M5 Exam Solutions

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1 Q1 - Structural Models

1.1 Part (a)

- Workings can be found in file **Q1_structural_models.xlsm** attached to the portal
- Model constant parameters:
  - $D = K = 5$ (Merton's notional or Black-Cox threshold)
  - $T = 1$ (maturity)
  - $r = 2\%$ (risk-free rate)
- For Merton's model Excel's Solver was used by setting $E_0 = 3$ as objective by changing the values of V_0 and σ_V with constrains $\sigma_E E_0 = N(d_1)\sigma_V V_0$, using the "GRG Nonlinear" algorithm. Using $\sigma_E = 50\%$ and the analytical result for d_1, d_2, $N(d_1)$, $N(d_2)$ and E_0, the resulting V_0 and σ_V were:

vol_E	V_0	vol_V
50%	7.8986	19.0804%

1.2 Part (b)

- To compare the sensitivity of the default probability (PD) for both Merton and Black-Cox against σ_E, both models were ran for different σ_E and $D = K = 5$. The workings can be found also in **Q1_structural_models.xlsm**
- The resulting plot of PD vs σ_E is shown below, where we see the difference becomes negligible as $\sigma_E \to 0$, so both models approximately agree for $\sigma_E \lesssim 50\%$. Then as $\sigma_E \to 1$, they appear to diverge exponentially. This is likely due to a different term structure of the hazard rate λ
- See excel workings for more details and VBA macro "loop_solver"

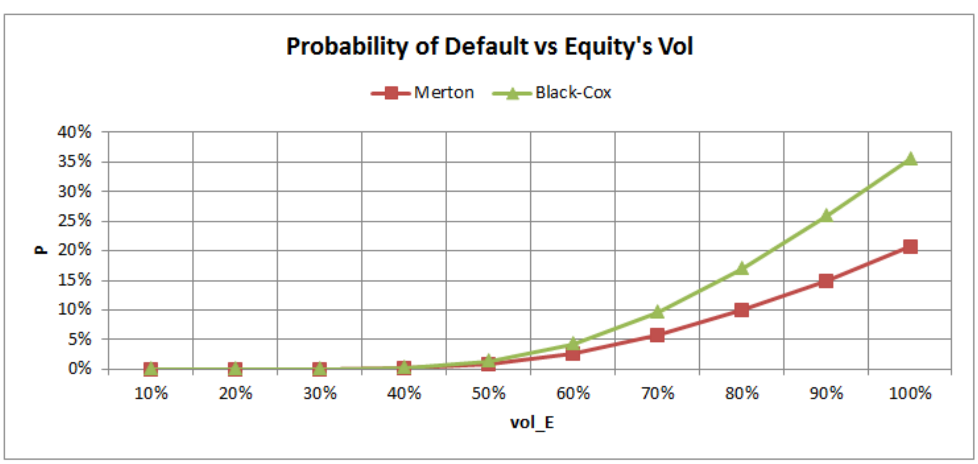

Figure 1: title

2 Q2 - Bivariate Call Pricing Using Copula

See hand-written workings attached

M5 EXAM

(Q2) Bivariate European binary call:

$$B(S_1, S_2) = e^{-r(T-t)} C(u_1, u_2)$$

- Using Frank Copula (Archimedean) for joint cumulative probability:

$$C(u_1, u_2) = \frac{1}{\alpha} \ln\left[1 + \frac{(e^{\alpha u_1}-1)(e^{\alpha u_2}-1)}{(e^{\alpha}-1)}\right]$$

- Instead of correlation, association parameter α determines dependance
- α related to Kendall's tau via:

$$\rho_K = 1 - \frac{4}{\alpha}[D_1(-\alpha) - 1] \quad \text{where } D_1(-\alpha) = \frac{1}{\alpha}\int_0^{\alpha} \frac{x}{e^x - 1}\,dx + \frac{\alpha}{2}$$

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a) To estimate α, we use Taylor Series expansion for

$$\frac{x}{e^x - 1} \approx 1 - \frac{x}{2} + \frac{x^2}{12} - O(x^4)$$

Ignoring higher order terms, solve for α analytically: using given value of $\rho_K = 0.35$

$$\int_0^{\alpha} \frac{x}{e^x-1}\,dx \approx \int_0^{\infty}\left(1 - \frac{x}{2} + \frac{x^2}{12} - O(\dots)\right) dx$$

$$\approx \left[x - \frac{x^2}{4} + \frac{1}{36}x^3\right]_0^{\alpha} = \alpha - \frac{\alpha^2}{4} + \frac{\alpha^3}{36}$$

$$\therefore \rho_K = 1 - \frac{4}{\alpha}[D_1(-\alpha) - 1]$$

$$\approx 1 - \frac{4}{\alpha}\left[\cancel{1} - \frac{\alpha}{4} + \frac{\alpha^2}{36} + \frac{\alpha}{2} \cancel{-1}\right]$$

$$\therefore \rho_K \approx \cancel{1} + 1 - \frac{\alpha}{9} - 2 \quad \rightarrow \quad \boxed{\alpha \approx -9\rho_K}$$

$$\therefore \alpha \approx -9 \times 0.35 = -3.15$$

$$\therefore \boxed{\alpha \approx -3.15}$$

b) We then need the risk-neutral prob. of a binary call option being ITM.

Using the Black-Scholes analytical result, instead of MC

ITM EU call $\boxed{u = N(d_2) \quad \text{where } d_2 = \frac{\ln(S/K) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}}$

c) Using above result for each individual binary call:

$$u_1 = N\left(\frac{\ln(90/120) + (0 - \frac{0.3^2}{2})(0.5)}{0.3\sqrt{0.5}}\right) \approx 0.0718$$

$$u_2 = N\left(\frac{\ln(110/120) + (0 - \frac{0.5^2}{2})(0.5)}{0.5\sqrt{0.5}}\right) \approx 0.3362$$

d) Inserting into $C(u_1, u_2)$ expression:

$$C(u_1, u_2) \approx -\frac{1}{3.15}\ln\left[1 + \frac{(e^{-3.15\times(0.0718)} - 1)(e^{-3.15\times 0.3362} - 1)}{(e^{-3.15} - 1)}\right]$$

≈ 0.04722 //

↳ See detailed workings in excel

3 Q3 - Credit Curve

All the workings can be found in the file **Q3_credit_curve.xlsx** attached to the portal

3.1 Part 1 - CDS pricing from hazard rate and DF data

- The workings can be found in the tab *part 1* in the excel file
- Constant parameters in model:
  - $\Delta t = 0.25$ (quarterly increments)
  - $N = 1$ (notional)
  - $R = 40\%$ (recovery rate), hence LGD $L = (1 - R)$
- Quarter interpolation from yearly data provided:
  - Discount factors using log-linear interpolation (same formula as given on exam)
  - Hazard rates λ using linear interpolation (same formula as given on exam)
- The cummulative survival probability $P(0, T)$ calculated using formula:

$$P(0,T) = \exp - \sum_{t=1}^{T} \lambda_t \Delta t$$

from this, the period probability of default (PD) was calculated

- The 'premium leg' (PL) was calculated as in the formula from the notes, as well as the contribution to it from accruals. The 'default leg' (DL) was also calculated as per the formula in the notes
- Excel's Solver was then used to calculate the spread on the assumption of a flat spread across all tenors, conditioned on the MTM = 0, i.e. PL = DL
- This resulted in a spread of 92.1368 bps, see workings for details

3.2 Part 2 - Bootstrapping Survival Probabilities

- The workings can be found in the tab *part 2 & 3* in the excel file
- The cumulative survival probability 'PrSurv' was bootstrapped from CDS yearly spread data using the formula in the lecture notes, for yearly tenors up to 5 years. This required:
  - CDS spreads: given for yearly increments up to 5Y tenor
  - Discount factors: approximated using continuous time formula $\exp(-rT_i)$ with $r = 0.8\%$ for each tenor
  - Loss given default: $L = 1 - R = 60\%$
- From PrSurv, the cumulative and period default probabilities ('PD' and 'PD_cum') were calculated. The results and plot are shown below

TIME (Years)	PD	PD_cum	P_cum (PrSurv)
0.0	0.00000%	0.0000%	100.0000%
1.0	2.30813%	2.3081%	97.6919%
2.0	2.99713%	5.3053%	94.6947%
3.0	3.61525%	8.9205%	91.0795%
4.0	3.91269%	12.8332%	87.1668%
5.0	3.73397%	16.5672%	83.4328%

3.3 Par 3 - Term Structure of Hazard Rates

- The workings can be found in the tab *part 2 & 3* in the file above

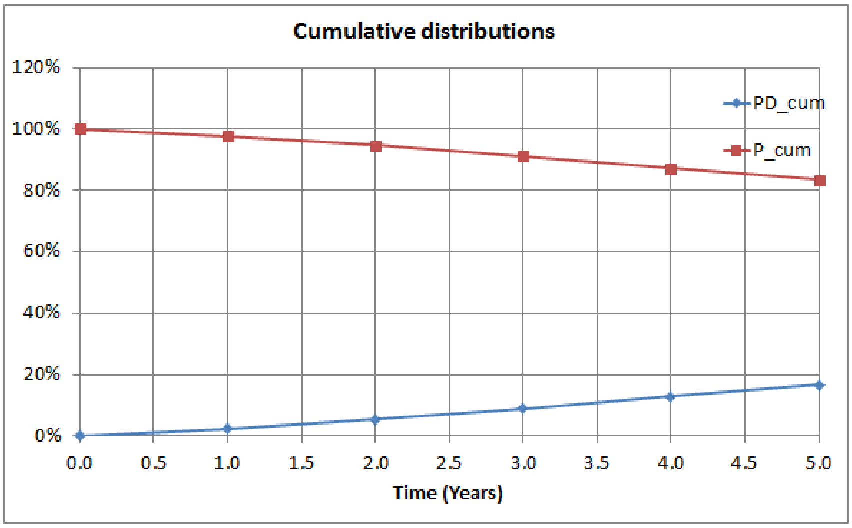

Figure 2:

- From PsSurv $P(0,T)$, the hazard rates can be calculated using the formula:

$$\lambda_m = -\frac{1}{\Delta t} \ln \frac{P(0,t_m)}{P(0,t_{m-1})}$$

The result is shown below - see excel workings for details

TIME (Years)	Lambda
0.0	-
1.0	2.33519%
2.0	3.11599%
3.0	3.89258%
4.0	4.39091%
5.0	4.37817%

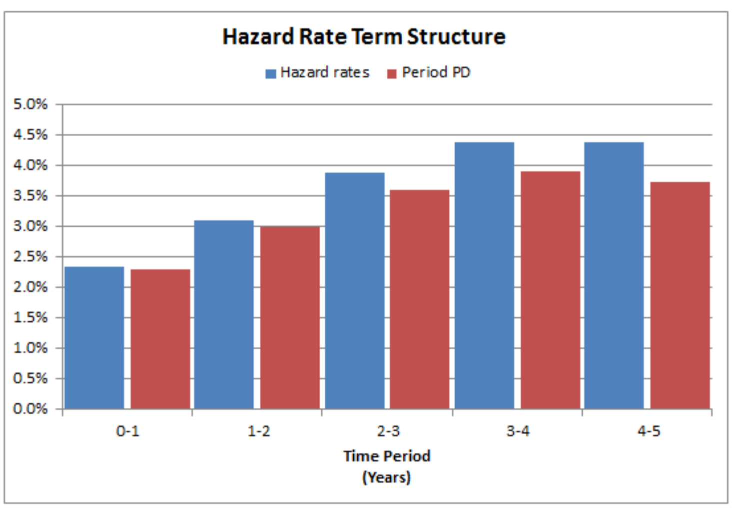

Figure 3:

- Plotting the exponential pdf $f(t) = \lambda e^{-\lambda t}$ for piecewise constant lambda the below plot is obtained. We note the instability at every tenor change, where we see 'jumps' instead of a smooth continous function. In theory each tenor period will have it's own f(t) but given we only get the hazard rate for a given period, we see this reflected as well

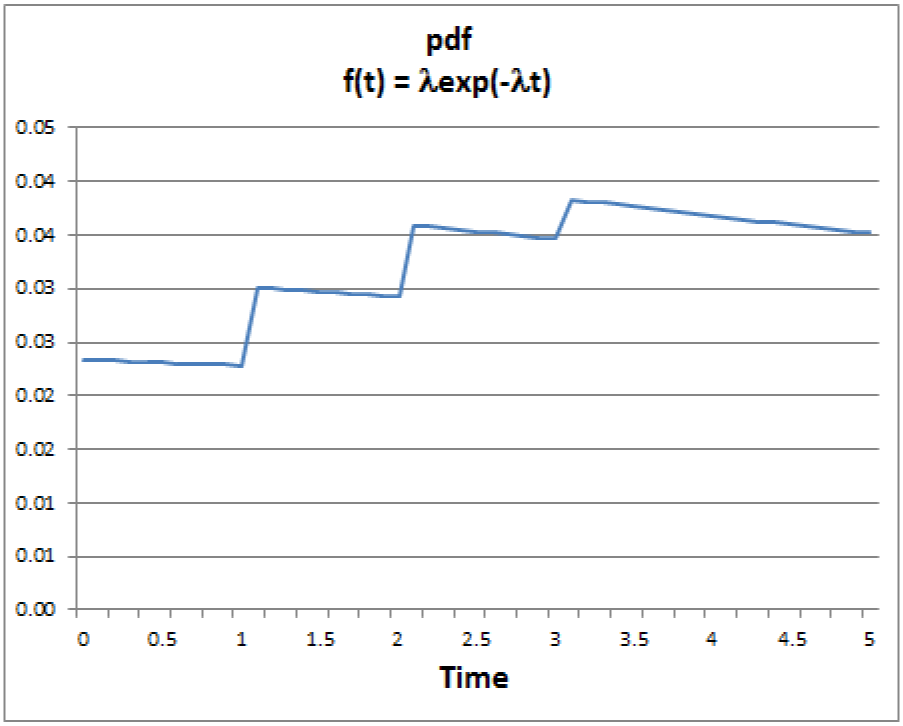

Figure 4: