Higher algebra in t-structured tensor triangulated ∞-categories (Draft)

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Abstract

Many concepts from higher algebra—such as finitely presented, flat, and étale morphisms of $\mathbb{E}_\infty$-rings—can be naturally generalized to the setting of t-structured tensor triangulated ∞-categories (ttt-∞-categories).

Under a natural structural condition we call "projective rigidity", we establish analogues of Lazard's theorem, étale rigidity, and the universal property of the derived category. We show that projective rigidity holds in many familiar examples, including the ∞-categories of spectra, filtered spectra, graded spectra, genuine G-spectra for finite groups G, and Artin–Tate motivic spectra over a perfect field—all equipped with their standard t-structures.

Contents

Introduction

The flatness of a module over a connective $\mathbb{E}_\infty$-ring can be characterized by the t-exactness of the tensor product functor. This observation suggests that many concepts from higher algebra can be meaningfully generalized to the setting of t-structured tensor triangulated ∞-categories (ttt-∞-categories), including (faithfully) flat morphisms, étale morphisms, and more. Similar ideas were developed in [KM25; BKK24; KKM22; Rak20; Man23].

This paper makes heavy use of the theory of ∞-categories and related notions, as developed in Jacob Lurie's three big books [HTT; HA; SAG].

Warning 0.1. This document is a draft in progress and should be used at your own risk. Specifically, the statements marked with "??" are likely true, but we have not yet figured out a proof of disproof. Any thoughts or ideas regarding these statements are welcome!

Convention 0.2. *Throughout this paper, we fix a t-structured tensor triangulated ∞-category (ttt-∞-category) $(\mathcal{A}^{\otimes}, \mathcal{A}_{\geq 0}, \mathcal{A}_{\leq 0})$. This consists of a presentably, stably, symmetric monoidal ∞-category $\mathcal{A}^{\otimes} \in \mathrm{CAlg}(\mathcal{P}\mathrm{r}^L_{\mathrm{st}})$ equipped with an accessible t-structure $(\mathcal{A}_{\geq 0}, \mathcal{A}_{\leq 0})$ that is compatible with the monoidal structure in the following sense:*

(1) *The unit $\mathbf{1} \in \mathcal{A}_{\geq 0}$;*

(2) *For any two connective objects $X, Y \in \mathcal{A}_{\geq 0}$, we have $X \otimes Y \in \mathcal{A}_{\geq 0}$.*

Convention 0.3. (1) *Since a t-structure is determined by its connective part, we will simply denote a ttt-∞-category by $(\mathcal{A}^{\otimes}, \mathcal{A}_{\geq 0})$ rather than $(\mathcal{A}^{\otimes}, \mathcal{A}_{\geq 0}, \mathcal{A}_{\leq 0})$.*

(2) *Some references refer to a presentably, stably, symmetric monoidal ∞-category as a "big" tt-∞-category. However, we do not deal with small ones in this paper and we will omit the prefix "big"; thus, when we use the terms tt-∞-category or ttt-∞-category, they are implicitly assumed to be big.*

(3) *We will often say $\mathcal{A}^{\otimes}$ (or simply $\mathcal{A}$ when the context is clear) satisfies a property P if the full structure $(\mathcal{A}^{\otimes}, \mathcal{A}_{\geq 0}, \mathcal{A}_{\leq 0})$ satisfies property P. For example, we will say $\mathcal{A}$ is right complete if its t-structure is right complete.*

We list our main results as follows.

Theorem 0.4 (Proposition 4.12, Theorem 4.18, and Proposition 5.7)**.** *Assume that $\mathcal{A}$ is Grothendieck (see Definition 1.14) and that $\mathcal{A}_{\geq 0}$ is projectively generated. Let $R \in \mathrm{Alg}(\mathcal{A}_{\geq 0})$. Then the following hold:*

(1) *The inclusion $\mathrm{LMod}_R(\mathcal{A}_{\geq 0})^{\mathrm{cproj}} \hookrightarrow \mathrm{LMod}_R(\mathcal{A}_{\geq 0})^{\mathrm{proj}}$ of the full subcategory of compact projective R-modules into that of all projective R-modules induces an equivalence*

$$\mathcal{P}^{\infty\sqcup,\mathrm{Idem}}_{\sqcup}(\mathrm{LMod}_R(\mathcal{A}_{\geq 0})^{\mathrm{cproj}}) \simeq \mathrm{LMod}_R(\mathcal{A}_{\geq 0})^{\mathrm{proj}},$$

where the left-hand side is the relative cocompletion obtained by formally adding small coproducts and idempotent completions while preserving existing finite coproducts. (See Theorem B.2 for the construction of the relative cocompletion.)

(2) *(Lazard's Theorem.) If $\mathcal{A}^{\otimes}_{\geq 0}$ is projectively rigid (see Definition 4.13), then the inclusion $\mathrm{LMod}_R(\mathcal{A}_{\geq 0})^{\mathrm{cproj}} \hookrightarrow \mathrm{LMod}_R(\mathcal{A})^{fl}$ of compact projective R-modules into flat R-modules induces an equivalence*

$$\mathcal{P}^{\sqcup,\mathrm{fil}}_{\sqcup}(\mathrm{LMod}_R(\mathcal{A}_{\geq 0})^{\mathrm{cproj}}) \simeq \mathrm{LMod}_R(\mathcal{A})^{fl},$$

where the left-hand side is the relative cocompletion obtained by formally adding small filtered colimits and finite coproducts while preserving existing finite coproducts. Alternatively, flat R-modules can be described as:

$$\mathrm{Ind}(\mathrm{LMod}_R(\mathcal{A}_{\geq 0})^{\mathrm{cproj}}) \simeq \mathrm{LMod}_R(\mathcal{A})^{fl}.$$

(3) *The inclusion $\mathrm{LMod}_R(\mathcal{A}_{\geq 0})^{\mathrm{cproj}} \hookrightarrow \mathrm{LMod}_R(\mathcal{A}_{\geq 0})^{\mathrm{aperf}}$ of compact projective R-modules into almost perfect R-modules induces an equivalence*

$$\mathcal{P}^{\sqcup,\Delta^{\mathrm{op}}}_{\sqcup}(\mathrm{LMod}_R(\mathcal{A}_{\geq 0})^{\mathrm{cproj}}) \simeq \mathrm{LMod}_R(\mathcal{A}_{\geq 0})^{\mathrm{aperf}},$$

where the left-hand side is the relative cocompletion obtained by formally adding geometric realizations and finite coproducts while preserving existing finite coproducts. Alternatively, almost perfect R-modules can be described as:

$$\mathcal{P}^{\Delta^{\mathrm{op}}}_{\emptyset}(\mathrm{LMod}_R(\mathcal{A}_{\geq 0})^{\mathrm{cproj}}) \simeq \mathrm{LMod}_R(\mathcal{A}_{\geq 0})^{\mathrm{aperf}}.$$

Theorem 0.5 (Theorem 4.19)**.** *Assume that $\mathcal{A}$ is Grothendieck and $\mathcal{A}_{\geq 0}$ is projectively generated. Then the following hold:*

(1) *For any discrete $R \in \mathrm{Alg}(\mathcal{A}^{\heartsuit})$ there exists a (unique up to contractible choices) equivalence in $\mathcal{P}\mathrm{r}_{\mathrm{st}}^{t\text{-}rex}$*

$$\mathcal{D}(\mathrm{LMod}_{\pi_0 R}(\mathcal{A}^{\heartsuit})) \xrightarrow{\sim} \mathrm{LMod}_R(\mathcal{A})$$

which induces the identity functor on the heart.

(2) *Assume that the $\mathcal{A}_{\geq 0}^{\otimes}$ is projectively rigid. Then for any discrete commutative algebra $R \in \mathrm{CAlg}(\mathcal{A}^{\heartsuit})$ there exists a (unique up to contractible choices) equivalence in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}_{\mathrm{st}}^{t\text{-}rex})$*

$$\mathcal{D}(\mathrm{Mod}_{\pi_0 R}(\mathcal{A}^{\heartsuit}))^{\otimes} \xrightarrow{\sim} \mathrm{Mod}_R(\mathcal{A})^{\otimes}$$

which induces the identity functor on the heart, where the symmetric monoidal structure on left-hand side is induced by projective model with tensor product of chain complexes.

Since our examples only admit enough projectives but not enough frees, the Zariski topology should admit a basis of "principle Cohn localizations".

Theorem 0.6 (Theorem 4.21, Cohn localization)**.** *Assume that $\mathcal{A}$ is Grothendieck and $\mathcal{A}_{\geq 0}^{\otimes}$ is projectively rigid. Let $R \in \mathrm{CAlg}(\mathcal{A}_{\geq 0})$, and let S be a set of morphisms between compact projective R-modules. Then there exists a Cohn localization, a map $R \to R[S^{-1}]$ in $\mathrm{CAlg}(\mathcal{A}_{\geq 0})$, satisfying the following universal property: For any $B \in \mathrm{CAlg}(\mathcal{A})$, the map*

$$\mathrm{Map}_{\mathrm{CAlg}(\mathcal{A})}(R[S^{-1}], B) \to \mathrm{Map}_{\mathrm{CAlg}(\mathcal{A})}(R, B),$$

induced by precomposition, is (-1)-truncated.

Furthermore, the image of the induced map on connected components consists of precisely those (homotopy classes of) maps $R \to B$ such that for each morphism $f \in S$, the map $B \otimes_R f$ is an equivalence of B-modules.

We also prove the following higher version of Nakayama's Lemma. (See [HS24, §2] for some applications in the context of almost $\mathbb{E}_{\infty}$-rings.)

Theorem 0.7 (Theorem 6.5, Nakayama's Lemma)**.** *Assume that $\mathcal{A}$ is hypercomplete. Let $\tilde{A} \to A$ be a nilpotent thickening in $\mathrm{CAlg}(\mathcal{A}_{\geq 0})$. Then the base change functor $(-) \otimes_{\tilde{A}} A$, when restricted to modules that are bounded below,*

$$\mathrm{Mod}_{\tilde{A}}(\mathcal{A})^{-} \to \mathrm{Mod}_A(\mathcal{A})^{-},$$

is conservative.

Theorem 0.8 (Theorem 6.19)**.** *Suppose $\mathcal{A}$ is Grothendieck and $\mathcal{A}_{\geq 0}^{\otimes}$ is projectively rigid. Let $f : A \to B$ be a faithfully flat morphism in $\mathrm{CAlg}(\mathcal{A})$ such that $\pi_0 B$ is an $\aleph_n$-compact $\pi_0 A$-module for some integer $n \geq 0$. Then f is descendable.*

We also proved the following version of étale rigidity.

Theorem 0.9 (Theorem 7.24, Étale rigidity)**.** *Assume that $\mathcal{A}$ is Grothendieck and left complete. Let $A \in \mathrm{CAlg}(\mathcal{A})$. Then:*

(1) *Suppose that A is connective. Let $\mathrm{CAlg}(\mathcal{A})_{A/}^{fl, L\text{-}et}$ denote the full subcategory of $\mathrm{CAlg}(\mathcal{A})_{A/}$ spanned by the flat L-étale maps $A \to B$. Then the functor π_0 induces an equivalence*

$$\mathrm{CAlg}(\mathcal{A})_{A/}^{fl, L\text{-}et} \xrightarrow{\sim} \mathrm{CAlg}(\mathcal{A}^{\heartsuit})_{\pi_0 A/}^{fl, L\text{-}et}$$

with (the nerve of) the discrete flat L-étale commutative $\pi_0 A$-algebras.

(2) *Suppose that $\mathcal{A}^{\otimes}_{\geq 0}$ is projectively rigid. Let $\mathrm{CAlg}(\mathcal{A})^{et}_{A/}$ denote the full subcategory of $\mathrm{CAlg}(\mathcal{A})_{A/}$ spanned by the étale maps $A \to B$. Then the functor π_0 induces an equivalence*

$$\mathrm{CAlg}(\mathcal{A})^{et}_{A/} \xrightarrow{\sim} \mathrm{CAlg}(\mathcal{A}^{\heartsuit})^{et}_{\pi_0 A/}$$

with (the nerve of) the discrete étale commutative $\pi_0 A$-algebras.

Theorem 0.10 (Theorem 8.11, Universal algebraic *ttt*-∞-category)**.** *The $\mathrm{CAlg}^{\mathrm{rig,at}}_{\mathrm{Sp}_{\geq 0}}$ admits a compact generator*

$$\mathrm{Fun}(\mathbf{Cob}_1^{\mathrm{op}}, \mathrm{Sp}_{\geq 0})^{\otimes}$$

where the symmetric monoidal structure is given by Day convolution and $\mathbf{Cob}_1$ denotes the 1-dimensional framed cobordism $(\infty, 1)$-category.

We list a collection of algebraic ttt-∞-categories here.

Example 0.11. Examples of algebraic *ttt*-∞-categories.

(1) The $\mathrm{Fun}(\mathcal{I}^{\mathrm{op}}, \mathrm{Sp})^{\otimes}$, where $\mathcal{I}^{\otimes}$ is a small rigid symmetric monoidal ∞-category, like
the $\mathrm{Fil}(\mathrm{Sp})^{\otimes} = \mathrm{Fun}(\mathbb{Z}, \mathrm{Sp})^{\otimes}$ the ∞-category of filtered spectra;
the $\mathrm{Gr}(\mathrm{Sp})^{\otimes} = \mathrm{Fun}(\mathbb{Z}^{disc}, \mathrm{Sp})^{\otimes}$ the ∞-category of graded spectra .

(2) the $\mathrm{Sp}(\mathcal{P}_{\Sigma}(\mathcal{I}))^{\otimes}$, where $\mathcal{I}^{\otimes}$ is a small rigid finite-coproduct cocompletely symmetric monoidal ∞-category, like $\mathrm{Sp}_G^{\otimes} = \mathrm{Sp}(\mathcal{P}_{\Sigma}(\mathrm{Fin}_G))^{\otimes}$ the genuine G-spectra over a finite group G.
As Proposition 8.8 indicates, actually every algebraic *ttt*-∞-category comes from this way.

(3) Universal example in $\mathrm{CAlg}^{rig,at}_{\mathrm{Sp}_{\geq 0}}$: the 1-dim cobordism $\mathrm{Fun}(\mathbf{Cob}_1^{\mathrm{op}}, \mathrm{Sp})^{\otimes}$

(4) The $\mathrm{Sh}_{\Sigma}(\mathcal{C})^{\otimes}$ where $\mathcal{C}$ is an excellent ∞-site, see [Pst23]. For example the synthetic spectra $\mathrm{Syn}_E^{\otimes}$?? (need certain conditions on E)

(5) The ∞-category $\mathrm{Shv}(X, \mathrm{Sp})^{\otimes}$ of sheaves on a stone space??

(6) The ∞-category $\mathrm{Shv}(\mathcal{X}, \mathrm{Sp})^{\otimes}$ of sheaves on an ∞-topos of locally homotopy dim=0??

(7) The ∞-category $\mathrm{SH}(k)^{A-T}_{\geq 0}$ of connective Artin-Tate motivic spectra over a perfect field k, see [BHS20].

(8) Cyclotomic spectra and Cartier modules [AN21]??

(9) [HP23][HP24], equivariant [Bar17], motivic [Bac+22] [BHS20], Beilinson t, Ban, condensed, Liquid, [Lur15]

(10) $\mathrm{Qcoh}(X)_{\geq 0}$, where X is an affine quotient stack, i.e. a stack of the form $\mathrm{Spec}(R)/G$ for a linearly reductive group G acting on $\mathrm{Spec}(R)$, this works: the compact projective objects are generated under taking retracts by pullbacks of G-representations and the dual is given by the pullback of the dual in this case.

(11) Voevodsky's category $\mathrm{DM}(k, \mathbb{Z}[1/p])$ (where p is characteristic of k or 1 if k is a $\mathbb{Q}$-algebra), then there is a Chow t-structure on it, generated by smooth projective varieties and their $\mathbb{P}^1$-desuspensions. The mapping spectra between smooth projective varieties are connective, so they are compact projective generators, and they are also dualizable within the retract closed-subcategory generated by it.

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1 Grothendieck prestable ∞-categories

1.1 Prestable ∞-categories

Now we recall some basic relations between prestable ∞-categories and stable ∞-categories equipped with t-structures. For a more complete exposition, we refer the reader to [SAG, Appendix C].

Definition 1.1 (See [SAG] C.1.2.2)**.** A prestable ∞-category is an ∞-category $\mathcal{C}$ satisfying the following properties:

(1) The initial and final objects of $\mathcal{C}$ agree (that is, $\mathcal{C}$ is pointed).

(2) Every cofiber sequence in $\mathcal{C}$ is also a fiber sequence.

(3) Every map in $\mathcal{C}$ of the form $f : X \to \Sigma(Y)$ is the cofiber of its fiber.

Moreover, we say $\mathcal{C}$ is a Grothendieck prestable ∞-category if it further satisfies that it is presentable and that filtered colimits and finite limits commute in $\mathcal{C}$. We let $\mathrm{Groth}_\infty \subset \mathcal{P}\mathrm{r}^L$ denote the full subcategory whose objects are Grothendieck prestable ∞-categories.

Example 1.2.

(1) Any stable ∞-category is prestable.

(2) Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$. Then the full subcategory $\mathcal{C}_{\geqslant 0} \subseteq \mathcal{C}$ is prestable.

Prestable ∞-categories have a quite close relation to t-structured stable ∞-categories.

Proposition 1.3. *Let $\mathcal{C}$ be an ∞-category. The following conditions are equivalent:*

(1) *The ∞-category $\mathcal{C}$ is prestable and admits finite limits.*

(2) *The ∞-category $\mathcal{C}$ is pointed and admits finite colimits, and the canonical map $\rho : \mathcal{C} \to \mathrm{SW}(\mathcal{C})$ is fully faithful. Moreover, the stable ∞-category $\mathrm{SW}(\mathcal{C})$ admits a t-structure $(\mathrm{SW}(\mathcal{C})_{\geqslant 0}, \mathrm{SW}(\mathcal{C})_{\leqslant 0})$ where $\mathrm{SW}(\mathcal{C})_{\geqslant 0}$ is the essential image of ρ.*

(3) *There exists a stable ∞-category $\mathcal{D}$ equipped with a t-structure $(\mathcal{D}_{\geqslant 0}, \mathcal{D}_{\leqslant 0})$ and an equivalence of ∞-categories $\mathcal{C} \simeq \mathcal{D}_{\geqslant 0}$.*

where $\mathrm{SW}(-)$ denotes the Spanier-Whitehead construction.

Proposition 1.4 (See [SAG] C.1.4.1)**.** *Let $\mathcal{C}$ be a presentable ∞-category. The following conditions are equivalent:*

(a) *The ∞-category $\mathcal{C}$ is prestable and filtered colimits in $\mathcal{C}$ are left exact* (see [HTT, Definition 7.3.4.2]).

(b) *The ∞-category $\mathcal{C}$ is prestable and the functor $\Omega : \mathcal{C} \to \mathcal{C}$ commutes with filtered colimits.*

(c) *The ∞-category $\mathcal{C}$ is prestable and the functor $\Omega^\infty : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ commutes with filtered colimits.*

(d) *There exists a presentable stable ∞-category $\mathcal{D}$, a t-structure $(\mathcal{D}_{\geqslant 0}, \mathcal{D}_{\leqslant 0})$ on $\mathcal{D}$ which is compatible with filtered colimits, and an equivalence $\mathcal{C} \simeq \mathcal{D}_{\geqslant 0}$.*

(e) *The suspension functor $\Sigma_+ : \mathcal{C} \to \mathrm{Sp}(\mathcal{C})$ is fully faithful and its essential image $\mathrm{Sp}(\mathcal{C})_{\geq 0}$ is the connective part of a t-structure on $\mathrm{Sp}(\mathcal{C})$ which is compatible with filtered colimits.*

Definition 1.5. We say a presentable prestable ∞-category is Grothendieck if it satisfies above equivalent conditions.

Theorem 1.6 (See [SAG] C.4.2.1)**.** *The full subcategory $\mathrm{Groth}_\infty \subset \mathcal{P}\mathrm{r}^L_{\mathrm{ad}}$ contains the unit object of $\mathrm{Sp}_{\geq 0}$ and is closed under Lurie tensor products. Consequently, Groth_∞ inherits a symmetric monoidal structure for which the inclusion $\mathrm{Groth}_\infty \hookrightarrow \mathcal{P}\mathrm{r}^L_{\mathrm{ad}}$ is symmetric monoidal.*

Definition 1.7 (See [SAG] C.3.1.3)**.** Let $\mathcal{C}$ be a presentable stable ∞-category. We define a full subcategory $\mathcal{C}_{\geqslant 0} \subseteq \mathcal{C}$ to be a core if it is closed under small colimits and extensions.
We will refer to $\mathcal{P}\mathrm{r}^+_{\mathrm{st}}$ as the ∞-category of cored stable ∞-categories. The objects of $\mathcal{P}\mathrm{r}^+_{\mathrm{st}}$ are pairs $(\mathcal{C}, \mathcal{C}_{\geqslant 0})$, where $\mathcal{C}$ is a presentable stable ∞-category and $\mathcal{C}_{\geqslant 0} \subseteq \mathcal{C}$ is a core. A morphism from $(\mathcal{C}, \mathcal{C}_{\geqslant 0})$ to $(\mathcal{D}, \mathcal{D}_{\geqslant 0})$ is given by a colimit-preserving functor $f : \mathcal{C} \to \mathcal{D}$ satisfying $f(\mathcal{C}_{\geqslant 0}) \subseteq \mathcal{D}_{\geqslant 0}$.

Remark 1.8.

(1) **Warning:** The $\mathcal{C}_{\geqslant 0}$ in this definition is not necessarily the connective part of an accessible t-structure unless it is presentable. If $\mathcal{C}_{\geqslant 0}$ is presentable, then we call the pair $(\mathcal{C}, \mathcal{C}_{\geqslant 0})$ a (big) t-structured stable ∞-category.

(2) In fact, we only care about the full subcategory $\mathcal{P}\mathrm{r}^{t\text{-rex}}_{\mathrm{st}} \subset \mathcal{P}\mathrm{r}^+_{\mathrm{st}}$ spanned by those (big) t-structured stable ∞-categories with right t-exact functors. However, the technical advantage of $\mathcal{P}\mathrm{r}^+_{\mathrm{st}}$ is that it admits good colimits and limits [SAG, Remark C.3.1.7].

(3) In [SAG, Remark C.3.1.3], our $\mathcal{P}\mathrm{r}^+_{\mathrm{st}}$ is denoted by Groth^+_∞.

Definition 1.9. There is a natural symmetric monoidal structure on $\mathcal{P}\mathrm{r}^+_{\mathrm{st}}$ given by the construction:

$$(\mathcal{C}, \mathcal{C}_{\geqslant 0}) \otimes (\mathcal{D}, \mathcal{D}_{\geqslant 0}) = (\mathcal{C} \otimes \mathcal{D}, m_!(\mathcal{C}_{\geqslant 0}, \mathcal{D}_{\geqslant 0}))$$

where $\mathcal{C} \otimes \mathcal{D}$ is the Lurie tensor product and $m_!(\mathcal{C}_{\geqslant 0}, \mathcal{D}_{\geqslant 0})$ is the smallest full subcategory of $\mathcal{C} \otimes \mathcal{D}$ which is closed under colimits and extensions and contains the objects $m(C, D)$ for each $C \in \mathcal{C}_{\geqslant 0}$ and $D \in \mathcal{D}_{\geqslant 0}$.

Remark 1.10.

(1) The full subcategory $\mathcal{P}\mathrm{r}^{t\text{-rex}}_{\mathrm{st}} \subset \mathcal{P}\mathrm{r}^+_{\mathrm{st}}$ is closed under tensor products since $m_!(\mathcal{C}_{\geqslant 0}, \mathcal{D}_{\geqslant 0})$ is presentable if both $\mathcal{C}_{\geqslant 0}$ and $\mathcal{D}_{\geqslant 0}$ are presentable.

(2) An object in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{t\text{-rex}}_{\mathrm{st}})$ can be identified with a ttt-∞-category.

Proposition 1.11 (See [SAG] C.3.1.1)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be Grothendieck prestable ∞-categories. Then the canonical map*

$$\theta : \mathrm{LFun}(\mathcal{C}, \mathcal{D}) \to \mathrm{LFun}(\mathrm{Sp}(\mathcal{C}), \mathrm{Sp}(\mathcal{D}))$$

is a fully faithful embedding, whose essential image consists of those functors $\mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{D})$ which preserve small colimits and are right t-exact (with respect to the canonical t-structure).

Proposition 1.12 (See [SAG] C.3.2.1)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be Grothendieck prestable ∞-categories and let $f : \mathcal{C} \to \mathcal{D}$ be a colimit-preserving functor. Then the following conditions are equivalent:*

(1) *The functor f is left exact.*

(2) *The functor f carries 0-truncated objects of $\mathcal{C}$ to 0-truncated objects of $\mathcal{D}$.*

(3) *The induced map $F : \mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{D})$ is left t-exact.*

Corollary 1.13.

(1) *The construction $\mathcal{C} \mapsto (\mathrm{Sp}(\mathcal{C}), \mathrm{Sp}(\mathcal{C})_{\geqslant 0})$ determines a fully faithful embedding*

$$\mathrm{Groth}_\infty \hookrightarrow \mathcal{P}\mathrm{r}^+_{\mathrm{st}}$$

from the ∞-category of Grothendieck prestable ∞-categories to the ∞-category of cored stable ∞-categories.

(2) *A pair $(\mathcal{C}, \mathcal{C}_{\geqslant 0})$ belongs to the essential image of $\mathrm{Groth}_\infty \hookrightarrow \mathcal{P}\mathrm{r}^+_{\mathrm{st}}$ if and only if it forms an accessible t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$ which is compatible with filtered colimits and is right complete.*

(3) *Furthermore, the embedding $\mathrm{Groth}_\infty \hookrightarrow \mathcal{P}\mathrm{r}^+_{\mathrm{st}}$ is symmetric monoidal, hence induces a fully faithful embedding*

$$\mathrm{CAlg}(\mathrm{Groth}_\infty) \hookrightarrow \mathrm{CAlg}(\mathcal{P}\mathrm{r}^+_{\mathrm{st}}).$$

Definition 1.14 (Grothendieck t-structured stable ∞-categories)**.** We say a t-structured stable ∞-category $(\mathcal{C}, \mathcal{C}_{\geq 0}) \in \mathcal{P}\mathrm{r}^{t\text{-rex}}_{\mathrm{st}}$ is Grothendieck if it lies in the essential image of the embedding $\mathrm{Groth}_\infty \hookrightarrow \mathcal{P}\mathrm{r}^{t\text{-rex}}_{\mathrm{st}}$, or equivalently, if the t-structure on $\mathcal{C}$ is right complete and compatible with filtered colimits (see [SAG, Remark C.3.1.5]).

Remark 1.15. By Proposition 3.7, we see that if $(\mathcal{A}^\otimes, \mathcal{A}_{\geq 0})$ is Grothendieck, then for any $R \in \mathrm{CAlg}(\mathcal{A}_{\geq 0})$, the pair $(\mathrm{Mod}_R(\mathcal{A})^\otimes, \mathrm{Mod}_R(\mathcal{A})_{\geq 0}) \in \mathrm{CAlg}(\mathcal{P}\mathrm{r}^{t\text{-rex}}_{\mathrm{st}})$ is also Grothendieck.

Definition 1.16 (See [SAG] C.1.2.12)**.** Let $\mathcal{C}$ be a prestable ∞-category which admits finite limits. We say that an object $X \in \mathcal{C}$ is ∞-connective if $\tau_{\leqslant n} X \simeq 0$ for every integer n.
We say $\mathcal{C}$ is separated if every ∞-connective object of $\mathcal{C}$ is a zero object.
We say $\mathcal{C}$ is complete if it is a homotopy limit of the tower of ∞-categories:

$$\cdots \to \tau_{\leqslant 2}\mathcal{C} \xrightarrow{\tau_{\leqslant 1}} \tau_{\leqslant 1}\mathcal{C} \xrightarrow{\tau_{\leqslant 0}} \tau_{\leqslant 0}\mathcal{C} = \mathcal{C}^\heartsuit.$$

In other words, $\mathcal{C}$ is complete if it is Postnikov complete.

Remark 1.17.

(1) If a prestable ∞-category $\mathcal{C}$ is complete, then it is separated.

(2) Let $\mathcal{C}$ be a stable ∞-category. Then $\mathcal{C}$ is separated if and only if $\mathcal{C} \simeq *$.

Proposition 1.18. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category. Then:*

(1) *The canonical t-structure $(\mathrm{Sp}(\mathcal{C})_{\geq 0}, \mathrm{Sp}(\mathcal{C})_{\leq 0})$ is hypercomplete if and only if $\mathcal{C}$ is separated.*

(2) *The canonical t-structure $(\mathrm{Sp}(\mathcal{C})_{\geq 0}, \mathrm{Sp}(\mathcal{C})_{\leq 0})$ is left complete if and only if $\mathcal{C}$ is complete.*

Proof. (1) The "only if" direction is obvious. Now assume that $\mathcal{C}$ is separated and that $X \in \mathrm{Sp}(\mathcal{C})$ satisfies $\tau_{\leq i} X = 0$ for every integer i. We need to show that $X = 0$. Since $\mathrm{Sp}(\mathcal{C})$ is right complete, we have $X \simeq \varinjlim \tau_{\geq -n} X$. However, each $\Sigma^n \tau_{\geq -n} X$ is ∞-connective as an object in $\mathcal{C}$, so $\tau_{\geq -n} X = 0$ by assumption, which implies that $X = 0$.

(2) The "only if" direction is obvious. Now assume that $\mathcal{C}$ is complete. Given a tower T

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & X_1 \arrow[d] \\
X \arrow[r] \arrow[ur] \arrow[uur] & X_0
\end{tikzcd}$$

in $\mathrm{Sp}(\mathcal{C})$ such that $\{X_n\}$ is a Postnikov tower in $\mathrm{Sp}(\mathcal{C})$, we need to show that $X \simeq \varprojlim X_i$ if and only if X is a Postnikov tower (see a similar argument in [HTT, Proposition 5.5.6.26]). Since $\mathrm{Sp}(\mathcal{C})$ is right complete, we have that

$$\mathrm{Sp}(\mathcal{C}) \xrightarrow{\varprojlim \tau_{\geq -n}} \varprojlim \mathrm{Sp}(\mathcal{C})_{\geq -n}$$

is an equivalence in $\mathcal{P}\mathrm{r}^R$. Thus $X \simeq \varprojlim X_i$ if and only if for each $n \geq 0$ we have $\tau_{\geq -n} X \simeq \varprojlim_i \tau_{\geq -n} X_i$. Also, the tower T is a Postnikov tower if and only if for each $n \geq 0$ the following tower

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & \tau_{\geq -n} X_1 \arrow[d] \\
\tau_{\geq -n} X \arrow[r] \arrow[ur] \arrow[uur] & \tau_{\geq -n} X_0
\end{tikzcd}$$

is a Postnikov tower. By the completeness of $\mathcal{C}$, the proof is complete. □

Proposition 1.19. *For any $\mathcal{C} \in \mathrm{Groth}_\infty$, we have $\mathrm{Shv}_{\mathcal{C}}(\mathcal{X}) \in \mathrm{Groth}_\infty$ too, where $\mathrm{Shv}_{\mathcal{C}}(\mathcal{X})$ is the $\mathcal{C}$-valued sheaf category on an ∞-topos $\mathcal{X}$. In particular, $\mathrm{Cond}(\mathcal{C}) \in \mathrm{Groth}_\infty$, and there is a fully faithful embedding: $\mathcal{C} \hookrightarrow \mathrm{Cond}(\mathcal{C})$.*

Proof. By definition, $\mathrm{Shv}_{\mathcal{C}}(\mathcal{X}) = \mathrm{Fun}^{\mathrm{R}}(\mathcal{X}^{\mathrm{op}}, \mathcal{C}) = \mathcal{X} \otimes \mathcal{C}$. By [HPT25, Corollary 3.7], $\mathcal{X} \otimes \mathcal{C} \in \mathrm{Groth}_\infty$. Taking $\mathcal{X} = \mathrm{Cond}(\mathcal{S})$, we know that $\mathrm{Cond}(\mathcal{C}) = \mathrm{Cond}(\mathcal{S}) \otimes \mathcal{C} \in \mathrm{Groth}_\infty$. To show the fully faithful embedding, note that $\mathrm{ED}_{\omega_0} \simeq \mathrm{Fin}$ (the category of discrete finite sets), thus $\mathrm{Cond}_{\omega_0}(\mathcal{S}) = \mathcal{P}_\Sigma(\mathrm{Fin}) \simeq \mathcal{S}$, where the identification is given by $F \mapsto F(*)$. Hence $\mathcal{C} \simeq \mathcal{S} \otimes \mathcal{C} \simeq \mathrm{Cond}_{\omega_0}(\mathcal{S}) \otimes \mathcal{C} \simeq \mathrm{Cond}_{\omega_0}(\mathcal{C})$. Since filtered colimits are left exact in $\mathcal{C}$, we use [Aok24, Proposition 2.5] to obtain the fully faithful embedding $\mathcal{C} \hookrightarrow \mathrm{Cond}(\mathcal{C})$ obtained from left Kan extension along $\mathrm{Fin} \to \mathrm{ED}_{\omega_1}$. □

An easy corollary follows, which is one of the fundamental results in condensed mathematics:

Corollary 1.20. *The category of Condensed Abelian groups, CondAb, is a Grothendieck Abelian category.*

Proof. Taking $\mathcal{C} = \mathrm{Sp}_{\geq 0}$ in Proposition 1.19, we know that $\mathrm{Cond}(\mathrm{Sp}_{\geq 0})$ is a Grothendieck prestable category. Thus, its heart $\mathrm{Cond}(\mathrm{Sp}_{\geq 0})^\heartsuit = \mathrm{Cond}(\mathrm{Sp}^\heartsuit) = \mathrm{CondAb}$ is a Grothendieck abelian category. □

1.2 Dualizable additive ∞-categories

Definition 1.21 (See [Ram24a] Definition 1.22, 1.27)**.** Let $\mathcal{V}^\otimes \in \mathrm{CAlg}(\mathcal{P}\mathrm{r}^L)$ and $\mathcal{M} \in \mathcal{P}\mathrm{r}^L_{\mathcal{V}} = \mathrm{Mod}_{\mathcal{V}}(\mathcal{P}\mathrm{r}^L)$.

(1) An object $x \in \mathcal{M}$ is called $\mathcal{V}$-atomic, or simply atomic if the base $\mathcal{V}$ is understood, if the $\mathcal{V}$-linear functor $\mathcal{V} \xrightarrow{-\otimes x} \mathcal{M}$ classifies an internal left adjoint, i.e. if

$$\underline{\mathrm{Map}}_{\mathcal{M}}(x, -) : \mathcal{M} \to \mathcal{V}$$

preserves small colimits and the canonical map

$$v \otimes \underline{\mathrm{Map}}_{\mathcal{M}}(x, y) \to \underline{\mathrm{Map}}_{\mathcal{M}}(x, v \otimes y)$$

is an equivalence for all $v \in \mathcal{V}, y \in \mathcal{M}$.

(2) The $\mathcal{M}$ is said to be $\mathcal{V}$-atomically generated if the smallest full $\mathcal{V}$-submodule of $\mathcal{M}$ closed under colimits and containing the atomics of $\mathcal{M}$ is $\mathcal{M}$ itself.

Proposition 1.22. *Let $\mathcal{M} \in \mathrm{Mod}_{\mathrm{Sp}_{\geq 0}}(\mathcal{P}\mathrm{r}^L) = \mathcal{P}\mathrm{r}^L_{\mathrm{ad}}$. Then an object $x \in \mathcal{M}$ is $\mathrm{Sp}_{\geq 0}$-atomic if and only if it is compact projective in $\mathcal{M}$.*

Proof. Because $\mathrm{Sp}^{\otimes}_{\geq 0}$ is a mode [see CSY21, §5], it only suffices to check the colimit-preserving property of the internal hom functor by [Ram24a, Example 1.24]. Now we note that the corepresentable functor $\mathrm{Map}_{\mathcal{M}}(x, -) : \mathcal{M} \to \mathcal{S}$ is the composition of $\underline{\mathrm{Map}}_{\mathcal{M}}(x, -) : \mathcal{M} \to \mathrm{Sp}_{\geq 0}$ and $\Omega^\infty : \mathrm{Sp}_{\geq 0} \to \mathcal{S}$. Since both $\mathcal{M}$ and $\mathrm{Sp}_{\geq 0}$ are additive, the connective mapping spectrum functor

$$\underline{\mathrm{Map}}_{\mathcal{M}}(x, -) : \mathcal{M} \to \mathrm{Sp}_{\geq 0}$$

preserves small colimits if and only if it preserves small sifted colimits. Therefore the result follows immediately from [HA, Proposition 1.4.3.9] that $\Omega^\infty : \mathrm{Sp}_{\geq 0} \to \mathcal{S}$ is conservative and preserves small sifted colimits. ∎

Definition 1.23. Let $\mathcal{P}\mathrm{r}^{L,\otimes}_{\mathrm{ad}}$ denote the symmetric monoidal ∞-category of presentable additive ∞-categories with colimit-preserving morphisms and Lurie tensor product. We say a presentable additive ∞-category $\mathcal{C}$ is dualizable if it is dualizable (see Appendix A.1) under Lurie tensor product.

We denote $\mathcal{P}\mathrm{r}^d_{\mathrm{ad}}$ as the full subcategory of $\mathcal{P}\mathrm{r}^L_{\mathrm{ad}}$ spanned by those dualizable presentable additive ∞-categories, $\mathcal{P}\mathrm{r}^{\mathrm{dbl}}_{\mathrm{ad}}$ as the (non-full) subcategory of $\mathcal{P}\mathrm{r}^d_{\mathrm{ad}}$ with the same objects but whose morphisms are internal left adjoints, i.e. those left adjoints such that their right adjoints are colimit-preserving.

Proposition 1.24. *Let $\mathcal{C} \in \mathcal{P}\mathrm{r}^d_{\mathrm{ad}}$. Then $\mathcal{C}$ is a complete Grothendieck prestable ∞-category.*

Proof. ∎

1.3 Additive rigidity

[Ram24a][Ram24b][BS24][CSY21] We will compare our projective rigidity with rigidity in enriched context.

Definition 1.25 (See [Ram24b] Definition 4.34)**.** Let $\mathcal{V}^{\otimes} \in \mathrm{CAlg}(\mathcal{P}\mathrm{r}^L)$ and $\mathcal{W} \in \mathrm{CAlg}_{\mathcal{V}} = \mathrm{CAlg}(\mathcal{P}\mathrm{r}^L_{\mathcal{V}})$. We say $\mathcal{W}$ is a rigid $\mathcal{V}$-algebra, if the following hold:

(1) The $\mathcal{W}$ is a dualizable $\mathcal{V}$-module.

(2) the multiplication map $\mathcal{W} \otimes_{\mathcal{V}} \mathcal{W} \to \mathcal{W}$ is an internal left adjoint in $\mathrm{Mod}_{\mathcal{W} \otimes_{\mathcal{V}} \mathcal{W}}(\mathcal{P}\mathrm{r}^L)$.

(3) The unit $\mathbf{1}_{\mathcal{W}} \in \mathcal{W}$ is $\mathcal{V}$-atomic.

Proposition 1.26 (See [Ram24b] Lemma 4.50)**.** *Let $\mathcal{V}^{\otimes} \in \mathrm{CAlg}(\mathcal{P}\mathrm{r}^L)$. Suppose $\mathcal{W}$ is a commutative $\mathcal{V}$-algebra, whose underlying $\mathcal{V}$-module is atomically generated. In this case, $\mathcal{W}$ is rigid if and only if its $\mathcal{V}$-atomics and dualizables coincide.*

Theorem 1.27. *Let $\mathcal{M} \in \mathcal{P}\mathrm{r}^L_{\mathrm{ad}}$. Then:*

(1) *The $\mathcal{M}$ is $\mathrm{Sp}_{\geq 0}$-atomically generated if and only if it is projectively generated.*

(2) *If $\mathcal{M}^{\otimes} \in \mathrm{CAlg}(\mathcal{P}\mathrm{r}^L_{\mathrm{ad}})$, then $\mathcal{M}^{\otimes}$ is $\mathrm{Sp}_{\geq 0}$-atomically generated and $\mathrm{Sp}_{\geq 0}$-rigid, if and only if, it is projectively rigid in the sense of Definition 4.13.*

Proof. The (1) follows immediately from Proposition 1.22.
The (2) follows by combining Proposition 1.22 and Proposition 1.26. □

We will discuss more about projective rigidity in Sections 4.2 and 8.

Example 1.28. We list a collection of ttt-∞-categories whose connective part satisfies projective rigidity in Example 8.20.

2 Grothendieck abelian categories

An important technique for learning derived algebra theory is to reduce to the "π_0" case, in the study of $\mathbb{E}_\infty$-rings and modules over them, after taking π_0, we are dealing with the rings and modules in the classical sense, since $\mathrm{Sp}^\heartsuit$ is the category of the abelian groups Ab. But in our setting, $\mathcal{A}^\heartsuit$ is an abelian category, our goal in this section is to establish the theory of "rings" and "module" in $\mathcal{A}^\heartsuit$. previous [TV09][Ban12][Ban17]

Definition 2.1. Let $\mathrm{Groth}_1 \subset \mathcal{P}\mathrm{r}^L_{\mathrm{ad},1}$ denote the full subcategory of presentable additive 1-categories spanned by those abelian categories such that filtered colimits commute with finite limits in it. We call an object in Groth_1 a Grothendieck abelian category.

Remark 2.2. For a Grothendieck prestable category $\mathcal{C} \in \mathrm{Groth}_\infty$, the heart $\mathcal{C}^\heartsuit$ is a Grothendieck Abelian category. However, the forgetful functor $\mathrm{Groth}_\infty \to \mathrm{Groth}_1$ is not one-to-one but many-to-one, e.g $\mathrm{Sp}^\heartsuit = D(\mathbb{Z})^\heartsuit$.

Example 2.3. The (light) Condensed Abelian Group **CondAb** and the full subcategory $\mathbf{Solid}_\mathbb{Z}$ are both Grothendieck Abelian categories.

Remark 2.4. The Groth_1 is closed under the Lurie tensor product on $\mathcal{P}\mathrm{r}^L$ by [SAG, Theorem C.5.4.16]. We call an object in $\mathrm{CAlg}(\mathrm{Groth}_1)$ a symmetric monoidal Grothendieck abelian category.

Throughout Section 2, we fix a symmetric monoidal Grothendieck abelian category $\mathbf{A}^\otimes \in \mathrm{CAlg}(\mathrm{Groth}_1)$.

2.1 Dualizable additive n-categories

Definition 2.5. Let $\mathbf{B} \in \mathrm{Groth}_1$ be a Grothendieck abelian category.

(1) We say an object $X \in \mathbf{B}$ is 1-projective if it is an projective in the ordinary sense in an abelian category.

(2) We say $\mathbf{B}$ is 1-projectively generated if $\mathbf{B}$ is generated by a small set of compact 1-projective objects under small colimits.

(3) If $\mathbf{B} \in \mathrm{CAlg}(\mathrm{Groth}_1)$ is a symmetric monoidal Grothendieck abelian category, then we say $\mathbf{B}^\otimes$ is 1-projectively rigid if $\mathbf{B}$ is 1-projectively generated and the dualizable objects in $\mathbf{B}$ coincide with compact 1-projective objects.

Remark 2.6. We use the terminology "1-projective" to distinguish it from the "projective" in the sense of [HTT, Definition 5.5.8.18] for an ∞-category. Generally they do not agree in an abelian 1-category [see HTT, Example 5.5.8.21].

Definition 2.7. Let $\mathcal{C}$ be an n-category. We say an object $X \in \mathcal{C}$ is n-projective if the corepresentable functor $\mathcal{C} \to \mathcal{S}_{\le n-1}$ preserves geometric realizations.

Proposition 2.8. *Let $\mathcal{M} \in \mathrm{Mod}_{\mathrm{Ab}}(\mathcal{P}\mathrm{r}^L) = \mathcal{P}\mathrm{r}^L_{\mathrm{ad},1}$ be an presentable additive 1-category. Then an object $x \in \mathcal{M}$ is* Ab*-atomic if and only if it is compact 1-projective in $\mathcal{M}$.*

Proof. It is parallel with Proposition 1.22. □

2.2 Linear Grothendieck abelian categories

Definition 2.9. We denote the $(2,2)$-category of $\mathbf{A}$-linear Grothendieck abelian category with $\mathbf{A}$-linear functors by $\mathrm{Mod}_{\mathbf{A}}(\mathrm{Groth}_1)$.

Example 2.10. Given an $R \in \mathrm{Alg}(\mathbf{A})$, the $\mathrm{LMod}_R(\mathbf{A})$ is an $\mathbf{A}$-linear Grothendieck abelian category.

2.3 Modules and algebras

Definition 2.11. Let $R \in \mathrm{Alg}(\mathbf{A})$.

(1) We say a left R-module M is finitely presented if M is a compact object in $\mathrm{Mod}_R(\mathbf{A})$.

(2) We say a left R-module M is (faithfully) flat if the relative tensor product functor $(-) \otimes_R M : \mathrm{RMod}_R(\mathbf{A}) \to \mathbf{A}$ is (conservative) left exact.

(3) We define an left ideal I of R as an left R-submodule of R.

Proposition 2.12. *Let $f : A \to B \in \mathrm{CAlg}(\mathbf{A})$ be a faithfully flat map. Then*

(1) *For any A-module M, the map $M \simeq M \otimes_A A \to M \otimes_A B$ is monomorphic. In particular, $A \to B$ is monomorphic.*

(2) *The $\mathrm{Coker}(f)$ is a flat A-module.*

Proof. (1) Let N be the kernel of $M \to M \otimes_A B$. Considering the following diagram.

$$\begin{tikzcd} N \arrow[r] \arrow[d, hook, "i"] & N \otimes_A B \arrow[d, "i \otimes_A B"] \\ M \arrow[r] & M \otimes_A B \end{tikzcd}$$

Then by the base change adjoint we conclude that the map $N \otimes_A B \to M \otimes_A B$ is zero. The faithful flatness implies $i = 0$ and hence $N = 0$.
(2) It follows immediately from (1) and snake lemma. □

Proposition 2.13. *Suppose that $\mathbf{A}$ is 1-projectively generated and $R \in \mathrm{CAlg}(\mathbf{A})$. Then:*

(1) *Any R-module M can be written as a pushout in $\mathrm{Mod}_R(\mathbf{A})$ as the following form*

$$\begin{tikzcd} P_1 \arrow[r] \arrow[d] & 0 \arrow[d] \\ P_2 \arrow[r] & M \end{tikzcd}$$

where $P_1, P_2 \in \mathrm{Mod}_R(\mathbf{A})$ are 1-projective R-modules.

(2) *An R-module M is finitely presented if and only if P_1, P_2 can be promoted to compact 1-projective R-modules.*

Proof. □

Proposition 2.14. *Suppose that $\mathbf{A}^{\otimes}$ is 1-projectively rigid. Let R be in $\mathrm{CAlg}(\mathcal{A}_{\geq 0})$. Then $\mathrm{Mod}_R(\mathbf{A})^{\otimes}$ is 1-projectively rigid too.*

Proof. Since the symmetric monoidal functor

$$\mathbf{A}^{\otimes} \xrightarrow{R\otimes(-)} \mathrm{Mod}_R(\mathbf{A})^{\otimes}$$

preserves compact 1-projective objects and dualizable objects, we conclude that

(*i*) The unit R is dualizable in $\mathrm{Mod}_R(\mathbf{A})$.

(*ii*) If $P \in \mathbf{A}$ is compact 1-projective, then $R \otimes P$ is dualizable in $\mathrm{Mod}_R(\mathbf{A})$.

So the full subcategory of dualizable objects $\mathrm{Mod}_R(\mathbf{A})^d$ contains $\{R \otimes X | X \in \mathbf{A}^{cproj}\}$. Then combining Lemma 3.1(2) and Proposition A.10(2)(3), we get $\mathrm{Mod}_R(\mathbf{A})^{cproj} \subset \mathrm{Mod}_R(\mathbf{A})^d$. Finally, the fact that the unit R is compact 1-projective implies the equality $\mathrm{Mod}_R(\mathbf{A})^{cproj} = \mathrm{Mod}_R(\mathbf{A})^d$. □

The [Ste23, Prop. 2.2.22] proves Lazard's theorem over commutative ring objects. We find the argument there also works in noncommutative settings.

Theorem 2.15 (Lazard's theorem)**.** *Suppose that $\mathbf{A}^{\otimes}$ is 1-projectively rigid. Let $R \in \mathrm{Alg}(\mathbf{A})$ and M be a left R-module of $\mathbf{A}$. Then:*

(1) *If M is 1-projective, then M is flat.*

(2) *The M is compact 1-projective if and only if it is left dualizable in $\mathrm{LMod}_R(\mathbf{A})$.*

(3) *The M is flat if and only if it is a filtered colimit of compact 1-projective left R-modules.*

Proof.
(1) Since flat modules are closed under small coproducts and retractions, we reduce to the case $M = R \otimes P$ where $P \in \mathbf{A}^{cproj}$ is compact 1-projective. It becomes easy because $(-) \otimes_R (R \otimes P) \simeq (-) \otimes P$ reduces to the case $R = \mathbf{1}$, which follows from the dualizability of P in $\mathbf{A}$.
(2) By Corollary A.14, we see that left dualizable objects are closed under finite coproducts and retracts. We observe that every $R \otimes P$ is left dualizable (given by $P^{\vee} \otimes R$), which proves "only if" direction. For the "if" direction, if M is left dualizable, then it follows from

$$\mathrm{Map}_{\mathrm{LMod}_R(\mathbf{A})}(M, -) \simeq \mathrm{Map}_{\mathbf{A}}(\mathbf{1}, {}^{\vee}M \otimes_R -)$$

and compact 1-projectivity of the unit.
(3) It is a parallel argument with Theorem 4.18. □

Proposition 2.16. *Suppose that $\mathbf{A}^{\otimes}$ is 1-projectively rigid. Let $R \in \mathrm{Alg}(\mathbf{A})$ and let M be a left R-module. Then the following are equivalent:*

(1) *The M is a compact 1-projective left R-module.*

(2) *The M is a finitely presented and flat left R-module.*

Proof. The (1) ⇒ (2) is obvious. For (2) ⇒ (1), by Lazard's theorem M can be written as the filtered colimit of a set of compact 1-projective left R-modules. Then the compactness implies M is the retraction of some compact 1-projective module, and hence compact 1-projective too. □

Definition 2.17. (1) We say a map $R \to S \in \mathrm{CAlg}(\mathbf{A})$ is of finite presentation if S is a compact object in $\mathrm{CAlg}(\mathbf{A})_{R/}$.

Proposition 2.18. *Suppose that $\mathbf{A}$ is compactly generated. Then a map $R \to S \in \mathrm{CAlg}(\mathbf{A})$ is of finite presentation if and only if S can be written as a pushout in $\mathrm{CAlg}(\mathbf{A})$ as the following form*

$$\begin{array}{ccc} \mathrm{Sym}^*_R(N) & \xrightarrow{\alpha} & R \\ \downarrow & & \downarrow \\ \mathrm{Sym}^*_R(M) & \longrightarrow & S \end{array}$$

where $M, N \in \mathrm{Mod}_R(\mathbf{A})^\omega$ are compact R-modules and α is the natural augmentation. (Note that the ϕ here is not necessarily induced by a map of $N \to M$.)
Furthermore, if $\mathbf{A}$ is 1-projectively generated, then M, N can be chosen as compact 1-projective R-modules.

Proof. The proof is parallel with the proof of Proposition 5.15. □

Proposition 2.19. *Let $R \to S \in \mathrm{CAlg}(\mathbf{A})$ be an epimorphism of commutative ring objects. Then the forgetful functor $\mathrm{Mod}_S(\mathbf{A})^\otimes \to \mathrm{Mod}_S(\mathbf{A})^\otimes$ is a symmetric monoidal embedding.*

Proof. Since $R \to S$ is an epimorphism is equivalent to that S is an idempotent commutative R-algebra, the result follows immediately from [HA, Prop. 4.8.2.10]. □

Proposition 2.20. *A faithfully flat epimorphism in $\mathrm{CAlg}(\mathbf{A})$ is an isomorphism.*

Proof. The epimorphism implies the map $R \otimes_R S \to S \otimes_R S$ is an isomorphism, so by the fully faithful flatness the $R \to S$ is an isomorphism. □

Definition 2.21. We say a map $f : R \to S \in \mathrm{CAlg}(\mathbf{A})$ is an open immersion if f is finitely presented, flat and epimorphic.

2.4 Cohn localizations

There is a notion of Cohn localization or Cohn localization [Coh71; Sch85] which forces not just elements of the ring to become invertible but forces more general some maps between finitely generated projective modules to become invertible, as the following.

Theorem 2.22 ([Sch85] Theorem 4.1)**.** *Let A be an associate ring. Let Σ be a set of morphisms between finitely generated projective right A-modules. Then there are a ring A_Σ and a morphism of rings $f_\Sigma : A \longrightarrow A_\Sigma$, called the universal localisation of A at Σ, such that*

(1) *f_Σ is Σ-inverting, i.e. if $\alpha : P \longrightarrow Q$ belongs to Σ, then $\alpha \otimes_A 1_{A_\Sigma} : P \otimes_A A_\Sigma \longrightarrow Q \otimes_A A_\Sigma$ is an isomorphism of right A_Σ-modules, and*

(2) *f_Σ is universal Σ-inverting, i.e. for any Σ-inverting ring homomorphism $\psi : A \longrightarrow B$, there is a unique ring homomorphism $\bar{\psi} : A_\Sigma \longrightarrow B$ such that $\bar{\psi} f_\Sigma = \psi$. Moreover, the homomorphism f_Σ is a ring epimorphism and $\mathrm{Tor}_1^A(A_\Sigma, A_\Sigma) = 0$.*

The statement above also works for commutative rings, which is the case we mainly care. See [Ang+20] for a relation between Cohn localizations and epimorphisms between commutative rings. We will generalize that to a symmetric monoidal Grothendieck abelian category. Our main result is the following.

Theorem 2.23. *Let $\mathbf{A}^\otimes$ be a 1-projectively rigid symmetric monoidal Grothendieck abelian category. Let $R \in \mathrm{CAlg}(\mathbf{A})$ and*

$$S = \{P_\beta \xrightarrow{f_\beta} Q_\beta\}$$

be a set of morphisms between compact 1-projective R-modules. Then there exists a Cohn localization $R \to R[S^{-1}] \in \mathrm{CAlg}(\mathbf{A})$ satisfying the following universal property:
For any $B \in \mathrm{CAlg}(\mathbf{A})$, the induced map

$$\mathrm{Hom}_{\mathrm{CAlg}(\mathbf{A})}(R[S^{-1}], B) \to \mathrm{Hom}_{\mathrm{CAlg}(\mathbf{A})}(R, B)$$

is an injection whose image consists those maps $R \to B$ such that for each $f_\beta \in S$ the $B \otimes_R P_\beta \to B \otimes_R Q_\beta$ is an equivalence of B-modules.

Proof. The proof is parallel with the proof of Theorem 4.21. Also see Remark 4.27. We just need to replace the Morita embedding $\mathrm{CAlg}(\mathcal{A}_{\geq 0}) \to \mathrm{CAlg}(\mathcal{P}\mathrm{r}^L_{\mathrm{ad}})_{\mathcal{A}^{\otimes}_{\geq 0}/}$ in the argument by

$$\mathrm{CAlg}(\mathbf{A}) \to \mathrm{CAlg}(\mathcal{P}\mathrm{r}^L_{\mathrm{ad},1})_{\mathbf{A}^{\otimes}/}$$

to adapt the 1-categorical setting. ∎

Proposition 2.24. *Let $f_S : R \to R[S^{-1}]$ be the Cohn localization at S in Theorem 2.23. Then:*

(1) *The $R[S^{-1}]$ is an idempotent commutative R-algebra.*

(2) *The f_S is flat???*

Proof.
(1) It suffices to show that the following diagram is a pushout in $\mathrm{CAlg}(\mathbf{A})$,

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R[S^{-1}] \arrow[d, equal] \\ R[S^{-1}] \arrow[r, equal] & R[S^{-1}] \end{tikzcd}$$

i.e. to show that f_S is an epimorphism in $\mathrm{CAlg}(\mathbf{A})$. That is implied by the description of the Hom set in Theorem 2.23.
(2) ??? ∎

Definition 2.25. Let $\mathbf{A}^{\otimes}$ be a 1-projectively rigid symmetric monoidal Grothendieck abelian category. We say a map $A \to B \in \mathrm{CAlg}(\mathbf{A})$ is a (finitary) Cohn localization if there exists a (finite) set $S = \{P_\beta \xrightarrow{f_\beta} Q_\beta\}$ of morphisms between compact 1-projective R-modules such that $B \simeq A[S^{-1}]$.

Remark 2.26. Note that if $S = \{P_i \xrightarrow{f_i} Q_i\}$ is finite, then $A[S^{-1}] = A[f^{-1}]$ can be written as the Cohn localization at a single element $f = \bigoplus_i f_i$.

Proposition 2.27. *Let $\mathbf{A}^{\otimes}$ be a 1-projectively rigid symmetric monoidal Grothendieck abelian category. Let $A \to B \in \mathrm{CAlg}(\mathbf{A})$ be a finitary Cohn localization. Then B is finitely presented over A.*

Proof. The proof is similar to Proposition 4.32 but we need to take a different strategy because the kernel is not preserved by base change.

Let $\mathcal{A}^{\otimes} = \mathcal{D}(\mathbf{A})^{\otimes}$. By Theorem 4.19 we have that $\mathcal{A}^{\heartsuit,\otimes} \simeq \mathbf{A}^{\otimes}$ and $\mathcal{A}^{cproj}_{\geq 0} = \mathbf{A}^{cproj}$. Let $A'[S^{-1}]$ be the (higher) Cohn localization at S in the sense of Theorem 4.21. Then $A'[S^{-1}]$ is compact in $\mathrm{CAlg}(\mathcal{A}_{\geq 0})_{A/}$ by Proposition 4.32. Therefore $\pi_0 A'[S^{-1}]$ is compact in $\mathrm{CAlg}(\mathbf{A})_{A/}$. However by Proposition 4.28 the $A[S^{-1}] = \pi_0 A'[S^{-1}]$. We are done. ∎

3 Flat and faithfully flat

We are inspired by equivalent conditions of the flatness over structured ring spectra appeared in [HA, Theorem 7.2.2.15].

3.1 t-structure on the category of modules

Lemma 3.1. *Let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ be an adjoint pair of ∞-categories.*

(1) *Assume κ is a regular cardinal, $\mathcal{C}$ is κ-presentable, and $\mathcal{D}$ is locally small and admits small colimits. If G is conservative and preserves small κ-filtered colimits, then $\mathcal{D}$ is κ-presentable. Furthermore, $\mathcal{D}^\kappa$ is the smallest full subcategory generated by $F(\mathcal{C}^\kappa)$ under κ-small colimits and retractions. Consequently, $\mathcal{D}$ is generated by the image of F under small colimits.*

(2) (See [HA, Corollary 4.7.3.18]).
Assume $\mathcal{D}$ admits small filtered colimits and geometric realizations, and G preserves both. Also, assume $\mathcal{C}$ is projectively generated (see [HTT, Definition 5.5.8.23]). *If the functor G is conservative, then $\mathcal{D}$ is projectively generated. An object $D \in \mathcal{D}$ is compact and projective if and only if there exists a compact projective object $C \in \mathcal{C}$ such that D is a retract of $F(C)$. Hence, $\mathcal{D}$ is generated by the image of F under small colimits.*

Proof. We first prove (1). Let $\mathcal{D}_0 \subset \mathcal{D}$ be the smallest full subcategory generated by $F(\mathcal{C}^\kappa)$ under finite colimits and retractions. Then the inclusion $\mathcal{D}_0 \subset \mathcal{D}$ extends to a fully faithful embedding $F_2 : \mathrm{Ind}_\kappa(\mathcal{D}_0) \hookrightarrow \mathcal{D}$ (by [HTT, p. 5.3.5.10]). Since F preserves small colimits, it admits a right adjoint H ([HTT, Proposition 5.5.1.9]). Thus, we have the following factorization of adjoint pairs:

$$\mathcal{C} \underset{G_1}{\overset{F_1}{\rightleftarrows}} \mathrm{Ind}_\kappa(\mathcal{D}_0) \underset{G_2}{\overset{F_2}{\rightleftarrows}} \mathcal{D}$$

It will therefore suffice to show that the functor G_2 is conservative. Let $\alpha : X \to Y$ be a morphism in $\mathcal{D}$ such that $G_2(\alpha)$ is an equivalence. We aim to show that α is an equivalence. For this, since $\mathcal{C}$ is κ-compactly generated, it will suffice to show that α induces a homotopy equivalence

$$\theta : \mathrm{Map}_{\mathcal{C}}\left(C, G(X)\right) \to \mathrm{Map}_{\mathcal{C}}\left(C, G(Y)\right)$$

for every κ-compact object $C \in \mathcal{C}$. This map can be identified with

$$\theta : \mathrm{Map}_{\mathrm{Ind}_\kappa(\mathcal{D}_0)}\left(F_1(C), G_2(X)\right) \to \mathrm{Map}_{\mathrm{Ind}_\kappa(\mathcal{D}_0)}\left(F_1(C), G_2(Y)\right)$$

Our assumption that $G_2(\alpha)$ is an equivalence guarantees that θ is a homotopy equivalence, as desired.

For (2), the argument is entirely parallel. See also [HA, Cor. 4.7.3.18]. ☐

Remark 3.2. The conditions of (2) guarantee that $\mathcal{D}$ is locally small because it is monadic over $\mathcal{C}$ by the Barr-Beck-Lurie theorem (see [HA, Thm. 4.7.3.5]).

Corollary 3.3. *Let $R \in \mathrm{Alg}(\mathcal{A})$. Applying Lemma 3.1 to the adjoint pair $\mathcal{A} \overset{R\otimes -}{\rightleftarrows} \mathrm{LMod}_R(\mathcal{A})$, we obtain:*

(1) *If $\mathcal{A}$ is κ-presentable for some regular cardinal κ, then so is $\mathrm{LMod}_R(\mathcal{A})$.*

(2) $\mathrm{LMod}_R(\mathcal{A})$ *is presentable.*

(3) $\mathrm{LMod}_R(\mathcal{A})$ *is generated by $\{R \otimes X \mid X \in \mathcal{A}\}$ under small colimits.*

Remark 3.4. By [HA, Proposition 7.1.1.4], $\mathrm{LMod}_R(\mathcal{A})$ is stable for any $R \in \mathrm{Alg}(\mathcal{A})$.

Definition 3.5. Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. We say it is hypercomplete if for an object $X \in \mathcal{C}$, the condition $\tau_{\leq n} X = 0$ for every integer n implies $X = 0$.

Example 3.6. Let $\mathcal{X}$ be a hypercomplete ∞-topos. Then the natural t-structure

$$(\mathrm{Shv}(\mathcal{X}, \mathrm{Sp})^{\otimes}, \mathrm{Shv}(\mathcal{X}, \mathrm{Sp})_{\geq 0})$$

developed in [SAG, Proposition 1.3.2.7] is hypercomplete by [SAG, Proposition 1.3.3.3].

Proposition 3.7. *Let R be in $\mathrm{Alg}(\mathcal{A}_{\geq 0})$. Then $\mathrm{LMod}_R(\mathcal{A})$ is a presentable stable ∞-category which admits a natural accessible t-structure $(\mathrm{LMod}_R(\mathcal{A})_{\geq 0}, \mathrm{LMod}_R(\mathcal{A})_{\leq 0})$ satisfying the following properties:*

(1) *$\mathrm{LMod}_R(\mathcal{A})_{\geq 0}$ and $\mathrm{LMod}_R(\mathcal{A})_{\leq 0}$ are the inverse images of $\mathcal{A}_{\geq 0}$ and $\mathcal{A}_{\leq 0}$ under the projection $\theta : \mathrm{LMod}_R(\mathcal{A}) \to \mathcal{A}$.*

(2) *The natural inclusion $\mathrm{LMod}_R(\mathcal{A}_{\geq 0}) \hookrightarrow \mathrm{LMod}_R(\mathcal{A})$ induces an equivalence $\mathrm{LMod}_R(\mathcal{A}_{\geq 0}) \xrightarrow{\sim} \mathrm{LMod}_R(\mathcal{A})_{\geq 0}$.*

(3) *The functor $\tau_{\leq n} : \mathcal{A}_{\geq 0} \to \mathcal{A}_{[0,n]}$ induces an equivalence $\mathrm{LMod}_R(\mathcal{A})_{[0,n]} \xrightarrow{\sim} \mathrm{LMod}_{\tau_{\leq n}R}(\mathcal{A}_{[0,n]})$. In particular, the π_0 functor induces an equivalence $\mathrm{LMod}_R(\mathcal{A})^{\heartsuit} \xrightarrow{\sim} \mathrm{LMod}_{\pi_0 R}(\mathcal{A}^{\heartsuit})$.*

(4) *If $\mathcal{A}$ is left (resp. right, resp. hyper) complete, then so is $\mathrm{LMod}_R(\mathcal{A})$.*

(5) *If the t-structure on $\mathcal{A}$ is compatible with filtered colimits, meaning $\mathcal{A}_{\leq 0} \subset \mathcal{A}$ is closed under filtered colimits, then so is the induced t-structure on $\mathrm{LMod}_R(\mathcal{A})$.*

(6) *If $\mathcal{A}_{\geq 0}$ is projectively generated, then so is $\mathrm{LMod}_R(\mathcal{A})_{\geq 0}$.*

Proof. We first prove (1). It follows immediately from the definitions that the full subcategory $\mathrm{LMod}_R(\mathcal{A})_{\geq 0} \subset \mathrm{LMod}_R(\mathcal{A})$ is closed under small colimits and extensions. Also, note that $\mathrm{LMod}_R(\mathcal{A})_{\geq 0}$ is presentable since the following is a pullback square in Pr^L:

$$\begin{tikzcd}
\mathrm{LMod}_R(\mathcal{A})_{\geq 0} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{LMod}_R(\mathcal{A}) \arrow[d] \\
\mathcal{A}_{\geq 0} \arrow[r] & \mathcal{A}
\end{tikzcd}$$

Using [HA, Prop. 1.4.4.11], we deduce the existence of an accessible t-structure

$$(\mathrm{LMod}_R(\mathcal{A})_{\geq 0}, \mathrm{LMod}_R(\mathcal{A})')$$

on $\mathrm{LMod}_R(\mathcal{A})$. To complete the proof, it will suffice to show that $\mathrm{LMod}_R(\mathcal{A})' = \mathrm{LMod}_R(\mathcal{A})_{\leq 0}$.

Suppose first that $N \in \mathrm{LMod}_R(\mathcal{A})'$. Then the mapping space $\mathrm{Map}_{\mathrm{LMod}_R(\mathcal{A})}(M, N)$ is discrete for every object $M \in \mathrm{LMod}_R(\mathcal{A})_{\geqslant 0}$. In particular, for every connective object $X \in \mathcal{A}_{\geqslant 0}$, the mapping space $\mathrm{Map}_{\mathrm{LMod}_R(\mathcal{A})}(R \otimes X, N) \simeq \mathrm{Map}_{\mathcal{A}}(X, \theta(N))$ is discrete, so that $\theta(N) \in \mathcal{A}_{\leqslant 0}$, and therefore $N \in \mathrm{LMod}_R(\mathcal{A})_{\leqslant 0}$.

Conversely, suppose that $N \in \mathrm{LMod}_R(\mathcal{A})_{\leqslant 0}$. We wish to prove that $N \in \mathrm{LMod}_R(\mathcal{A})'$. Let $\mathcal{C}$ denote the full subcategory of $\mathrm{LMod}_R(\mathcal{A})$ spanned by those objects $M \in \mathrm{LMod}_R(\mathcal{A})$ for which the mapping space $\mathrm{Map}_{\mathrm{LMod}_R(\mathcal{A})}(M, N)$ is discrete. We wish to prove that $\mathcal{C}$ contains $\mathrm{LMod}_R(\mathcal{A})_{\geq 0}$. Firstly, we have that θ induces a functor $\mathrm{LMod}_R(\mathcal{A}) \to \mathcal{A}_{\geqslant 0}$ which is conservative and preserves small colimits; moreover, this functor has a left adjoint Fr, given informally by the formula $Fr(X) \simeq R \otimes X$. Using Lemma 3.1, we conclude that $\mathrm{LMod}_R(\mathcal{A})_{\geq 0}$ is generated under small colimits by the essential image of Fr. Since $\mathcal{C}$ is stable under colimits, it will suffice to show that $\mathcal{C}$ contains the essential image of Fr. Unwinding the definitions, we are reduced to proving that the mapping space

$$\mathrm{Map}_{\mathrm{LMod}_R(\mathcal{A})}(F(X), N) \simeq \mathrm{Map}_{\mathcal{A}}(X, \theta(N))$$

is discrete for every connective object X in $\mathcal{A}_{\geq 0}$, which is equivalent to our assumption that $N \in \mathrm{LMod}_R(\mathcal{A})_{\leqslant 0}$. This completes the proof of (1).

For (2), the proof follows directly from the definition.

For (3), we observe that we have a natural factorization:

$$\begin{tikzcd}
 & \mathrm{LMod}_R(\mathcal{A})_{[0,n]} \arrow[dr, dashed, "F_0"] & \\
\mathrm{LMod}_R(\mathcal{A}_{\geq 0}) \arrow[ur] \arrow[rr, "F"] & & \mathrm{LMod}_{\tau_{\leq n}R}(\mathcal{A}_{[0,n]})
\end{tikzcd}$$

It suffices to prove that F_0 is fully faithful and essentially surjective. It is easy to see that F and F_0 preserve colimits. We wish to prove that, for a fixed $N \in \mathrm{LMod}_R(\mathcal{A})_{[0,n]}$, the full subcategory $\mathcal{D}$ of $\mathrm{LMod}_R(\mathcal{A})_{\geq 0}$ spanned by those objects M for which the map

$$\mathrm{Map}_{\mathrm{LMod}_R(\mathcal{A})}(M, N) \to \mathrm{Map}_{\mathrm{LMod}_{\tau_{\leq n}R}(\mathcal{A}_{[0,n]})}(F(M), F(N))$$

is an equivalence. It is easy to see that $\mathcal{D}$ is stable under colimits and contains $R \otimes X$ for all $X \in \mathcal{A}_{\geq 0}$. Lemma 3.1 shows that $\mathcal{D} = \mathrm{LMod}_R(\mathcal{A})_{\geq 0}$. In particular, F_0 is fully faithful.

It remains to show that F_0 is essentially surjective. Since F_0 is fully faithful and preserves small colimits, the essential image of F_0 is closed under small colimits. By applying Lemma Lemma 3.1 to $\mathcal{A}_{[0,n]} \rightleftarrows \mathrm{LMod}_{\tau_{\leq n}R}(\mathcal{A}_{[0,n]})$, it will therefore suffice to show that every free left $\tau_{\leq n}R$-module $\tau_{\leq n}R\overline{\otimes}Y$ where $Y \in \mathcal{A}_{[0,n]}$ belongs to the essential image of F_0, where $\overline{\otimes}$ denotes the tensor product in $\mathcal{A}_{[0,n]}$. We now conclude by observing that $F(R \otimes X) \simeq \tau_{\leq n}R\overline{\otimes}\tau_{\leq n}X$.

The (4) and (5) are concluded by the fact that $\theta : \mathrm{LMod}_R(\mathcal{A}) \to \mathcal{A}$ is t-exact, conservative, and preserves small colimits and limits.

The (6) is concluded by Lemma 3.1(2) applied to the adjoint pair $\mathcal{A}_{\geq 0} \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{LMod}_R(\mathcal{A})_{\geq 0}$. ☐

Remark 3.8. See also a brief discussion in [AN21, Appendix].

Corollary 3.9. *Let $R \in \mathrm{Alg}_{\mathbb{E}_{k+1}}(\mathcal{A}_{\geq 0})$ be a connective $\mathbb{E}_{k+1}$-algebra where $1 \leq k \leq \infty$. Then the presentably $\mathbb{E}_k$-monoidal category $\mathrm{LMod}_R(\mathcal{A})^{\otimes} \to \mathbb{E}_k^{\otimes}$ satisfies:*

1. *The natural t-structure $(\mathrm{LMod}_R(\mathcal{A})_{\geq 0}, \mathrm{LMod}_R(\mathcal{A})_{\leq 0})$ is compatible with the monoidal structure.*
2. *The natural inclusion $\mathrm{LMod}_R(\mathcal{A}_{\geq 0})^{\otimes} \hookrightarrow \mathrm{LMod}_R(\mathcal{A})^{\otimes}$ is an $\mathbb{E}_k$-monoidal functor which induces an equivalence $\mathrm{LMod}_R(\mathcal{A}_{\geq 0})^{\otimes} \xrightarrow{\sim} \mathrm{LMod}_R(\mathcal{A})^{\otimes}_{\geq 0}$ of $\mathbb{E}_k$-monoidal categories.*
3. *The symmetric monoidal functor $\tau^{\otimes}_{\leq n} : \mathcal{A}^{\otimes}_{\geq 0} \to \mathcal{A}^{\otimes}_{[0,n]}$ induces an equivalence $\mathrm{LMod}_R(\mathcal{A})^{\otimes}_{[0,n]} \xrightarrow{\sim} \mathrm{LMod}_{\tau_{\leq n}R}(\mathcal{A}_{[0,n]})^{\otimes}$ of $\mathbb{E}_k$-monoidal $(n+1)$-categories. In particular, the π_0 functor induces an equivalence of $\mathbb{E}_k$-monoidal 1-categories $\mathrm{LMod}_R(\mathcal{A})^{\heartsuit,\otimes} \xrightarrow{\sim} \mathrm{LMod}_{\pi_0 R}(\mathcal{A}^{\heartsuit})^{\otimes}$. Note that when $k > 1$, $\mathbb{E}_k$-algebras in $\mathcal{A}^{\heartsuit}$ are $\mathbb{E}_{\infty}$-algebras, so $\mathrm{LMod}_{\pi_0 R}(\mathcal{A}^{\heartsuit})^{\otimes}$ is symmetric monoidal in this case.*

Remark 3.10. (1) When $R \in \mathrm{CAlg}(\mathcal{A}_{\geq 0})$ is a connective $\mathbb{E}_{\infty}$-ring object, $\mathrm{LMod}_R(\mathcal{A})^{\otimes}$ becomes a new presentably stably symmetric monoidal ∞-category with an accessible t-structure $(\mathrm{LMod}_R(\mathcal{A})_{\geq 0}, \mathrm{LMod}_R(\mathcal{A})_{\leq 0})$ that is compatible with the monoidal structure.

(2) Proposition 3.7 also applies to the right module category $\mathrm{RMod}_R(\mathcal{A})$ and the bimodule category ${}_R\mathrm{BMod}_S(\mathcal{A})$ when R, S are connective.

Convention 3.11. *In the case where $R \in \mathrm{CAlg}(\mathcal{A})$ is commutative, we will simply denote $\mathrm{LMod}_R(\mathcal{A})^{\otimes}$ by $\mathrm{Mod}_R(\mathcal{A})^{\otimes}$.*

3.2 Flat modules and algebras

Definition 3.12 (connective case)**.** Let $R \in \mathrm{Alg}(\mathcal{A}_{\geq 0})$ be a connective $\mathbb{E}_1$-ring object.

1. We say a left R-module M is flat if the relative tensor product functor $(-) \otimes_R M : \mathrm{RMod}_R(\mathcal{A}) \to \mathcal{A}$ is t-exact.
2. We say a left R-module M is faithfully flat if the relative tensor product functor $(-) \otimes_R M : \mathrm{RMod}_R(\mathcal{A}) \to \mathcal{A}$ is t-exact and conservative.
3. If R is $\mathbb{E}_{\infty}$ and $f : R \to S$ is a morphism in $\mathrm{CAlg}(\mathcal{A})$, we say f is (faithfully) flat if S is a (faithfully) flat R-module.

Remark 3.13. If M is flat on a connective $\mathbb{E}_1$-ring object $R \in \mathrm{Alg}(\mathcal{A}_{\geq 0})$, then $M \simeq R \otimes_R M$ itself is connective.

Proposition 3.14. *Let $R \in \mathrm{Alg}(\mathcal{A}_{\geq 0})$ be a connective $\mathbb{E}_1$-ring object. Then:*

(1) *The full subcategory of flat modules $\mathrm{LMod}_R(\mathcal{A})^{fl} \subset \mathrm{LMod}_R(\mathcal{A})$ is closed under finite coproducts, retractions, and extensions. If the t-structure on $\mathcal{A}$ is compatible with filtered colimits, then $\mathrm{LMod}_R(\mathcal{A})^{fl} \subset \mathrm{LMod}_R(\mathcal{A})$ is furthermore closed under filtered colimits.*

(2) *If $R \in \mathrm{CAlg}(\mathcal{A}_{\geq 0})$ is $\mathbb{E}_\infty$, then the full subcategory of flat modules $\mathrm{Mod}_R(\mathcal{A})^{fl,\otimes} \subset \mathrm{Mod}_R(\mathcal{A})^{\otimes}$ contains the unit and is closed under tensor product and hence a symmetric monoidal full subcategory.*

(3) *If $R \in \mathrm{CAlg}(\mathcal{A}_{\geq 0})$ is $\mathbb{E}_\infty$ and $M \in \mathrm{Mod}_R(\mathcal{A})$ is a dualizable R-module, then M is flat if and only if both M and the dual $M^\vee$ are connective R-modules.*

Proof.
(1) and (2) are obvious by definition of flatness.
(3) Assume that M is flat, then M is connective by the remark above. Since the

$$\mathrm{Map}_{\mathrm{Mod}_R(\mathcal{A})}(M^\vee, N) \simeq \mathrm{Map}_{\mathrm{Mod}_R(\mathcal{A})}(R, M \otimes_R N)$$

is contractible for any (-1)-truncated N, the $M^\vee$ is connective too.
Now assume both M and the dual $M^\vee$ are connective. Since M is connective, the tensor product $(-) \otimes_R M$ is right t-exact. So it suffices to check the left t-exactness of $(-) \otimes_R M$. Let Q be a connective R-module and N is be a (-1)-truncated R-module. Then the

$$\mathrm{Map}_{\mathrm{Mod}_R(\mathcal{A})}(Q \otimes_R M^\vee, N) \simeq \mathrm{Map}_{\mathrm{Mod}_R(\mathcal{A})}(Q, M \otimes_R N)$$

is contractible. So the $(-) \otimes_R M$ is indeed left t-exact. $\square$

Proposition 3.15. *Assume that $\mathcal{A}$ is Grothendieck. Let $R \in \mathrm{Alg}(\mathcal{A}_{\geq 0})$ be a connective $\mathbb{E}_1$-ring object and M be a connective left R-module. Then the following conditions are equivalent:*

(1) *M is flat.*

(2) *The tensor product functor $(-) \otimes_R M$ is left t-exact, meaning it sends the negative part to negative part.*

(3) *The tensor product functor $(-) \otimes_R M$ sends discrete objects to discrete objects.*

Proof. The (1) $\Leftrightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious. Now we claim that (3) $\Rightarrow$ (2).
Given a coconnective right R-module $M \in \mathrm{RMod}_R(\mathcal{A})_{\leq 0}$, we wish to show that $N \otimes_R M \in \mathcal{A}_{\leq 0}$. Since $\mathcal{A}$ is right complete, we have that $M \simeq \varinjlim \tau_{\geq -i} M$. Now we will prove that $N \otimes_R \tau_{\geq -n} M \in \mathcal{A}_{[-n,0]}$ inductively. The case $n = 0$ is true by the assumption. Now assume that for $n - 1 \geq 0$ it is true, we need to show that $N \otimes_R \tau_{\geq -n} M \in \mathcal{A}_{[-n,0]}$, which is by observing that the first and third items in the following exact sequence

$$N \otimes_R \tau_{\geq -(n-1)} M \to N \otimes_R \tau_{\geq -n} M \to N \otimes_R \pi_{-n} M$$

belong to $\mathcal{A}_{[-n,0]}$. Since the t-structure is compatible with filtered colimits, we have that $N \otimes_R M \simeq \varinjlim N \otimes_R \tau_{\geq -i} M$ belong to $\mathcal{A}_{\leq 0}$. $\square$

Remark 3.16. If the $(\mathcal{A}^{\otimes}, \mathcal{A}_{\geq 0}) \in \mathrm{CAlg}(\mathcal{P}\mathrm{r}_{\mathrm{st}}^{t\text{-rex}})$ is Grothendieck, then a connective left R-module M for some $R \in \mathrm{Alg}(\mathcal{A}_{\geq 0})$ is flat if and only if the tensor product functor $(-) \otimes_R M : \mathrm{RMod}_R(\mathcal{A}_{\geq 0}) \to \mathcal{A}_{\geq 0}$ is left exact by Proposition 1.12.

Proposition 3.17. *If $R \to S \in \mathrm{Alg}(\mathcal{A}_{\geq 0})$ be a morphism of connective $\mathbb{E}_1$-ring objects, then*

(1) *The relative tensor product $S \otimes_R (-) : \mathrm{LMod}_R(\mathcal{A}) \to \mathrm{LMod}_S(\mathcal{A})$ sends (faithfully) flat modules to (faithfully) flat modules.*

(2) *If S is flat as a left R-module, then the forgetful functor $\theta : \mathrm{LMod}_S(\mathcal{A}) \to \mathrm{LMod}_R(\mathcal{A})$ sends flat modules to flat modules. If furthermore S is faithfully flat as a left R-module, then the forgetful functor $\theta : \mathrm{LMod}_S(\mathcal{A}) \to \mathrm{LMod}_R(\mathcal{A})$ preserves faithfully flat modules.*

Proof.
(1)Let $M \in \mathrm{LMod}_R(\mathcal{A})$. We observe that $(-) \otimes_S (S \otimes_R M) \simeq (-) \otimes_R M$.
(2) Given $N \in \mathrm{LMod}_S(\mathcal{A})$, we observe that $(-) \otimes_R \theta(N) \simeq (-) \otimes_R S \otimes_S N$. □

Definition 3.18 (Nonconnective case)**.** Let $R \in \mathrm{Alg}(\mathcal{A})$ and $\theta : \mathrm{LMod}_R(\mathcal{A}) \to \mathrm{LMod}_{\tau_{\geq 0}R}(\mathcal{A})$ be the forgetful functor.

(1) We say a left R-module M is flat if the counit map $R \otimes_{\tau_{\geq 0}R} \tau_{\geq 0}\theta(M) \to M$ with respect to the following composite adjunction

$$\mathrm{LMod}_{\tau_{\geq 0}R}(\mathcal{A})_{\geq 0} \underset{\tau_{\geq 0}}{\rightleftarrows} \mathrm{LMod}_{\tau_{\geq 0}R}(\mathcal{A}) \underset{\theta}{\rightleftarrows} \mathrm{LMod}_R(\mathcal{A})$$

is an equivalence and $\tau_{\geq 0}\theta(M)$ is flat over $\tau_{\geq 0}R$.

(2) We say a left R-module M is faithfully flat if it is flat over R and $\tau_{\geq 0}\theta(M)$ is faithfully flat over $\tau_{\geq 0}R$.

(3) If $f : R \to S$ is a morphism in $\mathrm{CAlg}(\mathcal{A})$, we say f is (faithfully) flat if S is a (faithfully) flat R-module.

Remark 3.19. Let $R \in \mathrm{Alg}(\mathcal{A})$ be an $\mathbb{E}_1$-ring object. Then

(1) The full subcategory of flat modules $\mathrm{LMod}_R(\mathcal{A})^{fl} \subset \mathrm{LMod}_R(\mathcal{A})$ is closed under finite coproducts and retractions. but under extension???

(2) If the t-structure on $\mathcal{A}$ is compatible with filtered colimits, then $\mathrm{LMod}_R(\mathcal{A})^{fl} \subset \mathrm{LMod}_R(\mathcal{A})$ is closed under filtered colimits.

(3) If M is a flat left R-module, then M is faithfully flat implies that the tensor product functor $(-) \otimes_R M$ is conservative. The converse holds if $\mathcal{A}$ is Grothendieck and hypercomplete???(probably wrong)

Proposition 3.20. *Let $R \to S$ be a map in $\mathrm{Alg}(\mathcal{A})$, where R is not necessarily connective.*

(1) *If $\tau_{\geq 0}R \to \tau_{\geq 0}S$ is an equivalence, then the relative tensor product $S \otimes_R (-) : \mathrm{LMod}_R(\mathcal{A}) \to \mathrm{LMod}_S(\mathcal{A})$ restricts to a categorical equivalence $\mathrm{LMod}_R(\mathcal{A})^{fl} \xrightarrow{\sim} \mathrm{LMod}_S(\mathcal{A})^{fl}$ between full subcategories of flat modules.* (See [HA, Prop. 7.2.2.16] for the case of spectra.)

(2) *If $R \in \mathrm{CAlg}(\mathcal{A})$, then the full subcategory of flat modules $\mathrm{Mod}_R(\mathcal{A})^{fl,\otimes} \subset \mathrm{Mod}_R(\mathcal{A})^{\otimes}$ is closed under tensor product and hence a full symmetric monoidal subcategory.*

(3) *If $R \to S$ is a morphism in $\mathrm{CAlg}(\mathcal{A})$ such that $\tau_{\geq 0}R \to \tau_{\geq 0}S$ is an equivalence, then $\mathrm{Mod}_R(\mathcal{A})^{fl,\otimes} \xrightarrow{\sim} \mathrm{Mod}_S(\mathcal{A})^{fl,\otimes}$ is an equivalence of symmetric monoidal categories.*

Proof. The (2), (3) are conclusions of (1). So it suffices to prove (1) in the case when $R = \tau_{\geq 0}S$. Since R is connective, the ∞-category $\mathrm{LMod}_R(\mathcal{A})$ admits a t-structure. Let F' denote the composite functor

$$\mathrm{LMod}_R(\mathcal{A})_{\geq 0} \subseteq \mathrm{LMod}_R(\mathcal{A}) \xrightarrow{F} \mathrm{LMod}_S(\mathcal{A})$$

Then F' has a right adjoint, given by the composition $G' = \tau_{\geq 0} \circ G$. Given M is a flat left R-module, we observe that $M \to G'F'(M)$ is equivalent by the flatness of M. Now we wish to prove F' preserves flatness, i.e. $F'G'F'(M) \to F'(M)$ is equivalent, which is obvious. Then we wish to prove G' preserves flatness too, which is by definition of flatness in nonconnective case. Consequently, F' and G' induce adjoint functors

$$\mathrm{LMod}_R^{fl}(\mathcal{A}) \underset{G'}{\overset{F'}{\leftrightarrows}} \mathrm{LMod}_S^{fl}(\mathcal{A})$$

It now suffices to show that the unit and counit of the adjunction are equivalences. In other words, we must show:
(i) For every flat left R-module M, the unit map $M \to G'F'(M)$ is an equivalence, which has been done by the argument above.
(ii) For every flat left S-module N, the counit map $F'G'(N) \to N$ is an equivalence, which is by definition of flatness in nonconnective case. □

Proposition 3.21 (Nonconnective case)**.** *If $R \to S \in \mathrm{Alg}(\mathcal{A})$ be a morphism of $\mathbb{E}_1$-ring objects, then*

(1) *The relative tensor product $S \otimes_R (-) : \mathrm{LMod}_R(\mathcal{A}) \to \mathrm{LMod}_S(\mathcal{A})$ sends (faithfully) flat modules to (faithfully) flat modules.*

(2) *If S is flat as a left R-module, then the forgetful functor $\theta : \mathrm{LMod}_S(\mathcal{A}) \to \mathrm{LMod}_R(\mathcal{A})$ sends flat modules to flat modules. If furthermore S is faithfully flat as a left R-module, then the forgetful functor $\theta : \mathrm{LMod}_S(\mathcal{A}) \to \mathrm{LMod}_R(\mathcal{A})$ preserves faithfully flat modules.*

Proof. The (1) is deduced by combination of Proposition 3.17 and Proposition 3.20.
For (2), we claim the following diagram is right adjointable,

$$\begin{tikzcd}
\mathrm{LMod}_{\tau_{\geq 0}R}(\mathcal{A}) \arrow[r, shift left] \arrow[d] & \mathrm{LMod}_{\tau_{\geq 0}S}(\mathcal{A}) \arrow[l, shift left] \arrow[d] \\
\mathrm{LMod}_R(\mathcal{A}) \arrow[r, shift left] & \mathrm{LMod}_S(\mathcal{A}) \arrow[l, shift left]
\end{tikzcd}$$

because $S \otimes_{\tau_{\geq 0}S} (-) \simeq R \otimes_{\tau_{\geq 0}R} \tau_{\geq 0}S \otimes_{\tau_{\geq 0}S} (-)$ by flatness of S over R. Then it reduces to the connective case, which is Proposition 3.17. □

Proposition 3.22.

(1) *If $f : R \to S$ is a morphism in $\mathrm{CAlg}(\mathcal{A})$, then f is flat if and only if $f_{\geq 0} : \tau_{\geq 0}R \to \tau_{\geq 0}S$ is flat and the following diagram*

$$\begin{tikzcd}
\tau_{\geq 0}R \arrow[r] \arrow[d] & \tau_{\geq 0}S \arrow[d] \\
R \arrow[r] & S
\end{tikzcd}$$

is a pushout diagram in $\mathrm{CAlg}(\mathcal{A})$.

(2) *If $f : R \to S$ is a flat map in $\mathrm{CAlg}(\mathcal{A}_{\geq 0})$, then the following diagram is a pushout diagram in $\mathrm{CAlg}(\mathcal{A}_{\geq 0})$.*

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & S \arrow[d] \\
\tau_{\leq n}R \arrow[r] & \tau_{\leq n}S
\end{tikzcd}$$

Hence $\tau_{\leq n}R \to \tau_{\leq n}S$ is also flat for any $n \geq 0$.

(3) *Let $f : R \to S$ be a (faithful) flat map in $\mathrm{CAlg}(\mathcal{A})$ and $R \to A$ be another map in $\mathrm{CAlg}(\mathcal{A})$. Then the map $A \to A \otimes_R S$ given by the following pushout diagram is (faithful) flat.*

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & S \arrow[d] \\
A \arrow[r] & A \otimes_R S
\end{tikzcd}$$

Proof.
(1) If f is flat, then we have $S \simeq R \otimes_{\tau_{\geq 0}R} \tau_{\geq 0}S$ by the flatness of S over R. If the converse is ture, then $R \to S$ is flat by Proposition 3.21(1).
(2) Since $(-) \otimes_R S$ is t-exact, the following diagram is a pushout diagram in $\mathrm{CAlg}(\mathcal{A})$.

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[d] \\ \tau_{\leq n}R \arrow[r] & \tau_{\leq n}S \end{tikzcd}$$

So $\tau_{\leq n}R \to \tau_{\leq n}S$ is flat by Proposition 3.21(1).
(3) It follows immediately from Proposition 3.21.

□

Proposition 3.23. *Given a diagram*

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "h"] & \\ B \arrow[rr, "g"] & & C \end{tikzcd}$$

in $\mathrm{CAlg}(\mathcal{A})$.

(1) *where f, g are flat morphisms, then so is the composition h.*

(2) *If h is flat and g is faithfully flat, then f is flat.*

Proof.
(1) It follows immediately from definition.
(2) Considering the following diagram

$$\begin{tikzcd} \tau_{\geq 0}A \arrow[r, "\tau_{\geq 0}f"] \arrow[d] & \tau_{\geq 0}B \arrow[r, "\tau_{\geq 0}g"] \arrow[d] & \tau_{\geq 0}C \arrow[d] \\ A \arrow[r, "f"] & B \arrow[r, "g"] & C \end{tikzcd}$$

in $\mathrm{CAlg}(\mathcal{A})$. Then the right square and outer square are pushouts by Proposition 3.22. The faithful flatness of g implies the tensor product functor $(-) \otimes_{\tau_{\geq 0}B} \tau_{\geq 0}C$ is conservative, which implies the left square is also a pushout. So we reduce to the case when A, B, C are connective.

Now given a coconnective A-module $M \in \mathrm{Mod}_A(\mathcal{A})_{\leq 0}$, we wish to show that $N = B \otimes_A M$ is also coconnective. However the $C \otimes_B N$ is coconnective by assumption, so N is coconnective by faithful flatness of g. □

Corollary 3.24. *Given a pushout diagram in* $\mathrm{CAlg}(\mathcal{A})$

$$\begin{tikzcd} A' \arrow[r, "\psi"] \arrow[d, "\phi"] & A \arrow[d, "\phi'"] \\ B' \arrow[r, "\psi'"] & B \end{tikzcd}$$

where ψ is faithfully flat. If B is flat over A, then B' is flat over A'.

Proof. Since ψ is faithfully flat, the morphism ψ' is also faithfully flat. By virtue of Proposition 3.23(2), it will suffice to show that the composition $\psi' \circ \phi \simeq \phi \circ \psi$ is flat. This also follows from Proposition 3.23, since ψ and ϕ are both flat. □

Proposition 3.25. *Let $R \in \mathrm{Alg}(\mathcal{A})$. Then:*

(1) *Let $M \in \mathrm{LMod}_R(\mathcal{A})$. If M is (faithfully) flat over R, then $\pi_0 M \in \mathrm{LMod}_{\pi_0 R}(\mathcal{A}^\heartsuit)$ is (faithfully) flat over $\pi_0 R$ in the sense of Definition 2.11.*

(2) *Assume that $\mathcal{A}$ is Grothendieck and hypercomplete. Let $f : M \to N$ be a map between flat left R-modules. If $\pi_0 f : \pi_0 M \to \pi_0 B$ is an equivalence, then so is $f : M \to N$.*

(3) *Let $M \in \mathrm{LMod}_R(\mathcal{A})$ be a flat left R-module. Then for any $n \in \mathbb{Z}$, we have $\pi_n(R)\overline{\otimes}_{\pi_0 R}\pi_0 M \to \pi_n M$ is a natural equivalence in $\mathrm{LMod}_{\pi_0 R}(\mathcal{A}^\heartsuit)$.*

(4) *Assume that $\mathcal{A}$ is Grothendieck. If $f : R \to S$ is a faithfully flat morphism in $\mathrm{CAlg}(\mathcal{A})$, then the* $\mathrm{cofib}(f)$ *is a flat R-module. The converse holds provided furthermore that $\mathcal{A}$ is hypercomplete.*

Proof. For (1), we have that $\tau_{\geq 0} M$ is flat over $\tau_{\geq 0} R$, so it suffices to show the case when R, M are connective. Therefore by t-exactness we have the following factorization

$$\begin{tikzcd}
\mathrm{RMod}_{\tau_{\geq 0}R}(\mathcal{A}) \arrow[r, "(-)\otimes_R M"] \arrow[d, "\tau_{\geq 0}"] & \mathcal{A} \arrow[d, "\tau_{\geq 0}"] \\
\mathrm{RMod}_{\tau_{\geq 0}R}(\mathcal{A})_{\geq 0} \arrow[r, dashed, "(-)\otimes_R M"] & \mathcal{A}_{\geq 0}
\end{tikzcd}$$

which implies that $(-) \otimes_R M : \mathrm{RMod}_R(\mathcal{A}_{\geq 0}) \to \mathcal{A}_{\geq 0}$ is left exact. Now since $\pi_0 : \mathcal{A}^{\otimes}_{\geq 0} \to (\mathcal{A}^\heartsuit)^{\otimes}$ is a symmetric monoidal functor preserving geometric realizations, we have the commutative diagram of relative tensor product functors.

$$\begin{tikzcd}
\mathrm{RMod}_R(\mathcal{A}_{\geq 0}) \arrow[r, "(-)\otimes_R M"] \arrow[d] & \mathcal{A}_{\geq 0} \arrow[d] \\
\mathrm{RMod}_{\pi_0 R}(\mathcal{A}^\heartsuit) \arrow[r, "(-)\otimes_{\pi_0 R}\pi_0 M"] & \mathcal{A}^\heartsuit
\end{tikzcd}$$

So we conclude that $(-)\overline{\otimes}_{\pi_0 R}\pi_0 M$ is left exact since the above horizontal functor preserves discrete objects. For the faithful flat case, it follows directly from definition.
(2) By definition of nonconnective flatness, without loss of generalization we can assume that R, M and N are connective. To do so it suffices to prove that $R \otimes_R f$ is an equivalence. Since both M and N are flat and $\mathcal{A}$ is hypercomplete we may reduce to proving that $\pi_0(\pi_n R \otimes_R f)$ is an equivalence for all $n \geq 0$. This agrees with $\pi_0(\pi_n R \otimes_R \pi_0 f)$, which is an equivalence by virtue of the fact that $\pi_0(f)$ is an equivalence.
(3) By definition we have that $\tau_{\geq 0} M$ is flat over $\tau_{\geq 0} R$ and $R \otimes_{\tau_{\geq 0} R} \tau_{\geq 0} M \simeq M$. Then it follows by combining the t-exactness of $(-) \otimes_{\tau_{\geq 0} R} \tau_{\geq 0} M$ and the natural equivalence $\pi_n(R) \otimes_{\tau_{\geq 0} R} \tau_{\geq 0} M \simeq \pi_n(R)\overline{\otimes}_{\pi_0 R}\pi_0 M$.
(4) Due to Proposition 3.22(1), we may reduce to the case where R and S are connective. Let C denote the $\mathrm{cofib}(f)$.

Now assume that C is flat over R. By Proposition 3.15, it suffices to show that $C \otimes_R M$ is discrete for any discrete R-module M. Since $R \otimes_R M$ and $S \otimes_R M$ are discrete, we have that the $C \otimes_R M \in \mathcal{A}_{\leq 1}$. Therefore it suffices to show that $\pi_i(C \otimes_R M) = 0$ when $i \neq 0$, which is deduced by combining the Proposition 3.25(1), Proposition 2.12 and the long exact sequence associated with $R \otimes_R M \to S \otimes_R M \to C \otimes_R M$.

For the converse implication, to see that S is flat over R, it suffices to show that $S \otimes_R M$ is discrete for any discrete R-module M, which is obvious by the cofiber sequence $R \otimes_R M \to S \otimes_R M \to C \otimes_R M$ with the first and third terms discrete. To see the faithfulness, due to the hypercompleteness, it is reduced to proving $M = 0$ if M is discrete and $S \otimes_R M = 0$, which is again by the cofiber sequence above. $\square$

4 Projective modules

Throughout Section 4, we assume that (*)

(1) The t-structure on $\mathcal{A}$ is right complete and compatible with filtered colimits, i.e., it is Grothendieck.

(2) The $\mathcal{A}_{\geq 0}$ is projectively generated, meaning $\mathcal{A}_{\geq 0} \simeq \mathcal{P}_{\Sigma}(\mathcal{A}_{\geq 0}^{cproj})$.

Remark 4.1.

(1) The assumption (1) implies that the t-structure is recovered by the Grothendieck prestable ∞-category $\mathcal{A}_{\geq 0}$ by Corollary 1.13 that $\mathrm{Sp}(\mathcal{A}_{\geq 0}) \simeq \mathcal{A}$.

(2) The assumptions (1) and (2) imply that the t-structure on $\mathcal{A}$ is **left complete** by combining Proposition 1.18 and [SAG, Remark C.1.5.9].

(3) Under the assumptions (1) and (2), for any $R \in \mathrm{Alg}(\mathcal{A}_{\geq 0})$, the $\mathrm{LMod}_R(\mathcal{A})_{\geq 0}$ is projectively generated and the induced t-structure on $\mathrm{LMod}_R(\mathcal{A})$ is right complete and compatible with filtered colimits too by Proposition 3.7.

4.1 Projective modules

Proposition 4.2. *The $\mathcal{A}$ is compactly generated. And hence for any $R \in \mathrm{Alg}(\mathcal{A})$, the $\mathrm{LMod}_R(\mathcal{A})$ is compactly generated by Lemma 3.1.*

Proof. We have $\mathcal{A} \simeq \varprojlim_n \mathcal{A}_{\geq -n}$ as an inverse limit diagram in $\mathcal{P}\mathrm{r}^R_{\omega}$ by the assumption, so $\mathcal{A} \in \mathcal{P}\mathrm{r}^R_{\omega}$ too. $\square$

Proposition 4.3. *Let $R \in \mathrm{Alg}(\mathcal{A}_{\geq 0})$, and let $\mathcal{C}$ be the smallest idempotent complete stable subcategory of $\mathrm{LMod}_R(\mathcal{A})$ which contains all compact projective left modules. Then $\mathcal{C} = \mathrm{LMod}_R(\mathcal{A})^c$ the full subcategory of compact modules.*

Proof. Since $\mathrm{LMod}_R(\mathcal{A})$ is right complete, the collection of connective cover functors $\{\tau_{\geq -n} | n \geq 0\}$ is jointly conservative. Therefore by Lemma 3.1 $\mathrm{LMod}_R(\mathcal{A})$ is generated by $\{R \otimes \Sigma^{-n} P | n \geq 0, P \in \mathcal{A}_{\geq 0}^{cproj}\}$ under small colimits. Then the compact generation of $\mathcal{A}$ implies $\mathcal{C} = \mathrm{LMod}_R(\mathcal{A})^c$. $\square$

Remark 4.4. We will see in Proposition 5.12 that under the stronger assumption of projective rigidity, the "idempotent-complete" condition above is removable, i.e. $\mathrm{LMod}_R(\mathcal{A})^c \subset \mathrm{LMod}_R(\mathcal{A})$ is the smallest stable subcategory which contains all compact projective left modules.

Definition 4.5. Let R be in $\mathrm{Alg}(\mathcal{A}_{\geq 0})$. We say $P \in \mathrm{LMod}_R(\mathcal{A})_{\geq 0}$ is a projective left R-module if it is a projective object in $\mathrm{LMod}_R(\mathcal{A})_{\geq 0}$, meaning that the corepresentable functor

$$\mathrm{Map}_{\mathrm{LMod}_R(\mathcal{A})_{\geq 0}}(P, -) : \mathrm{LMod}_R(\mathcal{A})_{\geq 0} \to \mathcal{S}$$

preserves geometric realizations.

We introduce a stronger version of [HA, Lemma 1.3.3.11(2)]. In there, it requires that both $\mathcal{C}$ and $\mathcal{C}'$ are left complete. However, we find the left complete condition on $\mathcal{C}$ is removable.

Lemma 4.6. *Let $\mathcal{C}$ and $\mathcal{C}'$ be stable ∞-categories equipped with t-structures. Then:*

(1) *If $F : \mathcal{C}_{\geq 0} \to \mathcal{C}'_{\geq 0}$ is a functor preserves finite colimits, then $\tau_{\leq n} \circ F \xrightarrow{\sim} \tau_{\leq n} \circ F \circ \tau_{\leq n}$ is a natural equivalence in $\mathrm{Fun}(\mathcal{C}_{\geq 0}, \mathcal{C}'_{[0,n]})$ for any $n \geq 0$.*

(2) *If $\mathcal{C}_{\geq 0}$ admits geometric realizations and $\mathcal{C}'$ is left complete, then a functor $F : \mathcal{C}_{\geq 0} \to \mathcal{C}'_{\geq 0}$ preserves finite colimits if and only if it preserves finite coproducts and geometric realizations.*

Proof.
(1) Since F is right exact, it preserves suspension. Given $X \in \mathcal{C}_{\geq 0}$, then we have that the sequence

$$F(\tau_{\geq n+1} X) \to F(X) \to F(\tau_{\leq n} X)$$

is a cofiber sequence in $\mathcal{C}'_{\geq 0}$ and that $F(\tau_{\geq n+1}X) \in \mathcal{C}'_{\geq n+1}$. Taking $\tau_{\leq n}$, we get the natural equivalence

$$\tau_{\leq n}F(X) \xrightarrow{\sim} \tau_{\leq n}F(\tau_{\leq n}X).$$

(2) If F preserves finite coproducts and geometric realizations of simplicial objects, then F is right exact [HA, Lemma 1.3.3.10]. Conversely, suppose that F is right exact; we wish to prove that F preserves geometric realizations of simplicial objects. It will suffice to show that each composition

$$\mathcal{C}_{\geq 0} \xrightarrow{F} \mathcal{C}'_{\geq 0} \xrightarrow{\tau_{\leq n}} \left(\mathcal{C}'_{\geq 0}\right)_{\leq n}$$

By the (1), in virtue of the right exactness of F, this functor is equivalent to the composition

$$\mathcal{C}_{\geq 0} \xrightarrow{\tau_{\leq n}} \left(\mathcal{C}_{\geq 0}\right)_{\leq n} \xrightarrow{\tau_{\leq n}\circ F} \left(\mathcal{C}'_{\geq 0}\right)_{\leq n}.$$

It will therefore suffice to prove that $\tau_{\leq n} \circ F$ preserves geometric realizations of simplicial objects, which follows from [HA, Lemma 1.3.3.10] since both the source and target are equivalent to n-categories. □

We also introduce a stronger version of [HA, Prop. 7.2.2.6]. In there, it requires that $\mathcal{C}$ is left complete, which is removable too.

Proposition 4.7. *Let $\mathcal{C}$ be a stable ∞-category with a t-structure such that $\mathcal{C}_{\geq 0}$ admits geometric realizations. Given $P \in \mathcal{C}_{\geq 0}$, then the following conditions are equivalent:*

(1) *The P is projective in $\mathcal{C}_{\geq 0}$.*

(2) *For any $Q \in \mathcal{C}_{\geq 0}$, the abelian group $\operatorname{Ext}^1(P,Q) = 0$.*

(3) *For any $Q \in \mathcal{C}_{\geq 0}$, the abelian group $\operatorname{Ext}^i(P,Q) = 0$ when $i > 0$.*

(4) *The mapping spectrum functor $\underline{\operatorname{Map}}_{\mathcal{C}}(P,-) : \mathcal{C} \to \operatorname{Sp}$ is t-exact.*

Proof. The implications (3) ⇒ (2) and (3) ⇔ (4) are obvious. The implication (2) ⇒ (3) follows by replacing Q by $Q[i-1]$.

We first show that (1) ⇒ (2). Let $f : \mathcal{C} \to \mathcal{S}$ be the functor corepresented by P. Let $M_\bullet$ be a Čech nerve for the morphism $0 \to Q[1]$, so that $M_n \simeq Q^n \in \mathcal{C}_{\geq 0}$. Then $Q[1]$ can be identified with the geometric realization $|M_\bullet|$. Since P is projective, $f(Q[1])$ is equivalent to the geometric realization $|f(M_\bullet)|$. We have a surjective map $* \simeq \pi_0 f(M_0) \to \pi_0 |f(M_\bullet)|$, so that $\pi_0 f(Q[1]) = \operatorname{Ext}^1_{\mathcal{C}}(P,Q) = 0$.

We now show that (3) ⇒ (1). That $\mathcal{C}$ is stable implies that f is homotopic to a composition

$$\mathcal{C} \xrightarrow{F} \operatorname{Sp} \xrightarrow{\Omega^\infty} \mathcal{S},$$

where F is an exact functor. Applying (3), we deduce that F is right t-exact (see [HA, Definition 1.3.3.1]). The Lemma 4.6 implies that the induced map $\mathcal{C}_{\geq 0} \to \operatorname{Sp}_{\geq 0}$ preserves geometric realizations of simplicial objects. Applying [HA, Proposition 1.4.3.9] that $\operatorname{Sp} \xrightarrow{\Omega^\infty} \mathcal{S}$ preserves small sifted colimits, we conclude that $f \mid \mathcal{C}_{\geq 0}$ preserves geometric realizations as well. □

Proposition 4.8 (See [Ste23] Proposition 2.4.8)**.** *Let $\mathcal{C}$ be a projectively generated Grothendieck prestable ∞-category. Then*

(1) *The truncation functor $H_0 : \mathcal{C} \to \mathcal{C}^\heartsuit$ sends projective objects to 1-projective objects and compact objects to compact objects.*

(2) *The 0-truncations of the compact projective objects of $\mathcal{C}$ provide a family of compact 1-projective generators for $\mathcal{C}^\heartsuit$.*

(3) *The functor* $\mathrm{h}(\pi_0) : \mathrm{h}(\mathcal{C}) \to \mathcal{C}^\heartsuit$ *induced at the level of homotopy categories restricts to an equivalence between the full subcategories of (compact) projective objects and (compact) 1-projective objects.*

Proof. We first prove (1). The fact that π_0 sends compact objects to compact objects follows directly from the fact that the inclusion $\mathcal{C}^\heartsuit \to \mathcal{C}$ preserves filtered colimits. The fact that π_0 sends projective objects to 1-projective objects follows from Proposition 4.7.

Item (2) follows directly from (1) together with the fact that π_0 is a localization. It remains to establish (3). We first prove fully faithfulness. Let X, Y be a pair of projective objects of $\mathcal{C}$. Then the map $\mathrm{Map}_{\mathcal{C}}(X,Y) \to \mathrm{Map}_{\mathcal{C}^\heartsuit}(\pi_0(X), \pi_0(Y))$ induced by π_0 is equivalent to the map $\eta_* : \mathrm{Map}_{\mathcal{C}}(X,Y) \to \mathrm{Map}_{\mathcal{C}}(X, \pi_0(Y))$ of composition with the unit $\eta : Y \to \pi_0(Y)$. The fact that X is projective and η induces an equivalence on π_0 implies that η_* is an effective epimorphism. Its fiber is given by $\mathrm{Map}_{\mathcal{C}}(X, \tau_{\geq 1}(Y))$ which is connected since X is projective. We conclude that η_* induces an equivalence on π_0, and therefore $\mathrm{h}(\pi_0)$ is fully faithful when restricts on the full subcategory of projective objects.

It remains to prove the essential surjectivity. In other words, we have to show that every (compact) 1-projective object of $\mathcal{C}^\heartsuit$ is the image under π_0 of a (compact) projective object of $\mathcal{C}$. We will establish the case of compact projective objects, and the proof in the projective case being similar. Let Y be a compact 1-projective object of $\mathcal{C}^\heartsuit$. Applying (2) we may find a compact projective object X in $\mathcal{C}$ such that Y is a retract of $\pi_0(X)$. Let $r : \pi_0(X) \to \pi_0(X)$ be the induced retraction. The fully faithfulness part of (3) allows us to lift r to an idempotent endomorphism ρ of X inside $\mathrm{h}(\mathcal{C})$. Let X' be a representative in $\mathcal{C}$ of the image of ρ. Then X' is a direct summand of X (see similar argument in [HA, Lem. 1.2.4.6]) and therefore it is compact projective. The proof finishes by observing that $\pi_0(X') = \mathrm{Im}(r) = Y$. ☐

Remark 4.9. We did not use $\mathcal{A}$ in the above because the proposition does not require a monoidal structure.

Proposition 4.10. *Let* R *be in* $\mathrm{Alg}(\mathcal{A}_{\geq 0})$ *and* $P \in \mathrm{LMod}_R(\mathcal{A})_{\geq 0}$. *Then:*

(1) *The* P *is a projective* R*-module if and pnly if every map* $X \to P$ *in* $\mathrm{LMod}_R(\mathcal{A})_{\geq 0}$ *which induces an epimorphism on* π_0 *admits a section.*

(2) *If* P *is a projective* R*-module, then* $\pi_0 P \in \mathrm{LMod}_{\pi_0 R}(\mathcal{A}^\heartsuit)$ *is a 1-projective discrete* $\pi_0 R$*-module in the sense for an abelian category.*

Proof.
(1) This is by equivalent conditions of Proposition 4.7 (1) and (2).
(2) Combining the (1) and the equivalence $\mathrm{Map}_{\mathrm{LMod}_R(\mathcal{A})_{\geq 0}}(P, M) \simeq \mathrm{Map}_{\mathrm{LMod}_{\pi_0 R}(\mathcal{A}^\heartsuit)}(\pi_0 P, M)$ for a discrete $\pi_0 R$-module M, we win. ☐

Corollary 4.11. *Let* $R \in \mathrm{Alg}(\mathcal{A}_{\geq 0})$. *Then the heart* $\mathrm{LMod}_{\pi_0 R}(\mathcal{A}^\heartsuit)$ *has enough projectives as an abelian category.*

Proposition 4.12. *Let* R *be in* $\mathrm{Alg}(\mathcal{A}_{\geq 0})$ *and* $P \in \mathrm{LMod}_R(\mathcal{A})_{\geq 0}$. *Then* P *is projective if and only if there exists a small collection of compact projective modules* $\{P_\alpha\}$ *in* $\mathrm{LMod}_R(\mathcal{A})_{\geq 0}$ *such that* P *is a retraction of* $\oplus_\alpha P_\alpha$.

Proof. Suppose first that P is projective. By the projective generation there exists an equivalence of left R-modules

$$\mathrm{colim}_\alpha M_\alpha \xrightarrow{\sim} P$$

where each M_α is compact projective. Then the induced map $\oplus_\alpha \pi_0 M_\alpha \to \pi_0 P$ is epimorphic. Invoking Proposition 4.7, we deduce that p admits a section (up to homotopy), so that P is a retract of M. To prove the converse, we observe that the collection of projective left R-modules is stable under small coproducts and retracts by Proposition 4.7. ☐

4.2 Projective rigidity and Lazard's theorem

Definition 4.13. We say a presentably symmetric monoidal Grothendieck prestable ∞-category $\mathcal{C}^{\otimes} \in \mathrm{CAlg}(\mathrm{Groth}_{\infty})$ is **projectively rigid** if it satisfies the following:

(1) The $\mathcal{C}$ is projectively generated.

(2) The $\mathcal{C}^d = \mathcal{C}^{cproj}$, i.e. the dualizable objects coincide with compact projective objects.

We also say a ttt-∞-category $(\mathcal{B}^{\otimes}, \mathcal{B}_{\geq 0})$ is algebraic if $\mathcal{B}$ is Grothendieck (see Definition 1.14) and $\mathcal{B}_{\geq 0}^{\otimes}$ is projectively rigid. We will discuss more details about projective rigidity and algebraic ttt-∞-categories in Section 8.

Remark 4.14.

(1) **Warning:** In general, $(\mathcal{A}_{\geq 0})^d \subsetneq \mathcal{A}^d \cap \mathcal{A}_{\geq 0}$ because the dual of an object $X \in \mathcal{A}^d \cap \mathcal{A}_{\geq 0}$ is not necessarily connective! However, that holds exactly when X is flat, see Proposition 3.14(3), which claims that $(\mathcal{A}_{\geq 0})^d = \mathcal{A}^d \cap \mathcal{A}^{fl}$.

(2) Suppose that $\mathcal{A}_{\geq 0}^{\otimes}$ is projectively rigid. Then the symmetric monoidal Grothendieck prestable ∞-category $\mathcal{A}_{\geq 0}^{\otimes}$ can be identified with the symmetric monoidal projective cocompletion $\mathcal{P}_{\Sigma}(\mathcal{A}_{\geq 0}^{cproj})^{\otimes}$. And the heart $(\mathcal{A}^{\heartsuit})^{\otimes}$ is 1-projectively rigid.

(3) The $\mathrm{Mod}_R(\mathrm{Sp}_{\geq 0})^{\otimes}$ is projectively rigid for any connective $\mathbb{E}_{\infty}$-ring R, see [SAG, Prop. 2.9.1.5].

Proposition 4.15. *Suppose that $\mathcal{A}_{\geq 0}^{\otimes}$ is projectively rigid. Then:*

(1) *The $\mathcal{A}^{\otimes}$ is compactly rigid, meaning the compact objects and dualizable objects in it coincide. Particularly we have $\mathcal{A}^{\otimes} \in \mathrm{CAlg}(\mathcal{P}\mathrm{r}^L_{\mathrm{st},\omega})$.*

(2) *Let R be in $\mathrm{CAlg}(\mathcal{A}_{\geq 0})$. Then $\mathrm{Mod}_R(\mathcal{A}_{\geq 0})^{\otimes}$ is projectively rigid too.*

(3) *Let R be in $\mathrm{CAlg}(\mathcal{A}_{\geq 0})$. Then $\mathrm{Mod}_R(\mathcal{A})^{\otimes}$ is compactly rigid too.*

Proof.
(1) Since $\mathcal{A}$ is right complete, the collection of connective cover functors $\{\tau_{\geq -n} | n \geq 0\}$ is jointly conservative. Therefore by Lemma 3.1, $\mathcal{A}$ is generated by $\{\Sigma^{-n} P | n \geq 0, P \in \mathcal{A}_{\geq 0}^{cproj}\}$ under small colimits. By Proposition A.10 (4), we have $\{\Sigma^{-n} P | n \geq 0, P \in \mathcal{A}_{\geq 0}^{cproj}\} \subset \mathcal{A}^d$ and hence $\mathcal{A}^c = \mathcal{A}^d$.
(2) Since the symmetric monoidal functor

$$\mathcal{A}_{\geq 0}^{\otimes} \xrightarrow{R \otimes (-)} \mathrm{Mod}_R(\mathcal{A}_{\geq 0})^{\otimes}$$

preserves compact projective objects and dualizable objects, we conclude that

(i) The unit R is dualizable in $\mathrm{Mod}_R(\mathcal{A}_{\geq 0})$.

(ii) The $R \otimes P$ is dualizable in $\mathrm{Mod}_R(\mathcal{A}_{\geq 0})$ if $P \in \mathcal{A}_{\geq 0}$ is compact projective.

So the full subcategory of dualizable objects $\mathrm{Mod}_R(\mathcal{A}_{\geq 0})^d$ contains $\{R \otimes X | X \in \mathcal{A}_{\geq 0}^{cproj}\}$. Then combining Lemma 3.1 (2) and Proposition A.10 (2)(3), we get $\mathrm{Mod}_R(\mathcal{A}_{\geq 0})^{cproj} \subset \mathrm{Mod}_R(\mathcal{A}_{\geq 0})^d$. And that the unit R is compact projective implies the equality $\mathrm{Mod}_R(\mathcal{A}_{\geq 0})^{cproj} = \mathrm{Mod}_R(\mathcal{A}_{\geq 0})^d$.
(3) Apply the (1) and (2) to $\mathrm{Mod}_R(\mathcal{A})^{\otimes}$, we win. $\square$

Proposition 4.16. *Suppose that $\mathcal{A}_{\geq 0}^{\otimes}$ is projectively rigid. Let $R \in \mathrm{CAlg}(\mathcal{A}_{\geq 0})$ and $M \in \mathrm{Mod}_R(\mathcal{A})_{\geq 0}$. Then M is compact projective if and only if it is dualizable in $\mathrm{Mod}_R(\mathcal{A})^{\otimes}$ and flat.*

Proof. This is directly by combining the projective rigidity and Proposition 3.14 (3). ☐

Proposition 4.17. *Suppose that $\mathcal{A}^{\otimes}_{\geq 0}$ is projectively rigid. Let $R \in \mathrm{Alg}(\mathcal{A}_{\geq 0})$ and M be a connective left R-module. Then:*

(1) *If M is projective, then M is flat.*

(2) *The M is compact projective if and only if it is left dualizable in $\mathrm{LMod}_R(\mathcal{A}_{\geq 0})$.*

(3) *The M is (compact) projective if and only if it is flat and $\pi_0 M$ is (compact) 1-projective in $\mathrm{LMod}_{\pi_0 R}(\mathcal{A}^{\heartsuit})$.*

(4) *Suppose that $R \in \mathrm{Alg}(\mathcal{A}^{\heartsuit})$ is discrete. Then M is flat if and only if M is discrete and flat over $\pi_0 R$ in the sense of Definition 2.11.*

Proof.
(1) Since flat modules are closed under small coproducts and retractions, we reduce to the case $M = R \otimes P$ where $P \in \mathcal{A}^{cproj}_{\geq 0}$. That is easy because $(-) \otimes_R (R \otimes P) \simeq (-) \otimes P$ reduces to the case $R = \mathbf{1}$, which is deduced by Proposition 4.16.
(2) By Corollary A.14, we see that left dualizable objects are closed under finite coproducts and retracts. We observe that every $R \otimes P$ is left dualizable (given by $P^{\vee} \otimes R$), which proves "only if" direction. For the "if" direction, if M is left dualizable, then it follows from

$$\mathrm{Map}_{\mathrm{LMod}_R(\mathcal{A}_{\geq 0})}(M, -) \simeq \mathrm{Map}_{\mathcal{A}_{\geq 0}}(\mathbf{1}, {}^{\vee}M \otimes_R -)$$

and compact projectivity of the unit.
(3) By the (1) we have that every projective left R-module is flat. Secondly, the fact that π_0 sends projective objects to 1-projective objects was already observed in Proposition 4.10. This finishes the proof of the "only if" direction.

Assume now that M is flat and $\pi_0 M$ is (compact) 1-projective. Applying Proposition 4.8 we may find a (compact) projective R-module M' and an isomorphism $\pi_0 M' = \pi_0 M$. The fact that M' is projective allows us to lift this isomorphism to a map $f : M' \to M$. We observe that f is an equivalence by Proposition 3.25(2).
(4) The "only if" direction follows from (1) and $M \simeq R \otimes_R M$. For the "if" direction, given a discrete right R-module N, we wish to show that $N \otimes_R M$ is discrete too. Take a map $f : P \to N$ of right R-module such that P is projective and f induces an epimorphism on π_0. Then we have exact sequence

$$0 \to \pi_0 \,\mathrm{fib}(f) \to \pi_0 P \to \pi_0 N \to 0$$

and hence $\mathrm{fib}(f)$ is discrete. Now tensoring with M, we get an exact sequence

$$0 \to \pi_0(\mathrm{fib}(f) \otimes_R M) \to \pi_0(P \otimes_R M) \to \pi_0(N \otimes_R M) \to 0$$

by 1-flatness and the commutative diagram of relative tensor product functors.

$$\begin{tikzcd} \mathrm{RMod}_R(\mathcal{A}_{\geq 0}) \arrow[r, "(-)\otimes_R M"] \arrow[d, "\pi_0"] & \mathcal{A}_{\geq 0} \arrow[d, "\pi_0"] \\ \mathrm{RMod}_{\pi_0 R}(\mathcal{A}^{\heartsuit}) \arrow[r, "(-)\otimes_{\pi_0 R}\pi_0 M"] & \mathcal{A}^{\heartsuit} \end{tikzcd}$$

Because P is also flat by (1), we see that $\pi_1(N \otimes_R M) = 0$. We actually have proved for any discrete right R-module N has the property $\pi_1(N \otimes_R M) = 0$. Then by induction on n we get that $\pi_n(\mathrm{fib}(f) \otimes_R M) = \pi_{n+1}(N \otimes_R M) = 0$ for all $n > 0$, which implies $N \otimes_R M$ is discrete. ☐

Theorem 4.18 (Lazard's Theorem)**.** *Suppose that $\mathcal{A}^{\otimes}_{\geq 0}$ is projectively rigid. Let $R \in \mathrm{Alg}(\mathcal{A}_{\geq 0})$ and $M \in \mathrm{LMod}_R(\mathcal{A})_{\geq 0}$. Then M is a flat left R-module if and only it it is equivalent to a filtered colimit of compact projective left R-modules.*

Proof. We take the strategy in [Ste23, Prop. 2.2.22]. The "if" direction can be concluded by combining Proposition 4.16 and Proposition 3.14 (1).
For the "only if" direction, assume now that M is flat. Let $L\mathcal{M}^{\mathrm{cp}}$ denote the full subcategory of $L\mathcal{M} = \mathrm{LMod}_R(\mathcal{A})_{\geq 0}$ spanned by compact projective objects and consider the functor $F(-) : (L\mathcal{M}^{\mathrm{cp}})^{\mathrm{op}} \to \mathcal{S}$ represented by M. We wish to show that this functor defines an ind-object of $L\mathcal{M}^{\mathrm{cp}}$. Let $(-)^{\vee} : R\mathcal{M}^{\mathrm{cp}} \to (L\mathcal{M}^{\mathrm{cp}})^{\mathrm{op}}$ be the dualization equivalence introduced in Corollary A.13. We will prove that $F((-)^{\vee}) : R\mathcal{M}^{\mathrm{cp}} \to \mathcal{S}$ defines a pro-object of $R\mathcal{M}^{\mathrm{cp}}$.

Let $p : \mathcal{E} \to R\mathcal{M}$ be the left fibration associated to the functor $\mathrm{Map}_{\mathcal{A}_{\geq 0}}(\mathbf{1}, - \otimes_R M) : R\mathcal{M} \to \mathcal{S}$. Then the base change of p to $R\mathcal{M}^{\mathrm{cp}}$ is the left fibration classifying $F((-)^{\vee})$. We have to show that every finite diagram $G : \mathcal{I} \to \mathcal{E} \times_{R\mathcal{M}} R\mathcal{M}^{\mathrm{cp}}$ admits a left cone. The fact that M is flat implies that the functor

$$\mathrm{Map}_{\mathcal{A}_{\geq 0}}(\mathbf{1}, - \otimes_R M) : R\mathcal{M} \to \mathcal{S}$$

is left exact, and therefore G extends to a left cone $G^{\triangleleft} : \mathcal{I}^{\triangleleft} \to \mathcal{E}$. Let $\overline{N} = (M, \rho : \mathbf{1} \to N \otimes_R M)$ be the value of $G^{\triangleleft}$ at the cone point. To show that G extends to a left cone in $\mathcal{E} \times_{R\mathcal{M}} R\mathcal{M}^{\mathrm{cp}}$ it is enough to prove that $\overline{N}$ receives a map from an object in $\mathcal{E} \times_{R\mathcal{M}} R\mathcal{M}^{\mathrm{cp}}$. This amounts to showing that there exists a map $N' \to N$ from a compact projective right R-module N' with the property that ρ factors through $N' \otimes_R M$. This follows from the fact that $\mathbf{1}$ is compact projective in $\mathcal{A}_{\geq 0}$. ☐

4.3 Modules over discrete algebras

Theorem 4.19. *We have:*

(1) *For any discrete $R \in \mathrm{Alg}(\mathcal{A}^{\heartsuit})$ there exists a (unique up to contractible choices) equivalence in $\mathcal{P}\mathrm{r}_{\mathrm{st}}^{t\text{-}rex}$*

$$\mathcal{D}(\mathrm{LMod}_{\pi_0 R}(\mathcal{A}^{\heartsuit})) \xrightarrow{\sim} \mathrm{LMod}_R(\mathcal{A})$$

which induces the identity functor on the heart.

(2) *Assume that the $\mathcal{A}_{\geq 0}^{\otimes}$ is projectively rigid. Then for any discrete commutative algebra $R \in \mathrm{CAlg}(\mathcal{A}^{\heartsuit})$ there exists a (unique up to contractible choices) equivalence in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}_{\mathrm{st}}^{t\text{-}rex})$*

$$\mathcal{D}(\mathrm{Mod}_{\pi_0 R}(\mathcal{A}^{\heartsuit}))^{\otimes} \xrightarrow{\sim} \mathrm{Mod}_R(\mathcal{A})^{\otimes}$$

which induces the identity functor on the heart, where the symmetric monoidal structure on left-hand side is induced by projective model with tensor product of chain complexes.

Proof.
(1) Since we have $\mathcal{P}_{\Sigma}(\mathrm{LMod}_R(\mathcal{A}_{\geq 0})^{cproj}) \simeq \mathrm{LMod}_R(\mathcal{A}_{\geq 0})$, the result follows from

$$\mathrm{LMod}_R(\mathcal{A}_{\geq 0})^{cproj} \simeq \mathrm{LMod}_{\pi_0 R}(\mathcal{A}^{\heartsuit})^{cproj}$$

and that the inclusion $\mathrm{LMod}_{\pi_0 R}(\mathcal{A}^{\heartsuit})^{cproj} \hookrightarrow \mathcal{D}(\mathrm{LMod}_{\pi_0 R}(\mathcal{A}^{\heartsuit}))_{\geq 0}$ induces an equivalence $\mathcal{P}_{\Sigma}(\mathrm{LMod}_{\pi_0 R}(\mathcal{A}^{\heartsuit})^{cproj}) \simeq \mathcal{D}(\mathrm{LMod}_{\pi_0 R}(\mathcal{A}^{\heartsuit}))_{\geq 0}$.
(2) By remark Remark 4.14(1), it suffices to show that $\mathcal{D}(\mathrm{Mod}_{\pi_0 R}(\mathcal{A}^{\heartsuit}))_{\geq 0}^{\otimes} \simeq \mathcal{P}_{\Sigma}(\mathrm{Mod}_{\pi_0 R}(\mathcal{A}^{\heartsuit})^{cproj})^{\otimes}$ is the symmetric monoidal projective cocompletion [see HA, Prop. 4.8.1.10]. That is to show the following:

(a) The natural inclusion $\mathrm{Mod}_{\pi_0 R}(\mathcal{A}^{\heartsuit})^{cproj} \hookrightarrow \mathcal{D}(\mathrm{Mod}_{\pi_0 R}(\mathcal{A}^{\heartsuit}))_{\geq 0}$ is a symmetric monoidal functor which preserves finite coproducts.

(b) The $\mathcal{D}(\mathrm{Mod}_{\pi_0 R}(\mathcal{A}^{\heartsuit}))_{\geq 0}^{\otimes}$ is presentably symmetric monoidal.

(c) The inclusion induces an equivalence $\mathcal{P}_{\Sigma}(\mathrm{Mod}_{\pi_0 R}(\mathcal{A}^{\heartsuit})^{cproj}) \simeq \mathcal{D}(\mathrm{Mod}_{\pi_0 R}(\mathcal{A}^{\heartsuit}))_{\geq 0}$.

The (a) and (c) follow directly from the construction of projective model on derived category. The (b) follows from [HA, Prop. 1.3.5.21] and the explicit internal hom construction in $\mathcal{D}(\mathrm{Mod}_{\pi_0 R}(\mathcal{A}^{\heartsuit}))$

$$\underline{\mathrm{Map}}_{\mathcal{D}}\left(M_*, N_*\right)_p = \prod_{n \in \mathbf{Z}} \underline{\mathrm{Hom}}_{\mathcal{M}}\left(M_n, N_{n+p}\right)$$

for each integer p, where we denote $\mathcal{D} = \mathcal{D}(\mathrm{Mod}_{\pi_0 R}(\mathcal{A}^{\heartsuit}))$ and $\mathcal{M} = \mathrm{Mod}_{\pi_0 R}(\mathcal{A}^{\heartsuit})$. We view $\underline{\mathrm{Map}}_{\mathcal{D}}\left(M_*, N_*\right)_*$ as a chain complex with values in $\mathcal{M}$, with differential given by the formula

$$(df)(x) = d(f(x)) - (-1)^p f(dx)$$

for $f \in \mathrm{Map}_{\mathcal{D}}\left(M_*, N_*\right)_p$. □

Remark 4.20. In fact, by our argument the uniqueness in above theorem can be promoted as which induces the identity functor on compact 1-projective $\pi_0 R$-modules in the heart.

4.4 Cohn localizations of $\mathbb{E}_\infty$-algebras

We introduced 1-categorical Cohn localizations in Theorem 2.23. Neeman constructed the Cohn localization in the derived category of a commutative ring in [Nee+06, §4]. That motivates us to give a higher categorical correspondence.

The Cohn localization is very useful in our abstract framework because most interesting cases are only projectively generated but not freely generated, unless it is the module category over an $\mathbb{E}_\infty$-ring spectrum. Our main result is the following.

Theorem 4.21. *Assume that $\mathcal{A}^{\otimes}_{\geq 0}$ is projectively rigid. Let $R \in \mathrm{CAlg}(\mathcal{A}_{\geq 0})$ and*

$$S = \{P_\beta \xrightarrow{f_\beta} Q_\beta\}$$

be a set of morphisms between compact projective R-modules. Then there exists a Cohn localization $R \to R[S^{-1}] \in \mathrm{CAlg}(\mathcal{A}_{\geq 0})$ satisfying the following universal property:
For any $B \in \mathrm{CAlg}(\mathcal{A})$, the induced map

$$\mathrm{Map}_{\mathrm{CAlg}(\mathcal{A})}(R[S^{-1}], B) \to \mathrm{Map}_{\mathrm{CAlg}(\mathcal{A})}(R, B)$$

is a (-1)-truncated map whose image on π_0 consists those maps $R \to B$ such that for each $f_\beta \in S$ the $B \otimes_R P_\beta \to B \otimes_R Q_\beta$ is an equivalence of B-modules.

Remark 4.22. (1) See [Hoy20, §3] and [Man23, §3.4] for discussions in the case where $Q_\beta = \mathbf{1}$ for each β, which is related to Moore objects in general settings.

(2) It is not hard to see that the Cohn localization is unique up to contractible choices.

Before the proof of theorem, we introduce some useful lemmas.

Lemma 4.23 (See [Ara25] B.5)**.** *Let $\mathcal{C} \xrightarrow{p} \mathcal{B}$ be a cocartesian fibration of ∞-categories. Let $\{S_b | b \in \mathcal{B}\}$ be given collections of morphisms such that $S_b \subset \mathrm{Fun}(\Delta^1, \mathcal{C}_b)$ for each $b \in \mathcal{B}$. We denote $S = \bigcup_b S_b$. If for any morphism $s \to t \in \mathcal{B}$ the cocartesian transformation $\mathcal{C}_s \to \mathcal{C}_t$ sends S_s into S_t, then the induced functor $q : \mathcal{D} = \mathcal{C}[S^{-1}] \to \mathcal{B}$ from the localization of $\mathcal{C}$ at S is a cocartesian fibration and canonical functor*

$$\begin{tikzcd} \mathcal{C} \arrow[rr] \arrow[dr, "p"'] & & \mathcal{D} \arrow[dl, "q"] \\ & \mathcal{B} & \end{tikzcd}$$

preserves cocartesian edges and exhibits $\mathcal{D}_b \simeq \mathcal{C}_b[S_b^{-1}]$ for each $b \in \mathcal{B}$. And for any cocartesian fibration $\mathcal{E} \to \mathcal{B}$, the composition induces a fully faithful embedding

$$\mathrm{Fun}^{\mathrm{coCar}}_{/\mathcal{B}}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{coCar}}_{/\mathcal{B}}(\mathcal{C}, \mathcal{E})$$

whose image consists of those cocartesian functors over $\mathcal{B}$ sending S to equivalences in $\mathcal{E}$.

Remark 4.24. (1) Note that a cocartesian functor $\mathcal{C} \to \mathcal{E}$ over $\mathcal{B}$ sends S to equivalences in $\mathcal{E}$ if and only if the induced functor on each fiber $\mathcal{C}_b \to \mathcal{E}_b$ sends S_b to equivalences in $\mathcal{E}_b$.

(2) The lemma above is a generalization of [HA, Prop. 2.2.1.9], which gives a construction in the case of reflective localization.

Corollary 4.25. *Let $\mathcal{C}^\otimes$ be a symmetric monoidal ∞-category and S be a collection of morphisms in $\mathcal{C}$ satisfying that $f \otimes g \in S$ if both $f, g \in S$. Then the localization $\mathcal{D} = \mathcal{C}[S^{-1}]$ inherits a natural symmetric monoidal structure and the localization can be promoted to a symmetric monoidal functor $\mathcal{C}^\otimes \to \mathcal{D}^\otimes$ satisfying the universal property that for any symmetric monoidal ∞-category $\mathcal{E}^\otimes$ the composition induces a fully faithful embedding*

$$\mathrm{Fun}^{\otimes}_{/\mathrm{N}(\mathrm{Fin}_*)}(\mathcal{D}^\otimes, \mathcal{E}^\otimes) \to \mathrm{Fun}^{\otimes}_{/\mathrm{N}(\mathrm{Fin}_*)}(\mathcal{C}^\otimes, \mathcal{E}^\otimes)$$

whose image consists of those symmetric monoidal functors sending S to equivalences in $\mathcal{E}$.

Lemma 4.26. *Let $\mathcal{C}^\otimes \xrightarrow{F^\otimes} \mathcal{D}^\otimes \in \mathrm{CAlg}(\mathcal{P}\mathrm{r}^L)$. Let $G^\otimes : \mathcal{D}^\otimes \to \mathcal{C}^\otimes$ be the relative right adjoint of $F^\otimes$. If $\mathcal{C}$ is generated by dualizables under small colimits and G is conservative and small-colimit-preserving, then the $G^\otimes$ is symmetric monoidal monadic, i.e. there exist an $R \in \mathrm{CAlg}(\mathcal{C})$ and a symmetric monoidal equivalence $\mathrm{Mod}_R(\mathcal{C})^\otimes \simeq \mathcal{D}^\otimes$ such that the following diagram is commutative.*

$$\begin{tikzcd} \mathrm{Mod}_R(\mathcal{C})^\otimes \arrow[rr, "\sim"] \arrow[dr] & & \mathcal{D}^\otimes \arrow[dl, "G^\otimes"] \\ & \mathcal{C}^\otimes & \end{tikzcd}$$

Proof. By [HA, Cor. 4.8.5.21], it suffices to show that G satisfies the projection formula, that is, for every object $C \in \mathcal{C}$ and $D \in \mathcal{D}$, the canonical map $C \otimes G(D) \to G(F(C) \otimes D)$ is an equivalence. By the assumption, it suffices to verify the case C is dualizable. In this case, for any $M \in \mathcal{C}$ we have

$$\begin{aligned} &\mathrm{Map}_{\mathcal{C}}(M, C \otimes G(D)) \simeq \mathrm{Map}_{\mathcal{C}}(C^\vee \otimes M, G(D)) \simeq \mathrm{Map}_{\mathcal{D}}\left(F\left(C^\vee \otimes M\right), D\right) \\ \simeq\ &\mathrm{Map}_{\mathcal{D}}\left(C^\vee \otimes F(M), D\right) \simeq \mathrm{Map}_{\mathcal{D}}(F(M), C \otimes D) \simeq \mathrm{Map}_{\mathcal{C}}(M, G(C \otimes D)). \end{aligned}$$

That indicates the projection formula holds. ☐

Proof of Theorem 4.21:
Let $S_1 \subset \mathrm{Fun}(\Delta^1, \mathrm{Mod}_R(\mathcal{A}_{\geq 0}))$ consists of morphisms $\{X_\alpha \otimes_R f_\beta | X_\alpha \in \mathrm{Mod}_R(\mathcal{A}_{\geq 0})^{cproj}, f_\beta \in S\}$. Then S_1 is small and thereby generates a strongly saturated class $\overline{S}_1$ of small generation (see [HTT, §5.5.4] for definition). Then the $\overline{S}_1 \subset \mathrm{Fun}(\Delta^1, \mathrm{Mod}_R(\mathcal{A}_{\geq 0}))$ satisfies conditions in Corollary 4.25, thereby it produces a symmetric monoidal localization $\mathrm{Mod}_R(\mathcal{A}_{\geq 0})^\otimes \xrightarrow{F^\otimes} \mathrm{Mod}_R(\mathcal{A}_{\geq 0})[\overline{S}_1^{-1}]^\otimes = D^\otimes$ such that $F^\otimes \in \mathrm{CAlg}(\mathcal{P}\mathrm{r}^L)$. Since $D \subset \mathrm{Mod}_R(\mathcal{A}_{\geq 0})$ closed under finite products, it lies in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^L_{\mathrm{ad}})$ and $F^\otimes \in \mathrm{CAlg}(\mathcal{P}\mathrm{r}^L_{\mathrm{ad}})$.

Now we wish to show that $F^\otimes$ satisfies conditions in Lemma 4.26. It suffices to verify that $D \subset \mathrm{Mod}_R(\mathcal{A}_{\geq 0})$ closed under small colimits, i.e. S_1-local objects are closed under small colimits. Unwinding the definition, a connective R-module M is S_1-local if and only if

$$\mathrm{Map}_{\mathrm{Mod}_R(\mathcal{A}_{\geq 0})}(X_\alpha \otimes_R Q_\beta, M) \to \mathrm{Map}_{\mathrm{Mod}_R(\mathcal{A}_{\geq 0})}(X_\alpha \otimes_R P_\beta, M)$$

is equivalent for any $X_\alpha \otimes_R f_\beta \in S_1$. However, this map can be identified with

$$\mathrm{Map}_{\mathrm{Mod}_R(\mathcal{A}_{\geq 0})}(X_\alpha, Q_\beta^\vee \otimes_R M) \to \mathrm{Map}_{\mathrm{Mod}_R(\mathcal{A}_{\geq 0})}(X_\alpha, P_\beta^\vee \otimes_R M).$$

So by the projective generation, M is S_1-local if and only if $f_\beta^\vee \otimes_R M : Q_\beta^\vee \otimes_R M \to P_\beta^\vee \otimes_R M$ is equivalent for each $f_\beta \in S$. That implies S_1-local objects are closed under small colimits. So there exist an $R[S^{-1}] \in$

$\mathrm{CAlg}(\mathcal{A}_{\geq 0})_{R/}$ and an equivalence $\mathrm{Mod}_{R[S^{-1}]}(\mathcal{A}_{\geq 0})^{\otimes} \simeq \mathcal{D}^{\otimes}$ such that the following diagram is commutative.

$$\begin{tikzcd} \mathrm{Mod}_{R[S^{-1}]}(\mathcal{A}_{\geq 0})^{\otimes} \arrow[rr, "\sim"] \arrow[dr] & & \mathcal{D}^{\otimes} \arrow[dl, "G^{\otimes}"] \\ & \mathrm{Mod}_R(\mathcal{A}_{\geq 0})^{\otimes} & \end{tikzcd}$$

Now given $B \in \mathrm{CAlg}(\mathcal{A})$, we need to show that the induced map

$$\mathrm{Map}_{\mathrm{CAlg}(\mathcal{A})}(R[S^{-1}], B) \to \mathrm{Map}_{\mathrm{CAlg}(\mathcal{A})}(R, B)$$

is a (-1)-truncated map whose image on π_0 consists those maps $R \to B$ such that for any $\beta \in J$ the $B \otimes_R P_\beta \to B \otimes_R Q_\beta$ is an equivalence of B-modules. Without loss of generality, we can assume that B is connective. By [HA, Cor. 4.8.5.21], we have the following Morita embedding,

$$\mathrm{CAlg}(\mathcal{A}_{\geq 0}) \to \mathrm{CAlg}(\mathcal{P}\mathrm{r}^L_{\mathrm{ad}})_{\mathcal{A}^{\otimes}_{\geq 0}/}$$

therefore it suffices to show that the $F^{\otimes}$ induces a fully faithful embedding

$$\mathrm{Fun}^{\otimes, L}_{/\mathrm{N}(\mathrm{Fin}_*)}(\mathrm{Mod}_{R[S^{-1}]}(\mathcal{A}_{\geq 0})^{\otimes}, \mathrm{Mod}_B(\mathcal{A}_{\geq 0})^{\otimes}) \to \mathrm{Fun}^{\otimes, L}_{/\mathrm{N}(\mathrm{Fin}_*)}(\mathrm{Mod}_R(\mathcal{A}_{\geq 0})^{\otimes}, \mathrm{Mod}_B(\mathcal{A}_{\geq 0})^{\otimes})$$

whose image consists of those functors sending S to equivalences in $\mathrm{Mod}_B(\mathcal{A}_{\geq 0})$, where $\mathrm{Fun}^{\otimes, L}_{/\mathrm{N}(\mathrm{Fin}_*)}$ denotes symmetric monoidal functors which preserve small colimits. However, that is implied by Corollary 4.25.

□

Remark 4.27. The argument above works for a set $S \subset \mathrm{Fun}(\Delta^1, \mathcal{C}^d)$ of morphisms between dualizables inside an arbitrary presentably symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ which is generated by dualizables under small colimits.

Proposition 4.28. *Let $f_S : R \to R[S^{-1}] \in \mathrm{CAlg}(\mathcal{A}_{\geq 0})$ be the Cohn localization at S in Theorem 4.21. Then the map $\pi_0 f_S : \pi_0 R \to \pi_0(R[S^{-1}])$ exhibits $\pi_0(R[S^{-1}]) \simeq (\pi_0 R)[(\pi_0 S)^{-1}]$ as the Cohn localization of $\pi_0 R$ at $\pi_0 S$ in the sense of Theorem 2.23, where $\pi_0 S = \{\pi_0 P_\beta \xrightarrow{\pi_0 f_\beta} \pi_0 Q_\beta | f_\beta \in S\}$.*

Proof. It follows immediately from the universal property of the Cohn localization. □

Proposition 4.29. *Let $f_S : R \to R[S^{-1}]$ be the Cohn localization at S in Theorem 4.21. Then:*

(1) *The $R[S^{-1}]$ is an idempotent commutative R-algebra.*

(2) *The f_S is flat???*

Proof.
(1) It suffices to show that the following diagram is a pushout in $\mathrm{CAlg}(\mathcal{A}_{\geq 0})$,

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R[S^{-1}] \arrow[d, equal] \\ R[S^{-1}] \arrow[r, equal] & R[S^{-1}] \end{tikzcd}$$

i.e. to show that f_S is an (∞-categorical) epimorphism in $\mathrm{CAlg}(\mathcal{A}_{\geq 0})$. That is implied by the description of mapping spaces in Theorem 4.21.
(2) ??? □

Definition 4.30. We say a map $A \to B \in \mathrm{CAlg}(\mathcal{A}_{\geq 0})$ is a (finitary) Cohn localization if there exists a (finite) set S of morphisms between compact projective R-modules such that $B \simeq A[S^{-1}]$.

Remark 4.31. Note that if $S = \{P_i \xrightarrow{f_i} Q_i\}$ is finite, then $A[S^{-1}] \simeq A[f^{-1}]$ is equivalent to the Cohn localization at the single element $f = \bigoplus_i f_i$.

Proposition 4.32. *Let $A \to B \in \mathrm{CAlg}(\mathcal{A}_{\geq 0})$ be a finitary Cohn localization. Then B is finitely presented over A.*

Proof. By the remark above, we can assume that $S = \{f\}$ consists of a single element. Now given a filtered colimit of connective commutative A-algebras $\varinjlim_\alpha C_\alpha = C$ we need to show that the natural map

$$\varinjlim_{\alpha} \mathrm{Map}_{\mathrm{CAlg}(\mathcal{A}_{\geq 0})_{A/}}(B, C_\alpha) \to \mathrm{Map}_{\mathrm{CAlg}(\mathcal{A}_{\geq 0})_{A/}}(B, C)$$

is an equivalence. By Proposition 4.29(1), each mapping space above is empty or a single point. If the $\mathrm{Map}_{\mathrm{CAlg}(\mathcal{A}_{\geq 0})_{A/}}(B, C) = \emptyset$, then nothing needs to prove.

Now assume that $\mathrm{Map}_{\mathrm{CAlg}(\mathcal{A}_{\geq 0})_{A/}}(B, C) \simeq \{*\}$, we wish to show that there exists an α such that $\mathrm{Map}_{\mathrm{CAlg}(\mathcal{A}_{\geq 0})_{A/}}(B, C_\alpha)$ is not empty. By assumption, the natural map $f \otimes_A C$ is an equivalence, thereby $\mathrm{cofib}(f) \otimes_A C = 0$. Since $\mathrm{cofib}(f)$ is a compact A-module, there exists an α such that the natural map $\mathrm{cofib}(f) \to \mathrm{cofib}(f) \otimes_A C_\alpha$ is zero. That implies $\mathrm{cofib}(f) \otimes_A C_\alpha = 0$ and we are done. $\square$

5 Finiteness properties

Throughout Section 5, we assume that $\mathcal{A}$ is Grothendieck and that $\mathcal{A}_{\geq 0} \in \mathcal{P}\mathrm{r}^L_\omega$.

5.1 Perfect and almost perfect modules

Definition 5.1. Let $R \in \mathrm{Alg}(\mathcal{A})$. We say a left R-module M is perfect if it is compact in $\mathrm{LMod}_R(\mathcal{A})$.

Proposition 5.2. *Let $R \in \mathrm{Alg}(\mathcal{A}_{\geq 0})$ and M be a left R-module. If M is perfect, then M is bounded-below.*

Proof. By the right completeness we have $M \simeq \varinjlim \tau_{\geq -n} M$, then the compactness of M implies that M is a retract of $\tau_{\geq -n} M$ for some n. $\square$

Definition 5.3. Let $\mathcal{C}$ be a presentable ∞-category. We will say an object $C \in \mathcal{C}$ is almost compact if $\tau_{\leq n} C$ is a compact object of $\tau_{\leq n}\mathcal{C}$ for all $n \geq 0$.

Remark 5.4. Let $\mathcal{C}$ be a compactly generated ∞-category. Then every compact object of $\mathcal{C}$ is almost compact by [HTT, Corollary 5.5.7.4].

Definition 5.5. Let $R \in \mathrm{Alg}(\mathcal{A}_{\geq 0})$ be a connective $\mathbb{E}_1$-ring object. We will say a left R-module M is almost perfect if there exists an integer k such that $M \in \mathrm{LMod}_R(\mathcal{A})_{\geq k}$ and is almost compact as an object of $\mathrm{LMod}_R(\mathcal{A})_{\geq k}$.

We let $\mathrm{LMod}_R(\mathcal{A})^{\mathrm{aperf}} \subset \mathrm{LMod}_R(\mathcal{A})$ denote the full subcategory spanned by the almost perfect left R-modules.

Proposition 5.6. *Let $R \in \mathrm{Alg}(\mathcal{A}_{\geq 0})$. Then:*

(1) *The full subcategory $\mathrm{LMod}_R(\mathcal{A})^{aperf} \subseteq \mathrm{LMod}_R(\mathcal{A})$ is closed under translations and finite colimits, and is therefore a stable subcategory of $\mathrm{LMod}_R(\mathcal{A})$.*

(2) *The full subcategory $\mathrm{LMod}_R(\mathcal{A})^{aperf} \subseteq \mathrm{LMod}_R(\mathcal{A})$ is closed under the formation of retracts.*

(3) *Every perfect left R-module is almost perfect.*

(4) *The full subcategory* $\mathrm{LMod}_R(\mathcal{A})^{aperf}_{\geq 0} \subseteq \mathrm{LMod}_R(\mathcal{A})$ *is closed under the formation of geometric realizations of simplicial objects.*

Proof. Proof. Assertions (1) and (2) are obvious, and (3) follows from Remark 5.4. To prove (4), it suffices to show that the collection of compact objects of $\mathrm{LMod}_R(\mathcal{A})_{[0,n]}$ is closed under geometric realizations, which follows from [HA, Lemma 1.3.3.10]. □

Proposition 5.7. *Assume that* $\mathcal{A}_{\geq 0}$ *is projectively generated. Let* $R \in \mathrm{Alg}(\mathcal{A}_{\geq 0})$ *and* $M \in \mathrm{LMod}_R(\mathcal{A})^{aperf}_{\geq 0}$ *be a left* R*-module which is connective and almost perfect. Then* M *can be obtained as the geometric realization of a simplicial left* R*-module* $P_\bullet$ *such that each* P_n *is a compact projective left* R*-module in* $\mathrm{LMod}_R(\mathcal{A})_{\geq 0}$*.*

Proof. We mimic the proof in [HA, Prop. 7.2.4.11] and carefully replace "free" by "projective". In view of ∞-categorical Dold-Kan correspondence, it will suffice to show that M can be obtained as the colimit of a sequence

$$D(0) \xrightarrow{f_1} D(1) \xrightarrow{f_2} D(2) \to \dots$$

where each $\mathrm{cofib}(f_n)[-n]$ is a compact projective left R-module; here we agree by convention that f_0 denotes the zero map $0 \to D(0)$. The construction goes by induction. Suppose that the diagram

$$D(0) \to \dots \to D(n) \xrightarrow{g} M$$

has already been constructed, and that $N = \mathrm{fib}(g)$ is n-connective. Part (1) of Proposition 5.6 implies that N is almost perfect, so that the bottom $\pi_n N$ is a compact object in the category of left $\pi_0 R$-modules. It follows that there exists a map $\beta : Q[n] \to N$, where Q is a compact projective left R-module because the $\mathrm{LMod}_R(\mathcal{A})_{\geq 0}$ is projectively generated. And β induces a surjection $\pi_0 Q \to \pi_n N$. We now define $D(n+1)$ to be the cofiber of the composite map $Q(n) \xrightarrow{\beta} N \to D(n)$, and construct a diagram

$$D(0) \to \dots \to D(n) \to D(n+1) \xrightarrow{g'} M$$

Using the octahedral axiom of triangulated category, we obtain a fiber sequence

$$Q[n] \to \mathrm{fib}(g) \to \mathrm{fib}(g')$$

and the associated long exact sequence in $\mathcal{A}^{\heartsuit}$ proves that $\mathrm{fib}(g')$ is $(n+1)$-connective. In particular, we conclude that for a fixed $m \geq 0$, the maps $\pi_m D(n) \to \pi_m M$ are isomorphisms for $n \gg 0$, so that the natural map $\varinjlim D(n) \to M$ is an equivalence of left R-modules by the left completeness, as desired. □

Proposition 5.8. *Assume that* $\mathcal{A}^{\otimes}_{\geq 0}$ *is projectively rigid. Let* $R \in \mathrm{Alg}(\mathcal{A}_{\geq 0})$ *and let* M *be a connective left* R*-module. Then the following are equivalent:*

(1) *The* M *is a compact projective left* R*-module.*

(2) *The* M *is a perfect and flat left* R*-module.*

(3) *The* M *is a almost perfect and flat left* R*-module.*

(4) *The* M *is a flat left* R*-module and* $\pi_0 M$ *is finitely presented over* $\pi_0 R$*.*

Proof. The (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are obvious. For (4) $\Rightarrow$ (1), by Proposition 2.16, we conclude that $\pi_0 M$ is compact 1-projective over $\pi_0 R$. Then by Proposition 4.17, we get that M is a compact projective left R-module. □

Definition 5.9. Let $R \in \mathrm{Alg}(\mathcal{A}_{\geq 0})$. We will say a left R-module M has Tor-amplitude $\leq n$ if, for every discrete right R-module N, the $\pi_i(N \otimes_R M)$ vanish for $i > n$. We will say M is of finite Tor-amplitude if it has Tor-amplitude $\leq n$ for some integer n.

Remark 5.10.

(1) In view of Proposition 3.15, a connective left R-module M has Tor-amplitude ≤ 0 if and only if M is flat.

(2) Assume that $\mathcal{A}$ is hypercomplete. Then a connective left R-module M has Tor-amplitude ≤ -1 if and only if $M = 0$.

Proposition 5.11. *Assume that $\mathcal{A}^{\otimes}_{\geq 0}$ is projectively rigid. Let $R \in \mathrm{Alg}(\mathcal{A}_{\geq 0})$. Then:*

(1) *If M is a left R-module of Tor-amplitude $\leq n$, then $M[k]$ has Tor-amplitude $\leq n+k$.*

(2) *Let*
$$M' \to M \to M''$$
be a fiber sequence of left R-modules. If M' and M'' have Tor-amplitude $\leq n$, then so does M.

(3) *Let M be a left R-module of Tor-amplitude $\leq n$. Then any retract of M has Tor-amplitude $\leq n$.*

(4) *Let M be an almost perfect left module over R. Then M is perfect if and only if M has finite Tor-amplitude.*

(5) *Let M be a left module over R having Tor-amplitude $\leq n$. Then for every $N \in \mathrm{RMod}_R(\mathcal{A})_{\leq 0}$, the $\pi_i\left(N \otimes_R M\right)$ vanishes for each $i > n$.*

Proof. We mimic the proof in [HA, Prop. 7.2.4.23] but carefully replace “free” by “projective”. The first three assertions follow immediately from the exactness of the functor $N \mapsto N \otimes_R M$. It follows that the collection left R-modules of finite Tor-amplitude is stable under retracts and finite colimits and desuspensions, and contains all compact projective left R-modules. This proves the “only if" direction of (4) by Proposition 4.3. For the converse, let us suppose that M is almost perfect and of finite Tor-amplitude. We wish to show that M is perfect. We first apply (1) to reduce to the case where M is connective. The proof now goes by induction on the Tor-amplitude n of M. If $n = 0$, then M is flat and we may conclude by applying Proposition 5.8. We may therefore assume $n > 0$.

Since M is almost perfect, there exists a compact projective left R-module P and a fiber sequence
$$M' \to P \xrightarrow{f} M$$
where f induces an epimorphism on π_0. To prove that M is perfect, it will suffice to show that P and M' are perfect. It is clear that P is perfect, and it follows from Proposition 5.6 that M' is almost perfect. Moreover, since $\pi_0 f$ is surjective, M' is connective. We will show that M' is of Tor-amplitude $\leq n-1$; the inductive hypothesis will then imply that M is perfect, and the proof will be complete.

Let N be a discrete right R-module. We wish to prove that $\pi_k(N \otimes_R M') \simeq 0$ for $k \geq n$. Since the functor $N \otimes_R \bullet$ is exact, we obtain for each k an exact sequence
$$\pi_{k+1}(N \otimes_R M) \to \pi_k(N \otimes_R M') \to \pi_k(N \otimes_R P)$$
The left entry vanishes in virtue of our assumption that M has Tor-amplitude $\leq n$. We now complete the proof of (4) by observing that $\pi_k\left(N \otimes_R P\right)$ vanishes because N is discrete and P is flat and $k \geq n > 0$.

We now prove (5). Assume that M has Tor-amplitude $\leq m$. Let $N \in \mathrm{RMod}_R(\mathcal{A})_{\leq 0}$; we wish to prove that $\pi_i(N \otimes_R M) \simeq 0$ for $i > n$. Since $N \simeq \varinjlim \tau_{\geq -m} N$, it will suffice to prove the vanishing after replacing N by $\tau_{\geq -m} N$ for every integer m. We may therefore assume that $N \in \mathrm{RMod}_R(\mathcal{A})_{[-m,0]}$ for some $m \geq 0$. We proceed by induction on m. When $m = 0$, the desired result follows immediately from our assumption on M. If $m > 0$, we have a fiber sequence
$$\tau_{\geq 1-m} N \to N \to \left(\pi_{-m} N\right)[-m]$$
hence an exact sequence
$$\pi_i\left(\left(\tau_{\geq 1-m} N\right) \otimes_R M\right) \to \pi_i\left(N \otimes_R M\right) \to \pi_{i+m}\left(\pi_{-m} N \otimes_R M\right)$$

If $i > n$, then the first group vanishes by the inductive hypothesis, and the third by virtue of our assumption that M has Tor-amplitude $\leq n$. $\square$

Proposition 5.12 (See [HA] Remark 7.2.4.24 for the case of spectra)**.**
Assume that $\mathcal{A}_{\geq 0}^{\otimes}$ is projectively rigid. Let $R \in \mathrm{Alg}(\mathcal{A}_{\geq 0})$, and let $\mathcal{C}$ be the smallest stable subcategory of $\mathrm{LMod}_R(\mathcal{A})$ which contains all compact projective modules. Then $\mathcal{C} = \mathrm{LMod}_R^{perf}(\mathcal{A})$.

Proof. The inclusion $\mathcal{C} \subseteq \mathrm{LMod}_R^{perf}(\mathcal{A})$ is obvious. To prove the converse, we must show that every object $M \in \mathrm{LMod}_R^{perf}(\mathcal{A})$ belongs to $\mathcal{C}$. Invoking Proposition 5.2, we may reduce to the case where M is connective. We then work by induction on the (necessarily finite) Tor-amplitude of M. If M is of Tor-amplitude ≤ 0, then M is flat and the desired result follows from Proposition 5.8. In the general case, we choose a compact projective R-module P and a map $f : P \to M$ which induces a surjection $\pi_0 P \to \pi_0 M$. We may conclude that that fiber K of f is a connective perfect module of smaller Tor-amplitude than that of M, so that $K \in \mathcal{C}$ by the inductive hypothesis. Since $P \in \mathcal{C}$ and $\mathcal{C}$ is stable under the formation of cofibers, we conclude that $M \in \mathcal{C}$ as desired. $\square$

5.2 Finite presentation and almost of finite presentation

Definition 5.13. Let $f : A \to B$ be a map in $\mathrm{CAlg}(\mathcal{A}_{\geq 0})$.

(1) We say $f : A \to B$ is locally of finite presentation (or finitely presented) if B is a compact object in $\mathrm{CAlg}(\mathcal{A}_{\geq 0})_{A/}$.

(2) We say $f : A \to B$ is almost of finite presentation if $\tau_{\leq n} B$ is a compact object in $\mathrm{CAlg}(\mathcal{A}_{[0,n]})_{\tau_{\leq n} A/}$.

Remark 5.14. Suppose given a commutative diagram in $\mathrm{CAlg}(\mathcal{A}_{\geq 0})$

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ B \arrow[rr] & & C \end{tikzcd}$$

where B is of locally of finite presentation over A. Then C is locally of finite presentation over B if and only if C is locally of finite presentation over A. This follows immediately from [HTT, Proposition 5.4.5.15].

Proposition 5.15. *Suppose further that $\mathcal{A}_{\geq 0}$ is projectively generated. Let $f : A \to B \in \mathrm{CAlg}(\mathcal{A}_{\geq 0})$ be a map of finite presentation. Then there exists compact projective A-modules M, N and a diagram*

$$\begin{tikzcd} \mathrm{Sym}_A^*(N) \arrow[r, "\alpha"] \arrow[d, "\phi"] & A \arrow[d] \\ \mathrm{Sym}_A^*(M) \arrow[r] & B \end{tikzcd}$$

such that the map $B' \to B$ induces an isomorphism on π_0, where B' is the pushout of above diagram in $\mathrm{CAlg}(\mathcal{A}_{\geq 0})$ and α is the natural augmentation. (Note that the ϕ here is not necessarily induced by a map of modules $N \to M$).

Proof. Firstly, by Corollary 4.11 there exists a set of compact projective A-modules $\{P_\alpha | \alpha \in I\}$ and a map $P = \oplus_\alpha P_\alpha \to \pi_0 B$ of A-modules which induces an epimorphism on π_0. Then there exists a lifting of A-module map

$$\begin{tikzcd} & B \arrow[d] \\ P \arrow[r] \arrow[ur, dashed] & \pi_0 B \end{tikzcd}$$

by Proposition 4.10. This lifting induces an A-algebra map $\mathrm{Sym}_A^*(P) \to B$, which induces an epimorphism on π_0 (as objects in $\mathcal{A}^\heartsuit$) by our construction. Since $f : A \to B$ is of finite presentation and $\mathcal{A}$ is Grothendieck, the $\pi_0 B$ is a compact object in $\mathrm{CAlg}(\mathrm{Mod}_{\pi_0 A}(\mathcal{A}^\heartsuit))$. That implies there exists finite collection $\{P_i\}$ such that the composition $\mathrm{Sym}_A^*(\oplus_i P_i) \to \mathrm{Sym}_A^*(P) \to B$ induces an epimorphism on π_0, by taking the filtration of images of $\pi_0 \mathrm{Sym}_A^*(\oplus_{j\in J} P_j) \to \pi_0 B$ where $J \subset I$ is a finite subset. We take $M = \oplus_i P_i$ and $N' = \mathrm{fib}(\mathrm{Sym}_A^*(M) \to B)$, then N' is a connective A-module by our construction. By similar argument as previous, there exists a set of compact projective A-modules $\{Q_\alpha | \alpha \in I_2\}$ and a map $Q = \oplus_\alpha Q_\alpha \to N'$ of A-modules which induces an epimorphism on π_0. Then there exists finite collection $\{Q_i\}$ such that the induced map $(\oplus_i Q_i) \otimes_A \mathrm{Sym}_A^*(M) \to N'$ of $\mathrm{Sym}_A^*(M)$-modules induces an epimorphism on π_0, by taking the filtered diagram of $\pi_0 \mathrm{Sym}_A^*(M)/\operatorname{Im} \pi_0(\oplus_i Q_i \otimes_A \mathrm{Sym}_A^*(M)) \to \pi_0 B$ where $J_2 \subset I_2$ is a finite subset. Take $N = \oplus_i Q_i$, we are done. □

6 Faithful and descendable algebras

6.1 Faithful algebras

Definition 6.1. Let $f : R \to S \in \mathrm{CAlg}(\mathcal{A})$.

(1) We say f is faithful if the tensor product functor $f_! = S \otimes_R (-) : \mathrm{Mod}_R(\mathcal{A}) \to \mathrm{Mod}_S(\mathcal{A})$ is conservative.

(2) We say f is bounded faithful if $f \in \mathrm{CAlg}(\mathcal{A}_{\geq 0})$ and the tensor product functor $f_! = S \otimes_R (-) : \mathrm{Mod}_R(\mathcal{A})^- \to \mathrm{Mod}_S(\mathcal{A})^-$ is conservative when restricting on bounded below modules.

Proposition 6.2. *Let $\eta : A \to M \in \mathrm{Der}$ and $\alpha : \tilde{A} \to A$ be the induced square-zero extension in $\mathrm{CAlg}(\mathcal{A})$ (see Definition 7.4). Then the α is faithful.*

Proof. We consider the following pullback diagram in $\mathrm{CAlg}(\mathcal{A})$, and hence also a pullback diagram in $\mathrm{Mod}_{\tilde{A}}(\mathcal{A})$.

$$\begin{tikzcd} \tilde{A} \arrow[r, "\alpha"] \arrow[d, "\alpha"] & A \arrow[d, "\delta"] \\ A \arrow[r, "\delta_0"] & A \oplus M \end{tikzcd}$$

Now given an $\tilde{A}$-module X such that $X \otimes_{\tilde{A}} A = 0$, we wish to show that $X = 0$ too. Because the following diagram is pullback in $\mathrm{Mod}_{\tilde{A}}(\mathcal{A})$,

$$\begin{tikzcd} X = X \otimes_{\tilde{A}} \tilde{A} \arrow[r] \arrow[d] & X \otimes_{\tilde{A}} A = 0 \arrow[d] \\ 0 = X \otimes_{\tilde{A}} A \arrow[r] & X \otimes_{\tilde{A}} (A \oplus M) = 0 \end{tikzcd}$$

we get $X = 0$. □

Remark 6.3. In fact, any square-zero extension $\alpha : \tilde{A} \to A$ in $\mathrm{CAlg}(\mathcal{A})$ is descendable. We will see that in Corollary 6.14.

Definition 6.4. Let $\alpha : \tilde{A} \to A$ be a map in $\mathrm{CAlg}(\mathcal{A}_{\geq 0})$. We say α is a nilpotent thickening if the $\mathrm{Ker}(\pi_0 \alpha)$ is a nilpotent ideal of $\pi_0 \tilde{A}$.

Theorem 6.5 (Nakayama lemma)**.** *Assume that $\mathcal{A}$ is hypercomplete. Let $\alpha : \tilde{A} \to A$ be a nilpotent thickening in $\mathrm{CAlg}(\mathcal{A}_{\geq 0})$. Then the base change functor is bounded faithful.*

Proof. Let I denote the $\mathrm{Ker}(\pi_0 \alpha)$. Suppose that $M \in \mathrm{Mod}_{\tilde{A}}(\mathcal{A})^-$ is a bounded below module such that $A \otimes_{\tilde{A}} M = 0$. We wish to show $M = 0$. Now suppose that $M \neq 0$, without loss of generalization, we can assume that M is connective and $\pi_0 M \neq 0$. Then $\pi_0 A \overline{\otimes}_{\pi_0 \tilde{A}} \pi_0 M \simeq \tau_{\leq 0}(A \otimes_{\tilde{A}} M) = 0$ where $\overline{\otimes}$ denotes the tensor product in the heart. Therefore $I \cdot \pi_0 M = \pi_0 M$. However I is nilpotent so $\pi_0 M = 0$, which leads to a contradiction. □

Remark 6.6. In general, a nilpotent thickening is not faithful, and hence not descendable. A basic counter-example is the truncation map $\mathbb{S} \to H\mathbb{Z}$ of $\mathbb{E}_\infty$-ring spectra, which is a nilpotent thickening but not faithful. Indeed, given a prime p and a natural number n, consider the spectrum $K(n)$ of the corresponding Morava K-theory. Then we have $H_*(K(n)) = 0$, while $\pi_*(K(n)) \neq 0$ (see [Rud98, Chap. IX.7.27]). That means the base change functor

$$\mathrm{Sp} \simeq \mathrm{Mod}_{\mathbb{S}}(\mathrm{Sp}) \to \mathrm{Mod}_{H\mathbb{Z}}(\mathrm{Sp}) \simeq \mathcal{D}(\mathbb{Z})$$

is not conservative.

6.2 Descendable algebras

Now let us recollect the notion of descendable algebras.

Definition 6.7. Let $\mathcal{C}^\otimes \in \mathrm{CAlg}(\mathcal{P}\mathrm{r}^L_{\mathrm{st}})$ and $\mathcal{I} \subset \mathcal{C}$ be a full subcategory. We say $\mathcal{I}$ is a thick ideal of $\mathcal{C}$ if $\mathcal{I} \subset \mathcal{C}$ is a stable subcategory, closed under retractions and the following condition holds:
For any $x \in \mathcal{C}$ and $y \in \mathcal{I}$ we have $x \otimes y \in \mathcal{I}$.

Definition 6.8 (See [Mat16] Definition 3.18)**.** Let $f : R \to S \in \mathrm{CAlg}(\mathcal{A})$. We say f is descendable if the smallest thick ideal of $\mathrm{Mod}_R(\mathcal{A})$ such that contains S is $\mathrm{Mod}_R(\mathcal{A})$ itself.

Proposition 6.9 (See [Mat16] 3.19)**.** *Given a map $f : A \to B$ in $\mathrm{CAlg}(\mathcal{A})$. If f is descendable, then f is faithful.*

Definition 6.10. Let $f : A \to B \in \mathrm{CAlg}(\mathcal{A})$. We define the augmented cosimplical object

$$B^\bullet : \Delta^+ \to \mathrm{CAlg}(\mathcal{A})_{A/}$$

be the Cech nerve in $(\mathrm{CAlg}(\mathcal{A})_{A/})^{\mathrm{op}}$.

Proposition 6.11 (See [Mat16] 3.20)**.** *Let $f : A \to B \in \mathrm{CAlg}(\mathcal{A})$. Then the following conditions are equivalent:*

(1) *The $A \to B$ is descendable.*

(2) *The $B^\bullet$ is a Δ-limit diagram in $\mathrm{Pro}(\mathrm{Mod}_A(\mathcal{A}))$.*

Proof. We first prove that (1) $\Rightarrow$ (2). Let $\mathcal{C}$ denote the full subcategory of $\mathrm{Mod}_A(\mathcal{A})$ spanned by those objects M for which the canonical map $\theta_M : M \otimes_A A \to M \otimes_A B^\bullet$ is an equivalence in $\mathrm{Pro}(\mathrm{Mod}_A(\mathcal{A}))$. By Corollary B.13, we see that the $\mathcal{C}$ is a thick ideal of $\mathrm{Mod}_A(\mathcal{A})$. It will suffice to show that $B \in \mathcal{C}$. This is clear, since $B \otimes_A B^\bullet$ can be identified with the split cosimplicial object $B^{\bullet+1}$.

Now suppose that (2) is satisfied. Let $\mathcal{D}$ denote the smallest thick ideal of $\mathrm{Mod}_A(\mathcal{A})$ which contains B. Then $B^\bullet$ is a cosimplicial object of $\mathcal{D}$ and each term $\mathrm{Tot}^n(B/A)$ in the tower $\mathrm{Tot}^\bullet(B/A)$ is in $\mathcal{D}$ too. Assumption (2) implies that $A \simeq \varprojlim_n \mathrm{Tot}^n(B/A)$ in $\mathrm{Pro}(\mathrm{Mod}_A(\mathcal{A}))$. However, the A is cocompact in $\mathrm{Pro}(\mathrm{Mod}_A(\mathcal{A}))$, so A is equivalent to a retract of $\mathrm{Tot}^n(B/A)$ for some integer n, so that $A \in \mathcal{D}$. That implies $\mathcal{D} = \mathrm{Mod}_A(\mathcal{A})$. □

Remark 6.12. We see that the descendable condition is stronger than that the $B^\bullet$ is a Δ-limit merely in $\mathrm{Mod}_A(\mathcal{A})$.

Corollary 6.13. *Let $f : A \to B \in \mathrm{CAlg}(\mathcal{A})$. Then $A \to B$ is descendable if and only if A as an A-module can be obtained as a retract of a finite colimit of a diagram of A-modules consisting of objects, each of which admits the structure of a module over B.*

Corollary 6.14. *Let $\eta : A \to M \in \mathrm{Der}$ and $\alpha : \tilde{A} \to A$ be the induced square-zero extension in $\mathrm{CAlg}(\mathcal{A})$. Then the α is descendable.*

Proposition 6.15 (See [Mat16] 3.21)**.** *Let $f : R \to S$ be a descendable morphism in $\mathrm{CAlg}(\mathcal{A})$ and $R \to A$ be another map in $\mathrm{CAlg}(\mathcal{A})$. Then the map $A \to A \otimes_R S$ given by the following pushout diagram is descendable.*

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[d] \\ A \arrow[r] & A \otimes_R S \end{tikzcd}$$

Proposition 6.16 (See [Mat16] 3.24)**.** *Let $A \to B \to C$ be maps in $\mathrm{CAlg}(\mathcal{A})$.*

(1) *If $A \to B$ and $B \to C$ admit descent, so does $A \to C$.*

(2) *If $A \to C$ admits descent, so does $A \to B$.*

Lemma 6.17 (See [SAG] D.3.3.6)**.** *Let n be a nonnegative integer, let J be a filtered partially ordered set of cardinality $\leqslant \aleph_n$, and let $\{X_j\}_{j \in J}$ be a diagram of spaces indexed by J^{op}. If each of the spaces X_j is m-connective for some integer m, then the inverse limit $\varprojlim_{j\in J} X_j$ is $(m-n)$-connective.*

Lemma 6.18. *Suppose that $\mathcal{A}$ is Grothendieck and that $\mathcal{A}_{\geq 0}^{\otimes}$ is projectively rigid. Let $A \in \mathrm{Alg}(\mathcal{A}_{\geq 0})$, let M be a flat left A-module, and let N be a connective left A-module. Assume that $\pi_0 M$ is an $\aleph_n$-compact object of the category of discrete $\pi_0 A$-modules for some $n \geq 0$. Then $\mathrm{Ext}_A^m(M, N) \simeq 0$ for $m > n$.*

Proof. The following argument is parallel with [SAG, Lem. D.3.3.7]. Let us identify N with the limit of its Postnikov tower

$$\cdots \to \tau_{\leqslant 2} N \to \tau_{\leqslant 1} N \to \tau_{\leqslant 0} N,$$

so that we have a Milnor exact sequence

$$0 \to \lim{}^1 \left\{ \mathrm{Ext}_A^{m-1}(M, \tau_{\leqslant k} N) \right\} \to \mathrm{Ext}_A^m(M, N) \to \varprojlim_k \left\{ \mathrm{Ext}_A^m(M, \tau_{\leqslant k} N) \right\} \to 0.$$

It will therefore suffice to show that the abelian groups $\lim^1 \left\{ \mathrm{Ext}_A^{m-1}(M, \tau_{\leqslant k} N) \right\}$ and $\varprojlim_k \left\{ \mathrm{Ext}_A^m(M, \tau_{\leqslant k} N) \right\}$ are trivial for $m > n$. To prove this, we will show that the maps

$$\mathrm{Ext}_A^{m-1}(M, \tau_{\leqslant k} N) \to \mathrm{Ext}_A^{m-1}(M, \tau_{\leqslant k-1} N)$$

are surjective for $k \geqslant 1$, and that the groups $\mathrm{Ext}^m(M, \tau_{\leqslant k} N)$ vanish for all $k \geq 0$. Using the exact sequences

$$\begin{gathered} \mathrm{Ext}_A^{m-1}(M, \tau_{\leqslant k} N) \to \mathrm{Ext}_A^{m-1}(M, \tau_{\leqslant k-1} N) \to \mathrm{Ext}_A^{m+k}(M, \pi_k N) \to \\ \mathrm{Ext}_A^{m}(M, \tau_{\leqslant k} N) \to \mathrm{Ext}_A^{m}(M, \tau_{\leqslant k-1} N) \to \mathrm{Ext}_A^{m+1+k}(M, \pi_k N) \end{gathered}$$

we are reduced to proving that the groups $\mathrm{Ext}_A^{m+k}(M, \pi_k N)$ vanish for all $k \geq 0$. Replacing m by $m+k$ and N by $\pi_k N$, we can further reduce to the case where N is discrete. In this case, we have a canonical isomorphism $\mathrm{Ext}_A^m(M, N) \simeq \mathrm{Ext}_{\pi_0 A}^m(\pi_0 A \otimes_A M, N)$. We may therefore replace A by $\pi_0 A$ (and M by $\pi_0 A \otimes_A M$) and thereby reduce to the case where A is discrete. Since M is flat over A, it follows that M is also discrete.

Since M is flat over A, it can be written as the colimit of a diagram $\{M_\alpha\}_{\alpha \in P}$ indexed by a filtered partially ordered set P, where each M_α is a compact projective left A-module (Theorem 4.18). In the case $n = 0$, it follows that M is compact projective and the conclusion is deduced by Proposition 4.7. In the case $n \geq 1$, for each $\aleph_n$-small filtered subset $P' \subseteq P$, let $M_{P'}$ denote the colimit $\varinjlim_{\alpha \in P'} M_\alpha$. Then by [Ker, 061J], when $n \geq 1$ the M can be written as a filtered colimit of the diagram $\{M_{P'}\}$, where P' ranges over all $\aleph_n$-small filtered subsets of P. Since M is $\aleph_n$-compact, the identity map $\mathrm{id}_M : \varinjlim_{P'} M_{P'} \to M$ factors through some $M_{P'}$, so that M is a retract of $M_{P'}$. We may therefore replace M by $M_{P'}$ and P by P', and thereby reduce to the case where P is $\aleph_n$-small. We have a canonical isomorphism

$$\mathrm{Ext}_A^m(M, N) \simeq \pi_0 \operatorname{Map}_{\mathrm{LMod}_A}(M, \Sigma^m N) \simeq \pi_0 \varprojlim_{\alpha \in P} \operatorname{Map}_{\mathrm{LMod}_A}(M_\alpha, \Sigma^m N)$$

To show that this group vanishes, it will suffice (by virtue of Lemma 6.17) to show that the mapping spaces $\operatorname{Map}_{\mathrm{LMod}_A}(M_\alpha, \Sigma^m N)$ are n-connective for each $\alpha \in P$. This is clear, since M_α is a compact projective left A-module and $\Sigma^m N$ is n-connective. $\square$

Theorem 6.19. *Suppose that $\mathcal{A}$ is Grothendieck and that $\mathcal{A}^{\otimes}_{\geq 0}$ is projectively rigid. Let $f : A \to B \in \mathrm{CAlg}(\mathcal{A})$ be a faithfully flat map such that $\pi_0 B$ is a $\aleph_n$-compact $\pi_0 A$-module for some $n \geq 0$. Then f is descendable.*

Proof. The following argument is parallel with the proof of [SAG, Prop. D.3.3.1]. Since B is flat over A, by Proposition 3.22 we can identify B with the image of its connective cover $\tau_{\geqslant 0} B$ under the base change functor $\mathrm{Mod}_{\tau_{\geqslant 0} A}(\mathcal{A}) \to \mathrm{Mod}_A(\mathcal{A})$. By virtue of Proposition 6.15, to prove that $A \to B$ is descendable, it will suffice to show that $\tau_{\geqslant 0} A \to \tau_{\geqslant 0} B$ is descendable. We may therefore replace ϕ by the induced map $\tau_{\geqslant 0} A \to \tau_{\geqslant 0} B$ and thereby reduce to the case where A is connective.

Let $\mathcal{C}$ denote the smallest stable subcategory of $\mathrm{Mod}_A(\mathcal{A})$ which contains all objects of the form $M \otimes_A B$ and is closed under retracts. It will suffice to show that A belongs to $\mathcal{C}$. Let K be the fiber of the map $\phi : A \to B$, and let $\rho : K \to A$ be the canonical map. For each integer $m \geqslant 0$, let $\rho(m) : K^{\otimes m} \to A^{\otimes m} \simeq A$ be the the m th tensor power of ρ, formed in the monoidal ∞-category LMod_A. Then $\rho(m+1)$ is given by the composition $K^{\otimes m+1} \xrightarrow{\mathrm{id}_{K^{\otimes m}}} K^{\otimes m} \xrightarrow{\rho(m)} A$, so we have a fiber sequence

$$K^{\otimes m} \otimes_A B \to \mathrm{cofib}(\rho(m+1)) \to \mathrm{cofib}(\rho(m))$$

It follows by induction on m that each $\mathrm{cofib}(\rho(m))$ belongs to $\mathcal{C}$. Consequently, to prove that $A \in \mathcal{C}$, it will suffice to show that A is a retract of $\mathrm{cofib}(\rho(m))$ for some $m \geqslant 0$. This condition holds if the homotopy class of $\rho(m)$ vanishes (when regarded as an element of $\mathrm{Ext}^0_A(K^{\otimes m}, A) \simeq \mathrm{Ext}^m_A((\Sigma K)^{\otimes m}, A)$. The Proposition 3.25(4) implies that $\Sigma K \simeq \mathrm{cofib}(\phi)$ is a flat A-module. Also we have that $\pi_0 \,\mathrm{cofib}(K)$ is $\aleph_n$-compact $\pi_0 A$-module. It follows that $(\Sigma K)^{\otimes m}$ has the same properties for each $m > 0$, so that $\mathrm{Ext}^m_A((\Sigma K)^{\otimes m}, A)$ vanishes for $m > n$ by virtue of Lemma 6.18. $\square$

6.3 Almost $\mathbb{E}_\infty$-algebras

7 Deformation theory and étale rigidity

Throughout Section 7, we assume that $\mathcal{A}$ is left complete.

7.1 The cotangent complex formalism

We will freely use the deformation theory developed in [HA, §7.3-7.5].

Definition 7.1 (See [HA] 7.3.2.14.)**.** Let $\mathcal{C}$ be a presentable ∞-category, and consider the associated diagram

$$\begin{tikzcd} T_{\mathcal{C}} \arrow[rr, "G"] \arrow[dr, "p"'] & & \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[dl, "q"] \\ & \mathcal{C} & \end{tikzcd}$$

where q is given by evaluation at $\{1\} \subseteq \Delta^1$. The functor G carries p-Cartesian morphisms to q-Cartesian morphisms, and for each object $A \in \mathcal{C}$ the induced map $G_A : \mathrm{Sp}(\mathcal{C}^{/A}) \to \mathcal{C}^{/A}$ admits a left adjoint Σ^∞. Applying [HA, Proposition 7.3.2.6], we conclude that G admits a left adjoint relative to $\mathcal{C}$, which we will denote by F. The absolute cotangent complex functor $L : \mathcal{C} \to T_{\mathcal{C}}$ is defined to be the composition

$$\mathcal{C} \to \mathrm{Fun}(\Delta^1, \mathcal{C}) \xrightarrow{F} T_{\mathcal{C}}$$

where the first map is given by the diagonal embedding. We will denote the value of L on an object $A \in \mathcal{C}$ by $L_A \in \mathrm{Sp}(\mathrm{C}^{/A})$, and will refer to L_A as the cotangent complex of A.

Definition 7.2 (See [HA] 7.4.1.1)**.** Let $\mathcal{C}$ be a presentable ∞-category, and let $p : \mathcal{M}^T(\mathcal{C}) \to \Delta^1 \times \mathcal{C}$ denote a tangent correspondence to $\mathcal{C}$ (see [HA, Definition 7.3.6.9]). A derivation in $\mathcal{C}$ is a map $f : \Delta^1 \to \mathcal{M}^T(\mathcal{C})$ such that $p \circ f$ coincides with the inclusion $\Delta^1 \times \{A\} \subseteq \Delta^1 \times \mathcal{C}$, for some $A \in \mathcal{C}$. In this case, we will identify f with a morphism $\eta : A \to M$ in $\mathcal{M}^T(\mathcal{C})$, where $M \in T_{\mathcal{C}} \times_{\mathcal{C}} \{A\} \simeq \mathrm{Sp}(\mathcal{C}^{/A})$. We will also say $\eta : A \to M$ is a derivation of A into M.

We let $\mathrm{Der}(\mathcal{C})$ denote the fiber product $\mathrm{Fun}(\Delta^1, \mathcal{M}^T(\mathcal{C})) \times_{\mathrm{Fun}(\Delta^1, \Delta^1 \times \mathcal{C})} \mathcal{C}$. We will refer to $\mathrm{Der}(\mathcal{C})$ as the ∞-category of derivations in $\mathcal{C}$.

Remark 7.3. We primarily care the case $\mathcal{C} = \mathrm{CAlg}(\mathcal{A})$. In this case, an object in $\mathrm{Der}(\mathcal{C})$ can be informally described as a triple data $(A, M, \eta : A \to M[1])$ where $A \in \mathrm{CAlg}(\mathcal{A}), M \in \mathrm{Mod}_A(\mathcal{A})$ and η is a derivation.

Definition 7.4 (See [HA] 7.4.1.3)**.** Let $\mathcal{C}$ be a presentable ∞-category, and let $p : \mathcal{M}^T(\mathcal{C}) \to \Delta^1 \times \mathcal{C}$ be a tangent correspondence for $\mathcal{C}$. An extended derivation is a diagram σ

$$\begin{tikzcd} \tilde{A} \arrow[r, "f"] \arrow[d] & A \arrow[d, "\eta"] \\ 0 \arrow[r] & M[1] \end{tikzcd}$$

in $\mathcal{M}^T(\mathcal{C})$ with the following properties:

(1) The object $0 \in T_{\mathcal{C}}$ is a zero object of $\mathrm{Sp}\left(\mathcal{C}^{/A}\right)$. Equivalently, 0 is a p-initial vertex of $\mathcal{M}^T(\mathcal{C})$.

(2) The diagram σ is a pullback square.

(3) The objects $\widetilde{A}$ and A belong to $\mathcal{C} \subseteq \mathcal{M}^T(\mathcal{C})$, while 0 and M belong to $T_{\mathcal{C}} \subseteq \mathcal{M}^T(\mathcal{C})$.

(4) Let $\overline{f} : \Delta^1 \to \mathcal{C}$ be the map which classifies the morphism f appearing in the diagram above, and let $e : \Delta^1 \times \Delta^1 \to \Delta^1$ be the unique map such that $e^{-1}\{0\} = \{0\} \times \{0\}$. Then the diagram is commutative.

$$\begin{tikzcd} \Delta^1 \times \Delta^1 \arrow[r, "\sigma"] \arrow[d, "e"] & \mathcal{M}^T(\mathcal{C}) \arrow[r, "p"] & \Delta^1 \times \mathcal{C} \arrow[d] \\ \Delta^1 \arrow[rr, "\overline{f}"] & & \mathcal{C} \end{tikzcd}$$

We let $\widetilde{\mathrm{Der}}(\mathcal{C})$ denote the full subcategory of

$$\mathrm{Fun}\left(\Delta^1 \times \Delta^1, \mathcal{M}^T(\mathcal{C})\right) \times_{\mathrm{Fun}(\Delta^1 \times \Delta^1, \Delta^1 \times \mathcal{C})} \mathrm{Fun}\left(\Delta^1, \mathcal{C}\right)$$

spanned by the extended derivations.

Definition 7.5. Throughout Section 7, we let $\mathrm{Der} = \mathrm{Der}(\mathrm{CAlg}(\mathcal{A}))$ denote the ∞-category of derivations in $\mathrm{CAlg}(\mathcal{A})$. We let $A^\eta = \mathrm{fib}(\eta)$ denote the corresponding square-zero extension of A.
We define a subcategory $\mathrm{Der}^+ \subseteq \mathrm{Der}$ as follows:

(1) An object $(\eta : A \to M[1]) \in \mathrm{Der}$ belongs to Der^+ if and only if both A and M are connective. Equivalently, η belongs to Der^+ if both A and A^η are connective, and the map $\pi_0 A^\eta \to \pi_0 A$ is an epimorphism in $\mathcal{A}^\heartsuit$.

(2) Let $f : (\eta : A \to M[1]) \to (\eta' : B \to N[1])$ be a morphism in Der between objects which belong to Der^+. Then f belongs to Der^+ if and only if the induced map $B \otimes_A M \to N$ is an equivalence of B-modules.

Proposition 7.6 (See [HA] 7.3.3.6)**.** *Let*

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "h"] & \\ B \arrow[rr, "g"] & & C \end{tikzcd}$$

be morphisms in $\mathrm{CAlg}(\mathcal{A})$*. Then there exists a canonical cofiber sequence*

$$C \otimes_B L_{B/A} \to L_{C/A} \to L_{C/B}$$

in $\mathrm{Mod}_C(\mathcal{A})$ *.*

Proposition 7.7 (See [HA] 7.3.3.7)**.** *Let*

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & A \arrow[d] \\ B' \arrow[r] & B \end{tikzcd}$$

be a pushout diagram in $\mathrm{CAlg}(\mathcal{A})$*. Then there exists a canonical equivalence* $L_{B/A} \simeq B \otimes_{B'} L_{B'/A'}$ *in* $\mathrm{Mod}_B(\mathcal{A})$*.*

Lemma 7.8. *Let* $f : (\eta : A \to M[1]) \to (\eta' : B \to N[1])$ *be a morphism in* Der^+*. If the induced map* $A^\eta \to B^{\eta'}$ *is an equivalence in* $\mathrm{CAlg}(\mathcal{A})$*, then* f *is an equivalence.* (See [HA, Lem. 7.4.2.9] for the case of spectra.)

Proof. The morphism f determines a map of fiber sequences

$$\begin{tikzcd} A^\eta \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "f_0"'] & M[1] \arrow[d, "f_1"'] \\ B^{\eta'} \arrow[r] & B \arrow[r] & N[1] \end{tikzcd}$$

Since the left vertical map is an equivalence, we obtain an equivalence $\alpha : \mathrm{cofib}(f_0) \simeq \mathrm{cofib}(f_1)$. To complete the proof, it will suffice to show that $\mathrm{cofib}(f_0)$ vanishes. Suppose otherwise. Since $\mathrm{cofib}(f_0)$ is connective and $\mathcal{A}$ is left complete , there exists some smallest integer n such that $\pi_n \,\mathrm{cofib}(f_0) \neq 0$. In particular, $\mathrm{cofib}(f_0)$ is n-connective.

Since f induces an equivalence $B \otimes_A M \to N$, the $\mathrm{cofib}(f_1)$ can be identified with $\mathrm{cofib}(f_0) \otimes_A M[1]$. Since M is connective, we deduce that $\mathrm{cofib}(f_1)$ is $(n+1)$-connective. Using the equivalence α, we conclude that $\mathrm{cofib}(f_0)$ is $(n+1)$-connective, which contradicts our assumption that $\pi_n \,\mathrm{cofib}(f_0) \neq 0$. ☐

Definition 7.9. We define a subcategory $\mathrm{Fun}^+(\Delta^1, \mathrm{CAlg}(\mathcal{A}))$ as follows:

(*i*) An object $f : \widetilde{A} \to A$ of $\mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathcal{A}))$ belongs to $\mathrm{Fun}^+(\Delta^1, \mathrm{CAlg}(\mathcal{A}))$ if and only if both A and $\widetilde{A}$ are connective, and f induces a surjection $\pi_0 \widetilde{A} \to \pi_0 A$.

(*ii*) Let $f, g \in \mathrm{Fun}^+(\Delta^1, \mathrm{CAlg}(\mathcal{A}))$, and let $\alpha : f \to g$ be a morphism in $\mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathcal{A}))$. Then α belongs to $\mathrm{Fun}^+(\Delta^1, \mathrm{CAlg}(\mathcal{A}))$ if and only if it classifies a pushout square in the ∞-category $\mathrm{CAlg}(\mathcal{A})$.

Theorem 7.10. *Let* $\Phi : \widetilde{\mathrm{Der}} \to \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathcal{A}))$ *be the functor given by* $(\eta : A \to M[1]) \mapsto (A^\eta \to A)$*. Then* Φ *induces a functor* $\Phi^+ : \widetilde{\mathrm{Der}}^+ \to \mathrm{Fun}^+(\Delta^1, \mathrm{CAlg}(\mathcal{A}))$*. Moreover, the functor* Φ^+ *is a left fibration.*

Proof. It is a parallel proof of [HA, Lem. 7.4.2.7]. ☐

Remark 7.11. The left complete condition that we assume at the beginning of the section is necessary in the proof of Lemma 7.8, which is the only part involving the left completeness in the proof of [HA, Lem. 7.4.2.7].

Corollary 7.12. *Let* $A \in \mathrm{CAlg}(\mathcal{A}_{\geq 0})$*,* M *a connective* A*-module, and* $\eta : A \to M[1]$ *a derivation. Then the functor* Φ *induces an equivalence of* ∞*-categories*

$$\mathrm{Der}^+_{\eta/} \to \mathrm{CAlg}^{\mathrm{cn}}_{A^\eta}$$

given on objects by $(\eta' : B \to N[1]) \mapsto B^{\eta'}$*.*

7.2 Deformation theory

Definition 7.13. We say a map of commutative ring objects $A \to B \in \mathrm{CAlg}(\mathcal{A})$ is L-étale (or formally étale) if the relative cotangent complex $L_{B/A}$ vanishes.

Remark 7.14. In deformation theory, the L-étale condition is only interesting in the connective case.

Lemma 7.15. *Let*

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "h"] & \\ B \arrow[rr, "g"] & & C \end{tikzcd}$$

be morphisms in $\mathrm{CAlg}(\mathcal{A})$*. Then:*

(1) *Suppose that f is L-étale. Then g is L-étale if and only if h is L-étale.*

(2) *If g is L-étale and faithfully flat. Then f is L-étale if and only if h is L-étale.*

Proof. It follows from the cofiber sequence

$$C \otimes_B L_{B/A} \to L_{C/A} \to L_{C/B}$$

in $\mathrm{Mod}_C(\mathcal{A})$. □

Proposition 7.16. *Let $f : R \to S \in \mathrm{CAlg}(\mathcal{A})$ be map such that S is an idempotent commutative R-algebra. Then f is L-étale.*

Proof. □

Corollary 7.17. *Let $R \to S \in \mathrm{CAlg}(\mathcal{A}^{\heartsuit})$ be a flat epimorphic map between discrete commutative ring objects. Then $R \to S$ is L-étale.*

Proof. □

Lemma 7.18. *Given a diagram of ∞-categories*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "p"'] & & \mathcal{D} \arrow[dl, "q"] \\ & \mathcal{E} & \end{tikzcd}$$

where p, q are cocartesian fibrations and F preserves cocartesian edges. Let $s \in \mathcal{E}$ and $\theta : K^{\triangleright} \to \mathcal{C}_s$ be a diagram. If for any morphism $f : s \to t \in \mathcal{E}$ the edge $f_! \circ \theta : K^{\triangleright} \to \mathcal{C}_t$ is an F_s-colimit, then θ is an F-colimit.

Proof. This is the relative version of [HTT, Prop. 4.3.1.10]. □

Corollary 7.19. *Given a diagram of ∞-categories*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "p"'] & & \mathcal{D} \arrow[dl, "q"] \\ & \mathcal{E} & \end{tikzcd}$$

where p, q are cocartesian fibrations and F preserves cocartesian edges. Let $s \in \mathcal{E}$ and $\theta : x \to y$ be a morphism in the fiber $\mathcal{C}_s$. If for any morphism $f : s \to t \in \mathcal{E}$, the edge $f_!(\theta) : f_!(x) \to f_!(y)$ is F_s-cocartesian, then θ is an F-cocartesian.

Proposition 7.20. *Let $f : A \to B \in \mathrm{CAlg}(\mathcal{A}) \subset \mathcal{M}^T(\mathrm{CAlg}(\mathcal{A}))$ be a morphism of $\mathbb{E}_\infty$-ring objects. Then f is F-cocartesian if and only if f is L-étale, where $F : \mathcal{M}^T(\mathrm{CAlg}(\mathcal{A})) \to \Delta^1 \times \mathrm{CAlg}(\mathcal{A})$ is the natural projection.*

Proof. Applying Corollary 7.19 to the following diagram, we win.

$$\begin{tikzcd} \mathcal{M}^T(\mathrm{CAlg}(\mathcal{A})) \arrow[rr, "F"] \arrow[dr, "p"'] & & \Delta^1 \times \mathrm{CAlg}(\mathcal{A}) \arrow[dl, "q"] \\ & \Delta^1 & \end{tikzcd}$$

□

Definition 7.21. We define a subcategory $\mathrm{Der}^{L\text{-}et} \subseteq \mathrm{Der}$ as follows:

(1) A derivation $\eta : A \to M[1]$ belongs to $\mathrm{Der}^{L\text{-}et}$ if and only if A and M are connective.

(2) Let $\phi : (\eta : A \to M[1]) \to (\eta' : B \to N[1])$ be a morphism between derivations belonging to $\mathrm{Der}^{L\text{-}et}$. Then ϕ belongs to $\mathrm{Der}^{L\text{-}et}$ if and only the map $A \to B$ is L-étale, and ϕ induces an equivalence $M \otimes_A B \to N$.

We define a subcategory $\mathrm{CAlg}(\mathcal{A}_{\geq 0})^{L\text{-}et} \subseteq \mathrm{CAlg}(\mathcal{A})$ as follows:

(1) An object $A \in \mathrm{CAlg}(\mathcal{A})$ belongs to $\mathrm{CAlg}(\mathcal{A}_{\geq 0})^{L\text{-}et}$ if and only if A is connective.

(2) A morphism $f : A \to B$ of connective $\mathbb{E}_\infty$-ring objects belongs to $\mathrm{CAlg}(\mathcal{A}_{\geq 0})^{L\text{-}et}$ if and only if f is L-étale.

Proposition 7.22. *Let $f : \mathrm{Der} \to \mathrm{CAlg}(\mathcal{A})$ denote the forgetful functor $(\eta : A \to M) \mapsto A$. Then f induces a left fibration $\mathrm{Der}^{L\text{-}et} \to \mathrm{CAlg}(\mathcal{A}_{\geq 0})^{L\text{-}et}$.*

Proof. Fix $0 \leq i < n$; we must show that every lifting problem of the form

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & \mathrm{Der}^{L\text{-}et} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "l"] & \mathrm{CAlg}(\mathcal{A}_{\geq 0})^{L\text{-}et} \end{tikzcd}$$

admits a solution l. Considering the following diagram,

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & \mathrm{Der}^{L\text{-}et} \arrow[r] \arrow[d] & \mathrm{Der} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Fun}(\Delta^1, \mathcal{M}^T(\mathrm{CAlg}(\mathcal{A}))) \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "l"] \arrow[urr, dashed, "l'"] \arrow[urrr, dashed, "l''"'] & \mathrm{CAlg}(\mathcal{A}_{\geq 0})^{L\text{-}et} \arrow[r] & \mathrm{CAlg}(\mathcal{A}) \arrow[r] & \mathrm{Fun}(\Delta^1, \Delta^1 \times \mathrm{CAlg}(\mathcal{A})) \end{tikzcd}$$

then there exists a lifting $l^{''}$ by Proposition 7.20, and hence there exists a lifting $l^{'}$. We observe that l' actually lies in $\mathrm{Der}^{L\text{-}et}$ by Lemma 7.15, hence we find a solution l. □

7.3 Étale rigidity

Definition 7.23. We say a map $f : A \to B$ in $\mathrm{CAlg}(\mathcal{A})$ is étale if f is flat, and $\tau_{\geq 0} f : \tau_{\geq 0} A \to \tau_{\geq 0} B$ is L-étale and finitely presented.

Our main result in this section is the following theorem (see [HA] §7.5 for the statement in case of spectra).

Theorem 7.24 (Étale rigidity)**.**
Assume that $\mathcal{A}$ is Grothendieck and left complete. Let $A \in \mathrm{CAlg}(\mathcal{A})$. Then:

(1) *Let $\mathrm{CAlg}(\mathcal{A})_{A/}^{fl,L\text{-}et}$ denote the full subcategory of $\mathrm{CAlg}(\mathcal{A})_{A/}$ spanned by the flat L-étale maps $A \to B$. If A is connective, then the functor π_0 induces an equivalence*
$$\mathrm{CAlg}(\mathcal{A})_{A/}^{fl,L\text{-}et} \xrightarrow{\sim} \mathrm{CAlg}(\mathcal{A}^\heartsuit)_{\pi_0 A/}^{fl,L\text{-}et}$$
with (the nerve of) the discrete flat L-étale commutative $\pi_0 A$-algebras.

(2) *Suppose further that $\mathcal{A}_{\geq 0}^{\otimes}$ is projectively rigid. Let $\mathrm{CAlg}(\mathcal{A})_{A/}^{et}$ denote the full subcategory of $\mathrm{CAlg}(\mathcal{A})_{A/}$ spanned by the étale maps $A \to B$. Then the functor π_0 induces an equivalence*
$$\mathrm{CAlg}(\mathcal{A})_{A/}^{et} \xrightarrow{\sim} \mathrm{CAlg}(\mathcal{A}^\heartsuit)_{\pi_0 A/}^{et}$$
with (the nerve of) the discrete étale commutative $\pi_0 A$-algebras.

The proof will occupy the remainder of this section.

Proposition 7.25. *Let $\eta : A \to M \in \mathrm{Der}^+$ and $\alpha : \tilde{A} \to A$ be the induced square-zero extension in $\mathrm{CAlg}(\mathcal{A}_{\geq 0})$. Now given a pushout diagram in $\mathrm{CAlg}(\mathcal{A}_{\geq 0})$.*
$$\begin{tikzcd} \tilde{A} \arrow[r, "\alpha"] \arrow[d, "f_0'"] & A \arrow[d, "f_0"] \\ \tilde{B} \arrow[r] & B \end{tikzcd}$$
Then:

(1) *The f_0' is L-étale if and only if f_0 is L-étale.*

(2) *Assume that $\mathcal{A}$ is Grothendieck. Then f_0' is flat if and only if f_0 is flat.*

(3) *If f_0' is L-étale, then f_0' is locally of finite presentation if and only if f_0 is locally of finite presentation.*

Proof.
(1) The “only if" direction is obvious. The “if” direction follows from the equivalences
$$L_{B/A} \simeq B \otimes_{\tilde{B}} L_{\tilde{B}/\tilde{A}} \simeq A \otimes_{\tilde{A}} L_{\tilde{B}/\tilde{A}}$$
and Proposition 6.2.
(2) The “only if" direction is obvious. For the converse, suppose that B is flat over $A:$ it suffices to show that for every discrete $\widetilde{A}$-module N, the relative tensor product $\widetilde{B} \otimes_{\widetilde{A}} N$ is discrete by Proposition 3.15(3). To prove this, let $I \subseteq \pi_0 \widetilde{A}$ be the kernel of the surjective map $\pi_0 \widetilde{A} \to \pi_0 A$, so that we have a short exact sequence of modules over $\pi_0 \widetilde{A}$:
$$0 \to IN \to N \to N/IN \to 0$$
It will therefore suffice to show that the tensor products $\widetilde{B} \otimes_{\widetilde{A}} IN$ and $\widetilde{B} \otimes_{\widetilde{A}} N/IN$ are discrete. Replacing N by IN or N/IN, we can reduce to the case where $IN = 0$, so that N has the structure of an A-module. Then $\widetilde{B} \otimes_{\widetilde{A}} N \simeq B \otimes_A N$ is discrete by virtue of the assumption that B is flat over A.
(3) The proof is a parallel argument as the proof of [DAGXIII, Lem. 2.5.4]. □

Proposition 7.26. *Let $A \in \mathrm{CAlg}(\mathcal{A}_{\geq 0})$, M be a connective A-module, and $\eta : A \to M[1]$ be a derivation. Then the square-zero extension $\tilde{A} \to A$ induces an equivalence*
$$(-) \otimes_{\tilde{A}} A : \mathrm{CAlg}(\mathcal{A}_{\geq 0})_{\tilde{A}/}^{L\text{-}et} \xrightarrow{\sim} \mathrm{CAlg}(\mathcal{A}_{\geq 0})_{A/}^{L\text{-}et}$$
between ∞-categories of connective L-étale commutative $\tilde{A}$-algebras and commutative A-algebras.

Proof. Any square-zero extension $\widetilde{A} \to A$ is associated to some derivation $(\eta : A \to M) \in \mathrm{Der}^{L\text{-}et}$. Let $\Phi : \mathrm{Der} \to \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathcal{A}))$ be the functor defined in Theorem 7.10. Let $\Phi_0, \Phi_1 : \mathrm{Der} \to \mathrm{CAlg}(\mathcal{A})$ denote the composition of Φ with evaluation at the vertices $\{0\}, \{1\} \in \Delta^1$. The functors Φ_0 and Φ_1 induce maps

$$\mathrm{CAlg}(\mathcal{A}_{\geq 0})^{L\text{-}et}_{\widetilde{A}/} \xleftarrow{\Phi_0'} \mathrm{Der}^{L\text{-}et} \xrightarrow{\Phi_1'} \mathrm{CAlg}(\mathcal{A}_{\geq 0})^{L\text{-}et}_{A/}$$

Moreover, the functor Φ exhibits Φ_1' as equivalent to the composition of Φ_0' with the relative tensor product $\otimes_{\widetilde{A}} A$. Consequently, it will suffice to prove the following:

(1) The functor Φ_0' is fully faithful, and its essential image consists precisely of the connective L-étale commutative $\widetilde{A}$-algebras.

(2) The functor Φ_1' is fully faithful, and its essential image consists precisely of the connective L-étale commutative A-algebras.

The (1) follows from Corollary 7.12 and Proposition 7.25 (1). And the (2) follows from Proposition 7.22. ☐

Definition 7.27 (See [HA] 7.4.1.18)**.** For $n \geq 0$, we say a morphism $f : A \to B$ in $\mathrm{CAlg}(\mathcal{A}_{\geq 0})$ is n-connective if $\mathrm{fib}(f)$ belongs to $\mathcal{A}_{\geq n}$. And we say f is an n-small extension if the following further conditions are satisfied:

(1) The fiber $\mathrm{fib}(f)$ belongs to $\mathcal{A}_{[n,2n]}$.

(2) The multiplication map $\mathrm{fib}(f) \otimes_A \mathrm{fib}(f) \to \mathrm{fib}(f)$ is nullhomotopic.

We let $\mathrm{Fun}_{n-con}\left(\Delta^1, \mathrm{CAlg}(\mathcal{A})\right)$ denote the full subcategory of $\mathrm{Fun}\left(\Delta^1, \mathrm{CAlg}(\mathcal{A})\right)$ spanned by the n-connective extensions, and $\mathrm{Fun}_{n-sm}\left(\Delta^1, \mathrm{CAlg}(\mathcal{A})\right)$ the full subcategory of $\mathrm{Fun}_{n-con}\left(\Delta^1, \mathrm{CAlg}(\mathcal{A})\right)$ spanned by the n-small extensions.

We let $\mathrm{Fun}_{n-sm}\left(\Delta^1, \mathrm{CAlg}(\mathcal{A})\right)$ denote the full subcategory of $\mathrm{Fun}\left(\Delta^1, \mathrm{CAlg}(\mathcal{A})\right)$ spanned by the n-small extensions.

Definition 7.28. For $A \in \mathrm{CAlg}(\mathcal{A})$, we let $L_A \in \mathrm{Sp}\left(\mathrm{CAlg}(\mathcal{A})_{/A}\right) \simeq \mathrm{Mod}_A(\mathcal{A})$ denote its cotangent complex.

Let Der denote the ∞-category $\mathrm{Der}\left(\mathrm{CAlg}(\mathcal{A})\right)$ of derivations in $\mathrm{CAlg}(\mathcal{A})$, so that the objects of Der can be identified with pairs $(A, \eta : L_A \to M[1])$ where A is an commutative algebra object of $\mathcal{A}$ and η is a morphism in $\mathrm{Mod}_A(\mathcal{A})$.

We let Der_{n-sm} denote the full subcategory of Der spanned by those pairs $(A, \eta : L_A \to M[1])$ such that A is connective and the image of M belongs to $\mathcal{A}_{[n,2n]}$.

Theorem 7.29 (See [HA] 7.4.1.26)**.** *Let* $\Phi : \mathrm{Der} \to \mathrm{Fun}\left(\Delta^1, \mathrm{CAlg}(\mathcal{A})\right)$ *be the functor given by* $(\eta : A \to B) \mapsto (A^\eta \to A)$*. Then the functor* $\Phi^{(k)}$ *restricts to an equivalence of* ∞*-categories*

$$\Phi_{n-sm} : \mathrm{Der}_{n-sm} \to \mathrm{Fun}_{n-sm}\left(\Delta^1, \mathrm{CAlg}(\mathcal{A})\right)$$

Corollary 7.30.

(1) *Every* n*-small extension in* $\mathrm{CAlg}(\mathcal{A}_{\geq 0})$ *is a square-zero extension.*

(2) *Let* $A \in \mathrm{CAlg}(\mathcal{A}_{\geq 0})$*. Then every map in the Postnikov tower*

$$\ldots \to \tau_{\leq 3} A \to \tau_{\leq 2} A \to \tau_{\leq 1} A \to \tau_{\leq 0} A$$

is a square-zero extension.

Theorem 7.31. *Let $f : A \to B \in \mathrm{CAlg}(\mathcal{A})$ be a flat map such that $\tau_{\geq 0} f : \tau_{\geq 0} A \to \tau_{\geq 0} B$ is L-étale, and let $C \in \mathrm{CAlg}(\mathcal{A})$. Then the canonical map*

$$\mathrm{Map}_{\mathrm{CAlg}(\mathcal{A})_{A/}}(B, C) \to \mathrm{Map}_{\mathrm{CAlg}(\mathcal{A})_{\pi_0 A/}}(\pi_0 B, \pi_0 C)$$

is a homotopy equivalence. In particular, $\mathrm{Map}_{\mathrm{CAlg}(\mathcal{A})_{A/}}(B, C)$ is homotopy equivalent to a discrete space.

Proof. The following proof is similar as [DAGIV, Prop. 3.4.13]. Let A_0, B_0, and C_0 be connective covers of A, B, and C, respectively. We have a pushout diagram

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d, "f_0"] & A \arrow[d, "f"] \\ B_0 \arrow[r] & B \end{tikzcd}$$

where f_0 is flat L-étale. It follows that the induced maps

$$\mathrm{Map}_{\mathrm{CAlg}(\mathcal{A})_{A/}}(B, C) \to \mathrm{Map}_{\mathrm{CAlg}(\mathcal{A})_{A_0/}}(B_0, C) \leftarrow \mathrm{Map}_{\mathrm{CAlg}(\mathcal{A})_{A_0/}}(B_0, C_0)$$

are homotopy equivalences. We may therefore replace A, B and C by their connective covers, and thereby reduce to the case where A, B, and C are connective.

We have a commutative diagram

$$\begin{tikzcd} & \mathrm{Map}_{\mathrm{CAlg}(\mathcal{A})_{A/}}(B, \pi_0 C) \arrow[dr, "\psi"] & \\ \mathrm{Map}_{\mathrm{CAlg}(\mathcal{A})_{A/}}(B, C) \arrow[ur, "\phi"] \arrow[rr] & & \mathrm{Map}_{\mathrm{CAlg}(\mathcal{A})_{\pi_0 A/}}(\pi_0 B, \pi_0 C) \end{tikzcd}$$

where the map ψ is a homotopy equivalence. It will therefore suffice to show that ϕ is a homotopy equivalence. Let us say a map $g : D \to D'$ of commutative A-algebras is good if the induced map $\phi_g : \mathrm{Map}_{\mathrm{CAlg}(\mathcal{A})_{A/}}(B, D) \to \mathrm{Map}_{\mathrm{CAlg}(\mathcal{A})_{A/}}(B, D')$ is a homotopy equivalence. Equivalently, g is good if $e_B(g)$ is an equivalence, where $e_B : \mathrm{CAlg}(\mathcal{A})_{A/} \to \mathcal{S}$ is the functor corepresented by B. We wish to show that the truncation map $C \to \pi_0 C$ is good. We will employ the following chain of reasoning:

- (i) Let D be a commutative A-algebra, let M be a D-module, and let $g : D \oplus M \to D$ be the projection. For every map of commutative A-algebras $h : B \to D$, the homotopy fiber of ϕ_g over the point h can be identified with $\mathrm{Map}_{\mathrm{Mod}_B}(L_{B/A}, M) \simeq \mathrm{Map}_{\mathrm{Mod}_D}(L_{B/A} \otimes_B D, M)$. Since f is L-étale, the homotopy fibers of ϕ_g are contractible. It follows that ϕ_g is a homotopy equivalence, so that g is good.
- (ii) The collection of good morphisms is stable under pullback. This follows immediately from the observation that e_B preserves limits.
- (iii) Any square-zero extension is good. This follows from (a) and (b).
- (iv) Suppose given a sequence of good morphisms

  $$\ldots \to D_2 \to D_1 \to D_0$$

  Then the induced map $\varprojlim \{D_i\} \to D_0$ is good. This follows again from the observation that e_B preserves limits.
- (v) For every connective commutative A-algebra C, the truncation map $C \to \pi_0 C$ is good. This follows by applying (d) to the Postnikov tower

  $$\ldots \to \tau_{\leq 2} C \to \tau_{\leq 1} C \to \tau_{\leq 0} C \simeq \pi_0 C$$

  which is a sequence of square-zero extensions.

□

Proposition 7.32. *Assume that $\mathcal{A}$ is Grothendieck. Let $A \in \mathrm{CAlg}(\mathcal{A}_{\geq 0})_{\leq n+1}$ be $(n{+}1)$-truncated connective. Then the truncation functor $\tau_{\leq n} : \mathrm{CAlg}(\mathcal{A})_{A/} \to \mathrm{CAlg}(\mathcal{A})_{\tau_{\leq n}A/}$ restricts to:*

(1) *An equivalence $\mathrm{CAlg}(\mathcal{A})^{fl,L\text{-}et}_{A/} \xrightarrow{\sim} \mathrm{CAlg}(\mathcal{A})^{fl,L\text{-}et}_{\tau_{\leq n}A/}$ from the ∞-category of flat L-étale commutative A-algebras to the ∞-category of flat L-étale commutative $\tau_{\leq n}A$-algebras.*

(2) *An equivalence $\mathrm{CAlg}(\mathcal{A})^{et}_{A/} \xrightarrow{\sim} \mathrm{CAlg}(\mathcal{A})^{et}_{\tau_{\leq n}A/}$ from the ∞-category of étale commutative A-algebras to the ∞-category of étale commutative $\tau_{\leq n}A$-algebras.*

Proof. It follows by combining Proposition 3.22(2), Proposition 7.25(2) and Proposition 7.26. □

Proposition 7.33 (See [HA] 7.4.3.17)**.** *Let $f : A \to B$ be a morphism in $\mathrm{CAlg}(\mathcal{A}_{\geq 0})$. Suppose that $n \geq 0$ and that f induces an equivalence $\tau_{\leq n}A \to \tau_{\leq n}B$. Then $\tau_{\leq n}L_{B/A} \simeq 0$.*

Corollary 7.34. *Let $f : A \to B$ be a map in $\mathrm{CAlg}(\mathcal{A}_{\geq 0})$. Assume that $\mathrm{cofib}(f)$ is n-connective, for $n \geq 0$. Then the relative cotangent complex $L_{B/A}$ is n-connective. The converse holds provided that f induces an isomorphism $\pi_0 A \to \pi_0 B$.*

Proposition 7.35. *Let $f : A \to B \in \mathrm{CAlg}(\mathcal{A}_{\geq 0})$. Then:*

(1) *The $f : A \to B$ is L-étale if $\tau_{\leq n}f : \tau_{\leq n}A \to \tau_{\leq n}B$ is L-étale for any $n \geq 0$.*

(2) *Assume that $\mathcal{A}$ is Grothendieck. Then the $f : A \to B$ is flat if and only if $\tau_{\leq n}f : \tau_{\leq n}A \to \tau_{\leq n}B$ is flat for any $n \geq 0$.*

Proof.
(1) For any $n \geq 0$, we have the cofiber sequence

$$\tau_{\leq n}B \otimes_B L_{B/A} \to L_{\tau_{\leq n}B/A} \to L_{\tau_{\leq n}B/B}.$$

Since $\tau_{\leq n}L_{\tau_{\leq n}B/B} = 0$ by Proposition 7.33, we get that $\tau_{\leq n-1}(\tau_{\leq n}B \otimes_B L_{B/A}) \simeq \tau_{\leq n-1}L_{\tau_{\leq n}B/A}$. Now consider another cofiber sequence

$$\tau_{\leq n}B \otimes_{\tau_{\leq n}A} L_{\tau_{\leq n}A/A} \to L_{\tau_{\leq n}B/A} \to L_{\tau_{\leq n}B/\tau_{\leq n}A}.$$

The $L_{\tau_{\leq n}B/\tau_{\leq n}A}$ above vanishes by the assumption and $\tau_{\leq n}L_{\tau_{\leq n}A/A} = 0$ by Proposition 7.33, so $\tau_{\leq n}L_{\tau_{\leq n}B/A} = \tau_{\leq n}(\tau_{\leq n}B \otimes_{\tau_{\leq n}A} L_{\tau_{\leq n}A/A}) = 0$. Then combining Lemma 4.6 and equations above we get

$$\tau_{\leq n-1}(L_{B/A}) \simeq \tau_{\leq n-1}(\tau_{\leq n}B \otimes_B L_{B/A}) \simeq \tau_{\leq n-1}L_{\tau_{\leq n}B/A} = 0.$$

By the left completeness, we get $L_{B/A} = 0$.
(2) The "only if" direction can be deduced by Proposition 3.22. Now suppose $\tau_{\leq n}f : \tau_{\leq n}A \to \tau_{\leq n}B$ is flat for any $n \geq 0$. Since B is connective, it suffices to show that given any discrete $M \in \mathrm{Mod}_A(\mathcal{A})^{\heartsuit}$ we have $B \otimes_A M \in \mathrm{Mod}_B(\mathcal{A})^{\heartsuit}$ is discrete too. Now we have

$$\tau_{\leq n}(B \otimes_A M) \simeq \tau_{\leq n}(\tau_{\leq n}B \otimes_A M)$$

by Lemma 4.6. Also we have

$$\tau_{\leq n}(\tau_{\leq n}B \otimes_A M) \simeq \tau_{\leq n}(\tau_{\leq n}B \otimes_{\tau_{\leq n}A} \tau_{\leq n}A \otimes_A M) \simeq \tau_{\leq n}B \otimes_{\tau_{\leq n}A} \tau_{\leq n}(\tau_{\leq n}A \otimes_A M)$$

where the second equality comes from the flatness of $\tau_{\leq n}f$. However note that by Lemma 4.6 we have

$$\tau_{\leq n}B \otimes_{\tau_{\leq n}A} \tau_{\leq n}(\tau_{\leq n}A \otimes_A M) \simeq \tau_{\leq n}B \otimes_{\tau_{\leq n}A} \tau_{\leq n}M.$$

Combining these we get an equivalence $\tau_{\leq n}(B \otimes_A M) \simeq \tau_{\leq n}B \otimes_{\tau_{\leq n}A} \tau_{\leq n}M$, then by the flatness of $\tau_{\leq n}f$ again we conclude that for any $n \geq 0$, the $\tau_{\leq n}(B \otimes_A M)$ is discrete. Hence $B \otimes_A M$ is discrete by the left completeness. □

We mimic the proof of [HA, Theorem 7.4.3.18] with light modification to get the following statements.

Proposition 7.36. *Suppose that $\mathcal{A}$ is Grothendieck and that $\mathcal{A}_{\geq 0}$ is compactly generated. Let $A \in \mathrm{CAlg}(\mathcal{A}_{\geq 0})$, and let B be a connective $\mathbb{E}_\infty$-algebra over A. Then:*

(1) *If B is locally of finite presentation over A, then $L_{B/A}$ is perfect as a B-module. The converse holds provided that $\mathcal{A}^{\otimes}_{\geq 0}$ is **projectively rigid** and that $\pi_0 B$ is finitely presented as a commutative $\pi_0 A$-algebra in the sense of Definition 2.11.*

(2) *If B is almost of finite presentation over A, then $L_{B/A}$ is almost perfect as a B-module. The converse holds provided that $\mathcal{A}_{\geq 0}$ is **projectively generated** and that $\pi_0 B$ is finitely presented as a commutative $\pi_0 A$-algebra in the sense of Definition 2.11.*

Proof. We first prove the forward implications. It will be convenient to phrase these results in a slightly more general form. Suppose given a commutative diagram σ:

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "h"] & \\ B \arrow[rr, "g"] & & C \end{tikzcd}$$

in $\mathrm{CAlg}(\mathcal{A}_{\geq 0})$, and let $F(\sigma) = L_{B/A} \otimes_B C$. We will show:
($1'$) If B is locally of finite presentation as an $\mathbb{E}_\infty$-algebra over A, then $F(\sigma)$ is perfect as a C-module.
($2'$) If B is almost of finite presentation as an $\mathbb{E}_\infty$-algebra over A, then $F(\sigma)$ is almost perfect as a C-module.

We will obtain the forward implications of (1) and (2) by applying these results in the case $B = C$. We first observe that the construction $\sigma \mapsto F(\sigma)$ defines a functor $\mathrm{CAlg}(\mathcal{A})_{A//C} \to \mathrm{Mod}_C(\mathcal{A})$. Note that the functor F can be identified with the fiber of the relative adjunction

$$\begin{tikzcd} \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathcal{A})_{A/}) \arrow[r, shift left] \arrow[dr] & T_{\mathrm{CAlg}(\mathcal{A})_{A/}} \arrow[l, shift left] \arrow[r, "\sim"] \arrow[d] & \mathrm{Mod}(\mathrm{Mod}_A(\mathcal{A})) \arrow[dl] \\ & \mathrm{CAlg}(\mathcal{A})_{A/} & \end{tikzcd}$$

on $C \in \mathrm{CAlg}(\mathcal{A})_{A/}$, we deduce that this functor preserves colimits. Since the collection of finitely presented C-modules is closed under finite colimits and retracts, it will suffice to prove ($1'$) in the case where $B = \mathrm{Sym}^*_A M$ for some connective perfect A-module M. Using Proposition [HA, Proposition 7.4.3.14], we deduce that $F(\sigma) \simeq M \otimes_A C$ is a perfect C-module, as desired.

We now prove ($2'$). By [HTT, Corollary 5.5.7.4], for any $n \geq 2$ there exists a finitely presented commutative A-algebra $B' \in \mathrm{CAlg}(\mathcal{A}_{\geq 0})_{A/}$ such that $\tau_{\leq n} B$ is a retraction of $\tau_{\leq n} B'$ as commutative A-algebras. Note that the retraction can be lifted in $\mathrm{CAlg}(\mathcal{A}_{\geq 0})_{A//\tau_{\leq n} C}$ by [Ker, 04KB], as the following.

$$\begin{tikzcd} \tau_{\leq n} B' \arrow[r, shift left, "r"] \arrow[dr] & \tau_{\leq n} B \arrow[l, shift left, "i"] \arrow[d] \\ & \tau_{\leq n} C \end{tikzcd}$$

Now consider the diagram

$$\begin{tikzcd} B' \arrow[r] & \tau_{\leq n} B' \arrow[r, "r"] & \tau_{\leq n} B \arrow[r] & \tau_{\leq n} C \\ & A \arrow[ul] \arrow[u] \arrow[r] & B \arrow[u] \arrow[r] & C \arrow[u] \end{tikzcd}$$

We claim that $\tau_{\leq n-2}(L_{B/A} \otimes_B C)$ is a retraction of $\tau_{\leq n-2}(L_{B'/A} \otimes_{B'} C)$. However, assertion ($1'$) implies that $L_{B'/A} \otimes_{B'} C$ will be perfect so long as B' is locally of finitely presentation as a commutative A-algebra.

Then $L_{B'/A} \otimes_{B'} C$ is perfect as a retraction of a perfect module and $L_{B/A} \otimes_B C$ is almost perfect. Now using Proposition 7.33, we see that $L_{\tau_{\leq n}B/B}$ and $L_{\tau_{\leq n}B'/B'}$ are n-connective, thus we have the natural equivalences

$$\tau_{\leq n-2}(L_{B/A} \otimes_B \tau_{\leq n}B) \xrightarrow{\sim} \tau_{\leq n-2}L_{\tau_{\leq n}B/A} \;,\; \tau_{\leq n-2}(L_{B'/A} \otimes_{B'} \tau_{\leq n}B') \xrightarrow{\sim} \tau_{\leq n-2}L_{\tau_{\leq n}B'/A}.$$

So

$$\tau_{\leq n-2}(L_{B/A} \otimes_B C) \xrightarrow{\sim} \tau_{\leq n-2}(L_{B/A} \otimes_B \tau_{\leq n}C) \xrightarrow{\sim} \tau_{\leq n-2}(L_{\tau_{\leq n}B/A} \otimes_{\tau_{\leq n}B} \tau_{\leq n}C)$$

are equivalences by Lemma 4.6 (1). By assumption we have that the $\tau_{\leq n-2}(L_{\tau_{\leq n}B/A} \otimes_{\tau_{\leq n}B} \tau_{\leq n}C)$ is a retraction of $\tau_{\leq n-2}(L_{\tau_{\leq n}B'/A} \otimes_{\tau_{\leq n}B'} \tau_{\leq n}C)$. Again by Lemma 4.6 (1), we get the equivalences

$$\tau_{\leq n-2}(L_{B'/A} \otimes_{B'} C) \xrightarrow{\sim} \tau_{\leq n-2}(L_{B'/A} \otimes_{B'} \tau_{\leq n}C) \xrightarrow{\sim} \tau_{\leq n-2}(L_{\tau_{\leq n}B'/A} \otimes_{\tau_{\leq n}B'} \tau_{\leq n}C).$$

Combining these, we in fact conclude that $\tau_{\leq n-2}(L_{B/A} \otimes_B C)$ is a retraction of $\tau_{\leq n-2}(L_{B'/A} \otimes_{B'} C)$.

We now prove the reverse implication of (2). Assume that $L_{B/A}$ is almost perfect and that $\pi_0 B$ is a finitely presented as a commutative $\pi_0 A$-algebra. To prove (2), it will suffice to construct a sequence of maps

$$A \to B(-1) \to B(0) \to B(1) \to \ldots \to B$$

such that each $B(n)$ is locally of finite presentation as a commutative A-algebra, and each map $f_n : B(n) \to B$ is $(n+1)$-connective. We begin by constructing $B(-1)$ with an even stronger property: the map f_{-1} induces an isomorphism $\pi_0 B(-1) \to \pi_0 B$. By Proposition 5.15, there exists compact projective A-modules M, N and a diagram

$$\begin{tikzcd} \mathrm{Sym}_A^*(N) \arrow[r, "\alpha"] \arrow[d, "\phi"] & A \arrow[d] \\ \mathrm{Sym}_A^*(M) \arrow[r] & B \end{tikzcd}$$

such that the map $B(-1) \to B$ induces an equivalence on $\pi_0 B$, where we take $B(-1)$ as the pushout of above diagram.

We now proceed in an inductive fashion. Assume that we have already constructed a connective commutative A-algebra $B(n)$ which is of finite presentation over A, and an $(n+1)$-connective morphism $f_n : B(n) \to B$ of commutative A-algebras. Moreover, we assume that the induced map $\pi_0 B(n) \to \pi_0 B$ is an isomorphism (if $n \geq 0$ this is automatic; for $n = -1$ it follows from the specific construction given above). We have a fiber sequence of B-modules

$$L_{B(n)/A} \otimes_{B(n)} B \to L_{B/A} \to L_{B/B(n)}$$

By assumption, $L_{B/A}$ is almost perfect. Assertion $(2')$ implies that $L_{B(n)/A} \otimes_{B(n)} B$ is perfect. Using Proposition 5.6, we deduce that the relative cotangent complex $L_{B/B(n)}$ is almost perfect. Moreover, Proposition 7.33 ensures that $L_{B/B(n)}$ is $(n+2)$-connective. It follows that $\pi_{n+2} L_{B/B(n)}$ is a compact module over $\pi_0 B$. Using [HA, Theorem 7.4.3.12] and the isomorphism $\pi_0 B(n) \to \pi_0 B$, we deduce that the canonical map

$$\pi_{n+1}\mathrm{fib}(f_n) \to \pi_{n+2} L_{B/B(n)}$$

is an isomorphism. Choose a compact projective $B(n)$-module M and a map $M[n+1] \to \mathrm{fib}(f_n)$ such that the composition

$$\pi_0 M \simeq \pi_{n+1} M[n+1] \to \pi_{n+1}\mathrm{fib}(f) \simeq \pi_{n+2} L_{B/B(n)}$$

is epimorphic. By construction, we have a commutative diagram of $B(n)$-modules

$$\begin{tikzcd} M[n+1] \arrow[r] \arrow[d] & 0 \arrow[d] \\ B(n) \arrow[r] & B \end{tikzcd}$$

Adjoint to this, we obtain a diagram in $\mathrm{CAlg}(\mathcal{A}_{\geq 0})_{A/}$.

$$\begin{tikzcd} \mathrm{Sym}^*_{B(n)} M[n+1] \arrow[r] \arrow[d] & B(n) \arrow[d] \\ B(n) \arrow[r] & B \end{tikzcd}$$

We now define $B(n+1)$ to be the pushout

$$B(n) \otimes_{\mathrm{Sym}^*_{B(n)} M[n+1]} B(n),$$

and $f_{n+1} : B(n+1) \to B$ to be the induced map. It is clear that $B(n+1)$ is locally of finite presentation over $B(n)$, and therefore locally of finite presentation over A (Remark 5.14). To complete the proof of (2), it will suffice to show that the fiber of f_{n+1} is $(n+2)$-connective.

By construction, we have a commutative diagram

$$\begin{tikzcd} & \pi_0 B(n+1) \arrow[dr, "e''"] & \\ \pi_0 B(n) \arrow[ur, "e'"] \arrow[rr, "e"] & & \pi_0 B \end{tikzcd}$$

where the map e' is epimorphic and e is isomorphic. It follows that e' and e'' are also isomorphic. In view of Corollary 7.34, it will now suffice to show $L_{B/B(n+1)}$ is $(n+3)$-connective. We have a fiber sequence of B-modules

$$L_{B(n+1)/B(n)} \otimes_{B(n+1)} B \to L_{B/B(n)} \to L_{B/B(n+1)}$$

Using [HA, Proposition 7.4.3.14] and Proposition 7.7, we conclude that $L_{B(n+1)/B(n)}$ is canonically equivalent to $M[n+2] \otimes_{B(n)} B(n+1)$. We may therefore rewrite our fiber sequence as

$$M[n+2] \otimes_{B(n)} B \to L_{B/B(n)} \to L_{B/B(n+1)}.$$

The inductive hypothesis and Corollary 7.34 guarantee that $L_{B/B(n)}$ is $(n+2)$-connective. The $(n+3)$-connectiveness of $L_{B/B(n+1)}$ is therefore equivalent to the surjectivity of the map

$$\pi_0 M \simeq \pi_{n+2}\left(M[n+2] \otimes_{B(n)} B\right) \to \pi_{n+2} L_{B/B(n)}$$

which is evident from our construction. This completes the proof of (2).

To complete the proof of (1), we use the same strategy but make a more careful choice of M. Let us assume that $L_{B/A}$ is perfect. It follows from the above construction that each cotangent complex $L_{B/B(n)}$ is likewise perfect. Using Proposition 5.11, we may assume $L_{B/B(-1)}$ is of Tor-amplitude $\leq k+2$ for some $k \geq 0$. Moreover, for each $n \geq 0$ we have a fiber sequence of B-modules

$$L_{B/B(n-1)} \to L_{B/B(n)} \to P[n+2] \otimes_{B(n)} B,$$

where P is compact projective by our construction, and therefore of Tor-amplitude ≤ 0. Using Proposition 5.11 and induction on n, we deduce that the Tor-amplitude of $L_{B/B(n)}$ is $\leq k+2$ for $n \leq k$. In particular, the B-module $\overline{M} = L_{B/B(k)}[-k-2]$ is connective and has Tor-amplitude ≤ 0. It follows from Remark 5.10 that $\overline{M}$ is a flat B-module. Invoking Proposition 5.8, we conclude that $\overline{M}$ is a compact projective B-module. Using Proposition 4.8, we can choose a compact projective $B(k)$-module M and an equivalance $M[k+2] \otimes_{B(k)} B \simeq L_{B/B(k)}$. Using this map in the construction outlined above, we guarantee that the relative cotangent complex $L_{B/B(k+1)}$ vanishes. It follows from Corollary 7.4.3.4 (which also works in our general setting) that the map $f_{k+1} : B(k+1) \to B$ is an equivalence, so that B is locally of finite presentation as an $\mathbb{E}_\infty$-algebra over A, as desired. □

Corollary 7.37. *Suppose that $\mathcal{A}$ is Grothendieck and that $\mathcal{A}^{\otimes}_{\geq 0}$ is projectively rigid. Let $f : A \to B \in \mathrm{CAlg}(\mathcal{A}_{\geq 0})$. Then f is étale if and only if $\tau_{\leq n} f : \tau_{\leq n} A \to \tau_{\leq n} B$ is étale for every $n \geq 0$.*

Proof. It follows immediately by combining Proposition 7.35 and Proposition 7.36. □

Proposition 7.38. *Given a diagram in* CAlg($\mathcal{A}$).

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "h"] & \\ B \arrow[rr, "g"] & & C \end{tikzcd}$$

(1) *If f are étale, then g étale if and only if h is étale ??*

(2) *Suppose that $\mathcal{A}$ is Grothendieck and $\mathcal{A}^{\otimes}_{\geq 0}$ is projectively rigid. If g is étale and faithfully flat and h is étale, then f is étale.??*

Proof.
(1) The "only if" direction follows from Proposition 3.23(1), Remark 5.14 and Proposition 7.6. For the "if" direction ??
(2) By Proposition 3.23 and Lemma 7.15 it suffices to show that f is finitely presented. ?? □

Now we can give the proof of our étale rigidity. Our proof is parallel with the proof of [DAGIV, Theorem 3.4.1].
Proof of Theorem 7.24:
(1) First, using Proposition 3.20(3), we may reduce to the case where A is connective. For each $0 \leq n \leq \infty$, let $\mathcal{C}_n$ denote the full subcategory of $\mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathcal{A}_{\geq 0}))$ spanned by those morphisms $f : B \to B'$ such that B and B' are connective and n-truncated, and let $\mathcal{C}_n^{fl,L\text{-}et}$ denote the full subcategory of $\mathcal{C}_n$ spanned by those morphisms which are also flat and L-étale. Using the left completeness, we deduce that $\mathcal{C}_\infty$ is the homotopy inverse limit of the tower

$$\ldots \to \mathcal{C}_2 \xrightarrow{\tau_{\leq 1}} \mathcal{C}_1 \xrightarrow{\tau_{\leq 0}} \mathcal{C}_0.$$

Using Proposition 7.35, we deduce that $\mathcal{C}_\infty^{fl,L\text{-}et}$ is the homotopy inverse limit of the restricted tower

$$\ldots \to \mathcal{C}_2^{fl,L\text{-}et} \to \mathcal{C}_1^{fl,L\text{-}et} \to \mathcal{C}_0^{fl,L\text{-}et}$$

Choose a Postnikov tower

$$A \to \ldots \to \tau_{\leq 2} A \to \tau_{\leq 1} A \to \tau_{\leq 0} A$$

For $0 \leq n \leq \infty$, let $\mathcal{D}_n$ denote the fiber product $\mathcal{C}_n^{fl,L\text{-}et} \times_{\mathrm{CAlg}(\mathcal{A}_{\geq 0})} \{\tau_{\leq n} A\}$, so that we can identify $\mathcal{D}_n$ with the full subcategory $\mathrm{CAlg}(\mathcal{A}_{\geq 0})^{fl,L\text{-}et}_{\tau_{\leq n} A/} \subset \mathrm{CAlg}(\mathcal{A}_{\geq 0})_{\tau_{\leq n} A/}$ spanned by the flat L-étale morphisms $f : \tau_{\leq n} A \to B$. It follows from Proposition 7.35 that $\mathcal{D}_\infty$ is the homotopy inverse limit of the tower

$$\ldots \to \mathcal{D}_2 \xrightarrow{g_1} \mathcal{D}_1 \xrightarrow{g_0} \mathcal{D}_0$$

We wish to prove that the truncation functor induces an equivalence $\mathcal{D}_\infty \to \mathcal{D}_0$. For this, it will suffice to show that each of the functors g_i is an equivalence. Consequently, it follows from Proposition 7.32.
(2) The proof is totally parallel with (1) by replacing "flat L-étale" with "étale" and replacing "by Proposition 7.35 " with "by Corollary 7.37 ". □

8 Algebraic *ttt*-∞-categories and algebraic transformations

8.1 The universal example

Definition 8.1. We say a ttt-∞-category $(\mathcal{B}^{\otimes}, \mathcal{B}_{\geq 0})$ is algebraic if $\mathcal{B}$ is Grothendieck and $\mathcal{B}^{\otimes}_{\geq 0}$ is projectively rigid. We denote a right t-exact colimit-preserving symmetric monoidal functor $(\mathcal{B}^{\otimes}, \mathcal{B}_{\geq 0}) \to (\mathcal{C}^{\otimes}, \mathcal{C}_{\geq 0})$ between algebraic ttt-∞-categories by an algebraic transformation.

Definition 8.2. Let $\mathcal{V} \in \mathrm{CAlg}(\mathcal{P}\mathrm{r}^L)$ and $\mathrm{CAlg}_{\mathcal{V}}^{\mathrm{rig,at}}$ denote the full subcategory of $\mathrm{CAlg}(\mathcal{P}\mathrm{r}_{\mathcal{V}}^L)$ spanned by rigid and atomically generated commutative $\mathcal{V}$-algebras.

Remark 8.3. By Corollary 1.13, a ttt-∞-category $(\mathcal{B}^{\otimes}, \mathcal{B}_{\geq 0})$ can be recovered from the $\mathrm{Sp}_{\geq 0}$-atomically generated rigid commutative algebra $\mathcal{B}_{\geq 0}^{\otimes}$. So there is an equivalence from the ∞-category of algebraic ttt-∞-categories

$$\mathrm{CAlg}^{\mathrm{alg}}(\mathcal{P}\mathrm{r}_{\mathrm{st}}^{t\text{-}rex}) \xrightarrow{\simeq} \mathrm{CAlg}_{\mathrm{Sp}_{\geq 0}}^{\mathrm{rig,at}}$$

given by $(\mathcal{B}^{\otimes}, \mathcal{B}_{\geq 0}) \mapsto \mathcal{B}_{\geq 0}^{\otimes}$. And the inverse is given by $\mathcal{C}^{\otimes} \mapsto (\mathrm{Sp}(\mathcal{C})^{\otimes}, \mathrm{Sp}(\mathcal{C})_{\geqslant 0})$, see Corollary 1.13.

Remark 8.4. In fact, the $\mathrm{CAlg}_{\mathcal{V}}^{\mathrm{rig,at}}$ is presentable for any $\mathcal{V} \in \mathrm{CAlg}(\mathcal{P}\mathrm{r}^L)$. To see that, firstly we have that $\mathrm{CAlg}_{\mathcal{V}}^{\mathrm{rig,at}} = \mathrm{CAlg}_{\mathcal{V}}^{\mathrm{rig}} \times_{\mathcal{P}\mathrm{r}_{\mathcal{V}}^{\mathrm{dbl}}} \mathcal{P}\mathrm{r}_{\mathcal{V}}^{\mathrm{at}}$ is accessible by combining [Ram24a, Corollary 3.15] and [Ram24b, Corollary 5.13, 5.14]. Then the presentability follows from that $\mathrm{CAlg}_{\mathcal{V}}^{\mathrm{rig,at}}$ admits small colimits, which is obtained by observing the inclusion $\mathrm{CAlg}_{\mathcal{V}}^{\mathrm{rig,at}} \subset \mathrm{CAlg}_{\mathcal{V}}$ is closed under small colimits.

Alternatively, we will give a more straightforward proof of the presentability of $\mathrm{CAlg}_{\mathcal{V}}^{\mathrm{rig,at}}$ in the case $\mathcal{V}^{\otimes} = \mathrm{Sp}_{\geq 0}^{\otimes}$, and even further give a compact generator which is linked to cobordism hypothesis. Before that, let us introduce a lemma, which we learned from Germán Stefanich.

Lemma 8.5. *Let $\mathrm{Cat}_{\infty,\mathrm{ad}}^{\times}$ denote the ∞-category of small additive ∞-categories with finite product preserving functors. Then the core functor $(-)^{\simeq} : \mathrm{Cat}_{\infty,\mathrm{ad}}^{\times} \to \mathcal{S}$ is conservative.*

Proof. □

Remark 8.6. Note that the additive condition in the above lemma can not be weakened to the semi-additive, otherwise the endmorphism $\left(\begin{smallmatrix} 1 & f \\ 0 & 1 \end{smallmatrix}\right)$ is not necessarily an automorphism.

Definition 8.7. We say a symmetric monoidal ∞-category is small rigid if it is small and every object in it is dualizable.

Proposition 8.8. *We have a natural equivalence*

$$\mathrm{CAlg}_{\mathrm{Sp}_{\geq 0}}^{\mathrm{rig,at}} \xrightarrow{\simeq} \mathrm{CAlg}^{\mathrm{rig}}(\mathrm{Cat}_{\infty,\mathrm{ad}}^{\mathrm{idem}})$$

given by $\mathcal{C}^{\otimes} \mapsto (\mathcal{C}^{cproj})^{\otimes}$, where $\mathrm{CAlg}^{\mathrm{rig}}(\mathrm{Cat}_{\infty,\mathrm{ad}}^{\mathrm{idem}})$ denotes the full subcategory of $\mathrm{CAlg}(\mathrm{Cat}_{\infty,\mathrm{ad}}^{\mathrm{idem}})$ spanned by small rigid idempotent-complete additively symmetric monoidal ∞-categories.

Proof. □

Now let us recall the (1-dimensional) cobordism hypothesis, which was originally formulated in [BD95] and was proved by Hopkins–Lurie in [Lur08].

Theorem 8.9 ([Lur08] Cobordism hypothesis of dimension 1)**.** *Let $\mathbf{Cob}_1^{\otimes}$ denote the oriented 1-dimensional cobordism $(\infty,1)$-category with the symmetric monoidal structure given by disjoint union. Then $\mathbf{Cob}_1^{\otimes}$ is small rigid and satisfies the following universal property:*
Let $\mathcal{C}$ be a symmetric monoidal $(\infty,1)$-category. Then the evaluation functor $Z \mapsto Z(*)$ induces an equivalence of ∞-categories

$$\mathrm{Fun}^{\otimes}(\mathbf{Cob}_1, \mathcal{C}) \to (\mathcal{C}^d)^{\simeq}$$

where $\mathrm{Fun}^{\otimes}$ denotes the ∞-category of symmetric monoidal functors.

Remark 8.10. Note that the 1-dimensional oriented and framed cobordism ∞-categories are equivalent $\mathbf{Cob}_1 \simeq \mathbf{Bord}_1^{\mathrm{fr}}$ [see Lur08, §4.2], but that does not hold in higher dimensional cases.

Now let us prove the main theorem of this section.

Theorem 8.11 (Universal example)**.** *The* $\mathrm{CAlg}^{\mathrm{rig,at}}_{\mathrm{Sp}_{\geq 0}}$ *is compactly generated by a single element*

$$\mathrm{Fun}(\mathbf{Cob}_1^{\mathrm{op}}, \mathrm{Sp}_{\geq 0})^{\otimes} \in \mathrm{CAlg}^{\mathrm{rig,at}}_{\mathrm{Sp}_{\geq 0}}$$

where the symmetric monoidal structure on $\mathrm{Fun}(\mathbf{Cob}_1^{\mathrm{op}}, \mathrm{Sp}_{\geq 0})$ *is given by Day convolution.*

Proof. □

Proposition 8.12. *Let* $\mathcal{I}^{\otimes}$ *be a small rigid symmetric monoidal* ∞*-category which admits finite coproducts and whose tensor product is compatible with finite coproducts. Then the natural symmetric monoidal functor*

$$\mathrm{Fun}(\mathcal{I}^{\mathrm{op}}, \mathrm{Sp}_{\geq 0})^{\otimes} \xrightarrow{L^{\otimes}} \mathrm{Fun}^{\times}(\mathcal{I}^{\mathrm{op}}, \mathrm{Sp}_{\geq 0})^{\otimes} \simeq \mathcal{P}_{\Sigma}(\mathcal{I})^{\otimes}$$

induced by the universal property of Yoneda embedding is a smashing localization, that is, there exists an idempotent commutative algebra $R \in \mathrm{CAlg}(\mathrm{Fun}(\mathcal{I}^{\mathrm{op}}, \mathrm{Sp}_{\geq 0}))$ *such that* $L(-) \simeq R \otimes (-)$*.*

Proof. By Lemma 4.26, it only suffices to show that the inclusion

$$\mathrm{Fun}^{\times}(\mathcal{I}^{\mathrm{op}}, \mathrm{Sp}_{\geq 0}) \subset \mathrm{Fun}(\mathcal{I}^{\mathrm{op}}, \mathrm{Sp}_{\geq 0})$$

is closed under small colimits. That is obvious because both sides are additive and the inclusion is closed under finite products and sifted colimits. □

8.2 Algebraic transformations

Proposition 8.13. *Let* $F : (\mathcal{B}^{\otimes}, \mathcal{B}_{\geq 0}) \to (\mathcal{C}^{\otimes}, \mathcal{C}_{\geq 0})$ *be an algebraic transformation between algebraic ttt-*∞*-categories, and let* $R \in \mathrm{Alg}(\mathcal{B}_{\geq 0})$*. Then the functor* $\mathrm{LMod}_R(\mathcal{B}_{\geq 0}) \to \mathrm{LMod}_{F(R)}(\mathcal{C}_{\geq 0})$ *preserves*

(1) *compact projectives;*

(2) *compacts;*

(3) *projectives;*

(4) *flats;*

(5) *almost perfects.*

Proof. □

Proposition 8.14. *Let* $F : (\mathcal{B}^{\otimes}, \mathcal{B}_{\geq 0}) \to (\mathcal{C}^{\otimes}, \mathcal{C}_{\geq 0})$ *be an algebraic transformation between algebraic ttt-*∞*-categories, and let* $R \in \mathrm{CAlg}(\mathcal{B}_{\geq 0})$*. Then the functor* $\mathrm{CAlg}(\mathcal{B}_{\geq 0})_{R/} \to \mathrm{CAlg}(\mathcal{C}_{\geq 0})_{F(R)/}$ *preserves*

(1) *finitely presented;*

(2) *almost finitely presented;*

(3) *flats;*

(4) *L-étale;*

(5) ∞*-epimorphisms.*

Proof. □

8.3 Synthetic objects

Throughout Section 8.3 we assume that the ttt-∞-category $(\mathcal{A}^{\otimes}, \mathcal{A}_{\geq 0})$ is algebraic.

Definition 8.15. Let $\mathcal{A}^{fp} \subset \mathcal{A}$ denotes the smallest full subcategory that contains all compact projectives and is closed under finite direct sums, shifts and retractions. We call an object in $\mathcal{A}^{fp}$ a graded finitely projective. We define $\mathrm{Syn}(\mathcal{A}) \overset{\text{def}}{=} \mathcal{P}_{\Sigma}(\mathcal{A}^{fp}; \mathrm{Sp})$ as the stable ∞-category of synthetic objects in $\mathcal{A}$. It admits a natural t-structure given by $\mathrm{Syn}(\mathcal{A})_{\geq 0} = \mathcal{P}_{\Sigma}(\mathcal{A}^{fp})$.

Proposition 8.16. *The $\mathcal{A}^{fp} \subset \mathcal{A}$ is closed under tensor products and inherits a small rigid additively symmetric monoidal ∞-category. Hence $\mathcal{P}_{\Sigma}(\mathcal{A}^{fp})$ is Grothendieck prestable and inherits a projectively rigid symmetric monoidal structure.*

We let $(\mathrm{Syn}(\mathcal{A})^{\otimes}, \mathrm{Syn}(\mathcal{A})_{\geq 0})$ denote the associated algebraic ttt-∞-category of $\mathcal{P}_{\Sigma}(\mathcal{A}^{fp})^{\otimes}$.

Proposition 8.17. *There exists a morphism in $\mathrm{CAlg}_{\mathrm{Sp}}^{\mathrm{rig,at}}$:*

$$\mathrm{Syn}(\mathcal{A})^{\otimes} \to \mathcal{A}^{\otimes}$$

which is a symmetric monoidal localization.

Remark 8.18. Beware that this is not an algebraic transformation.

8.4 Examples

Definition 8.19. Let $\mathbb{Z}^{\otimes}$ be the symmetric monoidal discrete category given by addition law. We define the symmetric monoidal ∞-category of graded spectra as $\mathrm{Fun}(\mathbb{Z}, \mathrm{Sp})^{\otimes}$.

Example 8.20. Examples of algebraic *ttt*-∞-categories.

(1) The $\mathrm{Fun}(\mathcal{I}^{\mathrm{op}}, \mathrm{Sp})^{\otimes}$, where $\mathcal{I}^{\otimes}$ is a small rigid symmetric monoidal ∞-category, like
the $\mathrm{Fil}(\mathrm{Sp})^{\otimes} = \mathrm{Fun}(\mathbb{Z}, \mathrm{Sp})^{\otimes}$ the ∞-category of filtered spectra;
the $\mathrm{Gr}(\mathrm{Sp})^{\otimes} = \mathrm{Fun}(\mathbb{Z}^{disc}, \mathrm{Sp})^{\otimes}$ the ∞-category of graded spectra .

(2) the $\mathrm{Sp}(\mathcal{P}_{\Sigma}(\mathcal{I}))^{\otimes}$, where $\mathcal{I}^{\otimes}$ is a small rigid finite-coproduct cocompletely symmetric monoidal ∞-category, like $\mathrm{Sp}_G^{\otimes} = \mathrm{Sp}(\mathcal{P}_{\Sigma}(\mathrm{Fin}_G))^{\otimes}$ the genuine G-spectra over a finite group G.
As Proposition 8.8 indicates, actually every algebraic *ttt*-∞-category comes from this way.

(3) Universal example in $\mathrm{CAlg}_{\mathrm{Sp}_{\geq 0}}^{rig,at}$: the 1-dim cobordism $\mathrm{Fun}(\mathbf{Cob}_1^{\mathrm{op}}, \mathrm{Sp})^{\otimes}$

(4) The $\mathrm{Sh}_{\Sigma}(\mathcal{C})^{\otimes}$ where $\mathcal{C}$ is an excellent ∞-site, see [Pst23]. For example the synthetic spectra $\mathrm{Syn}_E^{\otimes}$?? (need certain conditions on E)

(5) The ∞-category $\mathrm{Shv}(X, \mathrm{Sp})^{\otimes}$ of sheaves on a stone space??

(6) The ∞-category $\mathrm{Shv}(\mathcal{X}, \mathrm{Sp})^{\otimes}$ of sheaves on an ∞-topos of locally homotopy dim=0??

(7) The ∞-category $\mathrm{SH}(k)_{\geq 0}^{A-T}$ of connective Artin-Tate motivic spectra over a perfect field k, see [BHS20].

(8) Cyclotomic spectra and Cartier modules [AN21]??

(9) [HP23][HP24], equivariant [Bar17], motivic [Bac+22] [BHS20], Beilinson t, Ban, condensed, Liquid, [Lur15]

(10) $\mathrm{Qcoh}(X)_{\geq 0}$, where X is an affine quotient stack, i.e. a stack of the form $\mathrm{Spec}(R)/G$ for a linearly reductive group G acting on $\mathrm{Spec}(R)$, this works: the compact projective objects are generated under taking retracts by pullbacks of G-representations and the dual is given by the pullback of the dual in this case.

(11) Voevodsky's category $\mathrm{DM}(k, \mathbb{Z}[1/p])$ (where p is characteristic of k or 1 if k is a $\mathbb{Q}$-algebra), then there is a Chow t-structure on it, generated by smooth projective varieties and their $\mathbb{P}^1$-desuspensions. The mapping spectra between smooth projective varieties are connective, so they are compact projective generators, and they are also dualizable within the retract closed-subcategory generated by it.

Example 8.21. projectively generated but not projectively rigid

(1) The $\mathrm{Cond}(\mathrm{Sp})^{\otimes} = \mathrm{Sp}(\mathcal{P}_{\Sigma}(\mathbf{Stonean}))^{\otimes}$???

(2) The $\mathrm{Solid}(\mathrm{Sp})^{\otimes}$??

(3) The $\mathrm{Fun}(X, \mathrm{Sp})^{\otimes}$ the ∞-category of parametrized spectra on a small (??) ∞-groupoid X, with pointwise tensor product;

Example 8.22. Non-examples of projective generation:

(1) Let X be a projective variety over a field k with $\dim(X) > 0$, then the only projective object in the category $\mathrm{QCoh}(X)^{\heartsuit}$ of (discrete) quasi-coherent $\mathcal{O}_X$-modules is the zero object, see `Projectives in the category of quasi coherent sheaves Qch(X)`. That implies $\mathrm{QCoh}(X)_{\geq 0}$ is not projectively generated.

(2) The profinite equivariant $\mathrm{Sp}_G^{\otimes}$??

9 Applications and questions

9.1 Questions

(i) It is convenient to define flatness using t-exact in an abstract setting, but in practice many examples that people are interested in are flat in the abelian category i.e. π_0-flatness, so it's useful to have a comparison.

Consider that $M \in \mathrm{LMod}_R(\mathcal{A}_{\geq 0})$, $\pi_0(M) \in \mathrm{LMod}_{\pi_0 R}(\mathcal{A}^{\heartsuit})$, which we call $\pi_0(M)$ is flat in the following sense: [Ste23, Definition 2.2.20.]

Definition 9.1. We call $M \in \mathrm{LMod}_R(\mathcal{A}_{\geq 0})$ flat if it is π_0 -flat and periodic, i.e. $\pi_* M \simeq \pi_* A \otimes_{\pi_0 A} \pi_0 M$

But to show that it is equivalent to the original definition of flatness, we need to show that $- \otimes M$ is t-exact, it is right t-exact automatically because we assume the connective part closed under the tensor product, to see it is left t-exact, we need compute the homotopy group $\pi_i(M \otimes N)$, in HA this part is done by using Tor spectral sequence, we could do something similar here, the basic idea is using the truncation on connective part to construct the filtration, then deduce the useful spectral sequence [To-Do]

The following questions are what we did not figure out.

Question 9.2. Assume that $\mathcal{A}$ is Grothendieck and that $\mathcal{A}_{\geq 0}^{\otimes}$ is projectively rigid.

(1) If $f : A \to B \in \mathrm{CAlg}(\mathcal{A}_{\geq 0})$ is finite presented and L-étale, then f is flat? i.e. Is the flatness in the definition of "étale" removable?

(2) Given a diagram in $\mathrm{CAlg}(\mathcal{A}_{\geq 0})$

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "h"] & \\ B \arrow[rr, "g"] & & C \end{tikzcd}$$

If f, h are étale, then so is g? (This is true if (1) holds.)
If g is étale and faithfully flat, then f is étale if and only if h is étale?
If g is étale, then there exists a finitely presented commutative A-algebra B_0 and an étale map $B_0 \to C_0$ such that $C \simeq C_0 \otimes_{B_0} A$?

(3) If M is a flat left R-module, then M is faithfully flat implies that the tensor product functor $(-) \otimes_R M$ is conservative. Does The converse hold??(probably wrong)

(4) If $\mathcal{A}$ is ??, then flat is equivalent to that for any $n \in \mathbb{Z}$, we have $\pi_n(R) \otimes_{\pi_0 R} \pi_0 M \to \pi_n M$ is a natural equivalence. (A sufficient condition is that for any compact projective left R-module P we have that P_* is a projective left P_*-module).

(5) The P_* is projective as an object in $\mathrm{Mod}_{R_*}(\mathrm{Gr}^{anti}(\mathcal{A}^\heartsuit))$? where $P \in \mathrm{Mod}(\mathcal{A}_{\geq 0})^{cproj}$.

(6) The $\pi_0 \operatorname{Sym}^*(P)$ is projective?(wrong! considering dirac) where $P \in \mathcal{A}_{\geq 0}^{cproj}$.

(7) Let $f : A \to B \in \mathrm{CAlg}(\mathcal{A})$ be a flat morphism in the sense of Definition 3.18. Then f is L-étale if and only if $\tau_{\geq 0} f : \tau_{\geq 0} A \to \tau_{\geq 0} B$ is L-étale? (seems wrong, considering $ku \to KU$).

Question 9.3. Assume that $\mathcal{A}$ is Grothendieck and hypercomplete.

(1) Given a pushout diagram in $\mathrm{CAlg}(\mathcal{A}_{\geq 0})$.

$$\begin{tikzcd} \tilde{A} \arrow[r, "\alpha"] \arrow[d, "f_0'"] & A \arrow[d, "f_0"] \\ \tilde{B} \arrow[r] & B \end{tikzcd}$$

If f_0' is L-étale and flat, then f_0' is almost of finite presentation if and only if f_0 is almost of finite presentation?

(2) Let $\tilde{A} \to A$ be a nilpotent thickening in $\mathrm{CAlg}(\mathcal{A}_{\geq 0})$. Then the tensor product functor restricting on bounded below modules

$$\mathrm{Mod}_{\tilde{A}}(\mathcal{A})^- \to \mathrm{Mod}_A(\mathcal{A})^-.$$

reflects compacts?

Question 9.4. Let $\mathbf{A}^\otimes$ be a symmetric monoidal Grothendieck abelian category and $f : A \to B \in \mathrm{CAlg}(\mathbf{A})$.

(1) If $\mathrm{Coker}(f)$ is a flat A-module, then f is faithfully flat? (related with existence of monoidal prestable enhancement of $\mathbf{A}^\otimes$)

A Duality

A.1 Dualizable objects

We recollect some basic properties of dualizable objects in monoidal ∞-categories. Throughout Appendix A.1, we fix a symmetric monoidal ∞-category $\mathcal{C}^\otimes \to \mathrm{Comm}^\otimes$.

Definition A.1. We say an object $X \in \mathcal{C}$ is dualizable if there exists an object $X^\vee$ and a pair of morphisms

$$c : \mathbf{1} \to X \otimes X^\vee \quad e : X^\vee \otimes X \to \mathbf{1}$$

where $\mathbf{1}$ denotes the unit object of $\mathcal{C}$. These morphisms are required to satisfy the following conditions: The composite maps

$$X \xrightarrow{c\otimes \mathrm{id}} X \otimes X^\vee \otimes X \xrightarrow{\mathrm{id}\otimes e} X$$

$$X^\vee \xrightarrow{\mathrm{id}\otimes c} X^\vee \otimes X \otimes X^\vee \xrightarrow{e\otimes \mathrm{id}} X^\vee$$

are homotopic to the identity on X and $X^\vee$, respectively.

Definition A.2. We say an object $X \in \mathcal{C}$ is cotensorable if the tensor product functor $(-) \otimes X : \mathcal{C} \to \mathcal{C}$ admits a right adjoint. If so, we denote this right adjoint by $\underline{\mathrm{Map}}_{\mathcal{C}}(X, -)$.

Remark A.3. If $\mathcal{C}^{\otimes}$ is a presentably symmetric monoidal ∞-category, then any object in it is cotensorable.

Proposition A.4. *Let $X \in \mathcal{C}$ be an object. Then X is dualizable if and only if X is cotensorable and for any $Y \in \mathcal{C}$, the natural map $\underline{\mathrm{Map}}_{\mathcal{C}}(X, \mathbf{1}) \otimes Y \to \underline{\mathrm{Map}}_{\mathcal{C}}(X, Y)$ is an equivalence in $\mathcal{C}$.*

Proof. Assume that X is dualizable. Then X is cotensorable since we have $\underline{\mathrm{Map}}_{\mathcal{C}}(X, -) \simeq X^{\vee} \otimes (-)$. Particularly, $\underline{\mathrm{Map}}_{\mathcal{C}}(X, \mathbf{1}) \simeq X^{\vee}$. Now let $Y \in \mathcal{C}$. We wish to show that the composite map

$$\phi : \mathrm{Map}_{\mathcal{C}}\left(-, Y \otimes X^{\vee}\right) \to \mathrm{Map}_{\mathcal{C}}\left(- \otimes X, Y \otimes X^{\vee} \otimes X\right) \xrightarrow{e} \mathrm{Map}_{\mathcal{C}}(- \otimes X, Y)$$

is a homotopy equivalence. Let ψ denote the composition

$$\mathrm{Map}_{\mathcal{C}}(- \otimes X, Y) \to \mathrm{Map}_{\mathcal{C}}\left(- \otimes X \otimes X^{\vee}, Y \otimes X^{\vee}\right) \xrightarrow{c} \mathrm{Map}_{\mathcal{C}}\left(-, Y \otimes X^{\vee}\right).$$

Using the compatibility of e and c, we deduce that ϕ and ψ are homotopy inverses to one another. By the Yoneda lemma, ϕ can be identified with the map $\underline{\mathrm{Map}}_{\mathcal{C}}(X, \mathbf{1}) \otimes Y \to \underline{\mathrm{Map}}_{\mathcal{C}}(X, Y)$.

Assume that X is cotensorable and that for any $Y \in \mathcal{C}$, the natural map

$$\underline{\mathrm{Map}}_{\mathcal{C}}(X, \mathbf{1}) \otimes Y \to \underline{\mathrm{Map}}_{\mathcal{C}}(X, Y)$$

is an equivalence in $\mathcal{C}$. Particularly, we have an equivalence $\underline{\mathrm{Map}}_{\mathcal{C}}(X, \mathbf{1}) \otimes X \xrightarrow{\sim} \underline{\mathrm{Map}}_{\mathcal{C}}(X, X)$. Let $c : \mathbf{1} \to \underline{\mathrm{Map}}_{\mathcal{C}}(X, \mathbf{1}) \otimes X$ be the inverse image of the identity map $\mathrm{id} : \underline{\mathrm{Map}}_{\mathcal{C}}(X, X) \to \underline{\mathrm{Map}}_{\mathcal{C}}(X, X)$. Then it is straightforward to check that the counit $e : \underline{\mathrm{Map}}_{\mathcal{C}}(X, \mathbf{1}) \otimes X \to \mathbf{1}$ and $c : \mathbf{1} \to \underline{\mathrm{Map}}_{\mathcal{C}}(X, \mathbf{1}) \otimes X$ form a duality datum. $\square$

Proposition A.5. *Let $\mathcal{C}^d \subset \mathcal{C}$ be the full subcategory consisting of dualizable objects. Then the profunctor $\mathcal{C}^d \times \mathcal{C}^d \to \mathcal{S}$ given by* $\mathrm{Map}_{\mathcal{C}}(\mathbf{1}, - \otimes -)$ *is a balanced profunctor* (see [Ker, 03MM])*, which induces a natural equivalence of ∞-categories $(-)^{\vee} =: \underline{\mathrm{Map}}_{\mathcal{C}}(-, \mathbf{1}) : (\mathcal{C}^d)^{\mathrm{op}} \xrightarrow{\sim} \mathcal{C}^d$. Furthermore, $(-)^{\vee\vee} \simeq \mathrm{Id}_{\mathcal{C}^d}$ is equivalent to the identity functor.*

Proof. It suffices to observe that if $c : \mathbf{1} \to X \otimes Y$ is part of a duality datum for X, then it is also part of a duality datum for Y. $\square$

Remark A.6. In fact, this perfect pairing can be enhanced to a symmetric monoidal perfect pairing and hence induces an equivalence of symmetric monoidal ∞-categories $(-)^{\vee} =: \underline{\mathrm{Map}}_{\mathcal{C}}(-, \mathbf{1}) : (\mathcal{C}_d^{\mathrm{op}})^{\otimes} \xrightarrow{\sim} (\mathcal{C}^d)^{\otimes}$, see [ECI, Proposition 3.2.4].

Proposition A.7. *The full subcategory $\mathcal{C}^d \subset \mathcal{C}$ is closed under tensor product, hence it forms a symmetric monoidal full subcategory.*

Proof. Let $X, Y \in \mathcal{C}^d$. Choosing $c = c_X \otimes c_Y : \mathbf{1} \simeq \mathbf{1} \otimes \mathbf{1} \to (X \otimes X^{\vee}) \otimes (Y \otimes Y^{\vee}) \simeq (X \otimes Y) \otimes (Y^{\vee} \otimes X^{\vee})$, we see that c exhibits $Y^{\vee} \otimes X^{\vee}$ as a dual of $X \otimes Y$. $\square$

Definition A.8. Let $\mathcal{C}^{cot} \subset \mathcal{C}$ be the full subcategory consisting of cotensorable objects. We define the functor

$$\underline{\mathrm{Map}}_{\mathcal{C}}(-, -) : (\mathcal{C}^{cot})^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Fun}'(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \simeq \mathcal{C}$$

given by $(X, Y) \mapsto \mathrm{Map}_{\mathcal{C}}(- \otimes X, Y)$, where $\mathrm{Fun}'(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \simeq \mathcal{C}$ denotes the full subcategory of representable functors.

Lemma A.9. *Let $\mathcal{K}$ be a collection of simplicial sets. If $\mathcal{C}$ is $\mathcal{K}$-cocomplete and the monoidal structure on it is compatible with K-colimits for any $K \in \mathcal{K}$ (meaning the $- \otimes -$ preserves K-colimits separately), then for any $K \in \mathcal{K}$, the full subcategory $\mathcal{C}^{cot} \subset \mathcal{C}$ is closed under K-colimits and for any diagram $X_{(-)} : K \to \mathcal{C}^{cot}$, the natural map $\varprojlim_{\alpha \in K} \underline{\mathrm{Map}}_{\mathcal{C}}(X_{\alpha}, -) \simeq \underline{\mathrm{Map}}_{\mathcal{C}}(\varinjlim_{\alpha \in K} X_{\alpha}, -)$ is an equivalence in* $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$.

Proof. Consider the following diagram:

$$\begin{tikzcd}
(\mathcal{C}^{cot})^{\mathrm{op}} \arrow[r] \arrow[dr, dashed, "X\mapsto \underline{\mathrm{Map}}_{\mathcal{C}}(X,-)"'] & \mathcal{C}^{\mathrm{op}} \arrow[r, "X\mapsto(-)\otimes X"] \arrow[dr, "\phi"] & \mathrm{Fun}(\mathcal{C},\mathcal{C})^{\mathrm{op}} \arrow[d, hook, "i_L"] \\
& \mathrm{Fun}(\mathcal{C},\mathcal{C}) \arrow[r, hook, "i_R"] & \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}\times\mathcal{C},\mathcal{S})
\end{tikzcd}$$

where i_L is given by $F \mapsto \mathrm{Map}_{\mathcal{C}}(F(-),-)$ and i_R is given by $G \mapsto \mathrm{Map}_{\mathcal{C}}(-,G(-))$. An object $X \in \mathcal{C}$ is cotensorable if and only if $\phi(X)$ lies in the image of i_R. Now given a diagram $K \in \mathcal{K}$, it suffices to observe that:

(i) $(\mathcal{C}^{cot})^{\mathrm{op}} = \phi^{-1}(\mathrm{Im}(i_R))$.

(ii) ϕ preserves K^{op}-limits and the inclusion i_R is closed under K^{op}-limits.

□

Proposition A.10.

(1) *If $\mathcal{C}^{\otimes}$ is pointedly symmetric monoidal (meaning that $\mathcal{C}$ is pointed and the tensor product of the zero object with any object is zero), then the zero object $*$ is dualizable.*

(2) *If $\mathcal{C}$ is idempotent complete, then $\mathcal{C}^d \subset \mathcal{C}$ is closed under retractions.*

(3) *If $\mathcal{C}^{\otimes}$ is semiadditively symmetric monoidal, then $\mathcal{C}^d \subset \mathcal{C}$ is closed under finite coproducts and hence forms a full semiadditive subcategory.*

(4) *If $\mathcal{C}^{\otimes}$ is stably symmetric monoidal, then $\mathcal{C}^d \subset \mathcal{C}$ is closed under finite colimits and finite limits and hence forms a full stable subcategory.*

Proof. Applying Proposition A.4 and Lemma A.9 to $\mathcal{K} = \{\varnothing\}, \{\mathrm{N}(\mathrm{Idem})\}, \{\text{finite discrete diagrams}\}, \{\text{finite diagrams}\}$ respectively, we proved (1), (2), (3) and the "closed under finite colimits" part of (4). For the "closed under finite limits" part of (4), it suffices to show that $\mathcal{C}^d \subset \mathcal{C}$ is closed under desuspension. This follows from $\Sigma^{-1}X = (\Sigma X)^{\vee}$ for a dualizable object $X \in \mathcal{C}^d$. □

A.2 Duality of Bimodules

Throughout Appendix A.2, we fix a monoidal ∞-category $\mathcal{C}^{\otimes} \to \mathrm{Ass}^{\otimes}$ which admits geometric realizations of simplicial objects and such that the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations of simplicial objects.

Definition A.11. Let $X \in {}_A\mathrm{BMod}_B(\mathcal{C})$ and $Y \in {}_B\mathrm{BMod}_A(\mathcal{C})$. Let $c : B \to Y \otimes_A X$ be a map in ${}_B\mathrm{BMod}_B(\mathcal{C})$. We say c exhibits X as the right dual of Y, or c exhibits Y as the left dual of X, if there exists a map $e : X \otimes_B Y \to A$ in ${}_A\mathrm{BMod}_A(\mathcal{C})$ such that

$$X \simeq X \otimes_B B \xrightarrow{\mathrm{id}\otimes c} X \otimes_B Y \otimes_A X \xrightarrow{e\otimes\mathrm{id}} A \otimes_A X \simeq X$$
$$Y \simeq B \otimes_B Y \xrightarrow{c\otimes\mathrm{id}} Y \otimes_A X \otimes_B Y \xrightarrow{\mathrm{id}\otimes e} Y \otimes_A A \simeq Y$$

are homotopic to id_X and id_Y, respectively.

Proposition A.12 (See [HA] 4.6.2.18)**.** *Let $A \in \mathrm{Alg}(\mathcal{C})$, let $X \in \mathrm{LMod}_A(\mathcal{C})$, let $Y \in \mathrm{RMod}_A(\mathcal{C})$, and let $c : \mathbf{1} \to Y \otimes_A X$ be a map in $\mathcal{C}$. Then the following are equivalent:*

(1) *The map $c : \mathbf{1} \to Y \otimes_A X$ exhibits Y as a left dual of X.*

(2) *For each $C \in \mathcal{C}$ and each $M \in \mathrm{RMod}_A(\mathcal{C})$, the composite map*

$$\mathrm{Map}_{\mathrm{RMod}_A(\mathcal{C})}(C \otimes Y, M) \to \mathrm{Map}_{\mathcal{C}}(C \otimes Y \otimes_A X, M \otimes_A X) \xrightarrow{\circ c} \mathrm{Map}_{\mathcal{C}}(C, M \otimes_A X)$$

is a homotopy equivalence.

(3) *For each $C \in \mathcal{C}$ and each $N \in \mathrm{LMod}_A(\mathcal{C})$, the composite map*

$$\mathrm{Map}_{\mathrm{LMod}_A(\mathcal{C})}(X \otimes C, N) \to \mathrm{Map}_{\mathcal{C}}(Y \otimes_A X \otimes C, Y \otimes_A N) \xrightarrow{\circ c} \mathrm{Map}_{\mathcal{C}}(C, Y \otimes_A N)$$

is a homotopy equivalence.

Corollary A.13. *Let $A \in \mathrm{Alg}(\mathcal{C})$. Let $\mathrm{LMod}_A(\mathcal{C})^{ld} \subset \mathrm{LMod}_A(\mathcal{C})$ denote the full subcategory of left dualizable left A-modules, and let $\mathrm{RMod}_A(\mathcal{C})^{rd} \subset \mathrm{RMod}_A(\mathcal{C})$ denote the full subcategory of right dualizable right A-modules. Then the profunctor $\mathrm{RMod}_A(\mathcal{C})^{rd} \times \mathrm{LMod}_A(\mathcal{C})^{ld} \to \mathcal{S}$ given by $\mathrm{Map}_{\mathcal{C}}(\mathbf{1}, - \otimes_A -)$ is a balanced profunctor* (see [Ker, 03MM])*, which induces a natural equivalence of ∞-categories*

$$^{\vee}(-) : \mathrm{LMod}_A(\mathcal{C})^{ld} \overset{\sim}{\rightleftarrows} (\mathrm{RMod}_A(\mathcal{C})^{rd})^{\mathrm{op}} : (-)^{\vee}.$$

Proof. It suffices to observe that $c : \mathbf{1} \to Y \otimes_A X$ exhibits Y as a left dual of X if and only if it exhibits X as a right dual of Y. ☐

Corollary A.14. *Suppose that $\mathcal{C}^{\otimes}$ is a cocompletely symmetric monoidal (potentially large) ∞-category, i.e., $\mathcal{C}$ admits small colimits and the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately. Let $A \in \mathrm{Alg}(\mathcal{C})$, let $X \in \mathrm{LMod}_A(\mathcal{C})$, let $Y \in \mathrm{RMod}_A(\mathcal{C})$, and let $c : \mathbf{1} \to Y \otimes_A X$ be a map in $\mathcal{C}$. If $\mathcal{C}$ is generated by dualizable objects under small colimits, then the following are equivalent:*

(1) *The map $c : \mathbf{1} \to Y \otimes_A X$ in $\mathcal{C}$ exhibits Y as a left dual of X.*

(2) *The functor*

$$\mathrm{Map}_{\mathcal{C}}(\mathbf{1}, Y \otimes_A -) : \mathrm{LMod}_A(\mathcal{C}) \to \widehat{\mathcal{S}}$$

is corepresented by X with the element $c : \mathbf{1} \to Y \otimes_A X$.

(3) *The functor*

$$\mathrm{Map}_{\mathcal{C}}(\mathbf{1}, - \otimes_A X) : \mathrm{RMod}_A(\mathcal{C}) \to \widehat{\mathcal{S}}$$

is corepresented by Y with the element $c : \mathbf{1} \to Y \otimes_A X$.

(Note that we use the notation $\widehat{\mathcal{S}}$ above because $\mathcal{C}$ is not necessarily small here.)

B Ind(Pro)-completion of large ∞-categories

We fix three uncountable strongly inaccessible cardinals $\delta_0 < \delta_1 < \delta_2$. A set S is then defined as small if $S \in \mathcal{U}(\delta_0)$, large if $S \in \mathcal{U}(\delta_1)$, and very large if $S \in \mathcal{U}(\delta_2)$, where $\mathcal{U}(\delta_i)$ denotes the corresponding Grothendieck universe.

Definition B.1. Let $\widehat{\mathrm{Cat}}_\infty$ denote the very large ∞-category of large ∞-categories.

Theorem B.2 (See [HTT] 5.3.6.10)**.** *Let $\mathcal{K} \subseteq \mathcal{K}'$ be δ_1-small collections of simplicial sets. Let $\widehat{\mathrm{Cat}}_\infty^{\mathcal{K}}$ denote the subcategory spanned by those ∞-categories which admit $\mathcal{K}$-indexed colimits and those functors which preserve $\mathcal{K}$-indexed colimits, and let $\widehat{\mathrm{Cat}}_\infty^{\mathcal{K}'}$ be defined likewise. Then the inclusion*

$$\widehat{\mathrm{Cat}}_\infty^{\mathcal{K}'} \subseteq \widehat{\mathrm{Cat}}_\infty^{\mathcal{K}}$$

admits a left adjoint given by $\mathcal{C} \mapsto \mathcal{P}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{C})$.

Proposition B.3 (See [HP24] A.2)**.** *Let $\mathcal{C}$ be a coaccessible ∞-category (i.e. the $\mathcal{C}^{\mathrm{op}}$ is accessible). For a functor $X : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$, the following conditions are equivalent:*

(1) *The functor $X : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ is accessible.*

(2) *The functor $X : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ is the left Kan extension of a functor $Y : (\mathcal{C}_\kappa)^{\mathrm{op}} \to \mathcal{S}$ along the canonical inclusion $i : (\mathcal{C}_\kappa)^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ for some small regular cardinal κ, where $\mathcal{C}_\kappa \subset \mathcal{C}$ denotes the full subcategory of κ-cocompact objects.*

(3) *The functor $X : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ is a colimit in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ of a small diagram of representable functors.*

Corollary B.4 (See [HP24] A.4)**.** *Let $\mathcal{C}$ be a coaccessible ∞-category. Then the Yoneda embedding*

$$\mathcal{C} \to \mathrm{Fun}^{\mathrm{ac}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$$

exhibits $\mathrm{Fun}^{\mathrm{ac}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \simeq \mathcal{P}^{\mathrm{small}}_{\emptyset}(\mathcal{C})$, where $\mathrm{Fun}^{\mathrm{ac}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ denote the full subcategory of accessible functors.

Proposition B.5 (See [HP24] A.9)**.** *Let $\mathcal{C}$ be a coaccessible ∞-category. The ∞-category $\mathcal{P}^{\mathrm{ac}}(\mathcal{C})$ of accessible presheaves of anima on $\mathcal{C}$ admits all small limits and colimits, and both are calculated pointwise.*

Proposition B.6. *Let $\mathcal{C}$ be a copresentable ∞-category and κ be a small regular cardinal. Then the Yoneda embedding*

$$\mathcal{C} \to \mathrm{Fun}^{\mathrm{ac}}_{\kappa\text{-lex}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$$

exhibits $\mathrm{Fun}^{\mathrm{ac}}_{\kappa\text{-lex}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \simeq \mathcal{P}^{\mathrm{small}\,\kappa\text{-fil}}_{\emptyset}(\mathcal{C}) = \mathrm{Ind}_\kappa(\mathcal{C})$.

Proof. First we observe that $\mathrm{Ind}_\kappa(\mathcal{C}) = \mathrm{Fun}^{\mathrm{ac}}_{\kappa\text{-lex}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \subset \mathrm{Fun}_{\kappa\text{-lex}}(\mathcal{C}^{\mathrm{op}}, \widehat{\mathcal{S}}) = \widehat{\mathrm{Ind}}_\kappa(\mathcal{C})$ is closed under small κ-filtered colimits. We claim that any object in $\mathrm{Fun}^{\mathrm{ac}}_{\kappa\text{-lex}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ can be written as the retraction of a small κ-filtered colimit of representable functors. Then the result immediately follows.

Now let $F \in \mathrm{Fun}^{\mathrm{ac}}_{\kappa\text{-lex}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$. Since $F \in \widehat{\mathrm{Ind}}_\kappa(\mathcal{C})$, it can be written as a large κ-filtered colimit of representable functors $F \simeq \varinjlim_{i \in I} h_{X_i}$. For each small κ-filtered full subcategory $I' \subseteq I$, let $F_{I'}$ denote the colimit $\varinjlim_{\alpha \in I'} h_{X_\alpha}$. Then by [Ker, 0620] , the F can be written as a large δ_0-filtered colimit of the diagram $\{F_{I'}\}$, where I' ranges over all small κ-filtered full subcategory of I. However, by Proposition B.3 the F is largely δ_0-compact in $\widehat{\mathrm{Ind}}_\kappa(\mathcal{C})$, so F is a retraction of some $F_{I'}$. □

Proposition B.7. *Let $\mathcal{C}$ be a copresentable ∞-category and κ be a small regular cardinal. Then $\mathrm{Ind}_\kappa(\mathcal{C}) \simeq \mathcal{P}^{\mathrm{small}}_{\kappa\text{-small}}(\mathcal{C})$.*

Proof. By the construction in [HTT, Corollary 5.3.6.10], it is equivalent to prove that $\mathrm{Ind}_\kappa(\mathcal{C}) \subset \widehat{\mathrm{Ind}}_\kappa(\mathcal{C})$ is the smallest full subcategory which contains representables and closed under small colimits, i.e. to prove $\mathrm{Ind}_\kappa(\mathcal{C}) = \widehat{\mathrm{Ind}}_\kappa(\mathcal{C})^{\delta_0}$. Since the representables generates $\widehat{\mathrm{Ind}}_\kappa(\mathcal{C})$ under large colimits, it suffices to show that $\mathrm{Ind}_\kappa(\mathcal{C}) \subset \widehat{\mathrm{Ind}}_\kappa(\mathcal{C})^{\delta_0}$ and $\mathrm{Ind}_\kappa(\mathcal{C})$ is idempotent complete. Those are implied by Proposition B.3 and Proposition B.5. □

Definition B.8. Let $\widehat{\mathrm{Cat}}_\infty^{\kappa\text{-lex}}$ denote the subcategory spanned by those ∞-categories which admit finite limits and those functors which preserve κ-small limits, where $\kappa < \delta_1$ is a large regular cardinal.

Proposition B.9. *Let $\kappa < \lambda < \delta_1$ be two large regular cardinals. Then there exists an adjoint pair*

$$\widehat{\mathrm{Cat}}_\infty^{\kappa\text{-lex}} \overset{\mathrm{Pro}^\lambda_\kappa}{\rightleftarrows} \widehat{\mathrm{Cat}}_\infty^{\lambda\text{-lex}}$$

by the dual version of [HTT, Corollary 5.3.6.10].

Remark B.10. We have the identification $\mathrm{Pro}^\lambda_\kappa(\mathcal{C}) \simeq \mathrm{Ind}^\lambda_\kappa(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$.

Proposition B.11. *The above adjunction can be promoted to a symmetric monoidal adjunction*

$$\widehat{\mathrm{Cat}}_\infty^{\kappa\text{-lex},\otimes} \overset{\mathrm{Pro}_\kappa^\lambda}{\rightleftarrows} \widehat{\mathrm{Cat}}_\infty^{\lambda\text{-lex},\otimes}$$

by the dual version of [HA, Proposition 4.8.1.3].

Remark B.12 (Dual of [HA] 4.8.1.9)**.** The $\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty)$ can be identified with the very large ∞-category of large symmetric monoidal ∞-categories.

Unwinding the definitions, we see that $\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\kappa\text{-lex}})$ can be identified with the subcategory of $\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty)$ spanned by the symmetric monoidal ∞-categories which are compatible with κ-small limits (meaning the tensor product $-\otimes-$ preserves κ-small limits separately), and those symmetric monoidal functors which preserve κ-small limits.

Corollary B.13 (Dual version of [HA] 4.8.1.10)**.** *Let $\kappa < \lambda < \delta_1$ be two large regular cardinals. By the following adjunction,*

$$\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\kappa\text{-lex}}) \overset{\mathrm{Pro}_\kappa^\lambda}{\rightleftarrows} \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\lambda\text{-lex}})$$

we see that for any large symmetric monoidal ∞-category $\mathcal{C}^\otimes$ that the monoidal structure on $\mathcal{C}$ is compatible with κ-small limits, there exists a λ-completely large symmetric monoidal ∞-category $\mathcal{D}^\otimes$ and a symmetric monoidal functor $\mathcal{C}^\otimes \to \mathcal{D}^\otimes$ with the following properties:

(1) *The symmetric monoidal structure on $\mathcal{D}^\otimes$ is compatible with λ-small limits.*

(2) *The underlying functor $f : \mathcal{C} \to \mathcal{D}$ preserves κ-small limits.*

(3) *The f induces an identification $\mathcal{D} \simeq \mathrm{Pro}_\kappa^\lambda(\mathcal{C})$, and is therefore fully faithful.*

(4) *The $\mathcal{D}^\otimes$ is universal among those satisfying (1)-(3).*

C Ambidextrous subcategories

Definition C.1. Let $i : \mathcal{C} \hookrightarrow \mathcal{D}$ be a fully faithful functor of ∞-categories. We say $\mathcal{C}$ is an ambidextrous subcategory of $\mathcal{D}$ if i admits both left and right adjoints L, R and the composition $R(X) \to X \to L(X)$ for $X \in \mathcal{D}$ induces an natural equivalence $R(-) \simeq L(-)$ in $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$.

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