4. Segway Tours

(a). Since $\vec{x}[n+1] = A\vec{x}[n] + \vec{b}u[n]$, so $\vec{x}[1] = \boxed{A\vec{x}[0] + \vec{b}u[0]}$

(b). Similarly, $\vec{x}[2] = A\vec{x}[1] + \vec{b}u[1] = A(A\vec{x}[0] + \vec{b}u[0]) + \vec{b}u[1]$

so, $\boxed{\vec{x}[2] = A^2\vec{x}[0] + A\vec{b}u[0] + \vec{b}u[1]}$

and so $\vec{x}[3] = A\vec{x}[2] + \vec{b}u[2] = A(A^2\vec{x}[0] + A\vec{b}u[0] + \vec{b}u[1]) + \vec{b}u[2]$

so, $\boxed{\vec{x}[3] = A^3\vec{x}[0] + A^2\vec{b}u[0] + A\vec{b}u[1] + \vec{b}u[2]}$

and $\vec{x}[4] = A\vec{x}[3] + \vec{b}u[3] = A(A^3\vec{x}[0] + A^2\vec{b}u[0] + A\vec{b}u[1] + \vec{b}u[2]) + \vec{b}u[3]$

so, $\boxed{\vec{x}[4] = A^4\vec{x}[0] + A^3\vec{b}u[0] + A^2\vec{b}u[1] + A\vec{b}u[2] + \vec{b}u[3]}$

(c). Thus, we can derive that:

$$\vec{x}[N] = A^N\vec{x}[0] + A^{N-1}\vec{b}u[0] + A^{N-2}\vec{b}u[1] + \cdots + A^2\vec{b}[N-3] + A\vec{b}[N-2] + \vec{b}u[N-1]$$

(d). Since we have that $\vec{x}[2] - A^2\vec{x}[0] = A\vec{b}u[0] + \vec{b}u[1]$.

and that we wish to reach $\vec{x}_f = \vec{0}$ in two time steps, so that $\vec{x}[2] = \vec{0}$.

So we have a linear equation to plug into iPython notebook, and after Gaussian Elimination, we obtain:

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

Since we have a row of 0s with the right side being $1 \neq 0$, so there's no solution, which means that $\boxed{\text{No, I can't}}$ reach $\vec{x}_f$ in two time steps.

(e) Similarly, since we have that $\vec{x}[3] = A^3\vec{x}[0] + A^2\vec{b}u[0] + A\vec{b}u[1] + \vec{b}u[2]$

and that we wish to have $\vec{x}[3] = \vec{0}$, so $A^2\vec{b}u[0] + A\vec{b}u[1] + \vec{b}u[2] = -A^3\vec{x}[0]$

Using iPython notebook again, after Gaussian Elimination, we have:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Again, since we have a row of 0s with the right side being $1 \neq 0$, so there's no solution, which means that $\boxed{\text{No, I can't}}$ again.

(f). Similarly, we have: $\vec{x}[4] = A^4\vec{x}[0] + A^3\vec{b}u[0] + A^2\vec{b}u[1] + A\vec{b}u[2] + \vec{b}[3]$.

with $\vec{x}[4] = \vec{0}$, and using iPython notebook to solve by Gaussian Elimination, we have:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -57.054 \\ 0 & 1 & 0 & 0 & 15.915 \\ 0 & 0 & 1 & 0 & -28.298 \\ 0 & 0 & 0 & 1 & 4.191 \end{array}\right]$$

Thus, $\boxed{\text{Yes, I can.}}$ with $u[0] = -57.054$, $u[1] = 15.915$

$u[2] = -28.298$, $u[3] = 4.191$.

(g). As found and stated in part f. via Gaussian Elimination (with iPython notebooks). the control inputs are:

$$\boxed{\begin{cases} u[0] = -13.249 \\ u[1] = 23.733 \\ u[2] = -11.572 \\ u[3] = 1.465 \end{cases}}$$

Verified by the simulation.

(h). The condition is that span$\{\vec{b}, A\vec{b}, A^2\vec{b}, \cdots, A^{N-1}\vec{b}\}$ needs to contain $-A^N\vec{x}[0]$.

Since we have for $\vec{x}[N] = A^N\vec{x}[0] + A^{N-1}\vec{b}u[0] + A^{N-2}\vec{b}u[1] + \cdots + A\vec{b}u[N-2] + \vec{b}u[N-1]$

and we wish to have $\vec{x}[N] = \vec{x}_f = \vec{0}$.

Thus, we have.
$$\begin{bmatrix} | & | & \cdots & | & | \\ A^{N-1}\vec{b} & A^{N-2}\vec{b} & \cdots & A\vec{b} & \vec{b} \\ | & | & \cdots & | & | \end{bmatrix} \cdot \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[N-2] \\ u[N-1] \end{bmatrix} = \begin{bmatrix} | \\ -A^N\vec{x}[0] \\ | \end{bmatrix}$$

which, for this to have solution, we need to have that

$$\boxed{\text{span}\{\vec{b}, A\vec{b}, \cdots, A^{N-1}\vec{b}\} \text{ contains } (-A^N x[0])}$$

(i). Similarly. this time we have: $A^{N-1}\vec{b}u[0] + \cdots + A\vec{b}u[N-2] + \vec{b}u[N-1] = \vec{x}[N] - A^N\vec{x}[0]$.

Since $\vec{x}[N]$ is any valid state vector (being 4×1 vector), so this means that $(\vec{x}[N] - A^N\vec{x}[0])$ could be any vector in $\mathbb{R}^{4\times1}$.

which means that, since
$$\begin{bmatrix} | & | & \cdots & | & | \\ A^{N-1}\vec{b} & A^{N-2}\vec{b} & \cdots & A\vec{b} & \vec{b} \\ | & | & \cdots & | & | \end{bmatrix} \cdot \begin{bmatrix} u[0] \\ \vdots \\ u[N-2] \\ u[N-1] \end{bmatrix} = \begin{bmatrix} | \\ \vec{x}[N] - A^N\vec{x}[0] \\ | \end{bmatrix}$$

So this implies that $\boxed{\text{span}\{\vec{b}, A\vec{b}, \cdots, A^{N-1}\vec{b}\} = (\mathbb{R}^4, \mathbb{R}).}$