Elliptic cohomology theories and the σ-orientation

Jiacheng Liang

September 1, 2023

Contents

1. Sites, fppf sheaves and completion

Grothendieck topology and topoi are an important algebro-geometric machinery for homotopists since lots of algebro-geometric objects like schemes, algebraic spaces, formal groups and p-divisible groups all can fully faithfully embed into the category of fppf sheaves.

Now, let me give an introduction to Grothendieck topology and sheavs on sites. A good reference for them is stacks project [13].

1.1 Grothendieck topology

Definition 1.1. *[13] A site is given by a category $\mathcal{C}$ and a class $\mathrm{Cov}(\mathcal{C}) \subset 2^{Mor(C)}$ of families of morphisms with fixed target $\{U_i \to U\}_{i\in I}$ where I is a small set, called coverings of $\mathcal{C}$, satisfying the following axioms*
(1) If $V \to U$ is an isomorphism then $\{V \to U\} \in \mathrm{Cov}(\mathcal{C})$.
(2) If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ and for each i we have $\{V_{ij} \to U_i\}_{j\in J_i} \in \mathrm{Cov}(\mathcal{C})$, then $\{V_{ij} \to U\}_{i\in I, j\in J_i} \in \mathrm{Cov}(\mathcal{C})$.
(3) If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ and $V \to U$ is a morphism of $\mathcal{C}$ then $U_i \times_U V$ exists for all i and $\{U_i \times_U V \to V\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$.

Remark 1.2. *In axiom (3) we require the existence of the fibre products $U_i \times_U V$ for $i \in I$. Actually almost all sites appear in algebraic geometry have any pullback.*

Example 1.3. *(i)[Small Zariski site]*
Let X be a topological space. Let X_{Zar} be the category whose objects consist of all the open sets U in X and whose morphisms are just the inclusion maps. That is, there is at most one morphism between any two objects in X_{Zar}. Now define $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}\left(X_{Zar}\right)$ if and only if $\bigcup U_i = U$.

(ii)[Big τ site]
Let Sch be the category of schemes, and $\tau \in \{Zar, et, Smooth, fppf, fpqc\}$. Let T be a scheme. An τ covering of T is a family of morphisms $\{f_i : T_i \to T\}_{i\in I}$ of schemes such that each f_i is
(1)open immersion
(2)étale
(3)smooth
(4)flat, locally of finite presentation

(5)flat and such that for every affine open $U \subset T$ there exists $n \geq 0$, a map $a : \{1, \ldots, n\} \to I$ and affine opens $V_j \subset T_{a(j)}, j = 1, \ldots, n$ with $\bigcup_{j=1}^{n} f_{a(j)}(V_j) = U$, respectively, and such that $T = \bigcup f_i(T_i)$. We denote the corresponding site to be Sch_τ. Appearently we have

$$\mathrm{Cov}(Zar) \subset \mathrm{Cov}(et) \subset \mathrm{Cov}(Smooth) \subset \mathrm{Cov}(fppf) \subset \mathrm{Cov}(fpqc)$$

Definition 1.4 (Presheaf)**.** *Let $\mathcal{C}$ be a site. A presheaf of sets on $\mathcal{C}$ is a contravariant functor from $\mathcal{C}$ to Sets. Morphisms of presheaves are transformations of functors. The category of presheaves of sets is denoted $\mathrm{PSh}(\mathcal{C})$ or $Fun(C^{op}, Set)$. (Note $\mathcal{C}$ is not necessarily essentially small, so $\mathrm{PSh}(\mathcal{C})$ is not necessarily locally small)*

Definition 1.5 (Sheaf and topos)**.** *Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$. We say $\mathcal{F}$ is a sheaf if for every covering $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$ the diagram*

$$\mathcal{F}(U) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i) \overset{\mathrm{p}_0^*, \mathrm{p}_1^*}{\rightrightarrows} \prod_{(i_0, i_1) \in I \times I} \mathcal{F}(U_{i_0} \times_U U_{i_1})$$

represents the first arrow as the equalizer of p_0^* and p_1^*.

A topos is defined to be a category of sheaves on a site.

Definition 1.6 (Sheafification)**.** *Let $\mathcal{J}_U$ be the category of all coverings of U. The objects of $\mathcal{J}_U$ are the coverings of U in $\mathcal{C}$, and the morphisms are the refinements. Note that $\mathrm{Ob}(\mathcal{J}_U)$ is not empty since $\{\mathrm{id}_U\}$ is an object of it. We define*

$$\mathcal{F}^+(U) = \mathrm{colim}_{\mathcal{J}_U^{op}} H^0(\mathcal{U}, \mathcal{F})$$

where $H^0(\mathcal{U}, \mathcal{F}) = \left\{ (s_i)_{i \in I} \in \prod_i \mathcal{F}(U_i), s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j} \forall i, j \in I \right\}$. We can verify $\mathcal{F}^+$ is separated and $s\mathcal{F} = (\mathcal{F}^+)^+$ is a sheaf. We call $s\mathcal{F}$ by the sheafification.

Actually, this colimit is a direct colimit because we have the following lemma, which implies different refinements between 2 covers induce the same morphism of H^0.

Lemma 1.7. *Any two morphisms $f, g : \mathcal{U} \to \mathcal{V}$ of coverings inducing the same morphism $U \to V$ induce the same map $H^0(\mathcal{V}, \mathcal{F}) \to H^0(\mathcal{U}, \mathcal{F})$*

Proof: *Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ and $\mathcal{V} = \{V_j \to V\}_{j \in J}$. The morphism f consists of a map $U \to V$, a map $\alpha : I \to J$ and maps $f_i : U_i \to V_{\alpha(i)}$. Likewise, g determines a map $\beta : I \to J$ and maps $g_i : U_i \to V_{\beta(i)}$. As f and g induce the same map $U \to V$, the diagram*

$$\begin{tikzcd}
 & V_{\alpha(i)} \arrow[dr] & \\
U_i \arrow[ur, "f_i"] \arrow[dr, "g_i"'] & & V \\
 & V_{\beta(i)} \arrow[ur] &
\end{tikzcd}$$

is commutative for every $i \in I$. Hence f and g factor through the fibre product

$$\begin{tikzcd}
 & V_{\alpha(i)} \\
U_i \arrow[ur] \arrow[r] \arrow[dr] & V_{\alpha(i)} \times_V V_{\beta(i)} \arrow[u] \arrow[d] \\
 & V_{\beta(i)}
\end{tikzcd}$$

Now let $s = (s_j)_j \in H^0(\mathcal{V}, \mathcal{F})$. Then for all $i \in I$:

$$(f^* s)_i = f_i^* \left(s_{\alpha(i)}\right) = \varphi^* \operatorname{pr}_1^* \left(s_{\alpha(i)}\right) = \varphi^* \operatorname{pr}_2^* \left(s_{\beta(i)}\right) = g_i^* \left(s_{\beta(i)}\right) = (g^* s)_i$$

where the middle equality is given by the definition of $H^0(\mathcal{V}, \mathcal{F})$. This shows that the maps $H^0(\mathcal{V}, \mathcal{F}) \to H^0(\mathcal{U}, \mathcal{F})$ induced by f and g are equal.

□

Warning: $\mathcal{J}_U$ is not necessarily a (essentially) small catgory, so not any presheaf on any site can be sheafificated. **Actually, there exists a presheaf on Sch_{fpqc} which admits no fpqc sheafification!**
However if we remove *fpqc* and consider $\tau \in \{Zar, et, Smooth, fppf\}$, then all $\mathcal{J}_U$ in Sch_τ are essentially small and any presheaf in it can be sheafificated.

In the following context, we only consider the site whose $\mathcal{J}_U$ are essentially small and in which all pullbacks exists. (Actually, that holds for almost all sites in algebraic geometry except for fpqc ones.)

Proposition 1.8 (Adjoint). *$PSh(\mathcal{C}) \rightleftarrows Sh(\mathcal{C})$ is a pair of adjunction.*

Proposition 1.9. *The sheafification functor $s : PSh(\mathcal{C}) \to Sh(\mathcal{C})$ preserves any finite limit (because the sheafification can be witten as a filtered colimit of underlying sets).*

Proposition 1.10 (monomorphisms and epimorphisms)**.** *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of sheaves of sets or abelian groups, then*

(1) φ is monomorphism iff for every object U of $\mathcal{C}$ the map $\varphi : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective.

(2) φ is epimorphism iff for every object U of $\mathcal{C}$ and every section $s \in \mathcal{G}(U)$ there exists a covering $\{U_i \to U\}$ such that for all i the restriction $s|_{U_i}$ is in the image of $\varphi : \mathcal{F}(U_i) \to \mathcal{G}(U_i)$.

Proposition 1.11 (Adjoint)**.** *We denote $PAb(\mathcal{C})$ and $Ab(\mathcal{C})$ to be the categories of abelian presheaves and abelian sheaves on $\mathcal{C}$ respectively. Then $PAb(\mathcal{C}) \rightleftarrows Ab(\mathcal{C})$ is still a pair of adjunction.*

Proposition 1.12. *$PAbSh(\mathcal{C})$ and $AbSh(\mathcal{C})$ are abelian categories.*

Proof: First, the kernel and cokernel $PAb(\mathcal{C})$ are created objectwise, so it is abelian. For the $AbSh(\mathcal{C})$, we need the following lemma.

□

Lemma 1.13. *Let $\mathcal{C} \overset{b}{\underset{a}{\rightleftarrows}} \mathcal{D}$ be an adjoint pair of functors. Assume that*
(1) $\mathcal{C}, \mathcal{D}$ are additive categories, b, a are additive functors,
(2) $\mathcal{C}$ is abelian and b preserves finite limits,
(3) $b \circ a \cong id_{\mathcal{D}}$.
Then $\mathcal{D}$ is abelian.

Proof: *As $\mathcal{C}$ is abelian we see that all finite limits and colimits exist in $\mathcal{C}$. Since b is a left adjoint we see that b is also right exact and hence exact. Let $\varphi : B_1 \to B_2$ be a morphism of $\mathcal{C}$. In particular, if $K = \operatorname{Ker}(B_1 \to B_2)$, then K is the equalizer of 0 and φ and hence bK is the equalizer of 0 and $b\varphi$, hence bK is the kernel of $b\varphi$.*

Similarly, if $Q = \operatorname{Coker}(B_1 \to B_2)$, then Q is the coequalizer of 0 and φ and hence bQ is the coequalizer of 0 and $b\varphi$, hence bQ is the cokernel of $b\varphi$. Thus we see that every morphism of the form $b\varphi$ in $\mathcal{D}$ has a kernel and a cokernel. However, since $ba \cong id$ we see that every morphism of $\mathcal{D}$ is of this form, and we conclude that kernels and cokernels exist in $\mathcal{D}$. In fact, the argument shows that if $\psi : A_1 \to A_2$ is a morphism in $\mathcal{D}$ then

$$\operatorname{Ker}(\psi) = b \operatorname{Ker}(a\psi), \quad \textit{and} \quad \operatorname{Coker}(\psi) = b \operatorname{Coker}(a\psi).$$

Now we still have to show that $\operatorname{Coim}(\psi) = \operatorname{Im}(\psi)$. We do this as follows. First note that since $\mathcal{D}$ has kernels and cokernels it has all finite limits and colimits . Hence we see that a *is left exact and hence transforms kernels (=equalizers) into kernels.*

$$\begin{aligned}
\mathrm{Coim}(\psi) &= \mathrm{Coker}\,(\mathrm{Ker}(\psi) \to A_1) && \textit{by definition}\\
&= b\,\mathrm{Coker}\,(a\,(\mathrm{Ker}(\psi) \to A_1)) && \textit{by formula above}\\
&= b\,\mathrm{Coker}\,(\mathrm{Ker}(a\psi) \to aA_1)) && \textit{a preserves kernels}\\
&= b\,\mathrm{Coim}(a\psi) && \textit{by definition}\\
&= b\,\mathrm{Im}(a\psi) && \mathcal{C}\ \textit{is abelian}\\
&= b\,\mathrm{Ker}\,(aA_2 \to \mathrm{Coker}(a\psi)) && \textit{by definition}\\
&= \mathrm{Ker}\,(baA_2 \to b\,\mathrm{Coker}(a\psi)) && \textit{b preserves kernels}\\
&= \mathrm{Ker}\,(A_2 \to b\,\mathrm{Coker}(a\psi)) && ba = \mathrm{id}_{\mathcal{D}}\\
&= \mathrm{Ker}\,(A_2 \to \mathrm{Coker}(\psi)) && \textit{by formula above}\\
&= \mathrm{Im}(\psi) && \textit{by definition}
\end{aligned}$$

Thus the lemma holds.

□

Remark 1.14. *By the Yoneda lemma, if a presheaf of abelian groups is representable by an object H, then H admits a natural abelian group structure.*

1.2 Localization of topoi

In 1.2 we give some useful propositions about topoi.

Proposition 1.15. *Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. We turn $\mathcal{C}/U$ into a site by declaring a family of morphisms $\{V_j \to V\}$ of objects over U to be a covering of $\mathcal{C}/U$ if and only if it is a covering in $\mathcal{C}$. Consider the forgetful functor $j_U : \mathcal{C}/U \longrightarrow \mathcal{C}$. Then we have the following equivalence of categories*

$$Sh(\mathcal{C}/U) \rightleftarrows Sh(\mathcal{C})_{\downarrow U}$$

Proof: Actually we can give an equivalence

$$Sh(\mathcal{C}/S^{'}) \rightleftarrows Sh(\mathcal{C}/S)_{\downarrow S'}$$

for any morhism $S^{'} \to S$ in $\mathcal{C}$.

For a sheaf Y in $Sh(\mathcal{C}/S^{'})$ let Y_S denote the functor on $(\mathcal{C}/S)^{\mathrm{op}}$ sending an S-object T to the set of pairs (ϵ, y), where $\epsilon : T \to S'$ is an S-morphism and $y \in Y\,(\epsilon : T \to S')$ is an element. There is a natural morphism of functors $f_Y : Y_S \to S'$ sending (ϵ, y) to ϵ.

For a sheaf X in $Sh(\mathcal{C}/S)_{\downarrow S'}$, let $X_{S'}$ be the functor on $(\mathcal{C}/S')^{\mathrm{op}}$ whose value on $T \to S'$ is the set of morphisms $T \to X$ in $Sh(\mathcal{C}/S)_{\downarrow S'}$. It is easy to show these two functorial constructions give an equivalence of categories.

$\square$

Remark 1.16. *(1) In algebraic geometry, this equivalence tells us $Sh(Sch_{/S})_\tau$ is exactly the overcategory $Sh(Sch)_\tau \downarrow h_S$.*

(2) This equivalence still holds even if we replace U by any sheaf $\mathcal{F}$.

$$Sh(\mathcal{C}/\mathcal{F}) \rightleftarrows Sh(\mathcal{C})_{\downarrow \mathcal{F}}$$

Now let us focus on the big fppf site Sch_{fppf}. Actually any representable functor is an fppf sheaf.

Proposition 1.17. *[12] Let S be a base scheme, X be an S-scheme, then the representable functor $Hom_S(-, X)$ is an fppf sheaf on $Sch_{/S}$.*

Now we introduce a useful equivalence. The intuition is that a sheaf is a gluing result.

Lemma 1.18. *Let C be a site, and let $C' \subset C$ be a full subcategory such that the following hold:*
(i) For every $U \in C$ there exists a covering $\{U_i \to U\}_{i \in I}$ of U with $U_i \in C'$ for every i.
(ii) If $\{U_i \to U\}$ is a covering of an object $U \in C'$ with $U_i \in C'$ for all i, then for any morphism $V \to U$ in C' the fiber products $V \times_U U_i$ are in C'.
Then there is a Grothendieck topology on C' in which a collection of morphisms $\{U_i \to U\}$ in C' is a covering if and only if it is a covering in C. Furthermore, the topos defined by C' with this topology is equivalent to the topos defined by C.

Proposition 1.19. *For any $\tau \in \{Zar, et, Smooth, fppf\}$ (remove fpqc), $Aff \to Sch$ induces a natural equivalence of topoi*

$$Sh(Sch)_\tau \xrightarrow{\sim} Sh(Aff)_\tau$$

A τ-sheaf is determined by its values on affine schemes!

Corollary 1.20. *Note that any object in Aff_τ is compact, so the sheaf condition in it is a finite limit!*
So we get: In $Sh(Aff)_\tau$ any filtered colimit can be created in presheaf level, which commutes with any finite limit.

1.3 Completion of an fppf sheaf along a subsheaf

The most following definitions are from [11].

Definition 1.21. *Let $Y \subset X$ is an monomorphism of fppf sheaves on $Sch_{/S}$. We define $Inf_Y^k(X) \subset X$ to be the subsheaf whose value on an S-scheme T are given as follows: for a $t \in X(T)$, $t \in Inf_Y^k(X)(T)$ iff there is an fppf covering $\{T_i \to T\}$ and for each T_i associates a closed subscheme T_i' defined by an ideal whose $k+1$ power is (0) with the property that $t_{T_i'} \in X(T_i')$ is contained in $Y(T_i')$.*

This definition is somewhat general, in most cases we only involve the completion of a scheme along a subscheme.

Example 1.22. *(1) If X and Y are S-schemes and $Y \to U \subset X$ is an immersion, then $Inf_Y^k(X) = Inf_Y^k(U) \simeq \mathrm{Spec}(\mathcal{O}_U/\mathcal{I}^{k+1})$ where $\mathcal{I} \subset \mathcal{O}_U$ is the corresponding quasi-coherent ideal.*

(2) Let $Z \subset X$ be a closed immersion of S-schemes with corresponding quasi-coherent ideal $\mathcal{I}$, then the value of the sheaf $\hat{X}_Z = \varinjlim_k Inf_Z^k(X) = \varinjlim_k \mathrm{Spec}(\mathcal{O}_X/\mathcal{I}^{k+1})$ on a S-scheme T equals $\{t \in X(T) | t^*(\mathcal{I})$ is locally nilpotent$\}$.

We mostly consider the case when Y is a given base point, i.e. $Y(T) = \{*\} = h_S(T)$ for any S-scheme T. In this case we get an endfunctor $\widehat{(-)} : Sh(Sch_{/S})^* \to Sh(Sch_{/S})^*$ by $(X, e) \mapsto (\varinjlim_k Inf_e^k(X), e)$, where $Sh(Sch_{/S})^*$ is denoted as the category of fppf sheaves over S with a basepoint.

We say an $X \in Sh(Sch_{/S})^*$ is complete (ind-infinitesimal in [11]) iff $\hat{X} = X$. It is easy to check we have a natural inclusion $\hat{X} \subset X$, and that $\hat{\hat{X}} \subset \hat{X}$ is a natural isomorphism. So any completion of a pointed fppf sheaf is complete.

Proposition 1.23. *(a) The endfunctor $\widehat{(-)} : Sh(Sch_{/S})^* \to Sh(Sch_{/S})^*$ preserves finite limits. Let $CSh(Sch_{/S})^*$ be the category of complete pointed fppf sheaves, so $CSh(Sch_{/S})^*$ has finite limits, which are created in $Sh(Sch_{/S})^*$.*
(b) $CSh(Sch_{/S})^* \underset{\widehat{(-)}}{\overset{Forget}{\rightleftarrows}} Sh(Sch_{/S})^*$ is an adjoint pair.

(c) $CAb(Sch_{/S}) \underset{\widehat{(-)}}{\overset{Forget}{\rightleftarrows}} Ab(Sch_{/S})$ is an adjoint pair.

Proof: (a) We only need to check $\widehat{(-)}$ preserves final object and pullbacks. The case of final object is obvious.

For a pullback $X \times_Z Y$ we need to show $\widehat{X \times_Z Y} \to \hat{X} \times_{\hat{Z}} \hat{Y}$ is naturally isomorphic. Apparently this is a monomorphism of sheaves. It suffices to show it is an epimorphism. Let $(f,g) \in \Gamma\left(T, \hat{X} \times_{\hat{Z}} \hat{Y}\right)$ where T is affine. Then there is a (finite) covering family $\{T_i \to T\}$ and nilpotent immersions of order $k, \bar{T}_i \longrightarrow T_i$ such that $f \mid \bar{T}_{\mathrm{i}} = 0$. Similarly there is an fppf covering family $\left\{T'_j \to T\right\}$ and nilpotent immersions of order k: $\bar{T}'_j \hookrightarrow T'_j$ corresponding to g.

But $\left\{T_i \times_T T'_j \to T\right\}$ is a covering family, $\bar{T}_i \times_T \bar{T}'_j \to T_i \times_T T'_j$ is a nilpotent immersion of order $2k$ and obviously $(f,g) \mid \bar{T}_i \times_T \bar{T}_j = 0$. Thus $\widehat{X \times_Z Y} \to \hat{X} \times_{\hat{Z}} \hat{Y}$ is an epimorphism, and hence an isomorphism.

And (b),(c) are direct corollaries of (a).

□

2. Formal groups and p-divisible groups

All (big) sheaves involved in 2 will always mean fppf sheaves.

2.1 Linearly topological rings

Before the introduction of formal groups, we need some preliminary knowledge of linear topological rings. In the category of linear topological rings ([14] chap 4), we have an excellent framework to deal with the completion.

Definition 2.1. *A filtration of ideals $\mathfrak{I}$ in R is a non-empty collection of ideals of R such that $\forall I, J \in \mathfrak{I}$, $\exists I' \in \mathfrak{I}$, $I' \subset I \cap J$.*

Lemma 2.2. *Given a filtration of ideals $\mathfrak{I}$ in R, then*

(i) $\{a + I | a \in R, I \in \mathfrak{I}\}$ forms a topological basis in R, and we call it the topology induced by $\mathfrak{I}$.

(ii) The topology induced by $\mathfrak{I}$ makes R become a topological ring.

Proof: *Omitted.*

□

Definition 2.3. *A linearly topological ring R is a topological ring such that the topology induced by the filtrtion of open ideals in R is the same as its topology.*

Proposition 2.4. *A topological ring induced by a filtration of ideals is a linearly topological ring (note this is not a completely trivial statement).*

Example 2.5. *The linear topology induced by $\{I^n|n \geq 1\}$ for an ideal $I \in R$ is called I-adic topology. Note if $I = 0$, then this topology is discrete.*

Let us denote LRings to be the category of linearly topological rings with continuous ring maps.

Proposition 2.6. *[14] Let R, S and T be linearly topological rings, and let $R \to S$ and $R \to T$ be continuous homomorphisms. We then give $S \otimes_R T$ the linear topology defined by the ideals $I \otimes T + S \otimes J$, where I runs over open ideals in S and J runs over open ideals in T. This is easily seen to be the pushout of S and T under R in* LRings*. We conclude* LRings *has finite colimits since the initial object ($\mathbb{Z}$ with the discrete topology) and all pushouts exist in it.*

Proposition 2.7.

(i) Let $\{R_i \mid i \in \mathcal{J}\}$ be a family of objects in LRings*, and write $R = \prod_i R_i$. We give this ring the product topology, then it is the same as the linearly topology defined by the ideals of the form $\prod_i J_i$, where J_i is open in R_i and $J_i = R_i$ for almost all i. So it is easy to check $R = \prod_i R_i$ is the product in* LRings*.*

(ii) Given following morphisms in LRings

$$B \underset{g}{\overset{f}{\rightrightarrows}} C$$

then the subring $a = \{b \in B|f(b) = g(b)\}$ with the linear topology by filtration

$$\{J = I \cap B|I \text{ open in } B\}$$

is the equalizer in LRings*.*

(iii) So we conclude LRings *has any limit.*

Now we start to introduce the completion of linearly topological rings

Definition 2.8. *Let R be a linearly topological ring. The completion of R is the ring $\widehat{R} = \lim_{\leftarrow I} R/I$, where I runs over the open ideals in R. There is an evident* map $R \to \widehat{R}$, *and the composite $R \to \widehat{R} \to R/I$ is surjective so we have $R/I = \widehat{R}/\bar{I}$ for some ideal $\bar{I} \subset \widehat{R}$. These ideals form a filtered system, so we can give $\widehat{R}$ the linear topology for which they are a base of neighbourhoods of zero.*

It is easy to check that $\widehat{\widehat{R}} = \widehat{R}$. We say that R is complete, or that it is a formal ring, if $R = \widehat{R}$. Thus $\widehat{R}$ is always a formal ring. We write FRings *for the category of formal rings.*

Remark 2.9. *It is important to notice that the completion $\widehat{R}$ from an I-adic topology is not always the same as the $I\widehat{R}$-adic topology on $\widehat{R}$! But it is the case when I is finitely generated, see [13] Algebra 96.3.*

Proposition 2.10.

(i) A linearly topological ring with the discrete topology is always complete.

(ii) Let R, S and T be in FRings*, and let $R \to S$ and $R \to T$ be continuous homomorphisms, then $\widehat{S \otimes_R T}$ is easily seen to be the pushout of S and T under R in FLings. We conclude* FRings *has finite colimits since the initial object ($\mathbb{Z}$ with the discrete topology) and all pushouts exist in it.*

(iii) Any limit in FRings *exists and could be created in* LRings.

Definition 2.11. *Let $(R, \mathfrak{m})$ be a local ring, we have a natural linear topology in R by the $\mathfrak{m}$-adic topology. So we get a functor:* LocalRings $\longrightarrow$ LRings. *In fact this functor is fully faithful because of the following lemma, and base on that we will always treat local rings as linearly topological rings.*

Lemma 2.12. *Let $A, B \in$* LRings. *Suppose their linear topology is induced by filtrations $\mathfrak{A}$ and $\mathfrak{B}$ respectively. Let $f : A \longrightarrow B$ be a ring homomorphism. Then f is continuous if and only if $\forall J \in \mathfrak{B}$ there exists $I \in \mathfrak{A}$ such that $f(I) \subset J$.*

Proposition 2.13 ([13] Algebra chap 96,97)**.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring, then*

(i) $(\widehat{R}, \mathfrak{m}\widehat{R})$ is still Noetherian local, and $\widehat{\mathfrak{m}} = \lim_{\leftarrow n} \mathfrak{m}/\mathfrak{m}^n \simeq \mathfrak{m}\widehat{R}$.

(ii) $(R, \mathfrak{m})$ is regular if and only if $(\widehat{R}, \widehat{\mathfrak{m}})$ is.

(iii) The topology on the completion $\widehat{R}$ is the same as the $\widehat{\mathfrak{m}}$-adic topology on it, by 2.9.

Remark 2.14. *If a local ring $(R, \mathfrak{m})$ is not Noetherian, then $(\widehat{R}, \mathfrak{m}\widehat{R})$ is not necessarily local.*

2.2 Formal Lie varieties

We have known that the equivalence of topoi $Sh(Sch)_{fppf} \longrightarrow Sh(Aff)_{fppf}$, so we will be free to exchange things from each other.

Definition 2.15. *Let $\hat{\chi}$ be the full subcategory of $Fun(Rings, Sets)$ which consists of functors $X : Rings \to Sets$ that is a small filtered colimit of corepresentable functors. More precisely, there must be a small filtered category $\mathcal{J}$ and a functor $i \mapsto X_i = Hom(R_i, -)$ such that $X = \varinjlim_i X_i$.*

It is obvious that $\hat{\chi} \subset Sh(Aff)_{fppf}$. Actually $\hat{\chi}$ is the category of "formal schemes" in Strickland's sense [14], which equals $(Pro - Ring)^{op}$ or $Ind - Aff$. And we have fully faithful embeddings

$$FRing \to \hat{\chi}$$

by sending R to $Spf(R) = \varinjlim_{I\, open} \operatorname{Spec} R/I$ and natural inclusion

$$\hat{\chi} \to Sh(Aff)_{fppf}$$

Definition 2.16. *Let $X \in CSh(Sch_{/S})^*$, we call it a pointed formal Lie variety iff zariski locally on S, the F is isomorphic to $Spf(\mathcal{O}_S[[x_1, ..., x_n]])$ as pointed fppf sheaves for some $n \geq 0$.*

Proposition 2.17. *[11] Let $X \in CSh(Sch_{/S})^*$, the following are equivalent*
(1) X is a pointed formal Lie variety.
(2) Zariski locally on S, the X is isomorphic to $Spf(\mathcal{O}_S[[x_1, ..., x_n]])$ as sheaves (not necessarily pointed) for some $n \geq 0$.
(3)
(a) The $\mathrm{Inf}^k(X)$ is representable for all $k \geq 0$.
(b) The $\omega_X = e^*(\Omega_{\mathrm{Inf}^1(X)/S}) = e^*(\Omega_{\mathrm{Inf}^k(X)/S})$ is a finite locally free sheaf on S.
(c) Denoting by $gr_*^{inf}(X)$ the graded $\mathcal{O}_S$-algebra $\bigoplus_{k \geq 0} \mathcal{I}_k^k$, such that $gr_i^{\mathrm{inf}}(X) = gr_i(\mathrm{Inf}^i(X))$ holds for all $i \geq 0$. We have an isomorphism $Sym_S(\omega_X)_* \xrightarrow{\sim} gr_*^{inf}(X)$ induced by the canonical mapping $\omega_X \xrightarrow{\sim} gr_1^{\mathrm{inf}}(X)$.

Proposition 2.18. *Let $X \to S$ be a smooth S-scheme with a base point $e : S \to X \in X(S)$, then $\hat{X}$ is a formal Lie variety.*

Proof: Pick an affine open $U \subset S$ containing s Pick an affine open $V \subset f^{-1}(U)$ containing x. Pick an affine open $U' \subset e^{-1}(V)$ containing s. Note that $V' = f^{-1}(U') \cap V$ is affine as it is equal to the fibre product $V' = U' \times_U V$. Then $f : U' \to V'$ is separated smooth and $e : V' \to U'$ is an section (actually a closed immersion). Then we get that $\hat{X}_{V'} = \hat{U'}_{V'}$. The proposition can be easily deduced from the following lemma.

$\square$

Lemma 2.19. *[13](Algebra 139.4) Let $\varphi : R \to S$ be a smooth ring map. Let $\sigma : S \to R$ be a left inverse to φ. Set $I = \mathrm{Ker}(\sigma)$. Then*
(1) I/I^2 is a finite locally free R-module, and
(2) if I/I^2 is free, then $S^\wedge \cong R[[t_1, \ldots, t_d]]$ as R-linear topological rings, where $S^\wedge$ is the I-adic completion of S.

Proof: *By the exact sequence of Kahler differentials applied to $R \to S \to R$ we see that $I/I^2 = \Omega_{S/R} \otimes_{S,\sigma} R$. Since by definition of a smooth morphism the module $\Omega_{S/R}$ is finite locally free over S we deduce that (1) holds.*

If I/I^2 is free, then choose $f_1, \ldots, f_d \in I$ whose images in I/I^2 form an R-basis. Consider the R-algebra map defined by

$$\Psi : R[[x_1, \ldots, x_d]] \longrightarrow S^\wedge, \quad x_i \longmapsto f_i$$

Denote $P = R[[x_1, \ldots, x_d]]$ and $J = (x_1, \ldots, x_d) \subset P$. We write $\Psi_n : P/J^n \to S/I^n$ for the induced map of quotient rings. Note that $S/I^2 = \varphi(R) \oplus I/I^2$. Thus Ψ_2 is an isomorphism. Denote $\sigma_2 : S/I^2 \to P/J^2$ the inverse of Ψ_2. We will prove by induction on n that for all $n > 2$ there exists an inverse $\sigma_n : S/I^n \to P/J^n$ of Ψ_n. Namely, as S is formally smooth over R we see that in the solid diagram

$$\begin{tikzcd} S \arrow[r, dashed] \arrow[dr, "\sigma_{n-1}"] & P/J^n \arrow[d] \\ & P/J^{n-1} \end{tikzcd}$$

of R-algebras we can fill in the dotted arrow by some R-algebra map $\tau : S \to P/J^n$ making the diagram commute. This induces an R-algebra map $\bar{\tau} : S/I^n \to P/J^n$ which is equal to σ_{n-1} modulo J^n. By construction the map Ψ_n is surjective and now $\bar{\tau} \circ \Psi_n$ is an R-algebra endomorphism of P/J^n which maps x_i to $x_i + \delta_{i,n}$ with $\delta_{i,n} \in J^{n-1}/J^n$. It follows that Ψ_n is an isomorphism and hence it has an inverse σ_n. This proves the lemma.

$\square$

Actually, any formal Lie variety on an affine base can be from the completion of a pointed smooth scheme, as the following.

Proposition 2.20. *Let $X \in CSh(Sch_{/S})^*$ be a formal Lie variety. If $S = \mathrm{Spec}(R)$ is affine, then we have a (non-canonical) isomorphism $X \to Spf(\widehat{Sym}_S(\omega_X))$ as pointed sheaves.*

Proof: Let $I_k \subset \mathcal{O}_X$ be the quasi coherent ideal corresponding $S \to inf^k X$, and $I \to \omega_X \to 0$ be the projection of R-modules. Then we can lift following arrows one-by-one

$$\begin{tikzcd} & \cdots \arrow[d] \\ & I_2 \arrow[d] \\ & I_1 \arrow[d] \\ \omega_X \arrow[r, dash] \arrow[ur, dashed] \arrow[uur, dashed] \arrow[uuur, dashed] & I_0 \end{tikzcd}$$

Hence we get a sequence of isomorphisms

$$\begin{tikzcd} \cdots \arrow[r, "\simeq"] \arrow[d] & \cdots \arrow[d] \\ Sym(\omega_X)/(\omega_X^{k+1}) \arrow[r, "\simeq"] \arrow[d] & \mathcal{O}_{inf^k X} \arrow[d] \\ Sym(\omega_X)/(\omega_X^{k}) \arrow[r, "\simeq"] \arrow[d] & \mathcal{O}_{inf^{k-1} X} \arrow[d] \\ \cdots \arrow[r, "\simeq"] & \cdots \end{tikzcd}$$

which induces an isomorphism $X \to Spf(\widehat{Sym}_S(\omega_X))$.

□

Remark 2.21. *It is worth noting this theorem is based on the fact that a finite locally free sheaf on S is a projective object in $Qcoh(S)$ if S is affine.*

Corollary 2.22. *Let $X \in CSh(Sch_{/S})^*$ be a formal Lie variety (S here is not necessarily assumed to be affine), then X is a formally smooth fppf sheaf, which means $X(Spec(A)) \to X(Spec(A/I))$ is surjective for any $A \to A/I$ over S with a square-zero ideal I.*

Proof: To show that $X(Spec(A)) \to X(Spec(A/I))$ is surjective, we can assume $S = Spec(A)$ is affine. Then it is from the completion of a pointed smooth S-scheme $Y = Spec(Sym_S(\omega_X))$

by the proposition above. So it suffices to show the following is a pullback diagram of sets.

$$\begin{tikzcd}
\hat{Y}(Spec(A)) \arrow[r] \arrow[d, hook, "i"] & \hat{Y}(Spec(A/I)) \arrow[d, hook, "i"] \\
Y(Spec(A)) \arrow[r] & Y(Spec(A/I))
\end{tikzcd}$$

Let $u \in \hat{Y}(Spec(A/I))$, then $u \in Y(Spec(A/I))$ is from an element $v \in Y(Spec(A))$ by the formal smoothness of Y. Now we claim $v \in \hat{Y}(Spec(A))$.

There exists $n \geq 1$ such that $u : Spec(A/I) \to Y$ factors through $u : Spec(A/I) \to inf^k(Y)$ since $u \in \hat{Y}(Spec(A/I))$, then $u|Spec(A/I+J) = 0$ for some nilpotent ideal J. So $v \in \hat{Y}(Spec(A))$ by the fact $I+J$ is still nilpotent.

□

2.3 Formal Lie groups

Definition 2.23. *A formal Lie group is an abelian sheaf $X \in Ab(Sch_{/S})$ whose underlying pointed sheaf is a formal Lie variety.*

We more care about 1-dim formal Lie groups, which are called by "formal group" in most references. In 2.3 we will show that formal groups over an affine basis are equivalent to graded formal group laws on an even weakly periodic graded ring.

Definition 2.24 (EWP)**.** *A graded ring R_* is called EWP(even weakly periodic) iff it satisfies following conditions*
(a) $R_2 \otimes_{R_0} R_{-2} \to R_0$ is isomorphic;
(b) $R_1 = 0$.

Proposition 2.25. *From the definition, for an EWP ring R_* we immediately get*
(1) $R_2 \otimes_{R_0} R_n \to R_{n+2}$ is isomorphic for any $n \in \mathbb{Z}$.
(2) $R_{odd} = 0$.
(3) $R_2 \in Pic(R_0)$ with $(R_2)^{\otimes -1} = R_{-2}$.

Proof: We can directly check $R_* \simeq R[x^{\pm 1}], |x| = 2$ zariski locally on $Spec(R)$ and check these properties zariski locally.

□

Example 2.26. *Let R be a ring, and $L \in Pic(R)$. Then $Sym_R(L^{\pm 1})_* = \bigoplus_{i \in \mathbb{Z}} L^{\otimes i}$ is an EWP ring.*

Now let us calculate the data of a formal group.

Lemma 2.27. *For any $M, N \in Qcoh(S)$, we have*

$$Hom_{Sh(S)^*}(Spf(\widehat{Sym}_S(M)), Spf(\widehat{Sym}_S(N))) = \prod_{i=1}^{+\infty} Hom_{\mathcal{O}_S-Mod}(N, Sym_i(M))$$

Proof: Directly calculate by 1.23.

$\square$

Corollary 2.28. *Let $X, Y \in CSh(Sch_{/S})^*$ be a pointed formal Lie variety of $dim = 1$ over an affine base $S = \operatorname{Spec}(R)$, then*
(1) $Hom_{Sh(S)^*}(X \times X, X) \simeq \prod_{(i,j)|i+j\geq 1} Hom_{\mathcal{O}_S-Mod}(\omega_X, \omega_X^{i+j}) = \prod_{(i,j)|i+j\geq 1} \omega_X^{i+j-1}$ where $Sh(S)^*$ denotes pointed fppf sheaves over S. So any $F \in Hom_{Sh(S)^*}(X \times X, X)$ corresponds an element $F(x,y) \in R_*[[x,y]], |x| = |y| = -2$ where $R_* = Sym_R(\omega_X^{\pm 1})_*$.
If it satisfies the associated (commutative) law then it coincides with a graded formal (commutative) group law on the EWP ring $Sym_R(\omega_X^{\pm 1})_*$ or on $Sym_R(\omega_X)_*$.
(2) We have $Hom_{Sh(S)^*}(X, Y) = \prod_{i=1}^{+\infty} Hom_{\mathcal{O}_S-Mod}(\omega_Y, \omega_X^i)$ and

$$Isom_{Sh(S)^*}(X, Y) = Isom_{\mathcal{O}_S-Mod}(\omega_Y, \omega_X) \times \prod_{i=2}^{+\infty} Hom_{\mathcal{O}_S-Mod}(\omega_Y, \omega_X^i) =$$

$$Isom_{\mathcal{O}_S-Mod}(\omega_Y, \omega_X) \times \prod_{i=2}^{+\infty} \omega_X^{i-1} = Isom_{\mathcal{O}_S-Mod}(\omega_Y, \omega_X) \times \prod_{i=1}^{+\infty} \omega_X^i$$

Theorem 2.29. *Let $p : \mathcal{M}_{FGL_s(EWP)} \to Aff$ be the moduli stack of formal group laws on EWP rings whose objects are pairs (E_*, F) with F a formal group law on E_*, whose morphisms are (oppositely) pairs (ϕ, f) with $\phi : E_{1*} \to E_{2*}$ a morphism of graded rings and $f : \phi^* F_1 \xrightarrow{\simeq} F_2$ an isomorphism of formal group laws on E_{2*}. And $p(E_*, F) = Spec(E_0)$.*

Then The construction in last corollary actually gives an equivalence of moduli stacks

$$\begin{tikzcd} \mathcal{M}_{FG} \arrow[rr, "\sim"] \arrow[dr] & & \mathcal{M}_{FGL_s(EWP)} \arrow[dl] \\ & Aff & \end{tikzcd}$$

Remark 2.30. *This theorem provides a natural* ***graded*** *structure to a 1-dim formal group over an affine base, which is important when we consider the Landweber exact theorem.*

2.4 Barsotti-Tate groups (p-divisible groups)

Following Grothendieck, we prefer the term Barsotti-Tate group because the concept of ”p-divisible group” has a meaning for any abelian group object in an arbitrary category and does not indicate any relation with algebraic geometry.

Definition 2.31. *A Barsotti-Tate group over a base scheme S is an fppf abelian sheaf G in $Ab(Sch/S)$ satisfying the following conditions:*
(1) $\varinjlim_n G[p^n] \to G$ is naturally isomorphic. (p-torsion)
(2) $G \xrightarrow{p} G$ is an epimorphism of abelian sheaves. (p-divisible).
(3) $G[p^n]$ is representable by a scheme finite locally free over S for any $n \geq 1$.

Lemma 2.32. *Let G be an abelian sheaf over S satisfying (1) and (2). Then for any $m, n \geq 0$ we have a short exact sequence of abelian sheaf*

$$0 \to G[p^n] \to G[p^{m+n}] \xrightarrow{p^n} G[p^m] \to 0$$

So by fppf descent theory of finite group schemes [5], the (3) in the definition can be replaced by the following
$(3)'$ $G[p]$ is representable by a scheme finite locally free over S.

Proposition 2.33. *If $G_0 \to G_1 \to ... \to G_n \to ...$ be an sequence of morphisms of abelian sheaves over S satisfying the following conditions:*
(1) G_i is a scheme finite locally free of degree p^{hi} over S, where $h \geq 0$ is a number independent on i;
(2) $G_n \to G_{n+1}$ is a closed immersion for any $n \geq 0$;
(3) $0 \to G_n \to G_{n+1} \xrightarrow{p^n} G_{n+1}$ is exact for any $n \geq 0$,
then $G = \varinjlim G_n$ is a Barsotti-Tate group over S, and $G[p^n] = G_n$ for every $n \geq 0$.

Proof: The condition (3) implies $G_{n+1}[p^n] = G_n$, by induction we get $G_{n+m}[p^n] = G_n$, and hence $G[p^n] = G_n$ and $G = \varinjlim_n G[p^n]$.
On the other hand we get a new exact sequence $0 \to G_n \to G_{m+n} \xrightarrow{p^n} G_m$. We claim $G_{m+n} \xrightarrow{p^n} G_m$ is epimorphic. By fppf descent theory, we have a factorization

$$\begin{tikzcd}
G_{m+n} \arrow[rr, "p^n"] \arrow[dr] & & G_m \\
& G_{m+n}/G_n \arrow[ur, dashed, "i"'] &
\end{tikzcd}$$

where G_{m+n}/G_n is a finite locally free group of degree p^{mi} over S and i is a monomorphism. However, any proper monomorphism is a closed immersion. So i is a closed immersion between finite locally free schemes of the same degree over S, and hence an isomorphism. Let $n = 1$, we get $G_{m+1} \xrightarrow{p} G_m \to 0$. Therefore take the direct colimit about m we get $G \xrightarrow{p} G \to 0$.

□

Remark 2.34. *Actually, the proposition above is a local definition of the Barsotti-Tate group. Because for any BT group G and $s \in S$, $G[p]_s = G_s[p]$ is annihilated by p, which implies its rank must be p^{h_s} for some number h_s by the theory of algebraic groups.*

3. Thom spectrum functor and infinite loop space machine

Before getting into the σ-orientation we introduce two important topological settings which are infinite loop space machine and Thom spectrum functor respectively.

Here we only consider Thom spectra from a map into a classifying space of some topological **group**, from which Thom spectra admit more useful properties compared with from a topological monoid.

Definition 3.1 ([6] Thom spectrum functor)**.** *Let $(f : X \to BO) \in Top_{\downarrow BO}$, then the standard filtration $X_V = f^{-1}(BO(V))$ gives a Thom prespectrum*

$$M_p(f)(V) = Th(E(X_V) \to X_V) = E(X_V)_+ \wedge_{O(V)_+} S^V$$

The spectrification $M(f)$ of $M_p(f)$ is called the Thom spectrum corresponding f.

Remark 3.2. *(i) Actually, any filtration $\varinjlim_{V \subset \mathbb{R}^\infty} F_V X = X$ where $F_V X$ is a closed subspace of X such that $F_V X \subset X_V$ gives the same [6] Thom spectrum (though not the same prespectra).*
(ii) For $G = Sp(\infty), U(\infty), SU(\infty), O(\infty), SO(\infty)$, the construction above also applies.

3.1 Properties of the Thom spectrum functor

For any spectrum $E \in Sp$ and any $V \subset \mathbb{R}^\infty$, $\Omega^\infty E$ admits a right $O(V)$-action since $\Omega^\infty E = E_0 = \Omega^V E_V = F(S^V, E_V)$. These actions are coherent between different V, so we actually get a right O-action on $\Omega^\infty E$.

In the following content we always assume $G = Sp(\infty), U(\infty), SU(\infty), O(\infty)$ or $SO(\infty)$.

Theorem 3.3. *The Thom spectrum functor induces a continuous adjoint pair*

$$Top_{\downarrow BG} \underset{EG\times_G\Omega^\infty(-)}{\overset{M(-)}{\rightleftarrows}} Sp$$

Given a map $(f : X \to BG) \in \mathcal{U}/BG$ *and* $E \in Sp$ *, then*

$$\mathrm{Hom}_{Sp}(Mf, E) = \mathrm{Hom}_{\mathcal{U}[G]}(f^*EG, \Omega^\infty E) = \mathrm{Hom}_{\mathcal{U}/BG}(X, EG \times_G \Omega^\infty E)$$

Proof. Let us denote $\mathcal{U}$ and $\mathcal{S}$ to be the categories of unbased Topological spaces, and spectra respectively. First we have

$$\mathrm{Hom}_{\mathcal{S}}(MX, E) = \mathrm{Hom}_{\mathcal{S}}(\mathrm{colim}_V MX_V, E) = \lim_V \mathrm{Hom}_{\mathcal{S}}(MX_V, E)$$

Second we define EX_V and $Z(V)$ by pullback diagrams,

$$\begin{array}{ccccccc} EX_V & \longrightarrow & B(*, G(V), G(V)) & & Z_V & \longrightarrow & B(*, G(V), G) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X_V & \longrightarrow & B(*, G(V), *) & & X_V & \longrightarrow & B(*, G(V), *) \end{array}$$

then

$\lim_V \mathrm{Hom}_{\mathcal{S}}(MX_V, E) = \lim_V \mathrm{Hom}_{\mathcal{U}_*}(EX_{V+} \wedge_{G(V)} S^V, E_V) = \lim_V \mathrm{Hom}_{\mathcal{U}_*[G_{V+}]}(EX_{V+}, \Omega^V E_V)$

$= \lim_V \mathrm{Hom}_{\mathcal{U}[G_V]}(EX_V, \Omega^\infty E) = \lim_V \mathrm{Hom}_{\mathcal{U}[G]}(EX_V \times_{G_V} G, \Omega^\infty E) = \lim_V \mathrm{Hom}_{\mathcal{U}[G]}(Z_V, \Omega^\infty E) = \mathrm{Hom}_G(p^*X, \Omega^\infty E)$

Since equivariant maps from a principle G-bundle to a G-space are equivalent to the following sections, we can conclude

$$\mathrm{Hom}_G(p^*X, \Omega^\infty E) = \mathrm{Hom}_{\mathcal{U}/X}(X, p^*X \times_G \Omega^\infty E) = \mathrm{Hom}_{\mathcal{U}/BG}(X, EG \times_G \Omega^\infty E)$$

□

Proposition 3.4. *This adjunction* $Top_{\downarrow BG} \underset{EG\times_G\Omega^\infty(-)}{\overset{M(-)}{\rightleftarrows}} Sp$ *is actually a Quillen adjunction since* $M(S^{n-1} \to D^n)$ *is a cell pair of spectra and* $M(D^n \times 0 \to D^n \times I)$ *is a weak equivalent cell pair for those morphisms over BG.*

Proposition 3.5. *Let* $f : X \to BG$ *be a map and* A *a space. Let* g *be the composite* $X \times A \to X \to BG$*, where the first map is the projection away from* A*. Then* $T(g) = A_+ \wedge T(f)$*, which implies Thom spectrum functor preserves tensors, and hence is a topological Quillen functor.*

Proposition 3.6. *Thom spectrum functor* $T(-)$ *preserves weak equivalences. Any Thom spectrum* $T(f)$ *from a map* $F : X \to BG$ *is* (-1)*-connective.*

3.2 Monads and Thom spectrum functor

Proposition 3.7. *Let $\mathcal{V}_1, \mathcal{V}_2$ be two real universes.*
(i) Given maps $B \to \mathcal{L}(V_1, V_2)$ and $f : X \to BO(\mathcal{V}_1)$, denote g to be the composition $B \times X \to B \times BO(\mathcal{V}_1) \to BO(\mathcal{V}_2)$. Then we have the natural isomorphism $T(g) \cong B \ltimes T(f)$.
(ii) Given maps $f : X \to BO(\mathcal{V}_1)$ and $g : Y \to BO(\mathcal{V}_2)$, denote $f \times g$ to be the composition $X \times Y \to BO(\mathcal{V}_1) \times BO(\mathcal{V}_2) \to BO(\mathcal{V}_1 \oplus \mathcal{V}_2)$. Then $T(f \times g) \cong T(f) \,\bar{\wedge}\, T(g)$.

Proposition 3.8. *Let $\mathcal{L}(n) = \mathcal{L}(\mathbb{R}^{\infty \times n}, \mathbb{R}^\infty)$, then for any map $f : X \to BO$ we have*

$$T(g) = \bigvee_{n \geq 0} \mathcal{L}(n) \times_{\Sigma_n} T(f)^{\bar{\wedge} n}$$

where g is the composition $\coprod_{n \geq 0} \mathcal{L}(n) \times_{\Sigma_n} X^n \to \coprod_{n \geq 0} \mathcal{L}(n) \times_{\Sigma_n} BO^n \to BO$.

Now we introduce a quite useful lemma [3] which tells how to get the adjoint functor between monadic algebra categories.

Lemma 3.9. *Let $\mathcal{C}$ and $\mathcal{D}$ be topological bicomplete categories, and $\mathbb{A} : \mathcal{C} \to \mathcal{C}$ and $\mathbb{B} : \mathcal{D} \to \mathcal{D}$ be continuous monads. Further suppose that there is a continuous functor $F : \mathcal{C} \to \mathcal{D}$ which is coherent with the monad structure and therefore yields a functor $F : \mathcal{C}[\mathbb{A}] \to \mathcal{D}[\mathbb{B}]$.*

If $F : \mathcal{C} \to \mathcal{D}$ is left adjoint functor preserving tensors, and the monads $\mathbb{A}$ and $\mathbb{B}$ preserve reflexive coequalizers, then $F : \mathcal{C}[\mathbb{A}] \to \mathcal{D}[\mathbb{B}]$ is still a left adjoint functor preserving tensors.

Corollary 3.10. *Thom spectrum functor induces topological Quillen adjoint pairs*

$$Top[\mathcal{L}(1)]_{\downarrow BO} \rightleftharpoons Sp[\mathcal{L}(1)] \quad \text{and} \quad Top[E_\infty]_{\downarrow BO} \rightleftharpoons Sp[E_\infty]$$

where the $\mathcal{L}(1)$-spectrum is the $\mathbb{L}$-spectrum in EKMM [4] sense.

Remark 3.11. *This section 3.2 also applies to $G = U(\infty)$ or $G = Sp(\infty)$ if we replace real isometries operad by complex or symplectic isometries operads.*

3.3 Diagonal and Thom isomorphism

Definition 3.12 (coaction)**.** *For any map $f : X \to BG$, the diagonal induces a coaction $X \to X \times X$ in $Top_{\downarrow BG}$, where $X \times X \to BG$ is the projection of the second variable. It gives a natural coaction on Thom spectra: $Mf \to X_+ \wedge Mf$.*

Definition 3.13 (Thom morphism [6]). *With the same hypothesis above, given a homotopy commutative phantom ring spectrum (a commutative monoid in $Ho(Sp)/phantoms$) E and a morphism of spectra $Mf \to E$ we have a natural morphism $E \wedge Mf \to E \wedge X_+ \wedge Mf \to E \wedge X_+ \wedge E \to E \wedge X_+$ in $Ho(Sp)/phantoms$. It induces a natural homological morphism $\phi_f : E_*(Mf) \to E_*(X)$.*

Under certain condition ϕ_f will be an isomorphism, which is called Thom isomorphism.

Theorem 3.14 (Thom isomorphism). *Let $G = Sp(\infty), U(\infty), SU(\infty), O(\infty), SO(\infty)$ or $Spin(\infty)$. Let E be a homotopy commutative ring (phantom) spectrum.*
(i) Given a (phantom) ring spectrum morphism $MG \to E$, then for any map $X \to BG$ the Thom morphism $E_*(Mf) \to E_*(X)$ is an isomorphism.
Moreover, if X is E_∞ and f is an E_∞ map, then $E_*(Mf) \to E_*(X)$ is an isomorphism of E_*-algebras.
(ii) Given an E_∞ space X and an E_∞ map $f : X \to BG$. Let $Mf \to E$ be a (phantom) ring spectrum morphism. If X is 0-connected, then $E_*(Mf) \to E_*(X)$ is an isomorphism of E_*-algebras.

Example 3.15. *Let $MO \to H\mathbb{Z}/2$ and $MU \to H\mathbb{Z}$ be ring spectrum morphisms from the 0-th postnikov tower. Then we have natural Thom isomorphisms $H_*(MO; \mathbb{Z}/2) \to H_*(BO; \mathbb{Z}/2)$ and $H_*(MU) \to H_*(BU)$.*

3.4 Infinite loop space machine

Now we turn to the infinite loop space machine, which is an important technique in stable homotopy theory.

Definition 3.16. *(1). A commutative H-space space X i.e. a commutative monoid in $Ho(Top)$ is called group-like iff the monoid $\pi_0(X)$ is a group.*
(2). We define group-like E_∞-spaces as infinite loop spaces.
(3). Let $X \to Y$ be an H-map between commutative H-spaces, we call it the completion map of X iff $\pi_0(Y)$ is a group and $H_*(X)[(\pi_0 X)^{-1}] \to H_*(Y)$ is isomorphic.

Now let me introduce the existence and uniqueness of additive infinite loop space machine.

Theorem 3.17 ([1] Additive infinite loop space machine). *Let C be a cofibrant unital E_∞ operad in Top and $f : C_* \to \Omega^\infty\Sigma^\infty$ be a morphism of monads on Top_*. Then the Quillen pair (Σ^f, Ω^f) induces a equivalence of categories if we restrict it to the following Top-enriched* *subcategories (so actually an equivalence of ∞-categories)*

$$\text{group-like } Ho(E_\infty\text{-spaces}) \rightleftharpoons (-1)\text{-connective } Ho(Sp)$$

where $\Sigma^f(-) = \Sigma^\infty \otimes_{C_*} (-)$ is the coequalizer of the following diagram in Sp

$$\begin{tikzcd}
\Sigma^\infty C_* X \arrow[rr, shift left, "\Sigma^\infty \mu"] \arrow[rr, shift right] \arrow[dr] & & \Sigma^\infty X \arrow[r] & \Sigma^f X \\
& \Sigma^\infty \Omega^\infty \Sigma^\infty X \arrow[ur] & &
\end{tikzcd}$$

And $\Omega^f X = \Omega^\infty X$ is endowed with the C_*-action $C_*\Omega^\infty X \to \Omega^\infty\Sigma^\infty\Omega^\infty X \to \Omega^\infty X$.

Theorem 3.18 ([9] Uniqueness of additive infinite loop space machine)**.** *We define an (additive) infinite loop space machine to be an adjoint pair (F, G)*

$$Ho(E_\infty\text{-spaces}) \underset{G}{\overset{F}{\rightleftarrows}} (-1)\text{-connective } Ho(Sp)$$

such that
(1) The composition (-1)-connective $Ho(Sp) \xrightarrow{G} Ho(E_\infty$-spaces$) \to CMon(Ho(Top_*))$ is equivalent to Ω^∞;
(2) For any $X \in Ho(E_\infty$-spaces$)$, $X \to GF(X)$ is a group completion, which means $\pi_0 GF(X)$ is a group and $H_*(X)[(\pi_0 X)^{-1}] \to H_* GF(X)$ is isomorphic.

Now, if (F_1, G_1) and (F_2, G_2) are two infinite loop space machines, then there exists a natural equivalence between F_1 and F_2.

Remark 3.19. *The existence of an additive infinite loop space machine (F, G) implies that for any group-like E_∞-space X, the induced pointed H-space is actually an H-group because $X \cong \Omega^\infty FX$ in $CMon(Ho(Top_*))$ and $\Omega^\infty FX$ is a pointed H-group.*

Furthermore, beyond the additive, there exists multiplicative infinite loop space machine as the following constructed by May:

Theorem 3.20 ([10] Multiplicative infinite loop space machine)**.** *Let K be the Steiner E_∞ operad. We can construct a explicit morphism of monads $f : K_* \to \Omega^\infty\Sigma^\infty$ on Top_*, which further induces a morphism of monads on $Top_*[\mathcal{L}_+]$ where $\mathcal{L}$ is the real linear isometries operad. Then the Quillen pair (Σ^f_m, Ω^f_m) induces a equivalence of categories if we restrict it to the following subcategories (enriched in $Ho(Top)$.)*

$$\text{ring-like } Ho(E_\infty\text{-ring spaces}) \rightleftharpoons (-1)\text{-connective } Ho(E_\infty\text{-}Sp)$$

where E_∞-ring spaces means $(Top_*[\mathcal{L}_+])[K_*]$ and "ring like" means it is group-like after forgetting in $Top_*[K_*]$. The $\Sigma^f_m(-) = \Sigma^\infty \otimes_{K_*} (-)$ here should be the coequalizer of the following diagram in $Sp[\mathcal{L}]$ instead of in Sp in the additive case.

$$\begin{tikzcd}
\Sigma^\infty K_* X \arrow[rr, shift left, "\Sigma^\infty \mu"] \arrow[rr, shift right] \arrow[dr] & & \Sigma^\infty X \arrow[r] & \Sigma^f_m X \\
& \Sigma^\infty \Omega^\infty \Sigma^\infty X \arrow[ur] & &
\end{tikzcd}$$

And $\Omega^f_m X = \Omega^\infty X$ is endowed with the K_*-action $K_*\Omega^\infty X \to \Omega^\infty \Sigma^\infty \Omega^\infty X \to \Omega^\infty X$.

Remark 3.21. *(1) Note that for a unital operad C on Top, the C_* and C_+ are different constructions of operads on Top_*. The C_+ is added to an extra base point, while the $C_*(X)$ for an $X \in Top_*$ is defined as the following pushout diagram in $Top[C]$, which makes $C_*(X)$ become an object in Top_* by $C(\varnothing) = * \to C_*(X)$.*

$$\begin{tikzcd}
C(*) \arrow[r] \arrow[d] & C(\varnothing) = * \arrow[d] \\
C(X) \arrow[r] & C_*(X)
\end{tikzcd}$$

(2) An E_∞-ring space, i.e. an object in $(Top_*[\mathcal{L}_+])[K_*]$, can induce an additive monoid in $(Ho(Top_*), \times)$ and a multiplicative monoid in $(Ho(Top_*), \wedge)$, i.e. a semi-ring object in $(Ho(Top_*), \times, \wedge)$.

3.5 The E_∞-structure of $MString$ and $MU\langle 6 \rangle$

We also consider the connective complex K-theory bu. By strategy of [8], $bu = L\Sigma^f_m(\coprod_{i\geq 0} BU(i))$ 3.17 which means bu is a connective E_∞-ring and $bu^* = \mathbb{Z}[v], |v| = -2$.
We define $BU\langle 2k \rangle = R\Omega^f(\Sigma^{2k} bu)$, a group-like E_∞-space, then $bu^{2t}(X) = [X, BU\langle 2t \rangle]$. When $t = 0$, actually we have $BU\langle 0 \rangle = \mathbb{Z} \times BU$ in $Ho(Top)$.

Multiplication by $v^t : \Sigma^{2t} bu \to bu$ gives the $(2t-1)$-connective cover of bu. Under this identification, we get a sequence of morphisms in $Ho(Top[E_\infty])$ by the infinite loop space machine

$$\ldots \to BU\langle 2k \rangle \to \ldots \to BU\langle 6 \rangle \to BSU \to BU \to BU\langle 0 \rangle$$

derived from infinite loop space machine.

However, in order to get a Thom spectrum we need an actual over-map instead of a homotopy class of over-map which is what we only have now. The similar problem also appeared in [14]P87.

Lemma 3.22. *Let Sp denote the ∞-category of spectra, then the inclusions $Sp_{\geq n} \subset Sp_{\geq 0}, n \geq 0$ and $Sp_{\geq 0} \subset Sp$ are coreflective subcategories, which means the inclusion admits a left adjunction.*

Proof. It is a direct conclusion from the canonical t-structure on Sp. □

Corollary 3.23. *(1) By the infinite loop space machine, for any $n \geq 0$ the ∞-category of $(n-1)$-connective group-like E_∞-spaces $\mathcal{S}[E_\infty]^{gl}_{\geq n} \subset \mathcal{S}[E_\infty]^{gl}$ is a coreflective subcategory.*
(2) Given an $(n-1)$-connective covering $X_n \to X$ of group-like E_∞-spaces, $Y \in \mathcal{S}[E_\infty]^{gl}_{\geq n}$ and an arrow $f: Y \to X$, then $Map_{\mathcal{S}[E_\infty]^{gl}_{/X}}(Y, X_n)$ is contractible.

proof of (2): It follows from the following homotopy pullback diagram of spaces.

$$\begin{tikzcd} Map_{\mathcal{S}[E_\infty]^{gl}_{/X}}(Y, X_n) \arrow[r] \arrow[d] & Map_{\mathcal{S}[E_\infty]^{gl}}(Y, X_n) \arrow[d, "\sim"] \\ * \arrow[r, "\{f\}"] & Map_{\mathcal{S}[E_\infty]^{gl}}(Y, X) \end{tikzcd}$$

The corollary illustrates there a unique n-connective cover of a group like E_∞-space up to contractible choices.

Proposition 3.24 (E_∞ structure of $MO\langle n\rangle$ and $MU\langle 2k\rangle$)**.** *By the contractiblity above, we get for any group-like E_∞-space X the full sub ∞-category $Cov_n(X) \subset \mathcal{S}[E_\infty]^{gl}_{/X}$ is a contractible Kan complex.*

Using the E_∞ Thom spectrum functor3.10 over BO, we get contractible choices for $MO\langle n\rangle$ or $MU\langle 2k\rangle$ when we take $X = BO$ and $X = BU$ respectively. Moreover, there is a unique E_∞-map(up to contractible choices) $MU\langle 2k\rangle \to MO\langle 2k\rangle$ determined by the canonical E_∞ map $BU \to BO$.

$$\begin{tikzcd} \dots \arrow[r] \arrow[d] & MU\langle 6\rangle \arrow[r] \arrow[d] & MSU = MU\langle 4\rangle \arrow[r] \arrow[d] & MU \arrow[d] \arrow[dr] & \\ \dots \arrow[r] & Mstring = MO\langle 6\rangle = MO\langle 8\rangle \arrow[r] & MSpin = MO\langle 4\rangle \arrow[r] & MSO \arrow[r] & MO \end{tikzcd}$$

4. σ-orientation

We know that any commutative ring spectrum E with $E_{odd} = 0$ (actually $E_{2n+1} = 0$ for every $n \geq 1$ suffices) is complex orientable. So any elliptic cohomology theory is complex orientable. However, we can not find a canonical complex orientation on an elliptic cohomology theory without extra data.

But it can be well done if we consider $MU\langle 6\rangle$-orientation. The main result in [2] is that $MU\langle 6\rangle$-orientations of an EWP(2.24) ring spectrum E coincides with cubical structures of the bundle $\mathcal{I}(0)$ on $Spf(E^0CP^\infty)$.

Firstly define the map $\rho_0 : P \to 1 \times BU \subset BU\langle 0\rangle$ just to be the map classifying the tautological line bundle L.
As for $t > 0$, let $L_1, \ldots, L_t$ be the obvious line bundles over P^t. Let $x_i \in bu^2(P^t)$ be the bu-theory Euler class, given by the formula

$$vx_i = 1 - L_i.$$

Then we have the isomorphisms

$$bu^*\left(P^t\right) \cong \mathbb{Z}[v][[x_1, \ldots, x_t]]$$

The class $\prod_i x_i \in bu^{2t}(P^t)$ gives the map $\rho_t : P^t = (\mathbb{C}P^\infty)^t \to BU\langle 2t\rangle$.

Remark 4.1. *Note that the composition* $P^t \xrightarrow{\rho_t} BU\langle 2t\rangle \to BU\langle 0\rangle$ *classifies the bundle* $\prod_i (1 - L_i)$.

Definition 4.2. *We say a space X to be "even" iff $H_*(X)$ is concentrated in even degrees and $H_n(X)$ is free abelian for all n.*

Lemma 4.3 (Hatcher 4C.1)**.** *If X is even and simply-connected, then there exists a CW approximation $W \to X$ such that W only consists of cells of even degrees.*

Proposition 4.4. *(1) Let E be an EWP commutative ring (phantom-)spectrum. Then for any even space X, the A-T spectral sequence $H_*(X; E_*) \implies E_*(X)$ collapses . Therefore $E_*(X)$ is a free E_*-module and $E^*(X) \to Hom^*_{E_*}(E_*X, E_*)$ is bijective.*
(2) The $E_0(X)$ is a cocommutative E_0-coalgebra by kunneth theorem. If X is an even H-space, we define $X_E = \operatorname{Spf} E^0X$, then the natural Cartier morphism $\operatorname{Spec} E_0X \to \underline{Hom}_{Grp/E}(X_E, \mathbb{G}_{m,E})$ *is isomorphic, which is the special Cartier duality.*

4.1 n-cocycles in a Cartesian monoidal category

Definition 4.5. *Let C be a category admitting finite products. If A and T are abelian monoid objects in $CMon(C)$, we define $C^0(A,T)$ to be the group*

$$C^0(A,T) \overset{def}{=} \mathrm{Hom}_C(A,T)$$

and for $k \geq 1$ we let $C^k(A,T)$ be the subgroup of $f \in \mathrm{Hom}_C(A^k,T)$ such that
(a) $f(a_1,\ldots,a_{k-1},0) = 0$
(b) $f(a_1,\ldots,a_k)$ is symmetric in the a_i;
(c) $f(a_1,a_2,a_3,\ldots,a_k)+f(a_0,a_1+a_2,a_3,\ldots,a_k) = f(a_0+a_1,a_2,a_3,\ldots,a_k)+f(a_0,a_1,a_3,\ldots,a_k)$, whe
2.

Remark 4.6. *We refer to (c) as the "cocycle" condition for f. If T is an abelian group object, then in definition (a) can be replaced by (a)': $f(0,0,...,0) = 0$.*

4.2 n-cocycles in algebraic geometry and topology

From definition we can make n-cocycles a sheaf as the following: let X, Y are commutative momoid fppf sheaves over S, we define $\underline{C}^k(X,Y)(T) = C^k(X_T, Y_T)$. It is actually a representable commutative monoid sheaf in $Sh(Sch/S)_{fppf}$ in certain case.

Proposition 4.7. *Let G be a formal group over a scheme S. Then for all k, the functor $\underline{C}^k(G,\mathbb{G}_m)$ is an S-affine commutative group scheme.*

Proof: It suffices to work $k \geq 1$ and locally on S, so we may assume $S = \mathrm{Spec}(R)$ and choose a coordinate x on G. We define power series $g_0,\ldots,g_k$ by

$$g_i = \begin{cases} i = 0 & f(0,\ldots,0) \\ i < k & f(x_1,\ldots,x_{i-1},x_{i+1},x_i,\ldots,x_k)\, f(x_1,\ldots,x_k)^{-1} \\ i = k & f(x_1,\ldots,x_k)\, f(x_0 +_F x_1, x_2,\ldots)^{-1} f(x_0, x_1 +_F x_2,\ldots) f(x_0,x_1,x_3,\ldots)^{-1} \end{cases}$$

Let I be the ideal in R generated by all the coefficients of all the power series $g_i - 1$. It is not hard to check $\mathrm{Spec}(R/I)$ has the universal property that defines $\underline{C}^k(G,\mathbb{G}_m)$.

5. AHR E_∞-refinement

References

[1] Matthew Ando, Andrew J Blumberg, David Gepner, Michael J Hopkins, and Charles Rezk. Units of ring spectra, orientations, and thom spectra via rigid infinite loop space theory. *Journal of Topology*, 7(4):1077–1117, 2014. 3.17

[2] Matthew Ando, Michael J Hopkins, and Neil P Strickland. Elliptic spectra, the witten genus and the theorem of the cube. *Inventiones Mathematicae*, 146(3):595, 2001. 4

[3] Andrew J Blumberg. Topological hochschild homology of thom spectra which are e∞-ring spectra. *Journal of Topology*, 3(3):535–560, 2010. 3.2

[4] AD Elmendorf, I Kriz, MA Mandell, JP May, and JPC Greenlees. Rings, modules, and algebras in stable homotopy theory. *Bulletin of the London Mathematical Society*, 31(150):367–369, 1999. 3.10

[5] Gabriel and Demazure. *Groupes algébriques*. Springer, 1970. 2.32

[6] L Gaunce Jr, J Peter May, Mark Steinberger, et al. *Equivariant stable homotopy theory*, volume 1213. Springer, 2006. 3.1, 3.2, 3.13

[7] J. Lurie. *Kerodon*. version 2023.04.24.

[8] J Peter May. *E∞ ring spaces and E∞ ring spectra*. 1977. 3.5

[9] J Peter May and Robert Thomason. The uniqueness of infinite loop space machines. *Topology*, 17(3):205–224, 1978. 3.18

[10] JP May. What precisely are e∞ ring spaces and e∞ ring spectra? *Geometry & Topology Monographs*, 16:215–282, 2009. 3.20

[11] William Messing. The crystals associated to barsotti-tate groups. *The crystals associated to Barsotti-Tate groups: with applications to abelian schemes*, pages 112–149, 2006. 1.3, 1.3, 2.17

[12] Martin Olsson. *Algebraic spaces and stacks*, volume 62. American Mathematical Soc., 2016. 1.17

[13] The Stacks Project Authors. *Stacks Project*. 2018. 1, 1.1, 2.9, 2.13, 2.19

[14] Neil P Strickland. Formal schemes and formal groups. *Contemporary Mathematics*, 239:263–352, 1999. 2.1, 2.6, 2.2, 3.5