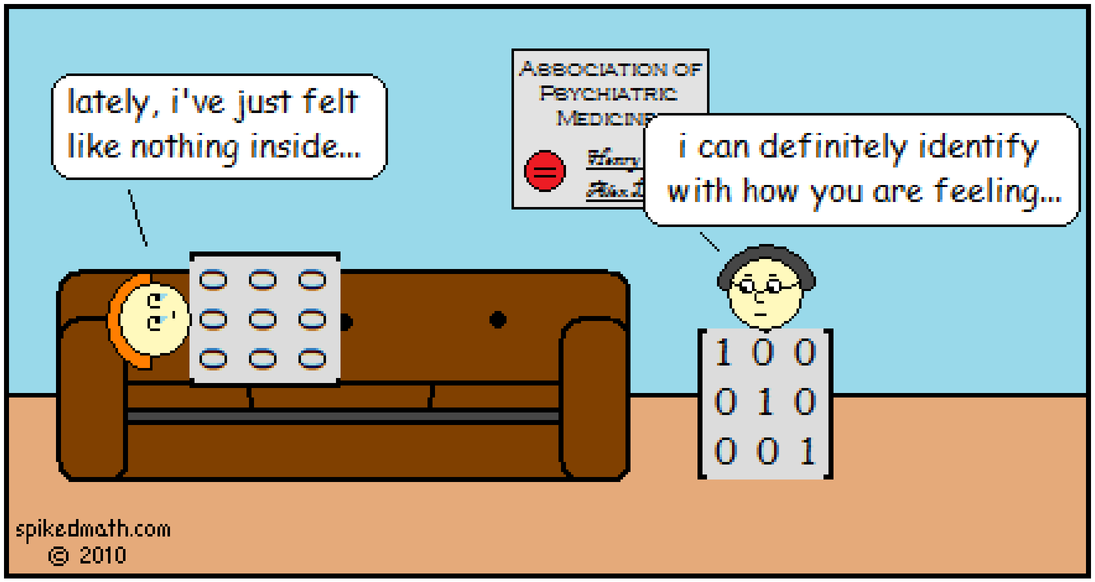
$$\mathbf{A} = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

identity matrix

$$\mathrm{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

upper triangular

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$


lately, i've just felt like nothing inside...
Association of Psychiatric Medicine
i can definitely identify with how you are feeling...
1 0 0
0 1 0
0 0 1
spikedmath.com
© 2010

Ways of representing linear systems of equations

$$
\begin{aligned}
ax_1 + ax_2 \phantom{{}+ax_3+ax_4} &= b_1 \\
ax_3 + ax_4 &= b_2 \\
ax_1 + ax_3 &= b_3 \\
ax_2 + ax_4 &= b_4
\end{aligned}
$$

Can also be represented as:

$$
\left[\begin{array}{cccc|c}
a & a & 0 & 0 & b_1 \\
0 & 0 & a & a & b_2 \\
a & 0 & a & 0 & b_3 \\
0 & a & 0 & a & b_4
\end{array}\right]
$$

Or:

$$
\begin{bmatrix} a & a & 0 & 0 \\ 0 & 0 & a & a \\ a & 0 & a & 0 \\ 0 & a & 0 & a \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}
$$

Or:

$$Ax = b$$

The lazy way

Today: Solving a linear system of equations

First, write in simple form:

1 $x + 4y = 6$

2 $-y + 2x = 3$

$\Longleftrightarrow$

$$\left[\begin{array}{cc|c} 1 & 4 & 6 \\ 2 & -1 & 3 \end{array}\right]$$

Don't forget to put x in one column and y in another

Now solve it. How?

Start plugging equations into each other….
See what happens?

e.g.

1) Solve 1 for x and plug into 2

2) 4 x 2 + 1

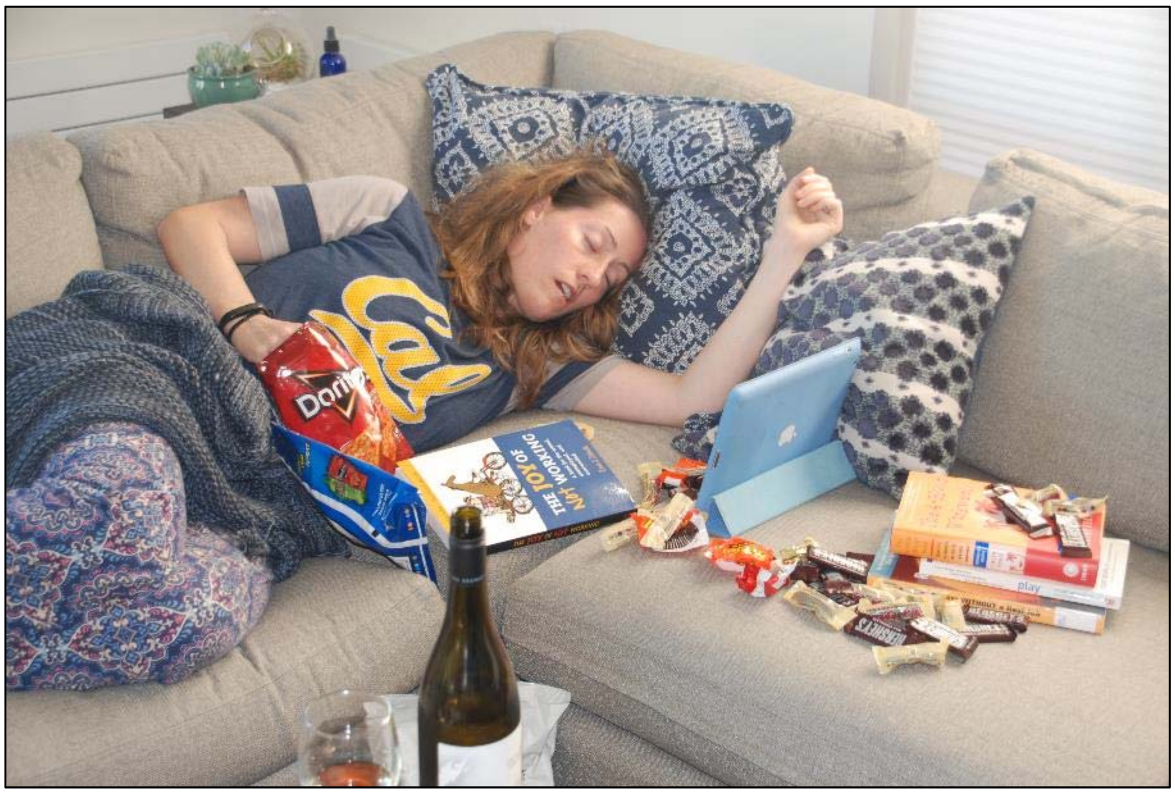
1 $x + 4y = 6$
2 $2x - y = 3$ Solving example $4 \times$ 2 $+$ 1 $\rightarrow$

$$\begin{array}{r} 8x - 4y = 12 \\ + \quad x + 4y = 6 \\ \hline 9x = 18 \end{array}$$

$\boxed{x = 2}$

plug into 1:
$(2) + 4y = 6$
$4y = 4$
$\boxed{y = 1}$

GOAL: to develop a <u>systematic</u> way of solving systems of equations with clear rules that *can be done by a computer*



(then I can be even lazier)

Gaussian Elimination for solving a linear system of equations

- Specifies the order in which you combine equations (rows) to “eliminate” (make zero) certain elements of the matrix

- Goal is to transform your system of equations into ***upper triangular***

$$\mathbf{A} = \begin{bmatrix} \boxed{a_{11}} & a_{12} & a_{13} & a_{14} \\ 0 & \boxed{a_{22}} & a_{23} & a_{24} \\ 0 & 0 & \boxed{a_{33}} & a_{34} \\ 0 & 0 & 0 & \boxed{a_{44}} \end{bmatrix}$$

diagonal elements are called *pivots*

How I solved it:

① $x + 4y = 6$

② $2x - y = 3$

$$② - 2① \longrightarrow 2x - y - 2x - 8y = 3 - 12$$

$$-9y = -9$$

$$\boxed{y = 1}$$

$\boxed{\text{use Eq. ① to eliminate } x \text{ from ②}}$

Then plug it in,

say to ①: $x + 4(1) = 6$

$$\boxed{x = 2}$$

This is <u>Gaussian Elimination</u>!

But order matters..

Go back to simple form:

$$\left[\begin{array}{cc|c} 1 & 4 & 6 \\ 2 & -1 & 3 \end{array}\right] \xrightarrow{②-2①}$$

Do this operation on simple form:

$$\left[\begin{array}{cc|c} 1 & 4 & 6 \\ 0 & -9 & -9 \end{array}\right]$$

Trying to 'eliminate' to make zeros so it is UPPER TRIANGULAR

UPPER TRIANGULAR means you can read off bottom row variable:

$$-9y = -9 \rightarrow \boxed{y=1}$$

Then plug into next row:

$$x + 4y = 6$$

$$x + 4(1) = 6 \rightarrow \boxed{x=2}$$

3D example

Could you solve this graphically? Intersection of 3 planes. Good luck with that!

$$\begin{aligned} 2y + z &= 1 & ① \\ 2x + 6y + 4z &= 10 & ② \\ x - 3y + 3z &= 14 & ③ \end{aligned} \longrightarrow \left[\begin{array}{ccc|c} 0 & 2 & 1 & 1 \\ 2 & 6 & 4 & 10 \\ 1 & -3 & 3 & 14 \end{array}\right]$$

Now follow the same procedure:

→ use 1st eqn to eliminate 1st variable from 2nd eqn.

Hmm.. But it's a ZERO! What to do?

We are allowed to swap rows, so swap Row 1 & 2

giving
$$\left[\begin{array}{ccc|c} 2 & 6 & 4 & 10 \\ 0 & 2 & 1 & 1 \\ 1 & -3 & 3 & 14 \end{array}\right]$$

Now what? divide 1st row by 2

Why is that allowed? doesn't change equation.

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 2 & 1 & 1 \\ 1 & -3 & 3 & 14 \end{array}\right]$$

Now what? Subtract Row 1 from Row 3

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 2 & 1 & 1 \\ 0 & -6 & 1 & 9 \end{array}\right]$$

zeros! Good

What to make zero (eliminate) first? the -6

Which row should I use? Row 2 because if we use Row 1 we'll lose a zero (= BAD!)

PIVOTS.

upper triangular ☺

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 4 & 12 \end{array}\right] \rightarrow 4z = 12 \rightarrow \boxed{z = 3}$$

To be systematic, we should instead make matrix manipulation so that 3rd row pivot is 1:

divide Row 3 by 4
$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

ELIMINATION PART

Now we read off $z = 3$

For the "Plug in" part, now we need to back-substitute upwards from bottom:

e.g. plug $z = 3$ into row 2 → $2y + 1(z) = 1$

$2y + 3 = 1$

$\boxed{y = -1}$

then plug $z = 3$ and $y = -1$ into Row 1

$x + 3y + 2z = 5$

$x + 3(-1) + 2(3) = 5$

$x = 5 + 3 - 6 = \boxed{2} = x$

To translate this into matrix manipulations:

Now how many letters have I saved myself from writing?!?

(Row 2 - Row 3)
$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

divide Row 2 by 2
$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$
Now we read off $y = -1$ ✓

Finally, (Row 1 − 3 Row 2 − 2 Row 3)
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$
Read off $x = 2$ ✓

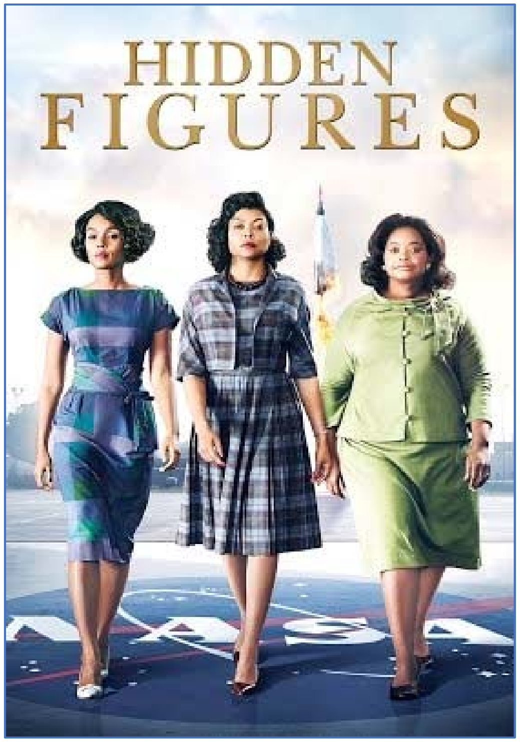
What is allowed in Gaussian elimination?

- Linear combinations of equations (adding scalar multiples of rows to other rows)
- Multiply a row by a scalar
- Swap rows

Goals of Gaussian Elimination algorithm

- Equation with i^{th} variable in the i^{th} row
- Coefficient of the i^{th} variable in the i^{th} row becomes 1
- For rows $j=i+1$ and higher, subtract row i times the entry in (j,i) to cancel variable i

Gaussian elimination was part of the work of human computers



What might be the variables/measurements in calculating rocket trajectories?

Position, direction of motion, tilt, power/thrust, weight…

Will it always work?

Example:

$x + 4y = 6$ ①
$2x + 8y = 12$ ②

$2① = ②$
No new info!

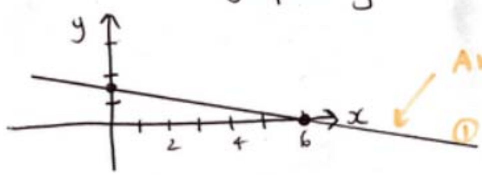
$$\left[\begin{array}{cc|c} 1 & 4 & 6 \\ 2 & 8 & 12 \end{array}\right] \xrightarrow[\text{Gauss. Elim.}]{\text{try}} \left[\begin{array}{cc|c} 1 & 4 & 6 \\ 0 & 0 & 0 \end{array}\right]$$

pivot is zero!

$0x + 0y = 0$

Can you solve this? No. (1 equation, 2 unknowns)

Let's look at it graphically:



Any point on this line is a solution.

Questions:

Is there a situation where infinitly many solutions is a good thing?
↳ Yes. In design, gives flexibility.

If you can't solve → UNDER DETERMINED, what should you do?
↳ take more meas!?

Example 2:

$x + 4y = 6$
$2x + 8y = 10$

$$\left[\begin{array}{cc|c} 1 & 4 & 6 \\ 2 & 8 & 10 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 4 & 6 \\ 0 & 0 & -2 \end{array}\right]$$

Pivot 0

Inconsistent
NO SOLUTION!
$0 = -2$
cannot solve!

In both cases, the pivot being zero was a red flag!

Possible situations

- Unique solution
- Infinitely many solutions (underdetermined)
- No solution (inconsistent)

Is it possible to have exactly 2 solutions?

No. consider graphically: two lines cannot intersect in exactly two places

Cats vs. Dogs

These measurements are different linear combinations of two images.

measurements

Can you guess what the measurements are?

Top: 0.6 (dog) + 0.4 (cat)
Top: 0.6 (cat) + 0.4 (dog)

Can I solve for both images from just these two linearly combined images? Just one? None? How many images do I need minimum?

Two images is enough if they're linearly independent at each pixel!

300x300 pixels = 90,000

Cats vs. Dogs

measurements

0.6  +0.4  = 

0.4  +0.6  = 

What are the ideal measurements?

Depends. Maybe direct measurements of cat and dog…

Cats vs. Dogs: Direct measurements

measurements

1  +0  = 

0  +1  = 

Very easy to solve!