토론 2 20211543 성수빈

1. $f(x,y = \begin{cases} \frac{x^2y}{x^2+y^2} &, (x,y) \neq (0.0) \\ 0 &, (x,y) = (0,0) \end{cases}$

(1) $f_x(0,0) = \lim_{h\to0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to0} \frac{0-0}{h} = 0$

$f_y(0,0) = \lim_{h\to0} \frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to0} \frac{0-0}{h} = 0$

(2) $f_x(x,y) = \begin{cases} 0, & (x,y) = (0,0) \\ \frac{2xy^3}{(x^2+y^2)^2} &, (x,y) \neq (0,0) \end{cases}$

$\lim_{(x,y)\to(0,0)} \frac{2xy^3}{(x^2+y^2)^2} \neq 0$

$f_y(x,y) = \begin{cases} 0, & (x,y) = (0,0) \\ \frac{x^2(x^2-y^2)}{(x^2+y^2)^2} &, (x,y) \neq (0,0) \end{cases}$

$\lim_{(x,y)\to(0,0)} \frac{x^2(x^2-y^2)}{(x^2+y^2)^2} \neq 0$ $\therefore$ f_x, f_y are not continuous at $(0,0)$.

(3) $D_{\vec{u}} f(0,0) = \lim_{h\to0} \frac{f(0+ha,0+hb)-f(0,0)}{h}$

$= \lim_{h\to0} \left(\frac{h^3a^2b}{h^2(a^2+b^2)} - 0 \right) \cdot \frac{1}{h}$

$= \frac{a^2b}{a^2+b^2}$

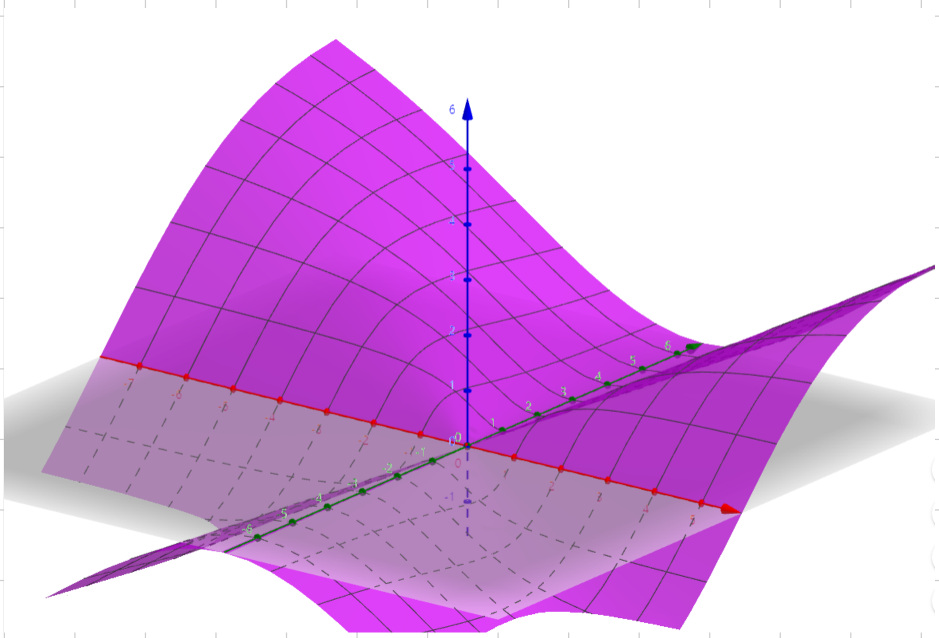
$= a^2b \quad (\because |\vec{u}| = \sqrt{a^2+b^2} = 1)$

2. $z = f(x,y)$가 (a,b)에서 미분가능하려면

f_x, f_y가 (a,b)에서 존재하여 연속이어야 한다.

$$z = f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

의 경우,



← 그림에서처럼
$f_x(0,0)$, $f_y(0,0)$은
존재하지만
$(0,0)$에서 불연속이므로
접평면이 생기지 않는다.
(= 미분가능하지 않다.)

3. $z = f(x,y)$가 (a,b)에서 미분가능하다면

f_x, f_y가 (a,b)에서 존재하고 연속이다.

$$\lim_{h\to 0}\left(f(a+h,b) - f(a,b)\right) = \lim_{h\to 0} h \cdot \frac{f(a+h,b)-f(a,b)}{h}$$
$$= 0 \cdot f_x(a,b) = 0.$$

$$\lim_{h\to 0}\left(f(a,b+h) - f(a,b)\right) = \lim_{h\to 0} h \cdot \frac{f(a,b+h)-f(a,b)}{h}$$
$$= 0 \cdot f_y(a,b) = 0.$$

$\therefore \lim_{h\to 0} f(a+h,b) = f(a,b)$, $\lim_{h\to 0} f(a,b+h) = f(a,b)$ 이므로

$z = f(x,y)$가 (a,b)에서 미분가능하면 (a,b)에서 연속이다.

4. $z=f(x,y)$, $x=r\cos\theta$, $y=r\sin\theta$

① $$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right)=\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial r}\right)$$

$$=\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right)\cos\theta+\frac{\partial z}{\partial r}\cancelto{0}{\frac{\partial}{\partial r}(\cos\theta)}+\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)\sin\theta+\frac{\partial z}{\partial y}\cancelto{0}{\frac{\partial}{\partial r}(\sin\theta)}$$

$$=\left(\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial r}+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial r}\right)\cos\theta+\left(\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial r}+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial r}\right)\sin\theta$$

$$=\left(\frac{\partial^2 z}{\partial x^2}\cos\theta+\frac{\partial^2 z}{\partial y\partial x}\sin\theta\right)\cos\theta+\left(\frac{\partial^2 z}{\partial x\partial y}\cos\theta+\frac{\partial^2 z}{\partial y^2}\sin\theta\right)\sin\theta$$

$$=\frac{\partial^2 z}{\partial x^2}\cos^2\theta+\frac{\partial^2 z}{\partial y^2}\sin^2\theta+\frac{\partial^2 z}{\partial y\partial x}2\sin\theta\cos\theta\qquad\left(\because\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial^2 z}{\partial x\partial y}\right)$$

② $$\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial\theta}\right)=\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial\theta}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial\theta}\right)$$

$$=\frac{1}{r^2}\left\{\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial x}\right)(-r\sin\theta)+\frac{\partial z}{\partial x}(-r\cos\theta)+\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial y}\right)(r\cos\theta)+\frac{\partial z}{\partial y}(-r\sin\theta)\right\}$$

$$=\frac{1}{r}\left\{\left(\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\underset{(-r\sin\theta)}{\cancel{\frac{\partial x}{\partial\theta}}}+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\underset{(r\cos\theta)}{\cancel{\frac{\partial y}{\partial\theta}}}\right)(-\sin\theta)+\left(\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\overset{(-r\sin\theta)}{\cancel{\frac{\partial x}{\partial\theta}}}+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\overset{(r\cos\theta)}{\cancel{\frac{\partial y}{\partial\theta}}}\right)\cos\theta\right.$$
$$\left.-\left(\frac{\partial z}{\partial x}(\cos\theta)+\frac{\partial z}{\partial y}(\sin\theta)\right)\right\}$$

$$=\frac{\partial^2 z}{\partial x^2}\sin^2\theta+\frac{\partial^2 z}{\partial y^2}\cos^2\theta-\frac{\partial^2 z}{\partial y\partial x}2\sin\theta\cos\theta-\frac{1}{r}\left(\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\sin\theta\right)$$

③ $$\frac{1}{r}\frac{\partial z}{\partial r}=\frac{1}{r}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial r}\right)$$

$$=\frac{1}{r}\left(\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\sin\theta\right)$$

① + ② + ③ $$=\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}.$$

$$\therefore f_{xx}+f_{yy}=f_{rr}+\frac{1}{r^2}f_{\theta\theta}+\frac{1}{r}f_r$$