1. $f(x,y) = \frac{x}{x+y}$ at $(2,1)$

$f_x(x,y) = \frac{y}{(x+y)^2}$ $\quad f_y(x,y) = \frac{-x}{(x+y)^2}$

$f_x(2,1) = \lim_{(x,y)\to(2,1)} f_x(x,y) = \frac{1}{9}$

$f_y(2,1) = \lim_{(x,y)\to(2,1)} f_y(x,y) = -\frac{2}{9}$

$\Bigg)$ f_x, f_y exist & continuous at $(2,1)$

$\Rightarrow$ differentiable at $(2,1)$

Linearization at $(2,1)$

$$L(x,y) = f_x(2,1)(x-2) + f_y(2,1)(y-1) + f(2,1)$$
$$= \frac{1}{9}(x-2) - \frac{2}{9}(y-1) + \frac{2}{3}$$
$$= \frac{1}{9}x - \frac{2}{9}y + \frac{2}{3}$$

2. (1) $f_x(x,y) = \frac{-\sin x\cos x}{\sqrt{y+\cos^2 x}}$ $\quad f_y(x,y) = \frac{1}{2\sqrt{y+\cos^2 x}}$

$$z = f_x(0,0)(x-0) + f_y(0,0)(y-0) + f(0,0)$$

tangent plane $z = \frac{1}{2}y + 1$

(2) $f_x(x,y) = \frac{1}{x}$ $\quad f_y(x,y) = -\frac{1}{y}$

$$z = f_x(1,2)(x-1) + f_y(1,2)(y-2) + f(1,2)$$
$$= x-1-\frac{1}{2}(y-2) + \ln\left(\frac{1}{2}\right)$$

tangent plane $z = x - \frac{1}{2}y - \ln 2$

3. $z=\sqrt{x^2+y^2}$ near $(3,4)$ $\quad \Delta x=-0.02$, $\Delta y=0.03$

$$f_x(x,y)=\frac{x}{\sqrt{x^2+y^2}}, \quad f_y(x,y)=\frac{y}{\sqrt{x^2+y^2}}$$

$$L(x,y)=f_x(3,4)(x-3)+f_y(3,4)(y-4)+f(3,4)\Big|_{\substack{x=2.98\\ y=4.03}}$$

$$\sqrt{(2.98)^2+(4.03)^2}\approx -\frac{6}{500}+\frac{12}{500}+5=5.012$$

4. $dz=f_x(x,y)dx+f_y(x,y)dy$

(1) $f_x(x,y)=2xye^{x^2y} \quad f_y(x,y)=x^2e^{x^2y}$

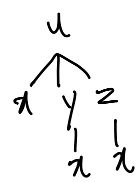
$$dz=2xye^{x^2y}dx+x^2e^{x^2y}dy$$

(2) $f_x(x,y)=\cos x\cos(x+y)-\sin x\sin(x+y)=\cos(2x+y)$

$f_y(x,y)=\cos x\cos(x+y)$

$$dz=\cos(2x+y)dx+\cos x\cos(x+y)dy$$

5.

(1) 

$$\frac{du}{dx}=\frac{du}{dx}\frac{dx}{dx}+\frac{du}{dy}\frac{dy}{dx}+\frac{du}{dz}\frac{dz}{dx}$$

$$=\frac{2x}{x^2+y^2+z^2}+\frac{2y(\sin x+x\cos x)}{x^2+y^2+z^2}+\frac{2z(\cos x-x\sin x)}{x^2+y^2+z^2}$$

$y=x\sin x$
$z=x\cos x$

$$=\frac{4x}{x^2+y^2+z^2}$$

(2)

$$\begin{array}{ccc} & z & \\ x & & y \\ r\ \ \theta & & r\ \ \theta \end{array}$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

$x = r\cos\theta$, $y = r\sin\theta$

$$= y e^{xy}\cos\theta + x e^{xy}\sin\theta$$

$$= r\sin\theta\cos\theta\, e^{r^2\sin\theta\cos\theta}$$

(3)

$$\begin{array}{ccc} & z & \\ x & & y \\ u\ \ v & & u\ \ v \end{array}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$x = u^2+v^2$, $y = 2uv$

$$= -\frac{y}{x\sqrt{x^2-y^2}}(2u) + \frac{1}{\sqrt{x^2-y^2}}(2v)$$

$$= -\frac{2uv}{(u^2+v^2)(u^2-v^2)}(2u) + \frac{1}{u^2-v^2}(2v)$$

$$= \frac{-4u^2v+2u^2v+2v^3}{(u^2+v^2)(u^2-v^2)} = \frac{-2v\cancel{(u^2-v^2)}}{(u^2+v^2)\cancel{(u^2-v^2)}}$$

$$= \frac{-2v}{u^2+v^2}$$

$(|x|>0)$
$$\frac{\partial z}{\partial x} = -\frac{y}{x\sqrt{x^2-y^2}} \qquad \frac{\partial x}{\partial u} = 2u \qquad x^2-y^2 = (u^2+v^2)^2-(2uv)^2$$

$$\frac{\partial z}{\partial y} = \frac{1}{\sqrt{x^2-y^2}} \qquad \frac{\partial y}{\partial u} = 2v \qquad = (u^2-v^2)^2 > 0 \quad (\because u^2 > v^2)$$

(4)
$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-2x^2-2xy}{\frac{3}{x}-2z} = \frac{2x^3+2x^2y}{3-2xz}$$

(5)
$$\frac{\partial x}{\partial y} = -\frac{F_y}{F_x} = -\frac{3y^2e^{x+z}-z\cos(y-x)}{y^3e^{x+z}+z\cos(y-x)}$$

6.

(1) $f_x = -\frac{y^2}{x^2}$ $f_y = \frac{2y}{x}$

$$D_u f(1,2) = \langle f_x(1,2), f_y(1,2) \rangle \cdot \langle \frac{2}{3}, \frac{\sqrt{5}}{3} \rangle$$

$$= \langle -4, 4 \rangle \cdot \langle \frac{2}{3}, \frac{\sqrt{5}}{3} \rangle = \frac{4\sqrt{5}-8}{3}$$

(2) $f_x = \frac{1}{x^2+1} y$ $f_y = \tan^{-1} x$ ($\tan^{-1} 1 = \frac{\pi}{4}$) $\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$

$$D_u f(1,2) = \langle f_x(1,2), f_y(1,2) \rangle \cdot \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$$

$$= \langle 1, \frac{\pi}{4} \rangle \cdot \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle = \frac{1}{\sqrt{5}} + \frac{\pi}{2\sqrt{5}}$$

(3) $f_x = y\cos(x+z)e^{y\sin(x+z)}$ $f_y = \sin(x+z)e^{y\sin(x+z)}$

$f_z = y\cos(x+z)e^{y\sin(x+z)}$ $\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle$

$$D_u f = \langle f_x(1,0,1), f_y(1,0,1), f_z(1,0,1) \rangle \cdot \langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle$$

$$= \langle 0, \sin 2, 0 \rangle \cdot \langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle = 0$$

(4) $f_x = e^y - y\sin(xy)$ $f_y = xe^y - x\sin(xy)$ $\frac{\vec{u}}{|\vec{u}|} = \langle \frac{4}{5}, \frac{3}{5} \rangle$

$$D_u f = \langle f_x(3,1), f_y(3,1) \rangle \cdot \langle \frac{4}{5}, \frac{3}{5} \rangle$$

$$= \langle e - \sin 3, 3e - 3\sin 3 \rangle \cdot \langle \frac{4}{5}, \frac{3}{5} \rangle$$

$$= \frac{13e - 13\sin 3}{5}$$

7. $\nabla f(x,y) = \langle 2x-2, 2y-4 \rangle$ direction of $\nabla f = \langle 1,1 \rangle$

$2x-2 = 2y-4$, $y = x+1$

all points are on line $y=x+1$

8.

(1) $F(x,y,z) = x^2 - y - z^2$

$\nabla F = \langle 2x, -1, -2z \rangle \Big|_{(4,7,3)} = \langle 8, -1, -6 \rangle$

(a) tangent plane : $8(x-4) - (y-7) - 6(z-3) = 0$

(b) normal line : $\frac{x-4}{8} = \frac{y-7}{-1} = \frac{z-3}{-6}$

(2) $F(x,y,z) = x^2 - xyz - z^3 - 1$

$\nabla F = \langle 2x-yz, -xz, -xy-3z^2 \rangle \Big|_{(1,1,1)} = \langle 1, -1, -4 \rangle$

(a) tangent plane : $(x-1) - (y-1) - 4(z-1) = 0$

(b) normal line : $\frac{x-1}{1} = \frac{y-1}{-1} = \frac{z-1}{-4}$

9. $f_x = e^{2y}$ $f_y = 2xe^{2y}$ $\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \rangle$

$f_{xx} = 0$ $f_{xy} = 2e^{2y}$ $f_{yy} = 4xe^{2y}$

$\vec{u} = \langle a.b \rangle$ 일 때

$D_u f(x.y) = \nabla f \circ \vec{u} = a f_x + b f_y$

$D_u(D_u f) = \langle a f_{xx} + b f_{yx},\ a f_{xy} + b f_{yy} \rangle \circ \langle a.b \rangle$

$= a^2 f_{xx} + 2ab f_{xy} + b^2 f_{yy}$

$a = \frac{2}{\sqrt{13}}$ $b = \frac{3}{\sqrt{13}}$

$\therefore D_u(D_u f) = \frac{4}{13} f_{xx} + \frac{12}{13} f_{xy} + \frac{9}{13} f_{yy}$

$= \frac{24}{13} e^{2y} + \frac{36}{13} x e^{2y}$

10. $f(x.y.z) = x^2 + y^2 - z^2 - 1$

$g(x.y.z) = x + y + z - 5$

$\nabla f = \langle 2x, 2y, -2z \rangle$ $\nabla f|_{(1.2.2)} = \langle 2.4.-4 \rangle$

$\nabla g = \langle 1, 1, 1 \rangle$ $\nabla g|_{(1.2.2)} = \langle 1.1,1 \rangle$

$\nabla f|_{(1,2,2)} \times \nabla g|_{(1,2,2)}$

$$= \begin{vmatrix} i & j & k \\ 2 & 4 & -4 \\ 1 & 1 & 1 \end{vmatrix} = (4+4)i - (2+4)j + (2-4)k$$

$$= \langle 8, -6, -2 \rangle$$

tangent line

$: \; 8(x-1) - 6(y-2) - 2(z-2) = 0$

$\therefore \; z = 4x - 3y + 4$