EECS 16A Designing Information Devices and Systems I
Fall 2018 Homework 4

This homework is due September 21, 2018, at 23:59.
Self-grades are due September 25, 2018, at 23:59.

Submission Format
Your homework submission should consist of **two** files.

- `hw4.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.

  If you do not attach a PDF of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible.

- `hw4.ipynb`: A single IPython notebook with all of your code in it.

  In order to receive credit for your IPython notebook, you must submit both a "printout" and the code itself.

Submit the file to the appropriate assignment on Gradescope.

1. Finding Null Spaces

(a) Consider the column vectors of any 3×5 matrix. What is the maximum possible number of linearly independent column vectors?

(b) Someone performed Gaussian elimination and got the following upper triangular matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & -2 & 3 \\ 0 & 0 & 2 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a set of vectors which span the column space of $\mathbf{A}$. How many unique vectors are required to span the column space of $\mathbf{A}$? (This is the dimension of the column space of $\mathbf{A}$)

(c) Recall that for every vector $\vec{x}$ in the null space of $\mathbf{A}$, we have $\mathbf{A}\vec{x} = \vec{0}$. The dimension of a the null space is the minimum number of vectors needed to span it. Find vectors that span the null space of $\mathbf{A}$ (the matrix in the previous part). What is the dimension of the null space of $\mathbf{A}$? Use the same $\mathbf{A}$ from part b.

(d) (**Practice**) Now consider the new matrix, B, which is related to A,

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \\ -2 & -2 & 0 \\ 3 & 2 & 0 \end{bmatrix}$$

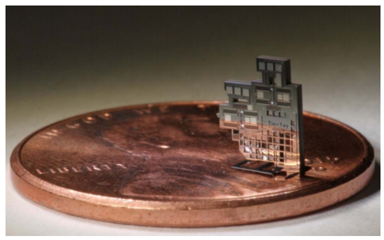
Find a set of vectors which span the column space of $\mathbf{B}$. How many unique vectors are required to span the column space of $\mathbf{B}$?

(e) Find vector(s) that span the null space of the following matrix:

$$\mathbf{C} = \begin{bmatrix} 2 & -4 & 4 & 8 \\ 1 & -2 & 3 & 6 \\ 2 & -4 & 5 & 10 \\ 3 & -6 & 7 & 14 \end{bmatrix}$$

2. Technical Issues

After a long and arduous search involving dozens of e-mails and way too many copies of your résumé, you've been offered a position with a research lab working on microelectromechanical systems (MEMS)!



Source: Contreras et al., *Transducers* (2017)

Your group has just starting working with another university, and your collaborators have sent you the design file of what they think is the next big break in microrobotics. The design file contains the layout information of the devices you're hoping to make; that is, it contains the shapes and polygons that will be fabricated in your real devices. **The design is a polygon defined by the coordinates of its corners (A,B,C,D)**. Unfortunately, the designs they sent don't make sense when you open them! When they see the rhombus in Figure 1

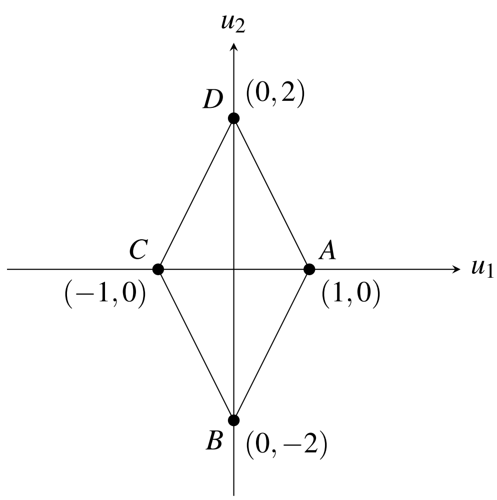


Figure 1: Collaborators' view (Basis: $\{\vec{u}_1, \vec{u}_2\}$)

you see the parallelogram in Figure 2 instead!

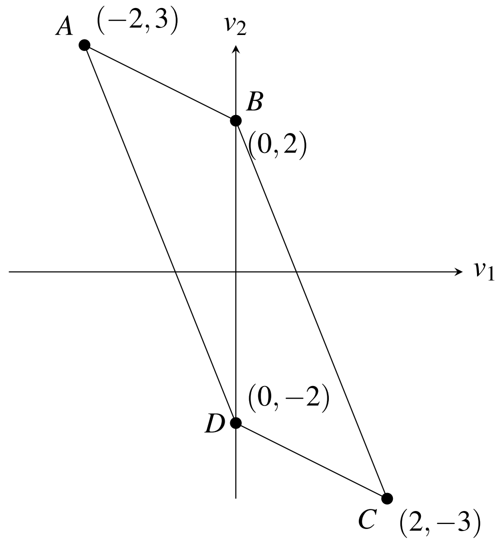


Figure 2: Your view (Basis: $\{\vec{v}_1, \vec{v}_2\}$)

(a) After some thought, you conclude that there must be a linear transformation between your view and your collaborators', i.e. you're viewing things in different coordinate systems. More specifically, your collaborators are drawing their designs using the basis $\{\vec{u}_1, \vec{u}_2\}$, and you're viewing in the basis $\{\vec{v}_1, \vec{v}_2\}$.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Using point A as an example, $(1,0)$ to your collaborators is the equivalent of saying $\vec{a} = 1\vec{u}_1 + 0\vec{u}_2$. Similarly, $(-2,3)$ in your basis is the equivalent of saying $\vec{a} = -2\vec{v}_1 + 3\vec{v}_2$. You want to modify the design files you receive so you can view the rest of the shapes as their senders intended. Find the basis vectors $\vec{u}_1$ and $\vec{u}_2$ your collaborators used.

(b) After making some modifications to your collaborators' designs, you now want to send them back for review. Unfortunately, their linear algebra is extremely rusty, so the task falls to you to convert from your encoding format in basis $\{\vec{v}_1, \vec{v}_2\}$ to their formatting basis $\{\vec{u}_1, \vec{u}_2\}$. Find the matrix $\mathbf{A}_{V \to U}$ which, when applied to your coordinate system, returns the design in your collaborators' coordinate system. In other words, find $\mathbf{A}_{V \to U}$ such that

$$\mathbf{A}_{V \to U} \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{bmatrix}$$

(c) Just as you fixed this issue, the company providing the software for design viewing introduced a feature (not a bug!) to only your version of the software. You know you're supposed to be seeing the left-leaning parallelogram in Figure 2, but now you see a right-leaning parallelogram in Figure 3! Once again your superior linear algebra skills come to the rescue, and you realize that your basis vectors have been changed from $\{\vec{v}_1, \vec{v}_2\}$ to a new set $\{\vec{w}_1, \vec{w}_2\}$. Solve for the new basis vectors $\{\vec{w}_1, \vec{w}_2\}$ your software now uses.

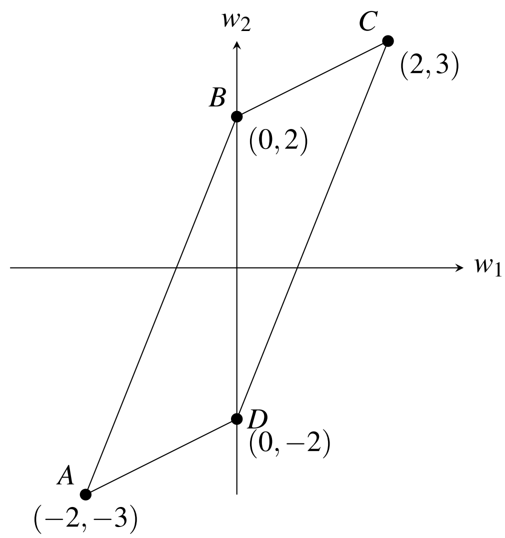


Figure 3: Your new view (Basis: $\{\vec{w}_1, \vec{w}_2\}$)

(d) Repeat part (b) for your new vectors $\{\vec{w}_1, \vec{w}_2\}$. That is, find $\mathbf{A}_{W\to U}$ such that

$$\mathbf{A}_{W\to U}\begin{bmatrix} | & | \\ \vec{w}_1 & \vec{w}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{bmatrix}$$

(e) **(PRACTICE)** Using your answers from the previous parts, show that

$$\mathbf{A}_{W\to U} = \begin{bmatrix} | & | \\ \vec{w}_1 & \vec{w}_2 \\ | & | \end{bmatrix}^{-1} \begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{bmatrix} \begin{bmatrix} | & | \\ \vec{w}_1 & \vec{w}_2 \\ | & | \end{bmatrix}$$

3. Traffic Flows

Your goal is to measure the flow rates of vehicles along roads in a town. However, it is prohibitively (too) expensive to place a traffic sensor along every road. You realize, however, that the number of cars flowing into an intersection must equal the number of cars flowing out. You can use this "flow conservation" to determine the traffic along all roads in a network by only measuring flow along only some roads. In this problem, we will explore this concept.

(a) Let's begin with a network with three intersections, A, B and C. Define the flow t_1 as the rate of cars (cars/hour) on the road between B and A, flow t_2 as the rate on the road between C and B, and flow t_3 as the rate on the road between C and A.

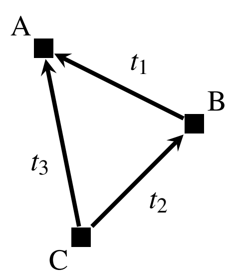


Figure 4: A simple road network.

(Note: The directions of the arrows in the figure are the way that we define positive flow by convention. For example, if there were 100 cars per hour traveling from A to C, then $t_3 = -100$.)

We assume the "flow conservation" constraints: the net number of cars per hour flowing into each intersection is zero. For example at intersection B, we have the constraint $t_2 - t_1 = 0$. The full set of constraints (one per intersection) is:

$$\begin{cases} t_1 + t_3 = 0 \\ t_2 - t_1 = 0 \\ -t_3 - t_2 = 0 \end{cases}$$

As mentioned earlier, we can place sensors on a road to measure the flow through it, but we have a limited budget, and we would like to determine all of the flows with the smallest possible number of sensors.

Suppose for the network above we have one sensor reading, $t_1 = 10$. Can we figure out the flows along the other roads? (That is, the values of t_2 and t_3). If we can, find the values of t_2 and t_3.

(b) Now suppose we have a larger network, as shown in Figure 5.

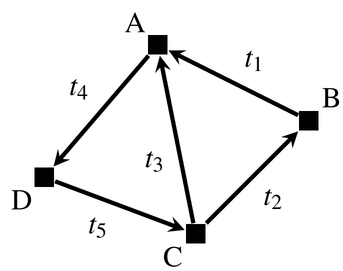


Figure 5: A larger road network.

We would again like to determine the traffic flows on all roads, using measurements from some sensors. A Berkeley student claims that we need two sensors placed on the roads AD (measuring t_4) and BA (measuring t_1). A Stanford student claims that we need two sensors placed on the roads CB (measuring t_2) and BA (measuring t_1). Write out the system of linear equations that represents this flow graph. Is it possible to determine all traffic flows, $\left[t_1, t_2, t_3, t_4, t_5\right]^T$, with the Berkeley student's suggestion? How about the Stanford student's suggestion?

(c) We would like a more general way of determining the possible traffic flows in a network. Suppose we write the traffic flow on all roads as a vector $\vec{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix}$. As a first step, let us try to write all the flow conservation constraints (one per intersection) as a matrix equation.

Construct a 4×5 matrix $\mathbf{B}$ such that the equation $\mathbf{B}\vec{t} = \vec{0}$

$$\begin{bmatrix} & & \mathbf{B} & & \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

represents the flow conservation constraints for the network in Figure 5.

Hint: Each row is the constraint of an intersection. You can construct **B** using only $0, 1$, and -1 entries. This matrix is called the **incidence matrix.** What constraint does each column of **B** represent?

(d) Again, suppose we write the traffic flow on all roads as a vector $\vec{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix}$. Then, determine the subspace of all valid traffic flows for the network of Figure 5. Specifically, express this space as the span of two linearly independent vectors.

Hint: Use the claim of the Berkeley student in part (b). Justify why you can use their claim. Then write all valid flows as a vector in terms of $t_1 = \alpha$ and $t_4 = \beta$.

(e) Notice that the set of all vectors $\vec{t}$ that satisfy $\mathbf{B}\vec{t} = \vec{0}$ is exactly the null space of the matrix **B**. That is, we can find all valid traffic flows by computing the null space of **B**. Use Gaussian elimination to determine the dimension of the null space of **B** and compute a basis for the null space. Does this match your answer to part (d)?

Challenge (optional): Can you interpret the dimension of the null space of **B** for the road networks of Figure 4 and Figure 5?

(f) Now let us analyze more general road networks. Say there is a road network graph G, with incidence matrix $\mathbf{B}_G$. If $\mathbf{B}_G$ has a k-dimensional null space, does this mean measuring the flows along ***any*** k **roads** is always sufficient to recover the exact flows? Prove or give a counterexample.

Hint: Consider the Stanford student from part (b).

(g) Let G be a network of n roads with the incidence matrix $\mathbf{B}_G$, which has a k-dimensional null space. We would like to characterize exactly when it is sufficent to measure a set of k roads to recover the exact flow along all roads.

To do this, it will help to generalize the problem and consider measuring *linear combinations* of flows. Let t_i be the flow on one road. We measure some linear combination of t_i's or $m_0 \cdot t_0 + m_1 \cdot t_1 + \cdots + m_n \cdot t_n$. Now we measure many of these linear combinations, which we will represent using matrix vector multiplication. Then, making k measurements is equivalent to observing the vector $\mathbf{M}\vec{t}$ for some $k \times n$ "measurement matrix" **M**.

For example, for the network of Figure 5, the measurement matrix corresponding to measuring t_1 and t_4 (as the Berkeley student suggests) is:

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Similarly, the measurement matrix corresponding to measuring t_1 and t_2 (as the Stanford student suggests) is:

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

For general networks G and measurements **M**, give a condition for when the exact traffic flows can be recovered in terms of the null space of **M** and the null space of $\mathbf{B}_G$.

Hint: Recovery will fail iff (if and only if) there are two valid flows with the same measurements, that is, there exist distinct $\vec{t}_1$ and $\vec{t}_2$ satisfying the flow conservation constraints, such that $\mathbf{M}\vec{t}_1 = \mathbf{M}\vec{t}_2$. Can you express this in terms of the null spaces of **M** and $\mathbf{B}_G$?

(h) **Challenge (optional):** If the incidence matrix $\mathbf{B}_G$ has a k-dimensional null space, does this mean we can **always pick a set of k roads** such that measuring the flows along these roads is sufficient to recover the exact flows? Prove or give a counterexample.

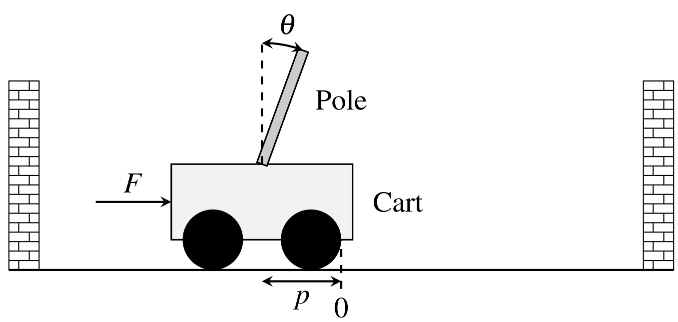
4. Segway Tours

Your friend has decided to start a new SF tour business, and you suggest he use segways.

He becomes intrigued by your idea and asks you how it works.

You let him know that a force (through the spinning wheel) is applied to the base of the segway, and this in turn controls both the position of the segway and the angle of the stand. As you push on the stand, the segway tries to bring itself back to the upright position, and it can only do this by moving the base.

Your friend is impressed, to say the least, but he is a little concerned that only one input (force) is used to control two outputs (position and angle). He finally asks if it's possible for the segway to be brought upright and to a stop from any initial configuration. He calls up a friend who's majoring in mechanical engineering, who tells him that a segway can be modeled as a cart-pole system:



A cart-pole system can be fully described by its position p, velocity $\dot{p}$, angle θ, and angular velocity $\dot{\theta}$. We write this as a "state vector":

$$\vec{x} = \begin{bmatrix} p \\ \dot{p} \\ \theta \\ \dot{\theta} \end{bmatrix}$$

The input to this system u will just be the force applied to the cart (or base of the segway).[1]

At time step n, we can apply scalar input $u[n]$. The cart-pole system can be represented by a linear model:

$$\vec{x}[n+1] = \mathbf{A}\vec{x}[n] + \vec{b}u[n], \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{4\times 4}$ and $\vec{b} \in \mathbb{R}^{4\times 1}$. The model tells us how the the state vector will evolve over (discrete) time as a function of the current state vector and control inputs.

[1] You might note that velocity and angular velocity are derivatives of position and angle respectively. Differential equations are used to describe continuous time systems, which you will learn more about in EE 16B. But even without these techniques, we can still approximate the solution to be a continuous time system by modeling it as a discrete time system where we take very small steps in time. We think about applying a force constantly for a given finite duration and we see how the system responds after that finite duration.

To answer your friend's question, you look at this general linear system and try to answer the following question: Starting from some initial state $\vec{x}_0$, can we reach a final desired state, $\vec{x}_f$, in N steps?

The challenge seems to be that the state is 4-dimensional and keeps evolving and that we can only apply a one dimensional control at each time. Is it possible to control something 4-dimensional with only one degree of freedom that we can control?

You will solve this problem by walking through several steps.

(a) Express $\vec{x}[1]$ in terms of $\vec{x}[0]$ and the input $u[0]$. (*Hint:* This is easy.)

(b) Express $\vec{x}[2]$ in terms of *only* $\vec{x}[0]$ and the inputs, $u[0]$ and $u[1]$. Then express $\vec{x}[3]$ in terms of *only* $\vec{x}[0]$ and the inputs, $u[0]$, $u[1]$, and $u[2]$, and express $\vec{x}[4]$ in terms of *only* $\vec{x}[0]$ and the inputs, $u[0]$, $u[1]$, $u[2]$, and $u[3]$.

(c) Now, derive an expression for $\vec{x}[N]$ in terms of $\vec{x}[0]$ and the inputs from $u[0],\ldots,u[N-1]$. (*Note: To obtain a compact expression, you can use a summation from* 0 *to* $N-1$.)

For the next four parts of the problem, you are given the matrix $\mathbf{A}$ and the vector $\vec{b}$:

$$\mathbf{A} = \begin{bmatrix} 1 & 0.05 & -0.01 & 0 \\ 0 & 0.22 & -0.17 & -0.01 \\ 0 & 0.10 & 1.14 & 0.10 \\ 0 & 1.66 & 2.85 & 1.14 \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} 0.01 \\ 0.21 \\ -0.03 \\ -0.44 \end{bmatrix}$$

Assume the cart-pole starts in an initial state $\vec{x}[0] = \begin{bmatrix} -0.3853493 \\ 6.1032227 \\ 0.8120005 \\ -14 \end{bmatrix}$, and you want to reach the desired state $\vec{x}_f = \vec{0}$ using the control inputs $u[0], u[1], \ldots$. The state vector $\vec{x}_f = \vec{0}$ corresponds to the cart-pole (or segway) being upright and stopped at the origin. (Reaching $\vec{x}_f = \vec{0}$ in N steps means that, given that we start at $\vec{x}[0]$, we can find control inputs, such that we get $\vec{x}[N]$, the state vector at the Nth time step, equal to $\vec{x}_f$.)

Note: You may use IPython to solve the next three parts of the problem. You may use the function we provided (`gauss_elim(matrix)`) to help you find the upper triangular form of matrices. An example of Gaussian Elimination using this code is provide in the iPython notebook. You may also use the function (`np.linalg.solve`) to solve the equations.

(d) Can you reach $\vec{x}_f$ in *two* time steps? (*Hint: Express* $\vec{x}[2] - \mathbf{A}^2\vec{x}[0]$ *in terms of the inputs* $u[0]$ *and* $u[1]$. *Then determine if the system of equations can be solved to obtain* $u[0]$ *and* $u[1]$. *If we obtain valid solutions for* $u[0]$ *and* $u[1]$, *then we can say we will reach* $\vec{x}_f$ *in two time steps.*)

(e) Can you reach $\vec{x}_f$ in *three* time steps?

(f) Can you reach $\vec{x}_f$ in *four* time steps?

(g) If you have found that you can get to the final state in 4 time steps, find the required correct control inputs using IPython and verify the answer by entering these control inputs into the *Plug in your controller* section of the code in the IPython notebook. The code has been written to simulate this system, and you should see the system come to a halt in four time steps! *Suggestion: See what* *happens if you enter all four control inputs equal to 0. This gives you an idea of how the system naturally evolves!*

(h) Let's return to a general matrix $\mathbf{A}$ and a general vector $\vec{b}$. What condition do we need on

$$\text{span}\left\{\vec{b}, \mathbf{A}\vec{b}, \mathbf{A}^2\vec{b}, \ldots, \mathbf{A}^{N-1}\vec{b}\right\}$$

for $\vec{x}_f = \vec{0}$ to be "reachable" from $\vec{x}_0$ in N steps?

(i) What condition would we need on span $\left\{\vec{b}, \mathbf{A}\vec{b}, \mathbf{A}^2\vec{b}, \ldots, \mathbf{A}^{N-1}\vec{b}\right\}$ for *any* valid state vector to be reachable from $\vec{x}_0$ in N steps?
Wouldn't this be cool?

5. (PRACTICE) Codes Revisited

Alice and Bob are back and they've successfully figured out how to avoid dropping symbols when sending messages. In this problem, Alice is using a similar encoding scheme as last time where she uses vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ to encode her message $\begin{bmatrix} a & b & c \end{bmatrix}^T$. (Assume Bob knows the vectors, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, that Alice is using.) Namely, she tries to send $\vec{k}$:

$$\vec{k} = \begin{bmatrix} - & \vec{v}_1^{\,T} & - \\ - & \vec{v}_2^{\,T} & - \\ - & \vec{v}_3^{\,T} & - \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{2}$$

Unfortunately, their arch-nemesis, Eve, is trying to interfere with Alice's messages to Bob and has found a way to add noise to the transmission! But, Eve's interference must pass through a linear transformation, $\mathbf{U}$, before the interference hits Alice's message. Now instead of seeing $\vec{k}$, Bob is receiving $\vec{y}$:

$$\vec{y} = \begin{bmatrix} - & \vec{v}_1^{\,T} & - \\ - & \vec{v}_2^{\,T} & - \\ - & \vec{v}_3^{\,T} & - \end{bmatrix} \left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} - & \vec{u}_1^{\,T} & - \\ - & \vec{u}_2^{\,T} & - \\ - & \vec{u}_3^{\,T} & - \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \right) \tag{3}$$

where Eve inserts $\begin{bmatrix} p & q & r \end{bmatrix}^T$ and it undergoes a transformation by the matrix $\mathbf{U}$ and then undergoes the same transformation as Alice's original 3-symbol vector. There are two ways Eve's meddling can mess up the transmission:

- If Bob receives $\vec{0}$, he doesn't even realize he's getting a message
- Bob receives a nonzero transmission but can't determine the original $\begin{bmatrix} a & b & c \end{bmatrix}^T$

(a) **(PRACTICE)** Alice is using the following vectors for her encoding scheme

$$\vec{v}_1^{\,T} = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}, \vec{v}_2^{\,T} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

If Eve is not interfering ($p = q = r = 0$), will Bob be able to uniquely determine what a, b, and c are?

(b) **(PRACTICE)** Eve decides to change her strategy—now she's sending her interference through its own transformation before adding it to Alice's message. Bob receives $\vec{y}$ according to Equation (3).
Eve's interference is transformed by the vectors:

$$\vec{u}_1^{\,T} = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}, \vec{u}_2^{\,T} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \tag{4}$$

Find the null space of $\mathbf{U} = \begin{bmatrix} - & \vec{u}_1^{\,T} & - \\ - & \vec{u}_2^{\,T} & - \\ - & \vec{u}_3^{\,T} & - \end{bmatrix}$

(c) **(PRACTICE)** Given **U** from Equation (4), find p, q, and r such that Bob is guaranteed to never realize he's receiving a message, regardless of Alice's encoding scheme, i.e. $\vec{y} = \vec{0}$ (this corresponds to the fact that the null space of any matrix always includes the zero vector).

State any constraints on a, b, and c which are necessary for Eve's cancellation to work. In other words, find $\begin{bmatrix} p & q & r \end{bmatrix}^T$ and define any restrictions on a, b, and c such that

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} - & \vec{u_1}^T & - \\ - & \vec{u_2}^T & - \\ - & \vec{u_3}^T & - \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \vec{0}$$

Do not take into account the vectors Alice is using for her encoding.

(d) **(PRACTICE)** Realizing her scheme from (a) was flawed, Alice has also chosen to change things up and is using new vectors for her encoding:

$$\vec{v_1}^T = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}, \vec{v_2}^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 1 & 3 & 1 \end{bmatrix}$$

For each of the following cases, determine whether Bob will receive a message at all and—if he does—whether he'll be able to correctly and uniquely determine what a, b, and c are. Justify your answer. **Do not use IPython to solve this problem**. *Hint: Use your answers from (b) and (c)*

i. $\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

ii. $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

iii. $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

6. Homework Process and Study Group

Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.) How did you work on this homework?