Postnikov-type convergence in ∞ -categories

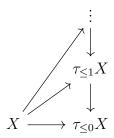
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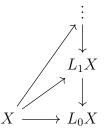
1. Postnikov-type decomposition

There are many examples of Postnikov-type tower in stable homotopy theory and chromatic homotopy theory such as

(1) The Postnikov tower of a space X



(2) The chromatic tower of a spectrum X



Although the tower could be constructed in corresponding homotopy category, the description of convergent condition is not well shaped in classical framework. However, Lurie provided a reasonable approach about Postnikov of truncation tower in [2], which actually can be generalized in any ascending sequence of reflective subcategories of any ∞ -category.

Throughout the following content, the \mathcal{C} is an ∞ -category, $I = \{\mathcal{C}_0 \subset \mathcal{C}_1 \subset ... \subset \mathcal{C}_n...\}$ is an ascending sequence of reflective replete full subcategories of \mathcal{C} .

Definition 1.1. An I-tower in C is a functor $N(\mathbf{Z}_{\geq 0}^{op})^{\triangleleft} \to C$, which we view as a diagram

$$X_{\infty} \to \cdots \to X_2 \to X_1 \to X_0.$$

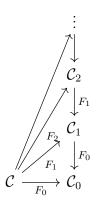
which satisfies that for each $n \geq 0$, the map $X_{\infty} \to X_n$ exhibits X_n as a \mathcal{C}_n -reflection of X_{∞} . We define a I-pretower to be a functor from $N(\mathbf{Z}_{\geq 0})^{op} \to \mathcal{C}$:

$$\cdots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0$$

which exhibits each X_n as a C_n -reflection of X_{n+1} .

We let $\operatorname{Post}_I^+(\mathcal{C})$ denote the full subcategory of $\operatorname{Fun}\left(\operatorname{N}(\mathbf{Z}_{\geq 0}^{op})^{\triangleleft},\mathcal{C}\right)$ spanned by the I-towers, and $\operatorname{Post}_I(\mathcal{C})$ the full subcategory of $\operatorname{Fun}\left(\operatorname{N}(\mathbf{Z}_{\geq 0})^{op},\mathcal{C}\right)$ spanned by the I-pretowers. We have an evident forgetful functor $\phi:\operatorname{Post}_I^+(\mathcal{C})\to\operatorname{Post}_I(\mathcal{C})$. We will say that I-towers in \mathcal{C} are convergent if ϕ is an equivalence of ∞ -categories.

Definition 1.2. Let \mathcal{E} denote the full subcategory of $\mathcal{C} \times \mathrm{N}(\mathbf{Z}_{\geq 0}^{op})^{\triangleleft}$ spanned by those pairs (C, n) where $C \in \mathcal{C}_n$ (by convention, we agree that this condition is always satisfied when $n = \infty$). Then we have a coCartesian fibration $p : \mathcal{E} \to \mathrm{N}(\mathbf{Z}_{\geq 0}^{op})^{\triangleleft}$, which classifies a tower of ∞ -categories



where F_n is the C_n -reflection functor.

Proposition 1.3. We can identify I-towers and with coCartesian sections of p, and I-pretowers with coCartesian sections of the induced fibration $\mathcal{E}' = N(\mathbf{Z}_{\geq 0}^{op}) \times_{N(\mathbf{Z}_{\geq 0}^{op})^{\triangleleft}} \mathcal{E}$:

$$\text{Post}_{I}^{+}(\mathcal{C}) \longrightarrow \text{Post}_{I}(\mathcal{C})$$

$$\downarrow = \qquad \qquad \downarrow =$$

$$Fun_{/\mathcal{N}(\mathbf{Z}_{>0}^{op})^{\triangleleft}}^{cCart}(\mathcal{N}(\mathbf{Z}_{\geq 0}^{op})^{\triangleleft}, \mathcal{E}) \longrightarrow Fun_{/\mathcal{N}(\mathbf{Z}_{>0}^{op})}^{cCart}(\mathcal{N}(\mathbf{Z}_{\geq 0}^{op}), \mathcal{E}')$$

According to [1] 7.4.1.1, the I-towers in C converge if and only if the tower above exhibits C as the homotopy limit of the sequence of ∞ -categories

$$\cdots \to \mathcal{C}_2 \xrightarrow{F_1} \mathcal{C}_1 \xrightarrow{F_0} \mathcal{C}_0.$$

Now we introduce a useful lemma which implies most diagrams of ∞ -categories are homotopy to strict diagrams.

Lemma 1.4. [2] 4.2.4.4. Let J be a small ordinary category, and QC at denote the simplicial category of (small) ∞ -categories, which is sSet-enriched by the form $Fun(C, D)^{\simeq}$, and $QC = N_{\Delta}(QCat)$ denote the ∞ -category of (small) ∞ -categories. Then the following induced map is an equivalence.

$$N_{\Delta}(F(J, sSet_{+})^{\circ}) \rightarrow \operatorname{Fun}(N_{*}(J), sSet_{+}^{\circ}) = \operatorname{Fun}(N_{*}(J), QC)$$

Proposition 1.5. If I-towers in C are convergent, then every I-tower in C is a limit diagram. Indeed, given objects $X, Y \in C$, we have natural homotopy equivalences

$$\operatorname{Map}_{\mathcal{C}}(X,Y) \simeq \operatorname{holim} \operatorname{Map}_{\mathcal{C}}(F_nX, F_nY) \simeq \operatorname{holim} \operatorname{Map}_{\mathcal{C}}(X, F_nY)$$
,

and the composition of these 2 equivalences is induced by the composition $Y \to F_n Y$. So Y is the limit of the I-pretower $\{F_n Y\}$.

Lurie gives this formula without a proof, which actually needs some straitening techniques.

Proof: Let $f: N(\mathbf{Z}_{\geq 0}^{op}) \to QC$ be the straitening presheaf by $p': \mathcal{E}' \to N(\mathbf{Z}_{\geq 0}^{op})$. By 1.4, it is homotopy to $N_{\Delta}(q)$ where q is a strict diagram $\mathbf{Z}_{\geq 0}^{op} \to QCat$. Without loss of generalization, we can assume q has the form ... $\to N_{\Delta}(\mathcal{D}_n) \xrightarrow{N_{\Delta}(G_n)} N_{\Delta}(\mathcal{D}_{n-1}) \to ... \to N_{\Delta}(\mathcal{D}_0)$ where $G_n: \mathcal{D}_n \to \mathcal{D}_{n-1}$ is an Joyal fibration of simplicial categories. Then q is an isofibrant diagram by [1] 4.5.6.6. So we have an (essentially unique) equivalence $\mathcal{C} \to N_{\Delta}(\mathcal{D}) = N_{\Delta}(\varprojlim \mathcal{D}_n)$ and

$$\underline{\operatorname{holim}} \operatorname{Map}_{\mathcal{C}}(F_nX, F_nY) = \underline{\varprojlim} \operatorname{Hom}_{\mathcal{D}_n}^*(G_nX, G_nY) = \operatorname{Hom}_{\underline{\varprojlim} \mathcal{D}_n}^*(X, Y) \simeq \operatorname{Map}_{\mathcal{C}}(X, Y)$$

Furthermore, we note that

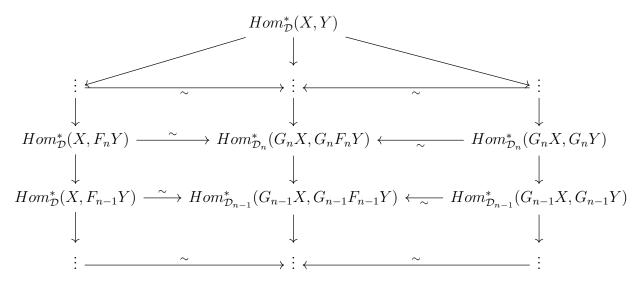
...
$$\to Hom_{\mathcal{D}_n}^*(G_n(-), G_n(-)) \to Hom_{\mathcal{D}_{n-1}}^*(G_{n-1}(-), G_{n-1}(-)) \to ...$$

gives an simplicial functor $(\mathbf{Z}_{\geq 0}^{op})^{\triangleleft} \times \mathcal{D}^{op} \times \mathcal{D} \to Kan$. Let

$$\{*\} \times \mathrm{N}(\mathbf{Z}_{>0}^{op})^{\triangleleft} \to \mathcal{C}^{op} \times \mathcal{C} \to N_{\Delta}(\mathcal{D}) \times N_{\Delta}(\mathcal{D})$$

be $(X, F_n Y)$ induced by the *I*-tower $\{Y \to F_n Y\}$ in \mathcal{C} . By Composition we get a diagram $N(\mathbf{Z}_{\geq 0}^{op})^{\triangleleft} \times N(\mathbf{Z}_{\geq 0}^{op})^{\triangleleft} \to \mathcal{S}$ which has the form $(m, n) \mapsto Hom_{\mathcal{D}_m}^*(G_m X, G_m(F_n Y))$. Take the

sub-diagram $(\Delta^2 \times N(\mathbf{Z}_{\geq 0}^{op}))^{\triangleleft} \subset N(\mathbf{Z}_{\geq 0}^{op})^{\triangleleft} \times N(\mathbf{Z}_{\geq 0}^{op})^{\triangleleft}$ we get



which gives

$$\operatorname{Map}_{\mathcal{C}}(X,Y) \simeq \operatorname{holim} \operatorname{Map}_{\mathcal{C}}(F_nX, F_nY) \simeq \operatorname{holim} \operatorname{Map}_{\mathcal{C}}(X, F_nY),$$

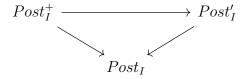
and the composition of these 2 equivalences is induced by the composition $Y \to F_n Y$.

Proposition 1.6. Let C is an ∞ -category, $I = \{C_0 \subset C_1 \subset ... \subset C_n...\}$ be an ascending sequence of reflective replete full subcategories of C. Then I-towers in C are convergent if and only if, for every diagram $X: N(\mathbf{Z}^{op}_{\geq 0})^{\triangleleft} \to \mathcal{C}$, the following conditions are equivalent:

- (1) The diagram X is a I-tower.
- (2) The diagram X is a limit in C, and the restriction $X \mid N(\mathbf{Z}_{\geq 0})^{op}$ is a I-pretower.

Proof. Let $\operatorname{Post}_I'(\mathcal{C})$ be the full subcategory of $\operatorname{Fun}\left(\operatorname{N}(\mathbf{Z}_{\geq 0}^{op})^{\triangleleft},\mathcal{C}\right)$ spanned by those towers which satisfy condition (2). Using Proposition [1] 7.3.6.13, we deduce that the restriction functor $\operatorname{Post}_I'(\mathcal{C}) \to \operatorname{Post}_I(\mathcal{C})$ is a trivial Kan fibration.

If conditions (1) and (2) are equivalent, then $\operatorname{Post}_I'(\mathcal{C}) = \operatorname{Post}_I^+(\mathcal{C})$, so that *I*-towers in \mathcal{C} are convergent. Conversely, suppose that I-towers in \mathcal{C} are convergent. Using 1.5, we deduce that $\operatorname{Post}_I^+(\mathcal{C}) \subseteq \operatorname{Post}_I'(\mathcal{C})$, so we have a commutative diagram



Since both of the vertical arrows are trivial Kan fibrations, we conclude that the inclusion $\operatorname{Post}_I^+(\mathcal{C}) \subseteq \operatorname{Post}_I'(\mathcal{C})$ is an equivalence, so that $\operatorname{Post}_I^+(\mathcal{C}) = \operatorname{Post}_I'(\mathcal{C})$ by repleteness. This proves that $(1) \Leftrightarrow (2)$.

References

- $[1]\ \ \mathrm{J.\ Lurie}.\ \mathit{Kerodon}.\ \mathrm{version\ 2023.04.24.}\ 1.3,\ 1,\ 1$
- [2] Jacob Lurie. Higher topos theory. Princeton University Press, 2009. 1, 1.4