

Postnikov-type convergence in ∞ -categories

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1. Postnikov-type decomposition

There are many examples of Postnikov-type tower in stable homotopy theory and chromatic homotopy theory such as

(1) The Postnikov tower of a space X

$$\begin{array}{ccc} & & \vdots \\ & \nearrow & \downarrow \\ & & \tau_{\leq 1} X \\ X & \nearrow & \downarrow \\ & & \tau_{\leq 0} X \end{array}$$

(2) The chromatic tower of a spectrum X

$$\begin{array}{ccc} & & \vdots \\ & \nearrow & \downarrow \\ & & L_1 X \\ X & \nearrow & \downarrow \\ & & L_0 X \end{array}$$

Although the tower could be constructed in corresponding homotopy category, the description of convergent condition is not well shaped in classical framework. However, Lurie provided a reasonable approach about Postnikov of truncation tower in [2], which actually can be generalized in any ascending sequence of reflective subcategories of any ∞ -category.

Throughout the following content, the \mathcal{C} is an ∞ -category, $I = \{\mathcal{C}_0 \subset \mathcal{C}_1 \subset \dots \subset \mathcal{C}_n \dots\}$ is an ascending sequence of reflective replete full subcategories of \mathcal{C} .

Definition 1.1. An I -tower in \mathcal{C} is a functor $N(\mathbf{Z}_{\geq 0}^{op})^\triangleleft \rightarrow \mathcal{C}$, which we view as a diagram

$$X_\infty \rightarrow \cdots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0.$$

which satisfies that for each $n \geq 0$, the map $X_\infty \rightarrow X_n$ exhibits X_n as a \mathcal{C}_n -reflection of X_∞ .

We define a I -pretower to be a functor from $N(\mathbf{Z}_{\geq 0})^{op} \rightarrow \mathcal{C}$:

$$\cdots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0$$

which exhibits each X_n as a \mathcal{C}_n -reflection of X_{n+1} .

We let $\text{Post}_I^+(\mathcal{C})$ denote the full subcategory of $\text{Fun}(N(\mathbf{Z}_{\geq 0}^{op})^\triangleleft, \mathcal{C})$ spanned by the I -towers, and $\text{Post}_I(\mathcal{C})$ the full subcategory of $\text{Fun}(N(\mathbf{Z}_{\geq 0})^{op}, \mathcal{C})$ spanned by the I -pretowers. We have an evident forgetful functor $\phi : \text{Post}_I^+(\mathcal{C}) \rightarrow \text{Post}_I(\mathcal{C})$. We will say that I -towers in \mathcal{C} are convergent if ϕ is an equivalence of ∞ -categories.

Definition 1.2. Let \mathcal{E} denote the full subcategory of $\mathcal{C} \times N(\mathbf{Z}_{\geq 0}^{op})^\triangleleft$ spanned by those pairs (C, n) where $C \in \mathcal{C}_n$ (by convention, we agree that this condition is always satisfied when $n = \infty$). Then we have a coCartesian fibration $p : \mathcal{E} \rightarrow N(\mathbf{Z}_{\geq 0}^{op})^\triangleleft$, which classifies a tower of ∞ -categories

$$\begin{array}{c} \vdots \\ \downarrow \\ \mathcal{C}_2 \\ \downarrow F_1 \\ \mathcal{C}_1 \\ \downarrow F_0 \\ \mathcal{C} \end{array} \begin{array}{c} \nearrow \\ \nearrow F_2 \\ \nearrow F_1 \\ \xrightarrow{F_0} \end{array} \begin{array}{c} \\ \mathcal{C}_1 \\ \mathcal{C}_0 \end{array}$$

where F_n is the \mathcal{C}_n -reflection functor.

Proposition 1.3. We can identify I -towers and with coCartesian sections of p , and I -pretowers with coCartesian sections of the induced fibration $\mathcal{E}' = N(\mathbf{Z}_{\geq 0}^{op}) \times_{N(\mathbf{Z}_{\geq 0}^{op})^\triangleleft} \mathcal{E}$:

$$\begin{array}{ccc} \text{Post}_I^+(\mathcal{C}) & \xrightarrow{\quad} & \text{Post}_I(\mathcal{C}) \\ \downarrow = & & \downarrow = \\ \text{Fun}_{/N(\mathbf{Z}_{\geq 0}^{op})^\triangleleft}^{c\text{Cart}}(N(\mathbf{Z}_{\geq 0}^{op})^\triangleleft, \mathcal{E}) & \xrightarrow{\quad} & \text{Fun}_{/N(\mathbf{Z}_{\geq 0}^{op})}^{c\text{Cart}}(N(\mathbf{Z}_{\geq 0}^{op}), \mathcal{E}') \end{array}$$

According to [1] 7.4.1.1, the I -towers in \mathcal{C} converge if and only if the tower above exhibits \mathcal{C} as the homotopy limit of the sequence of ∞ -categories

$$\cdots \rightarrow \mathcal{C}_2 \xrightarrow{F_1} \mathcal{C}_1 \xrightarrow{F_0} \mathcal{C}_0.$$

Now we introduce a useful lemma which implies any $N_*(J)$ -diagram in QC is homotopy to a strict diagram $J \rightarrow QCat$.

Lemma 1.4. [2] 4.2.4.4. *Let J be a small ordinary category, and $QCat$ denote the simplicial category of (small) ∞ -categories, which is $sSet$ -enriched by the form $Fun(C, D)^\simeq$, and $QC = N_\Delta(QCat)$ denote the ∞ -category of (small) ∞ -categories. Then the following induced map is an equivalence,*

$$N_\Delta(F(J, sSet_+)^{\circ}) \rightarrow Fun(N_*(J), sSet_+^{\circ}) = Fun(N_*(J), QC)$$

where $sSet_+$ is the Cartesian model category of marked simplicial sets, and $F(J, sSet_+)$ is endowed with projective or injective model, and $(-)^{\circ}$ means full subcategory of cofibrant-fibrant objects.

Proposition 1.5. *If I -towers in \mathcal{C} are convergent, then every I -tower in \mathcal{C} is a limit diagram. Indeed, given objects $X, Y \in \mathcal{C}$, we have natural homotopy equivalences*

$$Map_{\mathcal{C}}(X, Y) \simeq \operatorname{holim} Map_{\mathcal{C}}(F_n X, F_n Y) \simeq \operatorname{holim} Map_{\mathcal{C}}(X, F_n Y),$$

and the composition of these 2 equivalences is induced by the composition $Y \rightarrow F_n Y$. So Y is the limit of the I -pretower $\{F_n Y\}$.

Lurie gives this formula without a proof, which actually needs some straitening techniques.

Proof: Let $f : N(\mathbf{Z}_{\geq 0}^{op}) \rightarrow QC$ be the straitening presheaf by $p' : \mathcal{E}' \rightarrow N(\mathbf{Z}_{\geq 0}^{op})$. By 1.4, it is homotopy to $N_\Delta(q)$ where q is a strict diagram $\mathbf{Z}_{\geq 0}^{op} \rightarrow QCat$. Without loss of generalization, we can assume q has the form $\dots \rightarrow N_\Delta(\mathcal{D}_n) \xrightarrow{N_\Delta(G_n)} N_\Delta(\mathcal{D}_{n-1}) \rightarrow \dots \rightarrow N_\Delta(\mathcal{D}_0)$ where $G_n : \mathcal{D}_n \rightarrow \mathcal{D}_{n-1}$ is an Joyal fibration of simplicial categories. Then q is an isofibrant diagram by [1] 4.5.6.6. So we have an (essentially unique) equivalence $\mathcal{C} \rightarrow N_\Delta(\mathcal{D}) = N_\Delta(\varprojlim \mathcal{D}_n)$ and

$$\varprojlim Map_{\mathcal{C}}(F_n X, F_n Y) = \varprojlim Hom_{\mathcal{D}_n}^*(G_n X, G_n Y) = Hom_{\varprojlim \mathcal{D}_n}^*(X, Y) \simeq Map_{\mathcal{C}}(X, Y)$$

Furthermore, we note that

$$\dots \rightarrow Hom_{\mathcal{D}_n}^*(G_n(-), G_n(-)) \rightarrow Hom_{\mathcal{D}_{n-1}}^*(G_{n-1}(-), G_{n-1}(-)) \rightarrow \dots$$

gives an simplicial functor $(\mathbf{Z}_{\geq 0}^{op})^{\triangleleft} \times \mathcal{D}^{op} \times \mathcal{D} \rightarrow Kan$. Let

$$\{*\} \times N(\mathbf{Z}_{\geq 0}^{op})^{\triangleleft} \rightarrow \mathcal{C}^{op} \times \mathcal{C} \rightarrow N_\Delta(\mathcal{D}) \times N_\Delta(\mathcal{D})$$

be $(X, F_n Y)$ induced by the I -tower $\{Y \rightarrow F_n Y\}$ in \mathcal{C} . By Composition we get a diagram $N(\mathbf{Z}_{\geq 0}^{op})^\triangleleft \times N(\mathbf{Z}_{\geq 0}^{op})^\triangleleft \rightarrow \mathcal{S}$ which has the form $(m, n) \mapsto Hom_{\mathcal{D}_m}^*(G_m X, G_m(F_n Y))$. Take the sub-diagram $(\Delta^2 \times N(\mathbf{Z}_{\geq 0}^{op}))^\triangleleft \subset N(\mathbf{Z}_{\geq 0}^{op})^\triangleleft \times N(\mathbf{Z}_{\geq 0}^{op})^\triangleleft$ we get

$$\begin{array}{ccccc}
& & Hom_{\mathcal{D}}^*(X, Y) & & \\
& \swarrow & \downarrow & \searrow & \\
\vdots & \xleftarrow{\sim} & \vdots & \xleftarrow{\sim} & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
Hom_{\mathcal{D}}^*(X, F_n Y) & \xrightarrow{\sim} & Hom_{\mathcal{D}_n}^*(G_n X, G_n F_n Y) & \xleftarrow{\sim} & Hom_{\mathcal{D}_n}^*(G_n X, G_n Y) \\
\downarrow & & \downarrow & & \downarrow \\
Hom_{\mathcal{D}}^*(X, F_{n-1} Y) & \xrightarrow{\sim} & Hom_{\mathcal{D}_{n-1}}^*(G_{n-1} X, G_{n-1} F_{n-1} Y) & \xleftarrow{\sim} & Hom_{\mathcal{D}_{n-1}}^*(G_{n-1} X, G_{n-1} Y) \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & \xleftarrow{\sim} & \vdots & \xleftarrow{\sim} & \vdots
\end{array}$$

which gives

$$Map_{\mathcal{C}}(X, Y) \simeq \text{holim } Map_{\mathcal{C}}(F_n X, F_n Y) \simeq \text{holim } Map_{\mathcal{C}}(X, F_n Y),$$

and the composition of these 2 equivalences is induced by the composition $Y \rightarrow F_n Y$.

□

Proposition 1.6. *Let \mathcal{C} is an ∞ -category, $I = \{\mathcal{C}_0 \subset \mathcal{C}_1 \subset \dots \subset \mathcal{C}_{n\dots}\}$ be an ascending sequence of reflective replete full subcategories of \mathcal{C} . Then I -towers in \mathcal{C} are convergent if and only if, for every diagram $X : N(\mathbf{Z}_{\geq 0}^{op})^\triangleleft \rightarrow \mathcal{C}$, the following conditions are equivalent:*

- (1) *The diagram X is a I -tower.*
- (2) *The diagram X is a limit in \mathcal{C} , and the restriction $X \mid N(\mathbf{Z}_{\geq 0})^{op}$ is a I -pretower.*

Proof. Let $Post'_I(\mathcal{C})$ be the full subcategory of $\text{Fun}(N(\mathbf{Z}_{\geq 0}^{op})^\triangleleft, \mathcal{C})$ spanned by those towers which satisfy condition (2). Using Proposition [1] 7.3.6.13, we deduce that the restriction functor $Post'_I(\mathcal{C}) \rightarrow Post_I(\mathcal{C})$ is a trivial Kan fibration.

If conditions (1) and (2) are equivalent, then $Post'_I(\mathcal{C}) = Post_I^+(\mathcal{C})$, so that I -towers in \mathcal{C} are convergent. Conversely, suppose that I -towers in \mathcal{C} are convergent. Using 1.5, we deduce that $Post_I^+(\mathcal{C}) \subseteq Post'_I(\mathcal{C})$, so we have a commutative diagram

$$\begin{array}{ccc}
Post_I^+ & \xrightarrow{\quad} & Post'_I \\
& \searrow \quad \swarrow & \\
& Post_I &
\end{array}$$

Since both of the vertical arrows are trivial Kan fibrations, we conclude that the inclusion $\text{Post}_I^+(\mathcal{C}) \subseteq \text{Post}'_I(\mathcal{C})$ is an equivalence, so that $\text{Post}_I^+(\mathcal{C}) = \text{Post}'_I(\mathcal{C})$ by repleteness. This proves that (1) \Leftrightarrow (2). \square

References

- [1] J. Lurie. *Kerodon*. version 2023.04.24. 1.3, 1, 1
- [2] Jacob Lurie. *Higher topos theory*. Princeton University Press, 2009. 1, 1.4