Postnikov-type convergence in ∞-categories

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1. Postnikov-type decomposition

There are many examples of Postnikov-type tower in stable homotopy theory and chromatic homotopy theory such as
(1) The Postnikov tower of a space X

$$\begin{tikzcd} & \vdots \arrow[d] \\ & \tau_{\leq 1}X \arrow[d] \\ X \arrow[r] \arrow[ur] \arrow[uur] & \tau_{\leq 0}X \end{tikzcd}$$

(2) The chromatic tower of a spectrum X

$$\begin{tikzcd} & \vdots \arrow[d] \\ & L_1X \arrow[d] \\ X \arrow[r] \arrow[ur] \arrow[uur] & L_0X \end{tikzcd}$$

Although the tower could be constructed in corresponding homotopy category, the description of convergent condition is not well shaped in classical framework. However, Lurie provided a reasonable approach about Postnikov of truncation tower in [2], which actually can be generalized in any ascending sequence of reflective subcategories of any ∞-category.

Throughout the following content, the $\mathcal{C}$ is an ∞-category, $I = \{\mathcal{C}_0 \subset \mathcal{C}_1 \subset ... \subset \mathcal{C}_n ...\}$ is an ascending sequence of reflective replete full subcategories of $\mathcal{C}$.

Definition 1.1. *An I-tower in $\mathcal{C}$ is a functor $\mathrm{N}(\mathbf{Z}^{op}_{\geq 0})^{\triangleleft} \to \mathcal{C}$, which we view as a diagram*

$$X_\infty \to \cdots \to X_2 \to X_1 \to X_0.$$

which satisfies that for each $n \geq 0$, the map $X_\infty \to X_n$ exhibits X_n as a $\mathcal{C}_n$-reflection of X_∞. We define a I-pretower to be a functor from $\mathrm{N}\left(\mathbf{Z}_{\geq 0}\right)^{op} \to \mathcal{C}$:

$$\cdots \to X_2 \to X_1 \to X_0$$

which exhibits each X_n as a $\mathcal{C}_n$-reflection of X_{n+1}.
We let $\mathrm{Post}^+_I(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}\left(\mathrm{N}(\mathbf{Z}^{op}_{\geq 0})^{\triangleleft}, \mathcal{C}\right)$ spanned by the I-towers, and $\mathrm{Post}_I(\mathcal{C})$ the full subcategory of $\mathrm{Fun}\left(\mathrm{N}\left(\mathbf{Z}_{\geq 0}\right)^{op}, \mathcal{C}\right)$ spanned by the I-pretowers. We have an evident forgetful functor $\phi : \mathrm{Post}^+_I(\mathcal{C}) \to \mathrm{Post}_I(\mathcal{C})$. We will say that I-towers in $\mathcal{C}$ are convergent if ϕ is an equivalence of ∞-categories.

Definition 1.2. *Let $\mathcal{E}$ denote the full subcategory of $\mathcal{C} \times \mathrm{N}(\mathbf{Z}^{op}_{\geq 0})^{\triangleleft}$ spanned by those pairs (C, n) where $C \in \mathcal{C}_n$(by convention, we agree that this condition is always satisfied when $n = \infty$). Then we have a coCartesian fibration $p : \mathcal{E} \to \mathrm{N}(\mathbf{Z}^{op}_{\geq 0})^{\triangleleft}$, which classifies a tower of ∞-categories*

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & \mathcal{C}_2 \arrow[d, "F_1"] \\
 & \mathcal{C}_1 \arrow[d, "F_0"] \\
\mathcal{C} \arrow[r, "F_0"'] \arrow[ru, "F_1"] \arrow[ruu, "F_2"] \arrow[ruuu] & \mathcal{C}_0
\end{tikzcd}$$

where F_n is the $\mathcal{C}_n$-reflection functor.

Proposition 1.3. *We can identify I-towers and with coCartesian sections of p, and I-pretowers with coCartesian sections of the induced fibration $\mathcal{E}' = \mathrm{N}(\mathbf{Z}^{op}_{\geq 0}) \times_{\mathrm{N}(\mathbf{Z}^{op}_{\geq 0})^{\triangleleft}} \mathcal{E}$:*

$$\begin{tikzcd}
\mathrm{Post}^+_I(\mathcal{C}) \arrow[r] \arrow[d, "="] & \mathrm{Post}_I(\mathcal{C}) \arrow[d, "="] \\
Fun^{cCart}_{/\mathrm{N}(\mathbf{Z}^{op}_{\geq 0})^{\triangleleft}}(\mathrm{N}(\mathbf{Z}^{op}_{\geq 0})^{\triangleleft}, \mathcal{E}) \arrow[r] & Fun^{cCart}_{/\mathrm{N}(\mathbf{Z}^{op}_{\geq 0})}(\mathrm{N}(\mathbf{Z}^{op}_{\geq 0}), \mathcal{E}')
\end{tikzcd}$$

According to [1] 7.4.1.1, the I-towers in $\mathcal{C}$ converge if and only if the tower above exhibits $\mathcal{C}$ as the homotopy limit of the sequence of ∞-categories

$$\cdots \to \mathcal{C}_2 \xrightarrow{F_1} \mathcal{C}_1 \xrightarrow{F_0} \mathcal{C}_0.$$

Now we introduce a useful lemma which implies any $N_*(J)$-diagram in QC is homotopy to a strict diagram $J \to QCat$.

Lemma 1.4. *[2] 4.2.4.4. Let J be a small ordinary category, and $QCat$ denote the simplicial category of (small) ∞-categories, which is sSet-enriched by the form $Fun(C, D)^{\simeq}$, and $QC = N_\Delta(QCat)$ denote the ∞-category of (small) ∞-categories. Then the following induced map is an equivalence,*

$$N_\Delta(F(J, sSet_+)^\circ) \to \mathrm{Fun}(N_*(J), sSet_+^\circ) = \mathrm{Fun}(N_*(J), QC)$$

where $sSet_+$ is the Cartesian model category of marked simplicial sets, and $F(J, sSet_+)$ is endowed with projective or injective model, and $(-)^\circ$ means full subcategory of cofibrant-fibrant objects.

Proposition 1.5. *If I-towers in $\mathcal{C}$ are convergent, then every I-tower in $\mathcal{C}$ is a limit diagram. Indeed, given objects $X, Y \in \mathcal{C}$, we have natural homotopy equivalences*

$$\mathrm{Map}_{\mathcal{C}}(X, Y) \simeq \mathrm{holim}\, \mathrm{Map}_{\mathcal{C}}(F_n X, F_n Y) \simeq \mathrm{holim}\, \mathrm{Map}_{\mathcal{C}}(X, F_n Y),$$

and the composition of these 2 equivalences is induced by the composition $Y \to F_n Y$. So Y is the limit of the I-pretower $\{F_n Y\}$.

Lurie gives this formula without a proof, which actually needs some straitening techniques.

Proof: Let $f : \mathrm{N}(\mathbf{Z}^{op}_{\geq 0}) \to QC$ be the straitening presheaf by $p' : \mathcal{E}' \to \mathrm{N}(\mathbf{Z}^{op}_{\geq 0})$. By 1.4, it is homotopy to $N_\Delta(q)$ where q is a strict diagram $\mathbf{Z}^{op}_{\geq 0} \to QCat$. Without loss of generalization, we can assume q has the form $... \to N_\Delta(\mathcal{D}_n) \xrightarrow{N_\Delta(G_n)} N_\Delta(\mathcal{D}_{n-1}) \to ... \to N_\Delta(\mathcal{D}_0)$ where $G_n : \mathcal{D}_n \to \mathcal{D}_{n-1}$ is an Joyal fibration of simplicial categories. Then q is an isofibrant diagram by [1] 4.5.6.6. So we have an (essentially unique) equivalence $\mathcal{C} \to N_\Delta(\mathcal{D}) = N_\Delta(\varprojlim \mathcal{D}_n)$ and

$$\underleftarrow{\mathrm{holim}}\, \mathrm{Map}_{\mathcal{C}}(F_n X, F_n Y) = \varprojlim Hom^*_{\mathcal{D}_n}(G_n X, G_n Y) = Hom^*_{\varprojlim \mathcal{D}_n}(X, Y) \simeq \mathrm{Map}_{\mathcal{C}}(X, Y)$$

Furthermore, we note that

$$... \to Hom^*_{\mathcal{D}_n}(G_n(-), G_n(-)) \to Hom^*_{\mathcal{D}_{n-1}}(G_{n-1}(-), G_{n-1}(-)) \to ...$$

gives an simplicial functor $(\mathbf{Z}^{op}_{\geq 0})^{\triangleleft} \times \mathcal{D}^{op} \times \mathcal{D} \to Kan$. Let

$$\{*\} \times \mathrm{N}(\mathbf{Z}^{op}_{\geq 0})^{\triangleleft} \to \mathcal{C}^{op} \times \mathcal{C} \to N_\Delta(\mathcal{D}) \times N_\Delta(\mathcal{D})$$

be (X, F_nY) induced by the I-tower $\{Y \to F_nY\}$ in $\mathcal{C}$. By Composition we get a diagram $\mathrm{N}(\mathbf{Z}_{\geq 0}^{op})^{\triangleleft} \times \mathrm{N}(\mathbf{Z}_{\geq 0}^{op})^{\triangleleft} \to \mathcal{S}$ which has the form $(m, n) \mapsto Hom^*_{\mathcal{D}_m}(G_mX, G_m(F_nY))$. Take the sub-diagram $(\Delta^2 \times \mathrm{N}(\mathbf{Z}_{\geq 0}^{op}))^{\triangleleft} \subset \mathrm{N}(\mathbf{Z}_{\geq 0}^{op})^{\triangleleft} \times \mathrm{N}(\mathbf{Z}_{\geq 0}^{op})^{\triangleleft}$ we get

$$\begin{tikzcd}
 & Hom^*_{\mathcal{D}}(X,Y) \arrow[dl] \arrow[d] \arrow[dr] & \\
\vdots \arrow[r, "\sim"'] \arrow[d] & \vdots \arrow[d] & \vdots \arrow[l, "\sim"'] \arrow[d] \\
Hom^*_{\mathcal{D}}(X,F_nY) \arrow[r, "\sim"] \arrow[d] & Hom^*_{\mathcal{D}_n}(G_nX,G_nF_nY) \arrow[d] & Hom^*_{\mathcal{D}_n}(G_nX,G_nY) \arrow[l, "\sim"'] \arrow[d] \\
Hom^*_{\mathcal{D}}(X,F_{n-1}Y) \arrow[r, "\sim"] \arrow[d] & Hom^*_{\mathcal{D}_{n-1}}(G_{n-1}X,G_{n-1}F_{n-1}Y) \arrow[d] & Hom^*_{\mathcal{D}_{n-1}}(G_{n-1}X,G_{n-1}Y) \arrow[l, "\sim"'] \arrow[d] \\
\vdots \arrow[r, "\sim"] & \vdots & \vdots \arrow[l, "\sim"']
\end{tikzcd}$$

which gives

$$\mathrm{Map}_{\mathcal{C}}(X,Y) \simeq \operatorname{holim} \mathrm{Map}_{\mathcal{C}}(F_nX, F_nY) \simeq \operatorname{holim} \mathrm{Map}_{\mathcal{C}}(X, F_nY),$$

and the composition of these 2 equivalences is induced by the composition $Y \to F_nY$.

□

Proposition 1.6. *Let $\mathcal{C}$ is an ∞-category in which any I-pretower admits a limit, where $I = \{\mathcal{C}_0 \subset \mathcal{C}_1 \subset ... \subset \mathcal{C}_n ...\}$ be an ascending sequence of reflective replete full subcategories of $\mathcal{C}$. Then I-towers in $\mathcal{C}$ are convergent if and only if, for every diagram $X : \mathrm{N}(\mathbf{Z}_{\geq 0}^{op})^{\triangleleft} \to \mathcal{C}$, the following conditions are equivalent:*
(1) The diagram X is a I-tower.
(2) The diagram X is a limit in $\mathcal{C}$, and the restriction $X \mid \mathrm{N}(\mathbf{Z}_{\geq 0})^{op}$ is a I-pretower.

Proof. Let $\mathrm{Post}'_I(\mathcal{C})$ be the full subcategory of $\mathrm{Fun}\left(\mathrm{N}(\mathbf{Z}_{\geq 0}^{op})^{\triangleleft}, \mathcal{C}\right)$ spanned by those towers which satisfy condition (2). Using Proposition [1] 7.3.6.13, we deduce that the restriction functor $\mathrm{Post}'_I(\mathcal{C}) \to \mathrm{Post}_I(\mathcal{C})$ is a trivial Kan fibration.

If conditions (1) and (2) are equivalent, then $\mathrm{Post}'_I(\mathcal{C}) = \mathrm{Post}^+_I(\mathcal{C})$, so that I-towers in $\mathcal{C}$ are convergent.
Conversely, suppose that I-towers in $\mathcal{C}$ are convergent. Using 1.5, we deduce that $\mathrm{Post}^+_I(\mathcal{C}) \subseteq$

$\mathrm{Post}'_I(\mathcal{C})$, so we have a commutative diagram

$$\begin{tikzcd} Post_I^+ \arrow[rr] \arrow[dr] & & Post'_I \arrow[dl] \\ & Post_I & \end{tikzcd}$$

Since both of the vertical arrows are trivial Kan fibrations, we conclude that the inclusion $\mathrm{Post}_I^+(\mathcal{C}) \subseteq \mathrm{Post}'_I(\mathcal{C})$ is an equivalence, so that $\mathrm{Post}_I^+(\mathcal{C}) = \mathrm{Post}'_I(\mathrm{C})$ by repleteness. This proves that (1) $\Leftrightarrow$ (2). $\square$

References

[1] J. Lurie. *Kerodon.* version 2023.04.24. 1.3, 1, 1

[2] Jacob Lurie. *Higher topos theory.* Princeton University Press, 2009. 1, 1.4