Cointegration in R: Spot Rates Market Data

- Traditionally, cointegration is tested in the very long run
- Case Study A tests for an equilibrium between T-Bill rates and Treasury yields over the horizon of 1960-2010.

  HOWEVER

- As quants we have to look for co-movement in the current, frequent market data.

We will use this opportunity to get introduced to R.

Spot Curve

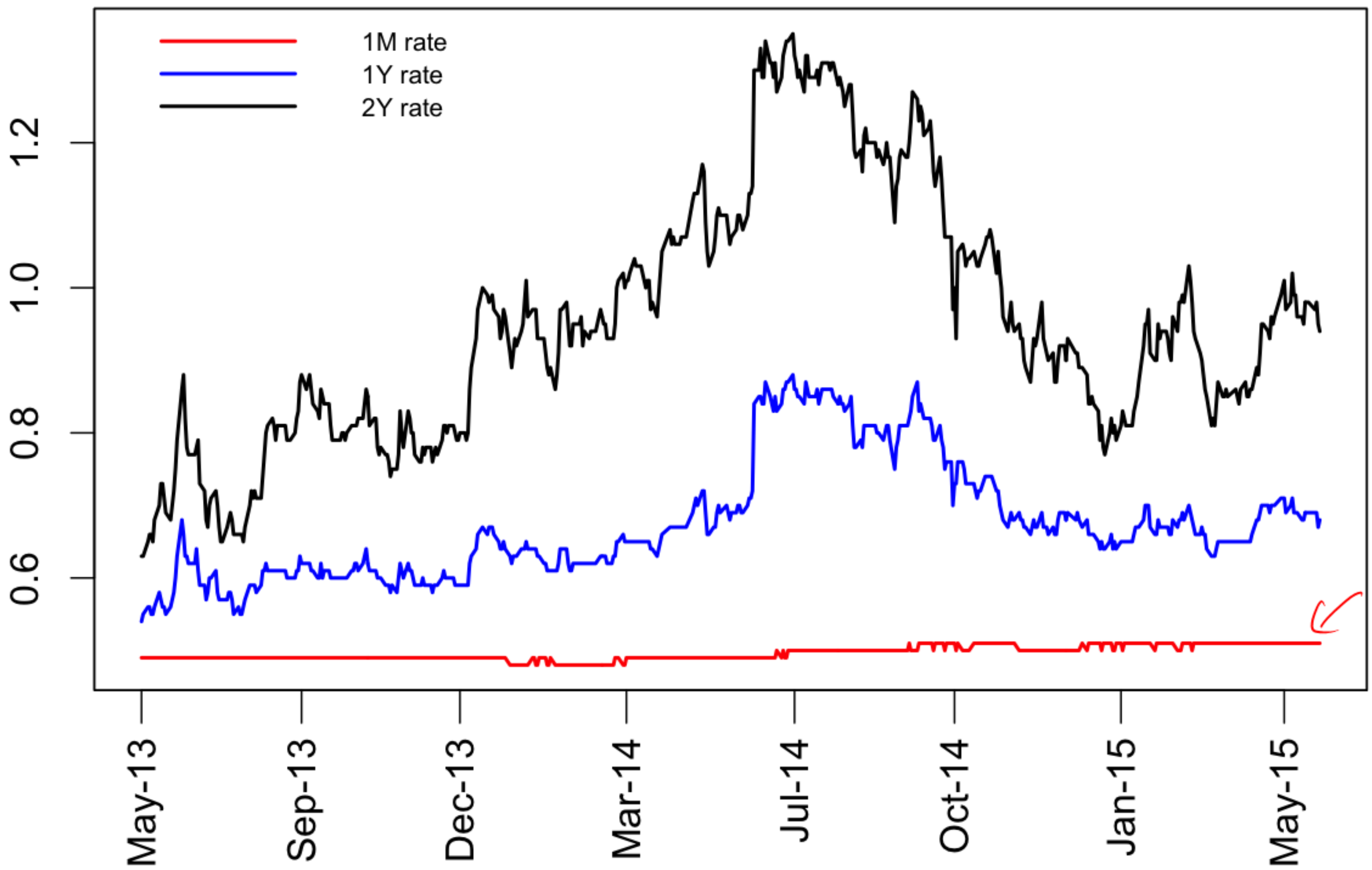
$r(t)$, yields on $Z(0,1M)$ $Z(0,6M)$, $Z(0,1Y)$, $Z(0,1.5Y)$

The Bank of England provides the daily yield curve data. It makes sense to consider smaller windows of the long timeframe:

- two-year window May 2013 – May 2015 (charts below) vs.
- all data from from Jan 2005 to May 2015.

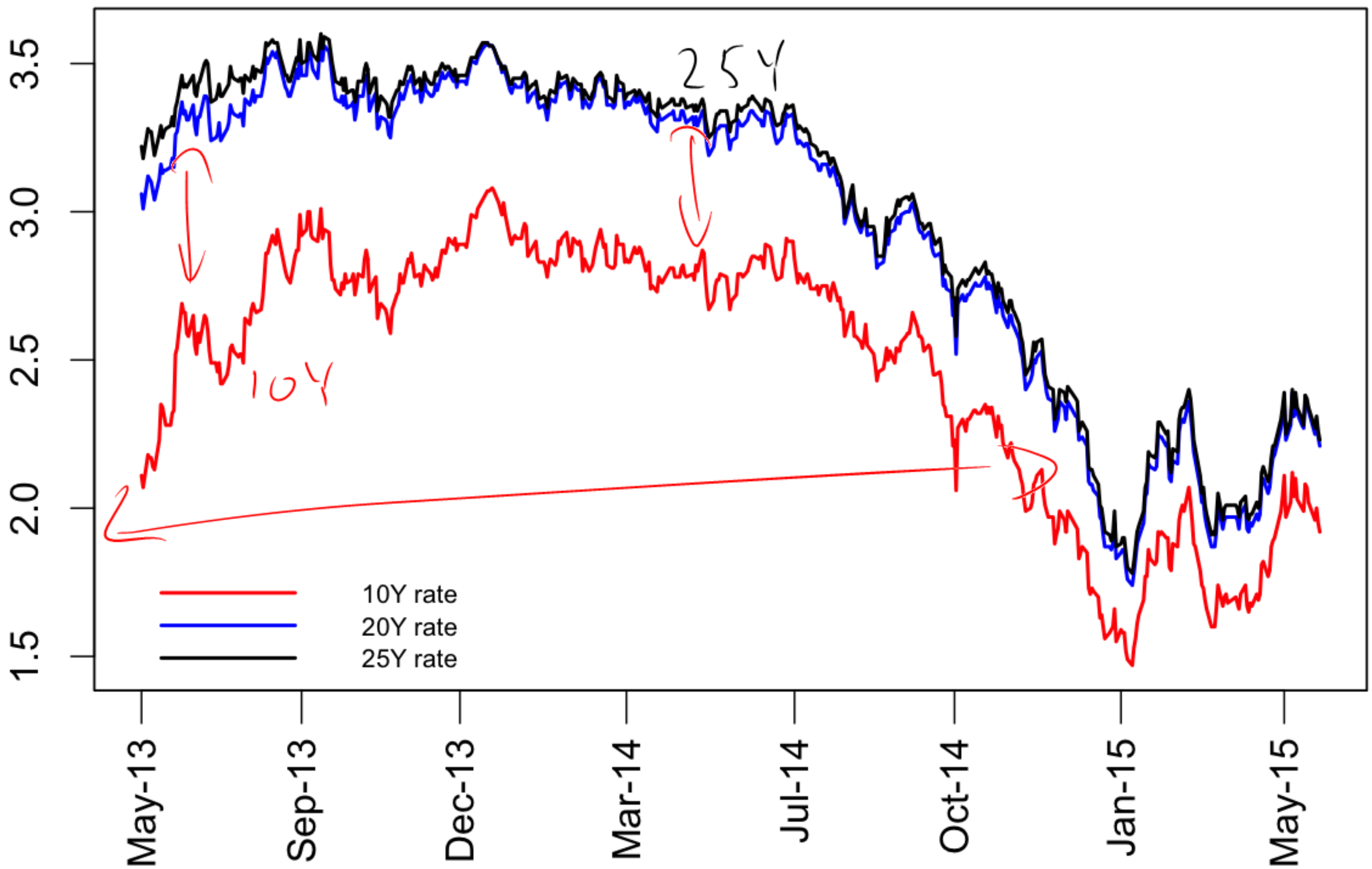
We have to learn the equilibrium-correction mechanics (**ECM**) but it's worthwhile to have a peek from the multivariate test for cointegration.

Spot Rates at Short End



Spot Rates at Long End

$r_{10Y} - \beta_c r_{25Y} = e_t$



Problems with curve data

1. In that kind of data r_t at "short end" and y_t at "long end" **do not** come as cointegrated.

   There is simply not enough horizon for a relationship to transpire. Daily movements are noisy.

2. Let's play a game: **Which long-end rates are co-integrated?** Choose pairs among 10Y, 20Y, 25Y.

   The answer is that cannot decouple movements at the long end that easy.

   Similar pattern comes up for the short end, if all data included in the testing. Also, short rates move independently.

Engle-Granger preview

Let's choose a model with cointegrated 10Y and 25Y tenors because of their importance as benchmarks.

- First, we test r_{10Y} and r_{25Y} for a unit root each.

- Then, we set up a naive cointegrating equation

$$r_{10Y} = \beta\, r_{25Y} + e_t \qquad \Rightarrow \qquad \hat{e}_t = r_{10Y} - \beta\, r_{25Y}$$

- We test this estimated residual $\hat{e}_t$ for stationarity.

  If the residual is stationary, it means that r_{10Y} and r_{25Y} have a unit root **in common** and it has been removed by the differencing

Dickey-Fuller Test reminder

Null Hypothesis: time series has a unit root

We assume a linear trend, so ΔY_t will have a constant

$$\Delta Y_t = \text{Const} + \phi Y_{t-1} + \phi_1 \Delta Y_{t-1}$$

If ϕ is not significant the time series has a unit root.

We can augment the test equation with more lags in $\phi_k \Delta Y_{t-k}$ or time-dependence $\phi_t t$ where ϕ_t is the drift.

That is likely to increase significance. However, beware you might be innocently introducing time dependence (growth/decrease) where there is none.

Unit root in r_{10Y} spot

```
###############################################
# Augmented Dickey-Fuller Test Unit Root Test #
###############################################

Test regression drift [This means Delta Y=Constant]

lm(formula = z.diff ~ z.lag.1 + 1 + z.diff.lag)

             Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.011484   0.012212   0.940    0.347
z.lag.1     -0.004753   0.004848  -0.9805   0.327
z.diff.lag  -0.053306   0.044634  -1.194    0.233

[p-Value of DF test-statistic for -0.9805 is 0.4922]
```

Unit root in r_{25Y} spot

```
################################################
# Augmented Dickey-Fuller Test Unit Root Test #
################################################

Test regression drift [This means Delta Y=Constant]

lm(formula = z.diff ~ z.lag.1 + 1 + z.diff.lag)

              Estimate Std. Error t value Pr(>|t|)
(Intercept) -0.0007984  0.0098612  -0.081    0.936
z.lag.1     -0.0003822  0.0032169  -0.1188   0.905
z.diff.lag  -0.0349222  0.0447136  -0.781    0.435

[p-Value of DF test-statistic for -0.1188 is 0.6514]
```

Long-run relationship r_{10Y} on r_{25Y}

```
lm(formula = curve2.this$X10 ~ curve2.this$X25)

                Estimate Std. Error t value Pr(>|t|)
(Intercept)      0.15878    0.03132    5.07  5.6e-07 ***
curve2.this$X25  0.76980    0.01023   75.28  < 2e-16 ***

Residual standard error: 0.1231 on 504 degrees of freedom
Multiple R-squared:  0.9183,Adjusted R-squared:  0.9182

Residuals:
     Min       1Q   Median       3Q      Max
-0.53675 -0.03449  0.01926  0.07920  0.18461
```

As usual, regressing one non-stationary series on another gives *extremely* significant coefficients. Large N_{obs} makes $R^2 \to 1$.

Long-run relationship if cointegrated

Step. E G.

$$\boxed{\hat{r}_{10Y} = 0.159 + 0.77\, r_{25Y} + \hat{e}_t}$$

This model is valid only if it produces stationary $\hat{e}_t$, so there is co-integration between r_{10Y} and r_{25Y}

It only works in the context of the equilibrium correction over the long-run, producing stationary and mean-reverting residual:

$$\boxed{\hat{e}_t = r_{10Y} - (0.159 + 0.77\, r_{25Y}).}$$

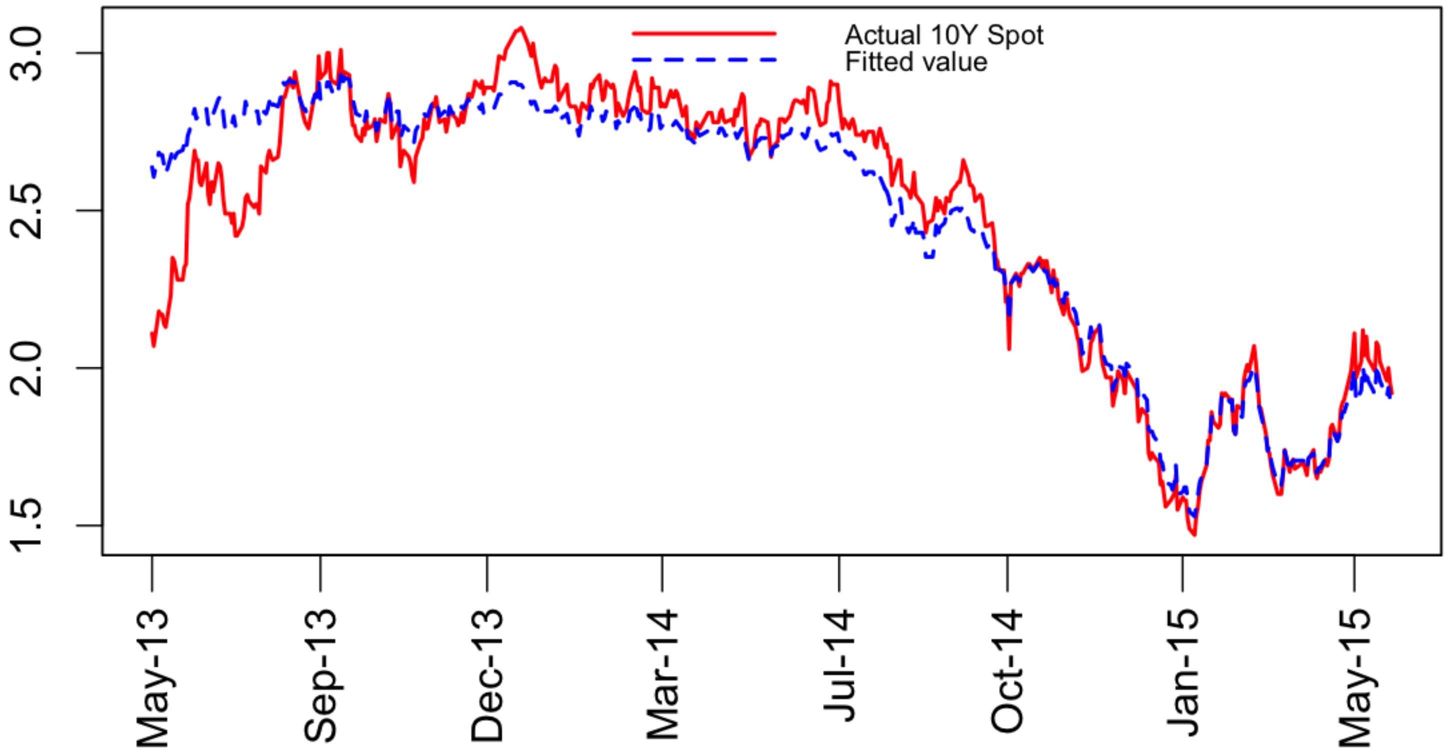
stationary

$\beta = ?$

$\hat{e}_t = r_{10Y} - \beta r_{25Y}$

Linear Regression fit for r_{10Y}

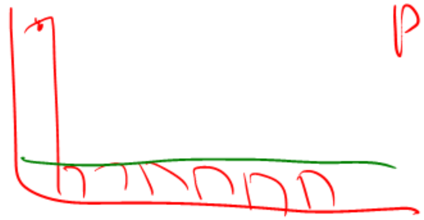
Our linear model aims to obtain $\widehat{e}_t$ so we would be differencing actual r_{10Y} with fitted $\widehat{r}_{10Y}$.

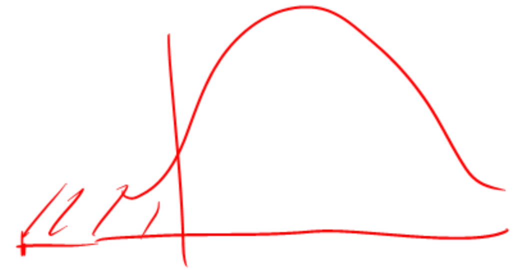


Corr(Δy_t vs. Δy_{t-1})

Stationarity test for $\hat{e}_t$

Corr(Δy_t vs Δy_{t-2})






```
############################################
# Augmented Dickey-Fuller Test Unit Root Test #
############################################
```

$\Delta y_t = \phi\, y_{t-1} - \text{const} + \beta\, \Delta y_{t-1}$

```
lm(formula = z.diff ~ z.lag.1 - 1 + z.diff.lag)
```

no βt "trend"

```
             Estimate Std. Error t value Pr(>|t|)
z.lag.1    -0.038559   0.008548  -4.511 8.06e-06 ***
z.diff.lag -0.042376   0.043711  -0.969    0.333
```

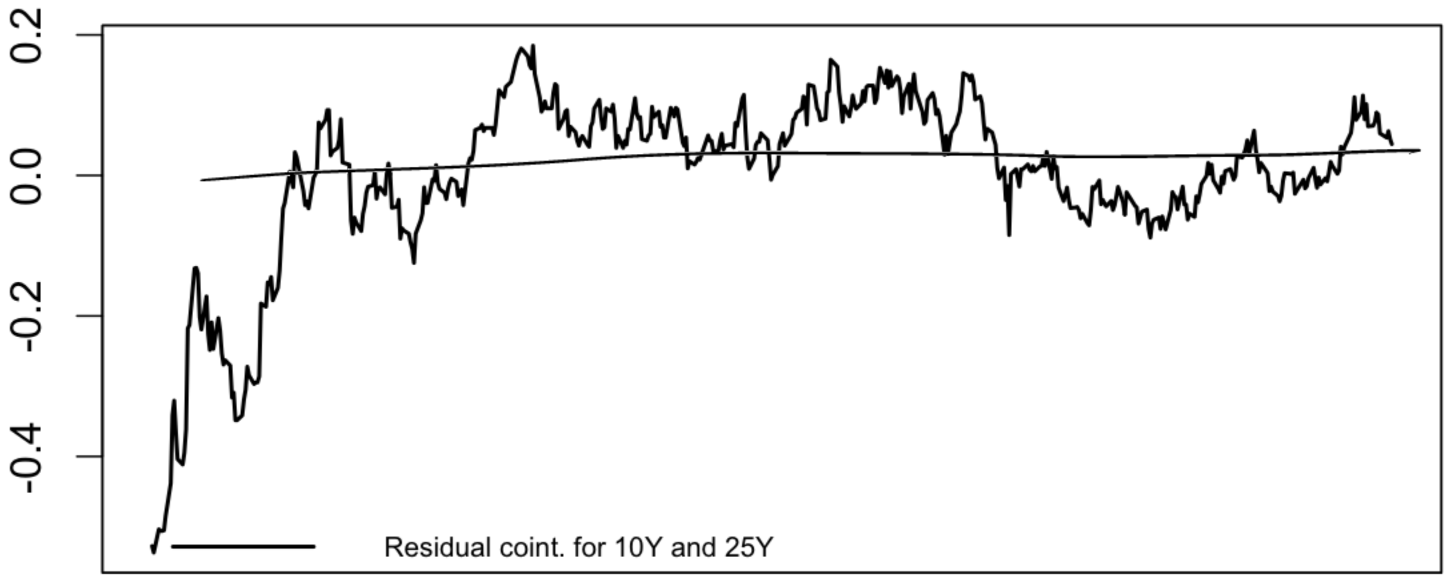
Reject H_0

```
[DF test-statistic is -4.5107, for which critical values]
      1pct  5pct 10pct
tau1 -2.58 -1.95 -1.62
```

DF Distribution

```
Residual standard error: 0.02318 on 502 degrees of freedom
Multiple R-squared:  0.04071,Adjusted R-squared:  0.03689
```

Stationary cointegrating residual $\widehat{e}_t$



We have confirmed the stationarity of residual, and therefore, cointegration according to the Engle-Granger procedure.

Long-run relationship (cointegrated)

ECM estimation [R code provided for your exploration] gives

- **the calbirated parameter** of interest is the speed of correction towards the equilibrium $(1 - \alpha)$

  It is inevitably small but **must be** significant for cointegration to exist.

- We have quite good correlation between differences Δr_{10Y} and Δr_{25Y}. There is co-movement on the short timescale.

  For the lower frequency samples, you might find that Δr_t (for the short rate) and Δy_t (for some long-term rate) are cointegrated but correlated weakly negatively.

Equilibrium Correction Model: two-way

$$\Delta r_{10Y} = 1.086 \Delta r_{25Y} \underbrace{- 0.02716}_{-(1-\alpha)} e_{t-1} + \epsilon_t \quad + \Delta_{t-1}(10Y)$$

```
              Estimate Std. Error t value Pr(>|t|)
tenorX.diff   1.085090   0.022986  47.206  < 2e-16 ***
eq_corr.lag -0.027164   0.007202  -3.772 0.000181 ***
```

```
Residual standard error: 0.01981   Multiple R-squared: 0.8202
```

$$\Delta r_{25Y} = 0.752 \Delta r_{10Y} \underbrace{- 0.01206}_{-(1-\alpha)} e_{t-1} + \epsilon_t$$

```
              Estimate Std. Error t value Pr(>|t|)
tenorY.diff   0.751627   0.015910  47.243   <2e-16 ***
eq_corr.lag -0.012059   0.004851  -2.486   0.0132 *
```

```
Residual standard error: 0.01649   Multiple R-squared: 0.8175
```

Summary

Please take away the following ideas...

- this case of evolution of spot rates at different tenors is a case of a basis relationship,
- so imposing a long-run relationship and using Engle-Granger procedure has more statistical power,
- r_{10Y} and r_{25Y} series each have a unit root,
- it turns out that by differencing these time series, the unit root got cancelled and a stationary residual obtained,
- that means the time series are co-integrated.

Extra Slides

- Restricted VECM from Johansen Procedure
- Engle-Granger Procedure for r_{25Y} on r_{10Y} (other way)
- Linear regression on differences Δr_{25Y}, Δr_{10Y}
- Hedging ratio puzzle

Restricted VECM for Δr_{10Y} and Δr_{25Y}

```
cajorls(johansen.test)

lm(formula = substitute(form1), data = data.mat)
```

Δr_{10Y} Δr_{25Y}

```
          X10.d     X25.d
ect1     -0.05842  -0.02647
X10.dl1  -0.13888  -0.09543
X25.dl1   0.07943   0.06495
```

$[1, -\beta_c]$

```
[Cointegrating Equation (EC term)]
                ect1
X10.l2     1.0000000
X25.l2    -0.7870489
constant  -0.1435463
```

$\begin{bmatrix} 1 \\ \beta_c \\ a \end{bmatrix}, \mu_e$ -0.77 0.159

Optimal

regress r_{10Y} on r_{25Y}

Δr_{10Y} Δr_{25Y}

Long-run relationship r_{25Y} on r_{10Y} (other way)

The linear model $r_{25Y} = \beta\, r_{10Y} + \epsilon_t$ only aims to obtain $\widehat{e}_t$.

```
lm(formula = curve2.this$X25 ~ curve2.this$X10)

                 Estimate Std. Error t value Pr(>|t|)
(Intercept)       0.05686    0.03989   1.425    0.155
curve2.this$X10  1.19295    0.01585  75.285   <2e-16 ***

Residual standard error: 0.1532 on 504 degrees of freedom
Multiple R-squared:  0.9183,Adjusted R-squared:  0.9182
F-statistic:  5668 on 1 and 504 DF,  p-value: < 2.2e-16

Residuals:
     Min        1Q    Median        3Q       Max
-0.18591 -0.08516 -0.03819  0.02177  0.65373
```

Stationarity test for $\hat{e}_t$ (other way)

```
###############################################
# Augmented Dickey-Fuller Test Unit Root Test #
###############################################

lm(formula = z.diff ~ z.lag.1 - 1 + z.diff.lag)

             Estimate Std. Error t value Pr(>|t|)
z.lag.1    -0.033920   0.007759  -4.372  1.5e-05 ***
z.diff.lag -0.038024   0.043779  -0.869    0.386

[DF test-statistic is -4.3718, for which critical values]
      1pct  5pct 10pct
tau1 -2.58 -1.95 -1.62

Residual standard error: 0.02619 on 502 degrees of freedom
Multiple R-squared:  0.03792,Adjusted R-squared:  0.03409
```

Comparison to linear regression

A regression between the simple differences Δr_{25Y} and Δr_{10Y} gives us a minimum variance relationship and shows that cointegration plays a completely separate role.

```
lm(formula = diff(curve2.this$X25) ~ diff(curve2.this$X10) + 0)

                      Estimate Std. Error t value Pr(>|t|)
diff(curve2.this$X10)  0.74570    0.01581   47.16   <2e-16 ***

Residual standard error: 0.01657 on 504 degrees of freedom
Multiple R-squared:  0.8153, Adjusted R-squared:  0.8149

Residuals:
      Min        1Q    Median        3Q       Max
-0.081683 -0.010172 -0.002371  0.007629  0.050172

cor(diff(curve2.this$X25), diff(curve2.this$X10))
[1] 0.903719
```

Hedging ratio puzzle

What would you use as a hedging ratio for assets r_{10Y} and r_{10Y}? Remember you are not investing in returns.

The choices are:

- 0.7698 from a linear regression of r_{10Y} on r_{25Y}
- 0.7457 from linear regression on differences Δr_{25Y} on Δr_{10Y}
- 0.7516 from ECM of r_{25Y} on r_{10Y}
- 0.7870 from a cointegrating regression (two-way VECM).