An overview of ∞-categories and higher algebra

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Why use ∞-categories?

Some phenomenons or propositions can not be stated clearly without ∞-categories.

Example

(1) Chromatic convergence and chromatic pullback:

$$\begin{tikzcd} & \vdots \arrow[d] \\ & L_1 X \arrow[d] \\ X \arrow[r] \arrow[ur] \arrow[uur] & L_0 X \end{tikzcd} \qquad \begin{tikzcd} L_n X \arrow[r] \arrow[d] & L_{K(n)} X \arrow[d] \\ L_{n-1} X \arrow[r] & L_{n-1} L_{K(n)} X \end{tikzcd}$$

Chromatic convergence and chromatic pullback should be described as homotopy limits of **homotopy coherent diagrams** $N_*(\mathbb{Z}_{\geq 0}^{op}) \to Sp$ and $\Lambda_2^2 \to Sp$ **instead of homotopy diagrams** $\mathbb{Z}_{\geq 0}^{op} \to h(Sp)$ or $\Lambda_2^2 \to h(Sp)$.

(2) Similarly, the Postnikov tower in the category $\mathcal{S}$ of spaces and its convergence.

Why use ∞-categories?

Example

(3) If $\mathcal{C}$ is a 1-category, then $Sp(\mathcal{C}) \simeq \{*\}$ is trivial. The stabilization for 1-categories is meaningless. "Stable homotopy" is a higher phenomenon.
(4) By ∞-categories we can define all kinds of **moduli spaces**, such as $CAlg(Sp) \times_{CAlg(hSp)} \{R\}$, the moduli space of E_∞-structures on a given homotopy commutative ring spectrum R. The E_∞-structures on a Lubin-Tate spectrum $E(n, \Gamma)$ is **unique**, meaning $CAlg(Sp) \times_{CAlg(hSp)} \{E(n, \Gamma)\}$ is a contractible Kan complex.
(5) **Bousfield localization** and **connective cover** of an E_∞-ring are still E_∞-rings. In the ∞-category theory, this is directly by the fact $L_E : Sp \rightleftarrows Sp_E : i$ and $i : Sp_{\geq 0} \rightleftarrows Sp : \tau_{\geq 0}$ are symmetric monoidal adjunctions, which will induce an adjunction $CAlg(Sp) \rightleftarrows CAlg(Sp_E)$ and $CAlg(Sp_{\geq 0}) \rightleftarrows CAlg(Sp)$.
However, the framework based on model category like EKMM can not provide such a machine.
(6) **Equivariant stable homotopy theory**: there are loads of model categories characterizing it, but all underlying ∞-categories of them are equivalent with $Fun(BG, Sp)$, which is both simple and intuitive.

Motivation of ∞-categories

Motivation

The most significant motivation is to change the morphism **set** $Hom_{\mathcal{C}}(X, Y)$ in a category $\mathcal{C}$ to a topological **space** $Map_{\mathcal{C}}(X, Y)$. Then we can have higher morphisms $\pi_n Map_{\mathcal{C}}(X, Y)$.

For example when considering the category of spectra, we have $\pi_n Map_{\mathcal{C}}(X, Y) = [\Sigma^n X, Y]$.

So the most intuitive model for the ∞-category theory should be $sSet$-enriched (or Top-enriched) categories. However all of these models are equivalent to Joyal model. Indeed we have Quillen equivalences $(sSet)_{Joyal} \rightleftarrows Cat_{sSet} \rightleftarrows Cat_{Top}$.

But Joyal model encodes information more concisely: the only data of an ∞-category is a simplicial set.

Information in an ∞-category

Underlying $\mathcal{H}$-enriched category

There are lots of different ways to extract the mapping space $Map_{\mathcal{C}}(X, Y)$ from an ∞-category $\mathcal{C}$.

But when we take their underlying $\mathcal{H} = Ho(sSet_{Kan})$-enriched categories, all of them are the same, written as $\underline{h\mathcal{C}}$.

Remark

The processes $\mathcal{C} \mapsto \underline{h\mathcal{C}} \mapsto h\mathcal{C}$ make it simpler to manage but meanwhile cause loss of homotopy coherent information. How to extract useful and discard redundant information of homotopy coherence in certain circumstances is an art in ∞-categories' world.

Universal properties in the category of all categories

We often **internalize** the ∞-category theory, meaning we often **characterize** a specific ∞-category by a universal property in the ∞-category $\mathcal{C}at_\infty$ of all ∞-categories.

Let $\mathcal{C}$ be an ∞-category.

Theorem (Cocompletion)

Let $\mathbb{K}$ be the collection of all small simplicial sets, then there exists a completion $\mathcal{C} \to \mathcal{P}^{\mathbb{K}}(\mathcal{C})$ such that for any $\mathbb{K}$-cocomplete ∞-category $\mathcal{D}$ the forgetful functor $\theta : \mathrm{Fun}^{\mathbb{K}}(\mathcal{P}^{\mathbb{K}}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}\,(\mathcal{C}, \mathcal{D})$ is an equivalence.

Actually, it is exactly Yoneda embedding, $P^{\mathbb{K}}(\mathcal{C}) = Fun(\mathcal{C}, \mathcal{S})$ and $P^{\mathbb{K}}(*) = \mathcal{S}$. So that means $\mathcal{S}$ is "freely generated" by the single-point space $* \in \mathcal{S}$.

Theorem (Stabilization)

If $\mathcal{C}$ admits finite limits, then there exists a stabilization $Sp(\mathcal{C}) \to \mathcal{C}$ such that for any stable ∞-category $\mathcal{D}$ the forgetful functor $\theta : \mathrm{Fun}^{Flim}(\mathcal{D}, Sp(\mathcal{C})) \to \mathrm{Fun}^{Flim}(\mathcal{D}, \mathcal{C})$ is an equivalence. That provides an adjunction $Cat_\infty^{Flim,st} \rightleftarrows Cat_\infty^{Flim}$.

For example, $Sp = Sp(\mathcal{S})$. In a sense, $\mathcal{S}$ and Sp are distinctive as elements in Cat_∞.

Postnikov-type tower

Let $\mathcal{C}$ be an ∞-category, and $I = \{\mathcal{C}_0 \subset \mathcal{C}_1 \subset \ldots \subset \mathcal{C}_n \subset \ldots \subset \mathcal{C}\}$ be an ascending sequence of reflective full subcategories of $\mathcal{C}$, where **reflective** means the inclusion functor admits a left adjunction.

Definition (Tower and pretower)

An I-tower in $\mathcal{C}$ is a functor $\mathrm{N}(\mathbb{Z}_{\geq 0}^{op})^{\triangleleft} \to \mathcal{C}$, which we view as a diagram

$$X_\infty \to \cdots \to X_2 \to X_1 \to X_0.$$

satisfying that for each $n \geq 0$, the map $X_\infty \to X_n$ exhibits X_n as a $\mathcal{C}_n$-reflection of X_∞. We define a I-pretower to be a functor from $\mathrm{N}\left(\mathbb{Z}_{\geq 0}^{op}\right) \to \mathcal{C}$:

$$\cdots \to X_2 \to X_1 \to X_0$$

which exhibits each X_n as a $\mathcal{C}_n$-reflection of X_{n+1}.

For example when taking $I = \{\mathcal{S}_{\leq 0} \subset \mathcal{S}_{\leq 1} \subset \ldots \subset \mathcal{S}_{\leq n} \subset \ldots \subset \mathcal{S}\}$, we back to the classical case.

Postnikov-type convergence

Let $\mathcal{C}$ be an ∞-category, and $I = \{\mathcal{C}_0 \subset \mathcal{C}_1 \subset \ldots \subset \mathcal{C}_n \ldots\}$ be an ascending sequence of reflective replete full subcategories of $\mathcal{C}$.

Definition

We let $\mathrm{Post}_I^+(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{N}(\mathbb{Z}_{\geq 0}^{op})^{\triangleleft}, \mathcal{C})$ spanned by the I-towers, and $\mathrm{Post}_I(\mathcal{C})$ the full subcategory of $\mathrm{Fun}(\mathrm{N}(\mathbb{Z}_{\geq 0}^{op}), \mathcal{C})$ spanned by the I-pretowers. We have an evident forgetful functor $\phi : \mathrm{Post}_I^+(\mathcal{C}) \to \mathrm{Post}_I(\mathcal{C})$. We will say that **$I$-towers in $\mathcal{C}$ are convergent** if ϕ is an equivalence of ∞-categories.

Theorem (Postnikov-type convergence)

If any I-pretower in $\mathcal{C}$ admits a limit, then I-towers in $\mathcal{C}$ are convergent
if and only if,
for every diagram $X : \mathrm{N}(\mathbb{Z}_{\geq 0}^{op})^{\triangleleft} \to \mathcal{C}$, the following conditions are equivalent:
(1) The diagram X is a I-tower.
(2) The diagram X is a limit in $\mathcal{C}$, and the restriction $X \mid \mathrm{N}(\mathbb{Z}_{\geq 0}^{op})$ is a I-pretower.

Higher commutative monoid

Definition (Reformulate ordinary commutative monoid)

A (3-)commutative monoid in an ordinary category $\mathcal{C}$ which admits finite products is a functor $M : (Fin_*)_{\leq 3} \to \mathcal{C}$ such that the canonical maps $M(\rho_i) : M(\langle n\rangle) \to M(\langle 1\rangle)$ exhibit $M(\langle n\rangle) \simeq \prod_{1\leq i\leq n} M(\langle 1\rangle)$ in the $\mathcal{C}$ for all $0 \leq n \leq 3$.

$$\begin{tikzcd} \langle 3\rangle \arrow[r] \arrow[d] \arrow[dr, phantom, "Assoc"] & \langle 2\rangle \arrow[d] \\ \langle 2\rangle \arrow[r] & \langle 1\rangle \end{tikzcd} \qquad \begin{tikzcd} \langle 1\rangle \arrow[r] \arrow[dr, "id"'] & \langle 2\rangle \arrow[d] & \langle 1\rangle \arrow[l] \arrow[dl, "id"] \\ & \langle 1\rangle & \end{tikzcd} \qquad \begin{tikzcd} \langle 2\rangle \arrow[rr, "\tau"] \arrow[dr] & \arrow[d, phantom, "comm"] & \langle 2\rangle \arrow[dl] \\ & \langle 1\rangle & \end{tikzcd}$$

Definition (∞-commutative monoid)

Let $\mathcal{C}$ be an ∞-category with finite products, we define a ∞-commutative monoid in $\mathcal{C}$ as a functor $M : N(Fin_*) \to \mathcal{C}$ such that the canonical maps $M(\rho_i) : M(\langle n\rangle) \to M(\langle 1\rangle)$ exhibit $M(\langle n\rangle) \simeq \prod_{1\leq i\leq n} M(\langle 1\rangle)$ in the $\mathcal{C}$ for all $n \geq 0$.

Symmetric monoidal ∞-category

Proposition (Baez-Dolan Stabilization)

Let $\mathcal{C}$ be an n-category with finite products, then $CMon^{\infty}(\mathcal{C}) \xrightarrow{\sim} CMon^{n+2}(\mathcal{C})$ is categorical equivalent.

Definition

A symmetric monoidal ∞-category is an (∞-)commutative monoid in Cat_{∞}.

Corollary

Particularly, if a symmetric monoidal ∞-category $\mathcal{C}$ is a 1-category, then it is an ∞-commutative monoid in the $(Cat_{\infty})_{\leq 1}$, which is a 2-category. So we have $CMon^{\infty}(Cat_{\leq 1}) \xrightarrow{\sim} CMon^{4}(Cat_{\leq 1})$.

It can be checked that the 4-commutativity in $Cat_{\leq 1}$ exactly corresponds with ordinary coherent conditions of a symmetric monoidal caregory.

Lurie's definition

By the (un)straightening equivalence $Fun(N(Fin_*), Cat_\infty) \simeq CoCart_{/N(Fin_*)}$, we get the following equivalent definition.

Definition

A symmetric monoidal ∞-category is a coCartesian fibration of simplicial sets $p : \mathcal{C}^\otimes \to N(Fin_*)$ with the following property:
For each $n \geq 0$, the maps $\{\rho^i : \langle n \rangle \to \langle 1 \rangle\}_{1 \leq i \leq n}$ induce functors $\rho^i_! : \mathcal{C}^\otimes_{\langle n \rangle} \to \mathcal{C}^\otimes_{\langle 1 \rangle}$ which determine an equivalence $\mathcal{C}^\otimes_{\langle n \rangle} \simeq (\mathcal{C}^\otimes_{\langle 1 \rangle})^n$.
We define $\mathcal{C}^\otimes_{\langle 1 \rangle}$ as its underlying ∞-category.

It has technical advantages in the framework of quasi-categories.

Symmetric monoidal colocalization

Proposition (Symmetric monoidal colocalization)

*Let $\mathcal{C}^{\otimes} \to N(Fin_*)$ be a symmetric monoidal ∞-category. Let $\mathcal{D} \subseteq \mathcal{C}$ be a full subcategory which is stable under equivalence. Suppose that the functor $- \otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ carries $\mathcal{D} \times \mathcal{D}$ into $\mathcal{D}$. (meaning $\mathcal{D}$ is **closed under tensor products**) Then:*
(1) The restricted map $\mathcal{D}^{\otimes} \to N(Fin_*)$ is a symmetric monoidal ∞-category.
(2) The inclusion $\mathcal{D}^{\otimes} \subseteq \mathcal{C}^{\otimes}$ is a symmetric monoidal functor.
(3) Suppose that the inclusion $\mathcal{D} \subseteq \mathcal{C}$ admits a right adjoint L (so that $\mathcal{D}$ is a colocalization of $\mathcal{C}$), then there exists a lax-symmetric-monoidal right adjunction $L^{\otimes} : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$.

Formally speaking, $L^{\otimes}$ is a right adjunction in the strict 2-category $h_2(Op_{/\mathcal{O}^{\otimes}})$.

Corollary

Under (3) above, a symmetric monoidal colocalization can induce a colocalization on algebras $CAlg(\mathcal{D}) \rightleftarrows CAlg(\mathcal{C})$.

Connective cover

Corollary (t-structure and symmetric monoidal structure)

Let $p : \mathcal{C}^{\otimes} \to N(Fin_*)$ be a symmetric monoidal ∞-category. Assume that the underlying ∞-category $\mathcal{C}$ is stable, and that $- \otimes -$ is exact in each variable. We will say that a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ is **compatible** with the symmetric monoidal structure if, the functor $\otimes$ carries $\mathcal{C}_{\geq 0} \times \mathcal{C}_{\geq 0}$ into $\mathcal{C}_{\geq 0}$.
Then the induced map $\mathcal{C}^{\otimes}_{\geq 0} \to N(Fin_*)$ is again a symmetric monoidal ∞-category, and $\mathcal{C}^{\otimes}_{\geq 0} \underset{\tau_{\geq 0}}{\overset{i}{\rightleftarrows}} \mathcal{C}^{\otimes}$ is a symmetric monoidal coclocalization. So it induces a colocalization $CAlg(\mathcal{C}_{\geq 0}) \underset{\tau_{\geq 0}}{\overset{i}{\rightleftarrows}} CAlg(\mathcal{C})$.

Example (Connective cover of an E_∞-ring)

When $\mathcal{C} = Sp$ we have $CAlg(Sp_{\geq 0}) \underset{\tau_{\geq 0}}{\overset{i}{\rightleftarrows}} CAlg(Sp)$, which means the connective cover of an E_∞-ring naturally inherits an E_∞-structure.

Symmetric monoidal localization

Proposition (Symmetric monoidal localization)

*Let $\mathcal{C}^{\otimes} \to N(Fin_*)$ be a symmetric monoidal ∞-category. Let $\mathcal{D} \subseteq \mathcal{C}$ be a full subcategory. Suppose that, the $\mathcal{D} \subset \mathcal{C}$ is a reflective subcategory (with localization $L : \mathcal{C} \to \mathcal{C}$). If for every pair of L-equivalences g_1, g_2 in $\mathcal{C}$, the morphism $g_1 \otimes g_2$ in $\mathcal{C}$ is also an L-equivalence (meaning **L-equivalences are closed under tensor products**), then:*

(1) The restricted map $\mathcal{D}^{\otimes} \to N(Fin_*)$ is lax-symmetric-monoidal.

(2) The inclusion $\mathcal{D}^{\otimes} \subseteq \mathcal{C}^{\otimes}$ is a symmetric monoidal functor.

(3) There exists a symmetric monoidal left adjunction $L^{\otimes} : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$.

Corollary

A symmetric monoidal localization can induce a localization on algebras $CAlg(\mathcal{C}) \rightleftarrows CAlg(\mathcal{D})$.

Bousfield localization

Let $\mathcal{C}^{\otimes}$ be a presentably symmetric monoidal category, i.e. an object in $CAlg(Pr^L)$.

Theorem (Bousfield localization)

Let $E \in \mathcal{C}$ be an object, then $W_E = \{X \to Y | X \otimes E \xrightarrow{\sim} Y \otimes E\} \subset Fun(\Delta^1, \mathcal{C})$ is a small-generated strongly saturated collection of morphisms, which means there exists an accessible localization functor $L_E : \mathcal{C} \to \mathcal{C}$.

Furthermore, Bousfield localization is compatible with its symmetric monoidal structure, meaning it forms a symmetric monoidal localization $\mathcal{C}^{\otimes} \underset{i^{\otimes}}{\overset{L_E^{\otimes}}{\rightleftarrows}} \mathcal{C}_E^{\otimes}$.

Example (Bousfield localization of an E_∞-ring)

When $\mathcal{C} = Sp$ we have $CAlg(\mathcal{C}) \underset{i}{\overset{CAlg(L_E)}{\rightleftarrows}} CAlg(\mathcal{C}_E)$, which means Bousfield localization of an E_∞-ring naturally inherits an E_∞-structure.

Idempotent object

Let $\mathcal{C}$ be a symmetric monoidal ∞-category.

Definition (idempotent object)

Let $e : 1_C \to E$ be a morphism in $\mathcal{C}$. We say e is idempotent iff $1_{\mathcal{C}} \otimes X \to X \otimes X$ is equivalent. (e.g. $\mathbb{Z} \to \mathbb{Z}[1/p]$ in Ab)

Theorem (Bousfield localization with respect to an idempotent object)

Let $e : 1_C \to E$ be a morphism in $\mathcal{C}$, then
(1) The e is an idempotent object of $\mathcal{C}$ iff the transformation $\alpha : \mathrm{id}_{\mathcal{C}} \to l_E$ exhibits l_E as a localization functor on $\mathcal{C}$, where $l_E : \mathcal{C} \to \mathcal{C}$ is given by the tensor product with E.
(2) If e is idempotent, then l_E is exactly the Bousfield localization with respect to E, which has the following properties:

(a) The l_E is compatible with $\otimes$, so induces a symmetric monoidal localization

$$\mathcal{C}^{\otimes} \underset{i^{\otimes}}{\overset{L_E^{\otimes}}{\rightleftarrows}} \mathcal{C}_E^{\otimes} \;;$$

(b) The inclusion $i^{\otimes}$ is also symmetric monoidal, meaning $\mathcal{C}_E$ is closed under tensor products.

Idempotent algebra

Definition

Let $\mathcal{C}$ be a symmetric monoidal ∞-category. We will say that a commutative algebra object $A \in \mathrm{CAlg}(\mathcal{C})$ is idempotent if unit map $e : \mathbf{1} \to A$ is idempotent.

Theorem

Let $\mathcal{C}$ be a symmetric monoidal ∞-category with unit object $\mathbf{1}$, which we regard as a trivial algebra object of $\mathcal{C}$. Then the functor

$$\theta : \mathrm{CAlg}^{\textit{idem}}(\mathcal{C}) \subseteq \mathrm{CAlg}(\mathcal{C}) \simeq \mathrm{CAlg}(\mathcal{C})_{1/} \to \mathcal{C}_{\mathbf{1}/}$$

is fully faithful, and its essential image are idempotent objects in $\mathcal{C}$, which gives an equivalence $\mathrm{CAlg}^{\textit{idem}}(\mathcal{C}) \xrightarrow{\sim} (\mathcal{C}_{\mathbf{1}/})^{\textit{idem}}$.
Furthermore, any mapping space in $(\mathcal{C}_{\mathbf{1}/})^{\textit{idem}}$ is either empty or contractible, i.e. $(\mathcal{C}_{\mathbf{1}/})^{\textit{idem}}$ is equivalent to a partial-order set $N(I)$.

Interesting applications after the internalization

Proposition

The full subcat $Pr^L \subset \widehat{Cat}_\infty(\mathbb{K})$ *is closed under tensor products* ($\mathcal{S}$ *is also the unit in* Pr^L) *and hence inherits a symmetric monoidal structure. In fact, for any* $\mathcal{C}, \mathcal{D} \in Pr^L$, *we have a natural equivalence* $\mathcal{C} \otimes D \simeq RFun(\mathcal{C}^{op}, \mathcal{D})$.

Theorem (Unique symmetric monoidal structure)

The following 4 colimit-preserving functors $\mathcal{S} \xrightarrow{\tau_{\leq n}} \tau_{\leq n}\mathcal{S}$, $\mathcal{S} \xrightarrow{(-)_+} \mathcal{S}_*$, $\mathcal{S} \xrightarrow{\Sigma^\infty_+} Sp$, *and* $\mathcal{S} \xrightarrow{* \mapsto \mathbb{Z}} N(Ab)$ *are idempotent objects in* Pr^L.
Hence by $\mathrm{CAlg}(Pr^L)^{idem} \xrightarrow{\sim} (Pr^L_{\mathcal{S}/})^{idem}$ *we conclude that* $\mathcal{S}$ *resp.* $\mathcal{S}_{\leq n}$, $\mathcal{S}_*$, Sp, $N(Ab)$ *only admits a unique cocomplete symmetric monoidal structure with the unit* $*$ *resp.* $*$, S^0, $\Sigma^\infty S^0$, $\mathbb{Z}$.

Interesting applications after the internalization

By Bousfield localization with respect to idempotent objects, we have:

Corollary

The following 4 full subcategories of Pr^L are closed under tensor products.
(a) $Pr^L_{\leq n+1}$: the ∞-category of presentable $(n+1)$-categories;

(b) Pr^L_{*} : the ∞-category of presentable pointed ∞-categories;

(c) Pr^L_{st} : the ∞-category of presentable stable ∞-categories, often called tensor-triangulated ∞-categories or tt-∞-categories;
(d) $Pr^L_{1\text{-}ad}$: the ∞-category of presentable additive 1-categories.

Corollary

The localization functors $Pr^L \xrightarrow{-\otimes\tau_{\leq n}\mathcal{S}} Pr^L_{\leq n+1}$, $Pr^L \xrightarrow{-\otimes\mathcal{S}_*} Pr^L_{*}$, $Pr^L \xrightarrow{-\otimes Sp} Pr^L_{st}$, and $Pr^L \xrightarrow{-\otimes N(Ab)} Pr^L_{1\text{-}ad}$ correspond with the n-truncation, copointedlization, costabilization, and 1-coadditivalization of presentable ∞-categories respectively.

My interests

(1) Using higher algebra and spectral algebraic geometry to reinterpret the chromatic homotopy theory.
For example, the Devinatz-Hopkins theorem $L_{K(n)}S \simeq E_n^{h\mathbb{G}_n}$ can be interpreted $\mathrm{Qcoh}(\mathrm{Spf}(E_n)/\mathbb{G}_n) \simeq Sp_{K(n)}$ in (formal) spectral algebraic geometry.

(2) In the framework of SAG, we can study spectral moduli problem: given an algebro-geomtric stack $\mathcal{M}_0$, can we give an E_∞ realization $\mathcal{M}$ making $\pi_0\mathcal{M} = \mathcal{M}_0$? It is true when $\mathcal{M}_0 = \mathcal{M}_{ell}$ for the moduli stack of elliptic curves and when $\mathcal{M}_0 = \mathcal{X}_{K^p}$ for some of Shimura stacks. Then take the global section of E_∞-stacks respectively, we get TMF and TAF, which are intriguing E_∞-rings.

(3) The orientation theory from a Thom spectrum to a (E_∞-)ring spectrum. There are all kinds of tools we can combine to use. For example, using the infinite loop space machine $\mathrm{Mon}^{gp}_{\mathbb{E}_\infty}(\mathcal{S}) \simeq Sp_{\geq 0}$ we have the following beautiful adjunction to use.

$$\begin{tikzcd}
Sp_{\geq 0} \arrow[r, "\sim"] & \mathrm{Mon}^{gp}_{\mathbb{E}_\infty}(\mathcal{S}) \arrow[r, hook, shift left] & \mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{S}) \arrow[l, shift left, "GL_1"] \arrow[r, shift left, "\Sigma^\infty_+"] & \mathrm{CAlg}(Sp) \arrow[l, shift left, "\Omega^\infty"] \arrow[lll, bend left, "gl_1"]
\end{tikzcd}$$

THANK YOU!