Credit Valuation Adjustment for an Interest Rate Swap

Tanya Sandoval

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Abstract

In this report we demonstrate some of the techniques currently used to calculate Credit Valuation Adjustments for fixed income instruments. We take the hypothetical scenario of an Interest Rate Swap entered by two counterparties. Through the simulation of forward rates using the HJM model on recent data and default probabilities using Credit Default Swaps spreads, we arrive at the fair price of the risk taken by counterparty A in entering the swap with counterparty B.

Contents

1 Introduction

To calculate the Credit Valuation Adjustment (CVA) taken by counterparty 'A' to the price of an Interest Rate Swap (IRS) with counterparty 'B', the main inputs are:

- Default Probabilities (PDs)
- Forward LIBORs
- Discount Factors (DFs)

Below we discuss each of these aspects separately. Additional assumptions for this scenario are:

- The IRS is assumed to be written on a 6M LIBOR L_{6M} expiring in 5Y, hence the payment frequency is $\tau = 0.5$
- The notional is assumed to be $N = 1$

2 Default Probabilities

The Default Probabilities can be implied from Credit Default Swaps (CDS) spreads using the bootstrapping technique.

2.1 CDS bootstrapping

The bootstrapping formula used was:

$$P(T_N) = \frac{\sum_{n=1}^{N-1} D(0,T_n)[LP(T_n-1) - (L + \Delta t_n S_N)P(T_n)]}{D(0,T_N)(L + \Delta t_n S_N)} + \frac{P(T_{N-1})L}{(L + \Delta t_N S_N)} \tag{1}$$

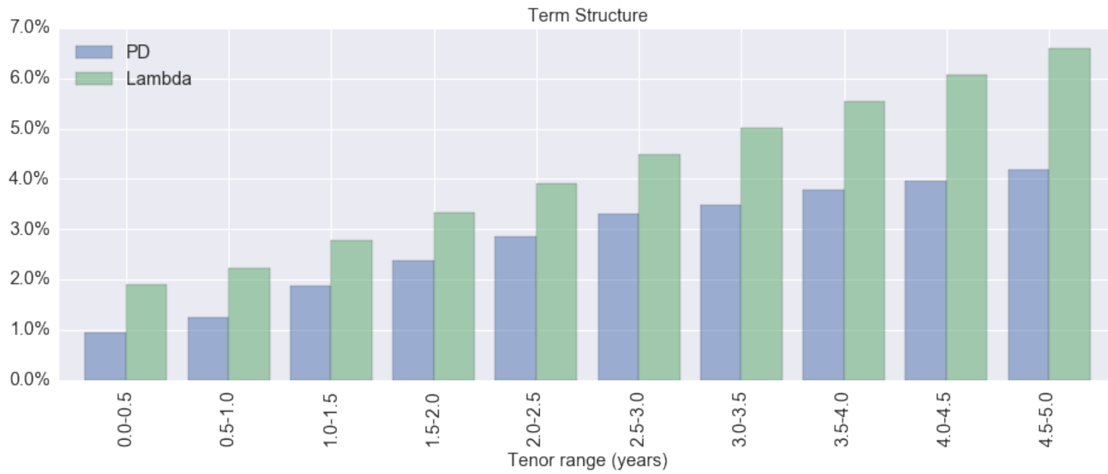
where P is the survival probability, $L = 1 - R$ is the expected loss calculated from the recovery rate R, Δt is the payment frequency, S is the CDS spread and D is the discount factor

- The counterparty B in this case was chosen to be an airline company - AirFrance ("AIRF")
- The CDS spreads were taken from Reuters on 27-Jun-2016 and are assumed to apply also for the period 31-May-2013 to 31-May-2016, which was used to calibrate the HJM model (see below)
- The Recovery Rate is assumed to be 40%
- Linear interpolation is used to approximate the CDS spreads at the half increments for which there was no market data available
- For consistency and simplicity, the discount factors used were the same as those implied from the HJM model after taking their average for each tenor (see methodology below)
- The full calculation can be found in the project repository file "*CVA/my CDS Bootstrapping v2.xlsx*"

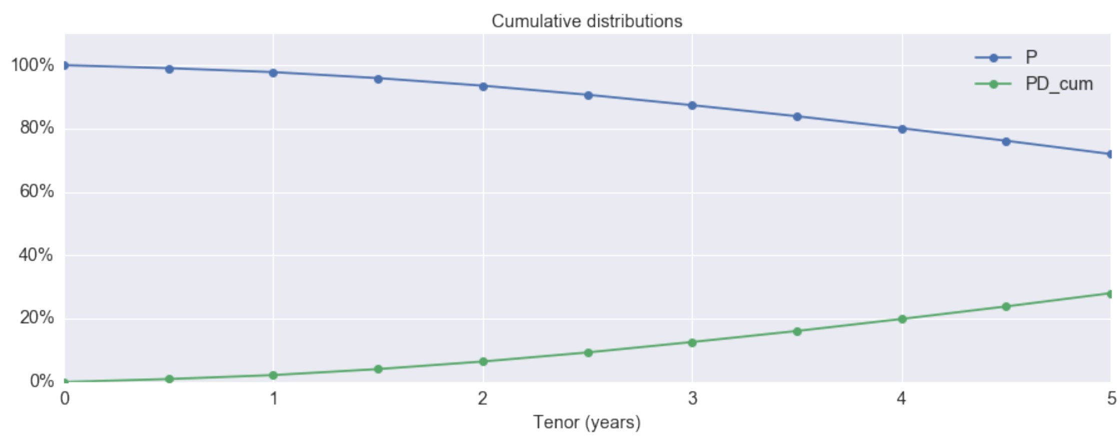
The table below summarises the results for each of the tenors $[0.0, 1.0, \ldots, 5.0]$, with 'DF' as the discount factor, 'Lambda' as the hazard rate λ_i, 'PD' as the default probability $PD(T_i, T_{i-1}) = P(T_{i-1}) - P(T_i)$ and 'P' as the survival probability. Note that by definition PD is over a period, whereas P is cumulative.

Tenor	CDS	DF	Lambda	PD	P
0.0	NaN	1.000000	nan%	nan%	100.0000%
0.5	114.400	0.995835	1.8976%	0.9443%	99.0557%
1.0	133.770	0.990963	2.2197%	1.2510%	97.8047%
1.5	167.180	0.985697	2.7798%	1.8887%	95.9161%
2.0	200.590	0.980105	3.3452%	2.3875%	93.5285%
2.5	233.965	0.974101	3.9174%	2.8579%	90.6707%
3.0	267.340	0.967832	4.4994%	3.2974%	87.3732%
3.5	296.545	0.961232	5.0170%	3.4776%	83.8957%
4.0	325.750	0.954239	5.5471%	3.7947%	80.1009%
4.5	353.200	0.946899	6.0576%	3.9605%	76.1405%
5.0	380.650	0.939187	6.5842%	4.1914%	71.9490%

The figures below show the term structure of the estimated default probability and hazard rates. As these aren't flat, this needs to be accounted in the CVA.



The plot below shows how the cumulative distribution for the survival probability P decreases with time, and vice versa for the cumulative PD since by definition this is equal to $1 - P$.



2.2 Forward LIBORs

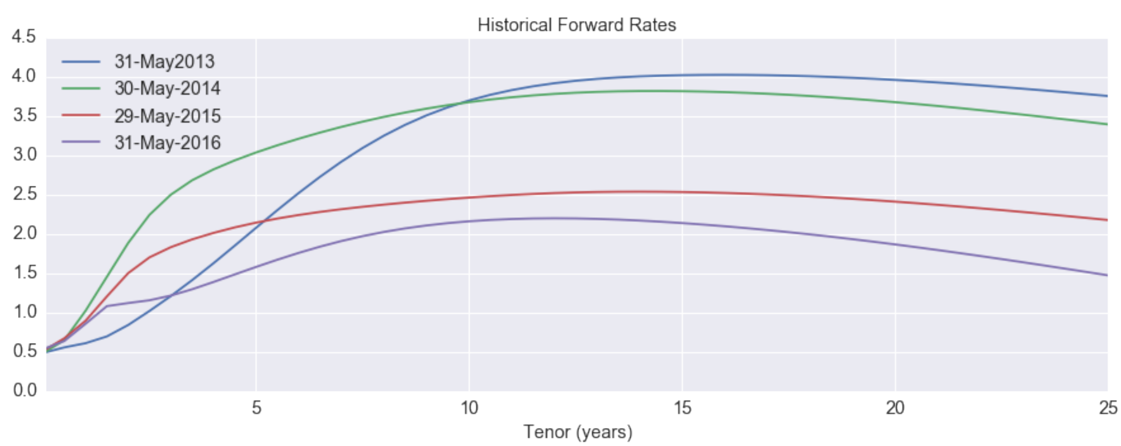
We then proceed to simulate the forward rates with the HJM model. To give a more realistic picture of the current CVA value, the HJM model was first calibrated to recent data . The steps taken are described in detail below.

2.2.1 PCA

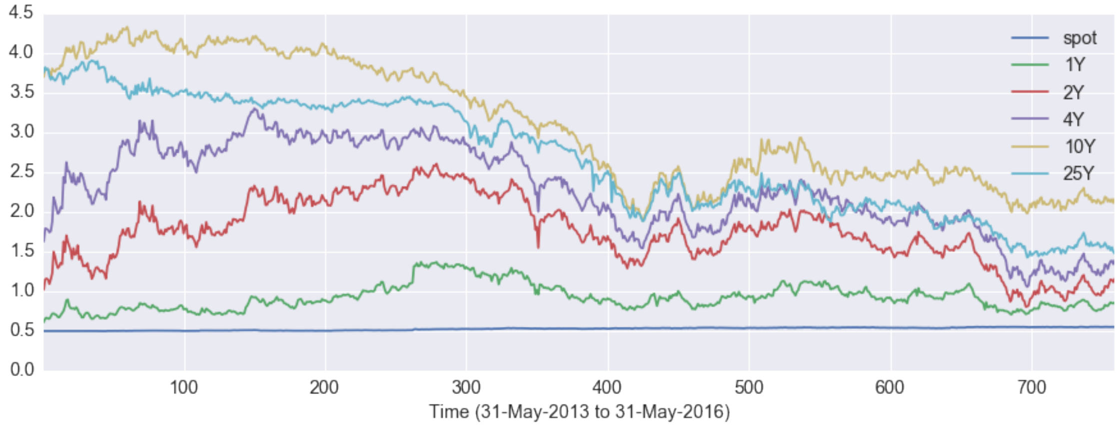
Dataset The HJM model requires a set of volatility functions which are estimated using Principal Component Analysis (PCA) on historical forward rates.

- To calibrate these functions to recent data, the forward rates were taken from the Bank of England (BoE) Bank Liability Curve (BLC) for the last 3 years (31-May-2013 to 31-May-2016). The data was taken from:
  - ukblc16_mdaily.xlsx
  - ukblc05_mdaily.xlsx
- Although we only need the short-end of the curve for the CVA calculation, we take the full curve to calibrate the HJM model, i.e. up the $25Y$ tenor. The dataset is hence constructed by taking the BLC forward curve short-end data (*"1. fwds,short end"* tab) for the tenors $[0.5Y, 1.0Y, 1.5Y, ..., 5Y]$ as it offers a better approximation to the short-end. For the remaining tenors $[5.5Y, ..., 25.0Y]$ we use the full approximation (*"2. fwd curve"* tab)
- The forward rate for the tenor $0.08Y$ in *"1. fwds,short end"* is used as a a proxy for the spot rate tenor $0.0Y$, i.e. $r(t) = f(t;t)$
- Dates with missing data values were removed
- The resulting dataset is found in the project repository under *CVA/PCA/my_ukblc_310513_310516.xlsx*

The figure below shows the forward curve for four sample dates spanning different years in the dataset. This shows how the curve can move for each of the tenors, for example, at the long-end of the curve the rates have decreased substantially in 2016 compared to 3 years ago. At the short-end of the curve we see the rates tend to increase with tenor as expected.



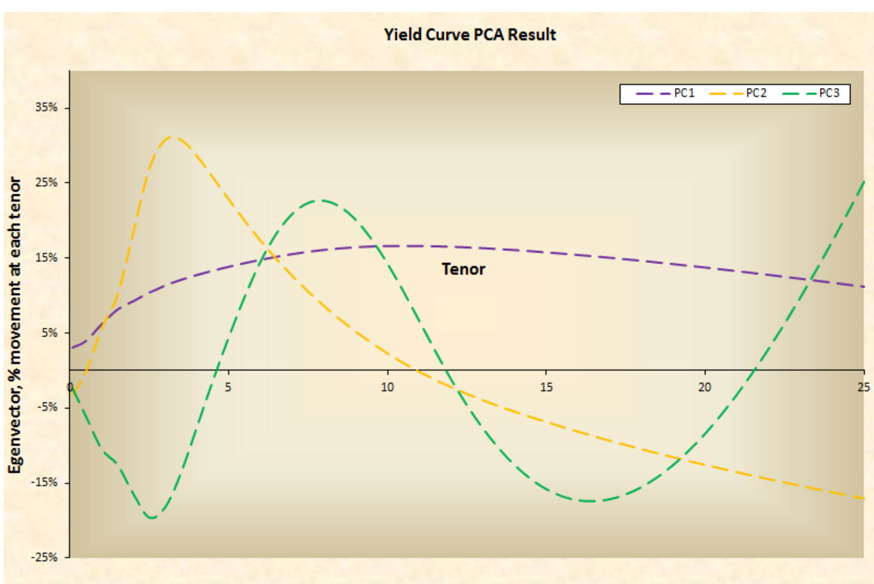
The next figure shows the historical evolution of the rate in the dataset for 3 example tenors. As mentioned, the $0.08Y$ tenor is a proxy for the spot rate. This shows the spot rate has remained quite constant throughout. The rate increases for the shorter tenors but then decreases for the longer tenors, .e.g. $25Y$ vs $10Y$ below. This could be interpreted as a lack of liquidity for the longer tenors.



Principal Components To obtain the volatility functions for the HJM model, the day-on-day changes (differences) for each tenor are calculated. This produces a set of independent random variables that can be used to get the principal components (PC), taken as the 3 largest eigenvalues and corresponding eigenvectors that explain 97.5% of the observed variance. The calculation details can be found in the project repository under "*PCA/my HJM Model - PCA.XLSM*". The results are summarised below:

	Tenor	Eigenvalue	Cum. R^2
1st largest PC	10.0	0.0076790	0.9107
2nd largest PC	3.0	0.0004034	0.9586
3rd largest PC	6.5	0.0001406	0.9752

The figure below shows the resulting eigenvectors of the principal components. In general, the largest component (PC1) is attributed to parallel shifts in the curve, the 2nd largest (PC2) to steepening/flattening (skewness) and 3rd largest to bending about specific maturity points (convexity).



We now proceed to use these eigenvectors to obtain the volatility functions for the HJM model.

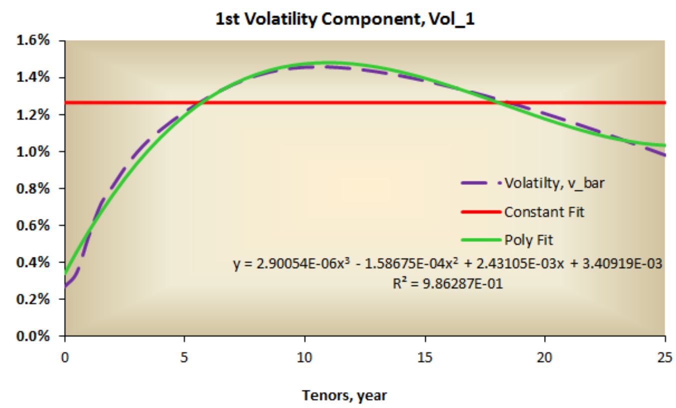
2.2.2 HJM

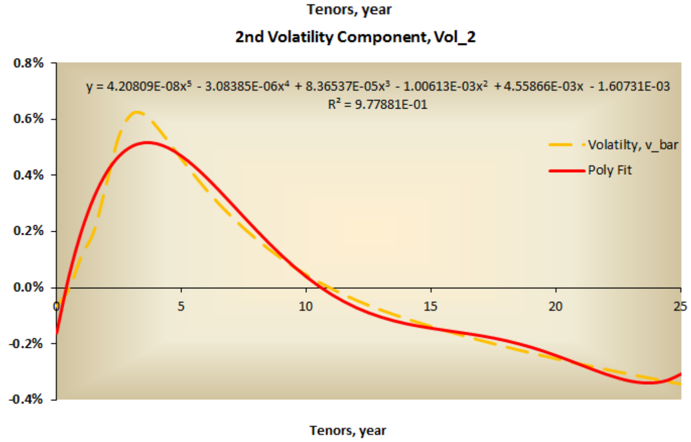
Volatility functions The volatility functions for the HJM model are defined as:

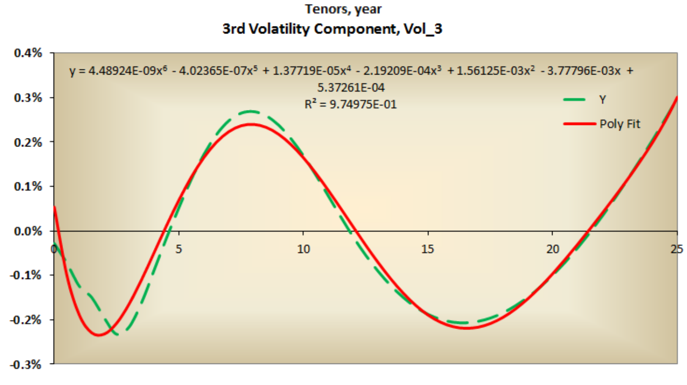
$$\text{Vol}_i = \sqrt{\lambda_i} e(i) \quad \forall \quad i = 1, 2, 3 \tag{2}$$

where λ_i is the eigenvalue and $e(i)$ the eigenvector. This is equivalent to one standard deviation move in the $e(i)$ direction.

The volatility functions have to then be fitted as we need analytical functions to carry out the integration to get the drift in the HJM model. In this case, 3rd, 5th and 6th degree polynomials were fitted to guarantee a goodness of fit of over 97%. These are shown in the figures below. In particular, for Vol_1 fitting a constant (red line) is not really suited and justifies the need for a polynomial fit (however, this could also lead to overfitting issues).







Drift The drift function $\mu(t)$ is obtained by integrating over the principal components and assuming that volatility is a function of time.

Monte Carlo The calibrated volatility and drift functions above are then entered into the HJM model Monte Carlo simulation script *my_HJM_model.py*, which evolves the whole forward curve according to the stochastic differential equation (SDE):

$$d\bar{f} = \mu(t)dt + \sum_{i=1}^{3} Vol_i \phi_i \sqrt{dt} + \frac{dF}{d\tau}dt \tag{3}$$

where ϕ_i is a random number drawn from the standard normal distribution and the last term is the *Musiela correction*. For brevity the details of the model are omitted here but additional details of the MC simulation are outlined below:

- The forward curve was initialised using the the last observed forward curve (last row in the BLC data)
- Time step taken was $dt = 0.01$
- Number of simulations used was $I = 1000$
- The random variables were taken from python's numpy module *np.random.standard_normal* which draws numbers from a standard normal distribution $N(\mu = 0, \sigma = 1)$
- Due to time constrains, the antithetic variance reduction technique was not implemented in the script, so the simulation error is less negligible

To obtain an expectation of LIBOR rate in the future $L(t; T_i, T_{i+1})$, the forward rate f is selected from the corresponding tenor column $\tau = T_{i+1} - T_i$ of the HJM output, from the correct simulated time t. This is then converted to a LIBOR rate using $L = \frac{1}{\tau}(e^{f\tau} - 1)$ where $\tau_i = 0.5\ \forall\ i$ in this case.

For the IRS in question (written on 6M LIBOR for 5Y), the relevant simulated rates we need are:

L(t; T_i, T_i+1) for 6M IRS
L(t; 0, 0.5)
L(t; 0.5, 1)
L(t; 1, 1.5)
L(t; 1.5, 2)
L(t; 2, 2.5)
L(t; 2.5, 3)
L(t; 3, 3.5)
L(t; 3.5, 4)
L(t; 4, 4.5)
L(t; 4.5, 5)

The figure below shows a sample of 100 simulations for the above LIBOR rates

3 Discount Factors

The DFs are implied from the HJM Forward LIBOR for each simulation via the formula:

$$DF(0, T_{i+1}) = \prod_i \frac{1}{1 + \tau_i L(t; T_i, T_{i+1})} \tag{4}$$

which is equivalent to 'integrating under the curve'. This then gives 1000 simulations of the DF for each tenor. For simplicity, an expectation across all simulations is taken to get a single value for $DF(0, T_{i+1})$. This is then used to obtain the forward-starting discount factors as a single number $DF(T_i, T_{i+1})$ for the IRS in question, where:

$$DF(T_i, T_{i+1}) = \frac{DF(0, T_{i+1})}{DF(0, T_i)} \tag{5}$$

and by definition $DF(T_{i+1}, T_{i+1}) = 1.0$

Tenor	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
0.5	1.0	DF(0.5, 1.0)	DF(0.5, 1.5)	DF(0.5, 2.0)	DF(0.5, 2.5)	DF(0.5, 3.0)	DF(0.5, 3.5)	DF(0.5, 4.0)	DF(0.5, 4.5)	DF(0.5, 5.0)
1.0	-	1.0	DF(1.0, 1.5)	DF(1.0, 2.0)	DF(1.0, 2.5)	DF(1.0, 3.0)	DF(1.0, 3.5)	DF(1.0, 4.0)	DF(1.0, 4.5)	DF(1.0, 5.0)
1.5	-	-	1.0	DF(1.5, 2.0)	DF(1.5, 2.5)	DF(1.5, 3.0)	DF(1.5, 3.5)	DF(1.5, 4.0)	DF(1.5, 4.5)	DF(1.5, 5.0)
2.0	-	-	-	1.0	DF(2.0, 2.5)	DF(2.0, 3.0)	DF(2.0, 3.5)	DF(2.0, 4.0)	DF(2.0, 4.5)	DF(2.0, 5.0)
2.5	-	-	-	-	1.0	DF(2.5, 3.0)	DF(2.5, 3.5)	DF(2.5, 4.0)	DF(2.5, 4.5)	DF(2.5, 5.0)
3.0	-	-	-	-	-	1.0	DF(3.0, 3.5)	DF(3.0, 4.0)	DF(3.0, 4.5)	DF(3.0, 5.0)
3.5	-	-	-	-	-	-	1.0	DF(3.5, 4.0)	DF(3.5, 4.5)	DF(3.5, 5.0)
4.0	-	-	-	-	-	-	-	1.0	DF(4.0, 4.5)	DF(4.0, 5.0)
4.5	-	-	-	-	-	-	-	-	1.0	DF(4.5, 5.0)
5.0	-	-	-	-	-	-	-	-	-	1.0

The numerical values are shown below:

Tenor	0.0	0.5	1.0	1.5	2.0	2.5	3.0	\
0.0	1	0.995835	0.990963	0.985697	0.980105	0.974101	0.967832	
0.5	0	1	0.995108	0.989819	0.984204	0.978175	0.97188	
1.0	0	0	1	0.994686	0.989043	0.982984	0.976658	
1.5	0	0	0	1	0.994327	0.988236	0.981876	
2.0	0	0	0	0	1	0.993874	0.987478	
2.5	0	0	0	0	0	1	0.993565	
3.0	0	0	0	0	0	0	1	
3.5	0	0	0	0	0	0	0	
4.0	0	0	0	0	0	0	0	
4.5	0	0	0	0	0	0	0	
5.0	0	0	0	0	0	0	0	

Tenor	3.5	4.0	4.5	5.0
0.0	0.961232	0.954239	0.946899	0.939187
0.5	0.965253	0.958231	0.950859	0.943115
1.0	0.969998	0.962941	0.955534	0.947752
1.5	0.975181	0.968086	0.960639	0.952816
2.0	0.980744	0.973609	0.96612	0.958252
2.5	0.986789	0.97961	0.972075	0.964158
3.0	0.993181	0.985955	0.978371	0.970403
3.5	1	0.992725	0.985088	0.977066
4.0	0	1	0.992308	0.984226
4.5	0	0	1	0.991856
5.0	0	0	0	1

4 Credit Valuation Adjustment

4.1 Mark-to-Market

The mark-to-market (MTM) value of the swap $V(T_i)$, i.e. the evolution of swap value over time, is obtained via:

$$
\begin{aligned}
V(T_i = 0) &= \sum_{i=1}^{11} N\tau D(0, T_i)(L_i - K) \\
V(T_i = 0.5) &= \sum_{i=2}^{11} N\tau D(0.5, T_i)(L_i - K) \\
V(T_i = 1.0) &= \sum_{i=3}^{11} N\tau D(1.0, T_i)(L_i - K) \\
\vdots \qquad & \qquad \vdots \qquad \qquad \vdots \\
V(T_i = 5.0) &= \sum_{i=11}^{11} N\tau D(5.0, T_i)(L_i - K)
\end{aligned}
$$

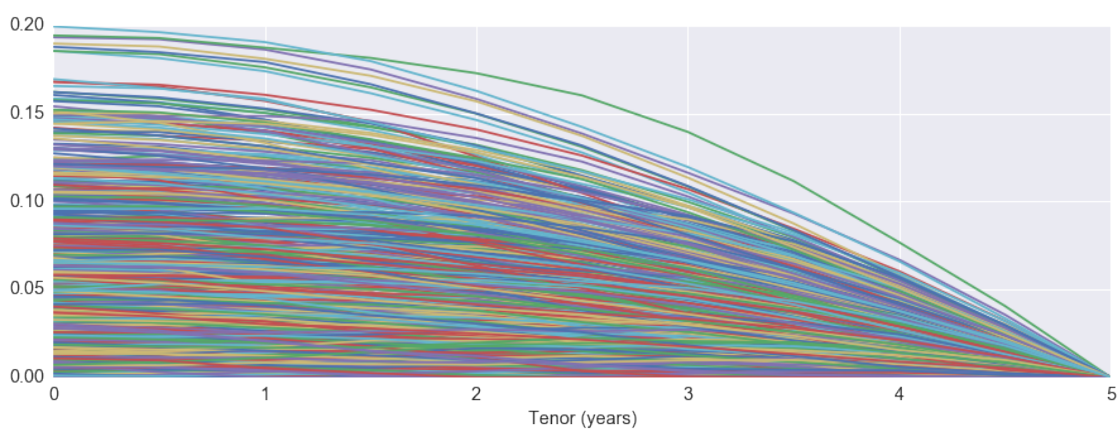
where N is the notional, K the fixed agreed rate and $L_i = L(t; T_{i-1}, T_i)$ the LIBOR effective for the period T_i. Here we assume a "par swap" (ATM) by choosing $K = L(t; 0, 0.5) \approx 0.0063845$ to have zero initial cashflows upon entering the swap.

4.2 Exposure

Finally, the exposure for each tenor E_i is calculated from the positive part of the MTM simulations as:

$$E_i = max(V_i, 0) \tag{6}$$

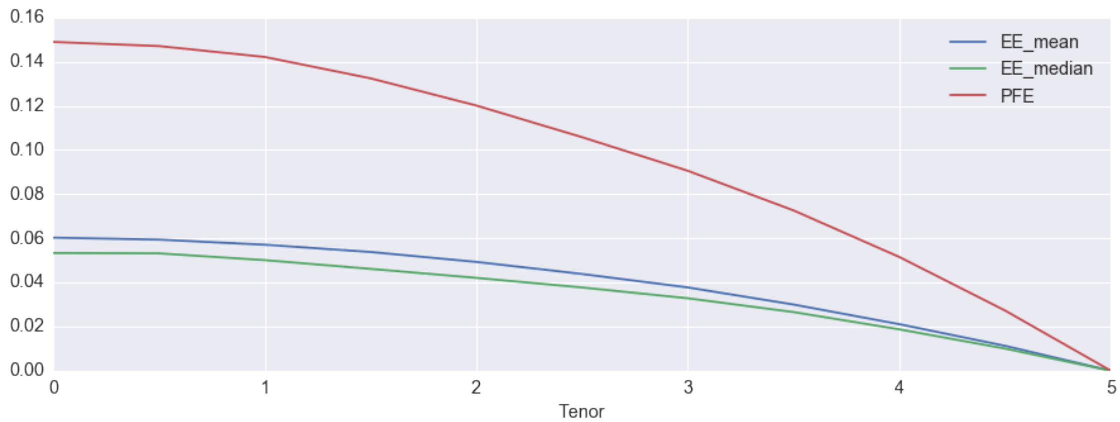
The figure below shows some sample simulations of the exposure profile.



4.3 Expected Exposure

The Expected Exposure (EE) is calculated as the median of the Exposure profile. The median is here preferred over the mean as the latter can be skewed by large observations and is more sensitive to outliers.

The figure below shows the period expected exposure EE_i as calculated from the mean and median, as well as the Potential Future Exposure (PFE), taken to be the 97.5 percentile of the E_i distribution. From this we see that using the median, the maximum EE and PFE are attained at the beginning of the swap in this case (equal to $\sim 5.33\%$ and $\sim 14.89\%$ of the notional respectively). Clearly the PFE is always bigger than the EE - in a way this tells us it is 97.5% probable that our exposure will not exceed $\sim 14.89\%$.

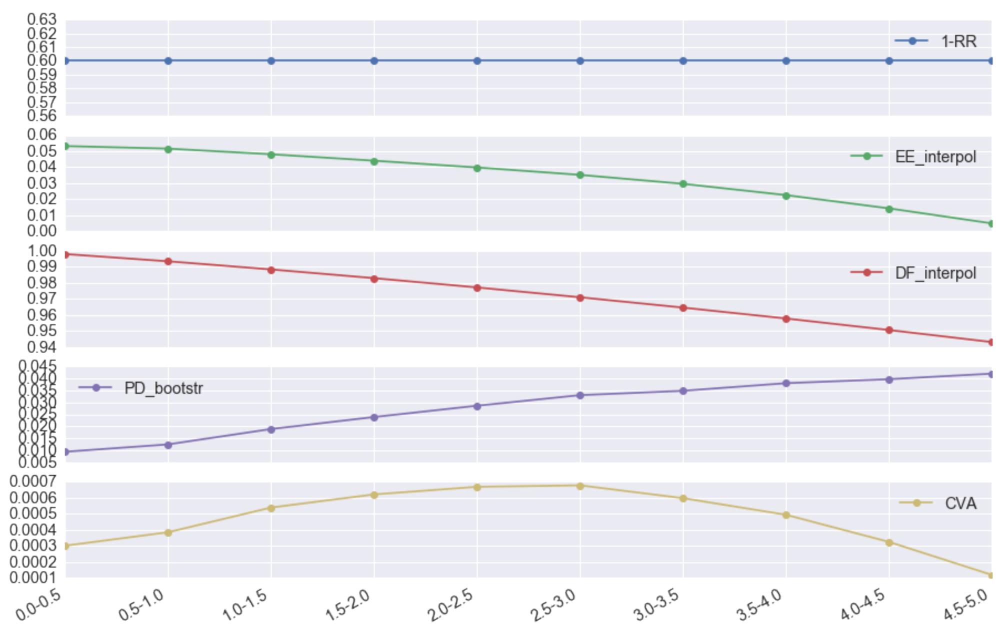


4.4 CVA

Lastly, the CVA is approximated by a linear interpolation across the tenors [1]:

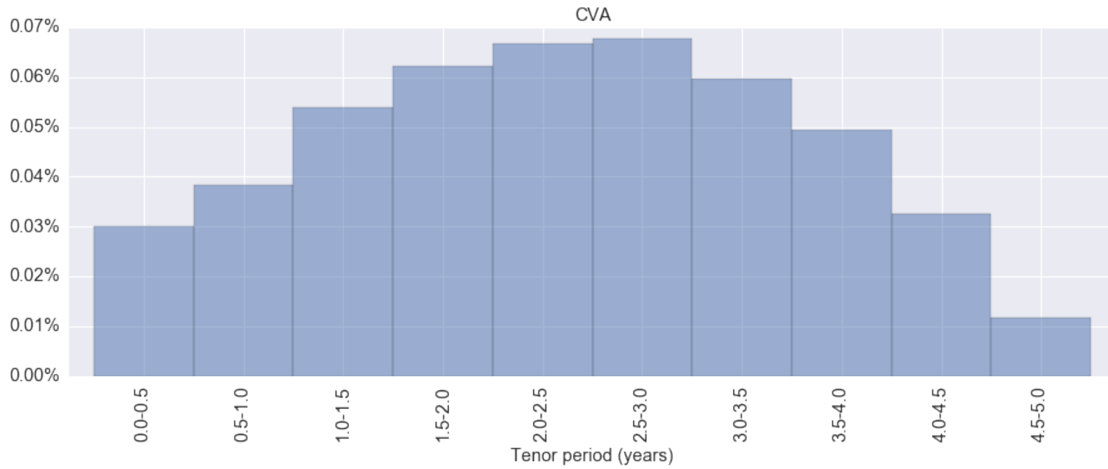
$$CVA \approx \sum_i (1-R)E(\frac{T_{i-1}-T_i}{2})DF(\frac{T_{i-1}-T_i}{2})PD(\frac{T_{i-1}-T_i}{2}) \tag{7}$$

The figure below shows each of the components in this equation, where we see their term structure isn't flat, except for the loss factor $(1-R)$.



[1] Note that the discount factors are more appropriately extrapolated using a log-linear extrapolation instead of purely linear as done here. Due to time constrains this wasn't implemented.

The figure below shows the final CVA result for each tenor range in percentage terms over the notional value. We see a 'hump'-shape, telling us the maximum is found in the $2.5 - 3.0Y$ tenor period. Based on this, the total CVA over the the lifetime of the swap amounts to 0.473% over the notional. So for example, if the notional was \$1m, the CVA would have amounted to $\sim \$4,730$, which although small is not negligible when it comes to pricing the true value of the swap.



5 Conclusion

For this hypothetical IRS scenario the CVA adjustment came out to be quite small. This was found a bit 'surprising', given AirFrance's relatively high probability of default compared to other companies CDS. Whether the CVA value arrived at is accurate or not is debatable, since through the exercise the following issues were noted:

- Typically a 'hump' structure should have been seen already in the Exposure profile, but this wasn't the case here and rather the maximum exposure was attained at the very start. This is atypical for the long-term contracts such as a $5Y$ IRS because the forward rates at these tenors are relatively high. One potential cause could be the HJM recalibration to recent data, given the rates have been historically low. Another possibility could be that the forward rates are too low compared to the estimated discount factors. On the other hand, the CVA plot did show a hump, but this is likely due to the default probability term structure. Overall this would require further investigation

- Due to time constrains, a 'shortcut' was taken when calculating the discount factors for the swap - an expectation across all simulations was taken to get a single value for each period. The effect on the valuation from this would need to be understood

- The above is in addition to all the pros and cons of the HJM model and MC associated errors. In particular, the antithetic reduction technique wasn't implemented here, yielding a higher simulation error

- The effect from accruals wasn't estimated either, although this is expected to be small

References

[1] Richard Diamond, *CQF Lectures - Heath Jarrow & Morton Model*, 2016

[2] Richard Diamond, *CQF Lectures - Final Project Workshop Part I*, 2016

[3] Riaz Ahmad, *Stochastic Interest Rate Modeling*, 2016

[4] *Bank of England Yield Curves*, 2016: http://www.bankofengland.co.uk/statistics/Pages/yieldcurve/archive.aspx