Some questions about $bu^*(bu)$

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The original problem is how to naturally give a canonical E_∞-ring structure of $MU\langle 2k\rangle$ when $k \geq 3$, which finally can be refined to 2 questions on calculating $bu^*(bu)$.

1 Whether $bu^{1-2k}(bu) = 0$?

We know the E_∞ Thom spectrum functor induces a Quillen adjunction $Top[E_\infty]_{/BU} \rightleftharpoons Sp[E_\infty]$.

Follow the paper AHS[01], E_∞-ring structure of $MU\langle 2k\rangle$ should be from the Thom spectrum of the map of infinite loop spaces $BU\langle 2k\rangle \to BU$, which takes from a homotopy class of morphism $\Sigma^2 bu \to bu$ (where bu is the connective complex K-theory). So $BU\langle 2k\rangle \to BU$ is a only a homotopy class of morphism in $Ho(Top[E_\infty])$, and hence give an object in $Ho(Top[E_\infty])_{/BU}$ but not in $Ho(Top[E_\infty]_{/BU})$! They are different.

We know there is a natural functor $Ho(Top[E_\infty]_{/BU}) \to Ho(Top[E_\infty])_{/BU}$. An natural question is that whether we can lift $BU\langle 2k\rangle \to BU$ uniquely up to unique isomorphism to $Ho(Top[E_\infty]_{/BU})$ in some way. If that is right, we can conclude the E_∞ ring $MU\langle 2k\rangle$ is independent on the choice of the lifting.

Actually, a homotopy between two E_∞ maps of $X \to BU$ give a following diagram.

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f"'] \arrow[rr, bend left, dotted] & X \otimes I \arrow[d] & X \arrow[l] \arrow[dl, "g"] \\ & BU & \end{tikzcd}$$

It means that a homotopy could give an isomorphism in $Ho(Top[E_\infty]_{/BU})$, which induces a functor from $\pi Map_{E_\infty}(X, BU) \to Ho(Top[E_\infty]_{/BU})$, where π is fundamental groupoid. (All mapping spaces involved should be considered as derived mapping spaces). Then the lifting problem will be solved if $Map_{E_\infty}(X, BU)$ is simply-connected. Now let $X = BU\langle 2k\rangle$, we have

$$\pi_1 Map_{E_\infty}(X, BU) = \pi_1 Map_{Sp}(\Sigma^{2k} bu, \Sigma^2 bu) = bu^{1-2k}(bu)$$

So the first question is: whether $bu^{1-2k}(bu) = 0$?

2 Is $bu^4(\Sigma^4 bu) \to bu^4(BSU)$ injective?

Is E_∞ structure of BSU unique up to unique isomorphism? Let $Ho(Top[E_\infty]) \to Ho(Top)$ be the derived functor induced by forgetful functor. Consider the category of the fiber F over BSU

$$\begin{tikzcd} F \arrow[r] \arrow[d] & * \arrow[d, "BSU"] \\ Ho(Top[E_\infty]) \arrow[r] & Ho(Top) \end{tikzcd}$$

Adams showed that F_{BSU} and F_{BSO} are connected, I hope further that it is simply-connected. By calculation $\pi_1 F = ker(\pi_0 Map_{E_\infty}(BSU, BSU) \to \pi_0 Map(BSU, BSU)) = Ker(bu^4(\Sigma^4 bu) \to bu^4(BSU))$.

So the Second question is: whether $bu^4(\Sigma^4 bu) \to bu^4(BSU)$ is injective?