## Postnikov-type convergence in $\infty$ -categories

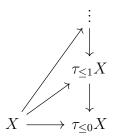
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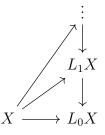
## 1. Postnikov-type decomposition

There are many examples of Postnikov-type tower in stable homotopy theory and chromatic homotopy theory such as

(1) The Postnikov tower of a space X



(2) The chromatic tower of a spectrum X



Although the tower could be constructed in corresponding homotopy category, the description of convergent condition is not well shaped in classical framework. However, Lurie provided a reasonable approach about Postnikov of truncation tower in [2], which actually can be generalized in any ascending sequence of reflective subcategories of any  $\infty$ -category.

Throughout the following content, the  $\mathcal{C}$  is an  $\infty$ -category,  $I = \{\mathcal{C}_0 \subset \mathcal{C}_1 \subset ... \subset \mathcal{C}_n...\}$  is an ascending sequence of reflective replete full subcategories of  $\mathcal{C}$ .

**Definition 1.1.** An I-tower in C is a functor  $N(\mathbf{Z}_{\geq 0}^{op})^{\triangleleft} \to C$ , which we view as a diagram

$$X_{\infty} \to \cdots \to X_2 \to X_1 \to X_0.$$

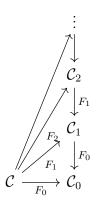
which satisfies that for each  $n \geq 0$ , the map  $X_{\infty} \to X_n$  exhibits  $X_n$  as a  $\mathcal{C}_n$ -reflection of  $X_{\infty}$ . We define a I-pretower to be a functor from  $N(\mathbf{Z}_{\geq 0})^{op} \to \mathcal{C}$ :

$$\cdots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0$$

which exhibits each  $X_n$  as a  $C_n$ -reflection of  $X_{n+1}$ .

We let  $\operatorname{Post}_I^+(\mathcal{C})$  denote the full subcategory of  $\operatorname{Fun}\left(\operatorname{N}(\mathbf{Z}_{\geq 0}^{op})^{\triangleleft},\mathcal{C}\right)$  spanned by the I-towers, and  $\operatorname{Post}_I(\mathcal{C})$  the full subcategory of  $\operatorname{Fun}\left(\operatorname{N}(\mathbf{Z}_{\geq 0})^{op},\mathcal{C}\right)$  spanned by the I-pretowers. We have an evident forgetful functor  $\phi:\operatorname{Post}_I^+(\mathcal{C})\to\operatorname{Post}_I(\mathcal{C})$ . We will say that I-towers in  $\mathcal{C}$  are convergent if  $\phi$  is an equivalence of  $\infty$ -categories.

**Definition 1.2.** Let  $\mathcal{E}$  denote the full subcategory of  $\mathcal{C} \times \mathrm{N}(\mathbf{Z}_{\geq 0}^{op})^{\triangleleft}$  spanned by those pairs (C, n) where  $C \in \mathcal{C}_n$  (by convention, we agree that this condition is always satisfied when  $n = \infty$ ). Then we have a coCartesian fibration  $p : \mathcal{E} \to \mathrm{N}(\mathbf{Z}_{\geq 0}^{op})^{\triangleleft}$ , which classifies a tower of  $\infty$ -categories



where  $F_n$  is the  $C_n$ -reflection functor.

**Proposition 1.3.** We can identify I-towers and with coCartesian sections of p, and I-pretowers with coCartesian sections of the induced fibration  $\mathcal{E}' = N(\mathbf{Z}_{\geq 0}^{op}) \times_{N(\mathbf{Z}_{\geq 0}^{op})^{\triangleleft}} \mathcal{E}$ :

$$\text{Post}_{I}^{+}(\mathcal{C}) \longrightarrow \text{Post}_{I}(\mathcal{C})$$

$$\downarrow = \qquad \qquad \downarrow =$$

$$Fun_{/\mathcal{N}(\mathbf{Z}_{>0}^{op})^{\triangleleft}}^{cCart}(\mathcal{N}(\mathbf{Z}_{\geq 0}^{op})^{\triangleleft}, \mathcal{E}) \longrightarrow Fun_{/\mathcal{N}(\mathbf{Z}_{>0}^{op})}^{cCart}(\mathcal{N}(\mathbf{Z}_{\geq 0}^{op}), \mathcal{E}')$$

According to [1] 7.4.1.1, the I-towers in C converge if and only if the tower above exhibits C as the homotopy limit of the sequence of  $\infty$ -categories

$$\cdots \to \mathcal{C}_2 \xrightarrow{F_1} \mathcal{C}_1 \xrightarrow{F_0} \mathcal{C}_0.$$

Now we introduce a useful lemma which implies any  $N_*(J)$ -diagram in QC is homotopy to a strict diagram  $J \to QCat$ .

**Lemma 1.4.** [2] 4.2.4.4. Let J be a small ordinary category, and QC at denote the simplicial category of (small)  $\infty$ -categories, which is sSet-enriched by the form  $Fun(C, D)^{\simeq}$ , and  $QC = N_{\Delta}(QCat)$  denote the  $\infty$ -category of (small)  $\infty$ -categories. Then the following induced map is an equivalence,

$$N_{\Delta}(F(J, sSet_{+})^{\circ}) \rightarrow \operatorname{Fun}(N_{*}(J), sSet_{+}^{\circ}) = \operatorname{Fun}(N_{*}(J), QC)$$

where  $sSet_+$  is the Cartesian model category of marked simplicial sets, and  $F(J, sSet_+)$  is endowed with projective or injective model, and  $(-)^{\circ}$  means full subcategory of cofibrant-fibrant objects.

**Proposition 1.5.** If I-towers in C are convergent, then every I-tower in C is a limit diagram. Indeed, given objects  $X, Y \in C$ , we have natural homotopy equivalences

$$\operatorname{Map}_{\mathcal{C}}(X,Y) \simeq \operatorname{holim} \operatorname{Map}_{\mathcal{C}}(F_nX, F_nY) \simeq \operatorname{holim} \operatorname{Map}_{\mathcal{C}}(X, F_nY)$$
,

and the composition of these 2 equivalences is induced by the composition  $Y \to F_n Y$ . So Y is the limit of the I-pretower  $\{F_n Y\}$ .

Lurie gives this formula without a proof, which actually needs some straitening techniques.

Proof: Let  $f: N(\mathbf{Z}_{\geq 0}^{op}) \to QC$  be the straitening presheaf by  $p': \mathcal{E}' \to N(\mathbf{Z}_{\geq 0}^{op})$ . By 1.4, it is homotopy to  $N_{\Delta}(q)$  where q is a strict diagram  $\mathbf{Z}_{\geq 0}^{op} \to QCat$ . Without loss of generalization, we can assume q has the form ...  $\to N_{\Delta}(\mathcal{D}_n) \xrightarrow{N_{\Delta}(G_n)} N_{\Delta}(\mathcal{D}_{n-1}) \to ... \to N_{\Delta}(\mathcal{D}_0)$  where  $G_n: \mathcal{D}_n \to \mathcal{D}_{n-1}$  is an Joyal fibration of simplicial categories. Then q is an isofibrant diagram by [1] 4.5.6.6. So we have an (essentially unique) equivalence  $\mathcal{C} \to N_{\Delta}(\mathcal{D}) = N_{\Delta}(\varprojlim \mathcal{D}_n)$  and

$$\underline{\operatorname{holim}} \operatorname{Map}_{\mathcal{C}}(F_{n}X, F_{n}Y) = \underline{\lim} \operatorname{Hom}_{\mathcal{D}_{n}}^{*}(G_{n}X, G_{n}Y) = \operatorname{Hom}_{\underline{\lim} \mathcal{D}_{n}}^{*}(X, Y) \simeq \operatorname{Map}_{\mathcal{C}}(X, Y)$$

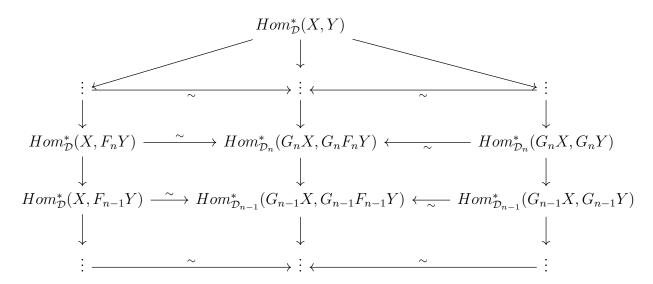
Furthermore, we note that

... 
$$\to Hom_{\mathcal{D}_n}^*(G_n(-), G_n(-)) \to Hom_{\mathcal{D}_{n-1}}^*(G_{n-1}(-), G_{n-1}(-)) \to ...$$

gives an simplicial functor  $(\mathbf{Z}_{\geq 0}^{op})^{\triangleleft} \times \mathcal{D}^{op} \times \mathcal{D} \to Kan$ . Let

$$\{*\}\times\mathrm{N}(\mathbf{Z}^{op}_{\geq 0})^{\triangleleft}\to\mathcal{C}^{op}\times\mathcal{C}\to N_{\Delta}(\mathcal{D})\times N_{\Delta}(\mathcal{D})$$

be  $(X, F_n Y)$  induced by the *I*-tower  $\{Y \to F_n Y\}$  in  $\mathcal{C}$ . By Composition we get a diagram  $N(\mathbf{Z}_{\geq 0}^{op})^{\triangleleft} \times N(\mathbf{Z}_{\geq 0}^{op})^{\triangleleft} \to \mathcal{S}$  which has the form  $(m, n) \mapsto Hom_{\mathcal{D}_m}^*(G_m X, G_m(F_n Y))$ . Take the sub-diagram  $(\Delta^2 \times N(\mathbf{Z}_{\geq 0}^{op}))^{\triangleleft} \subset N(\mathbf{Z}_{\geq 0}^{op})^{\triangleleft} \times N(\mathbf{Z}_{\geq 0}^{op})^{\triangleleft}$  we get



which gives

$$\operatorname{Map}_{\mathcal{C}}(X, Y) \simeq \operatorname{holim} \operatorname{Map}_{\mathcal{C}}(F_n X, F_n Y) \simeq \operatorname{holim} \operatorname{Map}_{\mathcal{C}}(X, F_n Y),$$

and the composition of these 2 equivalences is induced by the composition  $Y \to F_n Y$ .

**Proposition 1.6.** Let C is an  $\infty$ -category in which any I-pretower admits a limit, where  $I = \{C_0 \subset C_1 \subset ... \subset C_n...\}$  be an ascending sequence of reflective replete full subcategories of C. Then I-towers in C are convergent if and only if, for every diagram  $X : \mathbb{N}(\mathbf{Z}_{\geq 0}^{op})^{\triangleleft} \to C$ , the following conditions are equivalent:

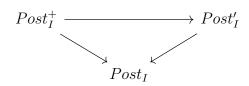
- (1) The diagram X is a I-tower.
- (2) The diagram X is a limit in C, and the restriction  $X \mid N(\mathbf{Z}_{>0})^{op}$  is a I-pretower.

*Proof.* Let  $\operatorname{Post}'_I(\mathcal{C})$  be the full subcategory of  $\operatorname{Fun}\left(\operatorname{N}(\mathbf{Z}^{op}_{\geq 0})^{\triangleleft}, \mathcal{C}\right)$  spanned by those towers which satisfy condition (2). Using Proposition [1] 7.3.6.13, we deduce that the restriction functor  $\operatorname{Post}'_I(\mathcal{C}) \to \operatorname{Post}_I(\mathcal{C})$  is a trivial Kan fibration.

If conditions (1) and (2) are equivalent, then  $\operatorname{Post}_I'(\mathcal{C}) = \operatorname{Post}_I^+(\mathcal{C})$ , so that *I*-towers in  $\mathcal{C}$  are convergent.

Conversely, suppose that I-towers in  $\mathcal{C}$  are convergent. Using 1.5, we deduce that  $\operatorname{Post}_I^+(\mathcal{C}) \subseteq$ 

 $\operatorname{Post}_I'(\mathcal{C})$ , so we have a commutative diagram



Since both of the vertical arrows are trivial Kan fibrations, we conclude that the inclusion  $\operatorname{Post}_I^+(\mathcal{C}) \subseteq \operatorname{Post}_I'(\mathcal{C})$  is an equivalence, so that  $\operatorname{Post}_I^+(\mathcal{C}) = \operatorname{Post}_I'(\mathcal{C})$  by repleteness. This proves that  $(1) \Leftrightarrow (2)$ .

## References

- [1] J. Lurie. Kerodon. version 2023.04.24. 1.3, 1, 1
- [2] Jacob Lurie. Higher topos theory. Princeton University Press, 2009. 1, 1.4