One quick note about the stage we are in and our other stages:
1)Undirected network with diagonal cluster compatibility matrix: Friendship networks
2)Directed Network with diagonal cluster compatibility matrix: simple followeship networks
3)Directed Network with flexible cluster compatibility matrix: expertise/learning network.
The network generation is as follows:

Algorithm 1 Data generation process for the directed network

$for\ a \in \mathcal{N}:$
$\quad \theta_a \sim Dir(\alpha_{[K]})$
$for\ (a,b) \in \mathcal{N} \times \mathcal{N}:$
$\quad z_{a\to b} \sim Mult(\theta_a)$
$\quad z_{a\leftarrow b} \sim Mult(\theta_b)$
$\quad y(a,b) \sim Bern(z_{a\to b}^T B\, z_{a\leftarrow b})$

The NIPS paper by Airoldi et al 2008 states that:

The indicator vector $z_{a\to b}$ denotes the specific block membership of node p when it connects to node q , while $z_{a\leftarrow b}$ denotes the specific block membership of node q when it is connected from node p.

So this means that the order of indexes indicates the order of potential link, and the direction of the arrow indicates the potential behavior upon initiation versus reception(I am still waiting to hear from Airoldi et al to make sure, no luck yet!). Consider the scenario of how an opion leader mayy interact with a follower versus follower with an opinion leader(or expert or novice relationship). Although the possibility of link in one direction should be very much higher that the other way around in these scenarios if the group memberships differ.

We begin by writing down the ELBO:

$$
\begin{aligned}
\mathcal{L} &= \sum_a \sum_{b \in sink(a)} \sum_k \phi_{a\rightarrow b,k}\phi_{a\leftarrow b,k}\Big(\Psi(\tau_{k0}) - \Psi(\tau_{k0}+\tau_{k1})\Big) \\
&+ \sum_a \sum_{b \in sink(a)} \sum_k \Big(1 - \phi_{a\rightarrow b,k}\phi_{a\leftarrow b,k}\Big) log\,\epsilon \\
&+ \sum_a \sum_{b \notin sink(a)} \sum_k \phi_{a\rightarrow b,k}\phi_{a\leftarrow b,k}\Big(\Psi(\tau_{k1}) - \Psi(\tau_{k0}+\tau_{k1})\Big) \\
&+ \sum_a \sum_{b \notin sink(a)} \sum_k \Big(1 - \phi_{a\rightarrow b,k}\phi_{a\leftarrow b,k}\Big)\Big(log\,(1-\epsilon)\Big) \\
&+ \sum_a \sum_b \sum_k \phi_{a\rightarrow b,k}\Big(\Psi(\gamma_{a,k}) - \Psi(\sum_h \gamma_{a,h})\Big) \\
&+ \sum_a \sum_b \sum_k \phi_{a\leftarrow b,k}\Big(\Psi(\gamma_{b,k}) - \Psi(\sum_h \gamma_{b,h})\Big) \\
&+ \sum_a log\,\Gamma(\sum_k \alpha_k) - \sum_a \sum_k log\,\Gamma(\alpha_k) + \sum_a \sum_k (\alpha_k - 1)\Big(\Psi(\gamma_{a,k}) - \Psi(\sum_h \gamma_{a,h})\Big) \\
&+ \sum_k log\,\Gamma(\eta_0+\eta_1) - \sum_k log\,\Gamma(\eta_0) - \sum_k log\,\Gamma(\eta_1) \\
&+ \sum_k (\eta_0 - 1)\Big(\Psi(\tau_{k0}) - \Psi(\tau_{k0}+\tau_{k1})\Big) + \sum_k (\eta_1 - 1)\Big(\Psi(\tau_{k1}) - \Psi(\tau_{k0}+\tau_{k1})\Big) \\
&- \sum_a \sum_b \sum_k \phi_{a\rightarrow b,k} log\,\phi_{a\rightarrow b,k} - \sum_a \sum_b \sum_k \phi_{a\leftarrow b,k} log\,\phi_{a\leftarrow b,k} \\
&- \sum_a log\,\Gamma(\sum_k \gamma_{a,k}) + \sum_a \sum_k log\,\Gamma(\gamma_{a,k}) - \sum_a \sum_k (\gamma_{a,k} - 1)\Big(\Psi(\gamma_{a,k}) - \Psi(\sum_h \gamma_{a,h})\Big) \\
&- \sum_k log\,\Gamma(\tau_{k0}+\tau_{k1}) + \sum_k log\,\Gamma(\tau_{k0}) + \sum log\,\Gamma(\tau_{k1}) \\
&- \sum_k (\tau_{k0} - 1)\Big(\Psi(\tau_{k0}) - \Psi(\tau_{k0}+\tau_{k1})\Big) - \sum_k (\tau_{k1} - 1)\Big(\Psi(\tau_{k1}) - \Psi(\tau_{k0}+\tau_{k1})\Big)
\end{aligned}
$$

This can be further simplified dividing expressions between links and non links as follows:

$$
\begin{aligned}
\mathcal{L} \;=\;& \sum_a \sum_{b\in sink(a)} \sum_k \phi_{a\to b,k}\phi_{a\leftarrow b,k}\Big(\Psi(\tau_{k0})-\Psi(\tau_{k0}+\tau_{k1})\Big)\\
+& \sum_a \sum_{b\in sink(a)} \sum_k \Big(1-\,\phi_{a\to b,k}\phi_{a\leftarrow b,k}\Big) log\,\epsilon\\
+& \sum_a \sum_{b\notin sink(a)} \sum_k \phi_{a\to b,k}\phi_{a\leftarrow b,k}\Big(\Psi(\tau_{k1})-\Psi(\tau_{k0}+\tau_{k1})\Big)\\
+& \sum_a \sum_{b\notin sink(a)} \sum_k \Big(1-\phi_{a\to b,k}\phi_{a\leftarrow b,k}\Big)\Big(log\,(1-\epsilon)\Big)\\
+& \sum_a \sum_{b\in sink(a)} \sum_k \phi_{a\to b,k}\Big(\Psi(\gamma_{a,k})-\Psi(\sum_h \gamma_{a,h})\Big)\\
+& \sum_a \sum_{b\notin sink(a)} \sum_k \phi_{a\to b,k}\Big(\Psi(\gamma_{a,k})-\Psi(\sum_h \gamma_{a,h})\Big)\\
+& \sum_a \sum_{b\in source(a)} \sum_k \phi_{b\leftarrow a,k}\Big(\Psi(\gamma_{a,k})-\Psi(\sum_h \gamma_{a,h})\Big)\\
+& \sum_a \sum_{b\notin source(a)} \sum_k \phi_{b\leftarrow a,k}\Big(\Psi(\gamma_{a,k})-\Psi(\sum_h \gamma_{a,h})\Big)\\
+& \sum_a log\,\Gamma(\sum_k \alpha_k)-\sum_a\sum_k log\,\Gamma(\alpha_k)+\sum_a\sum_k(\alpha_k-1)\Big(\Psi(\gamma_{a,k})-\Psi(\sum_h \gamma_{a,h})\Big)\\
+& \sum_k log\,\Gamma(\eta_0+\eta_1)-\sum_k log\,\Gamma(\eta_0)-\sum_k log\,\Gamma(\eta_1)\\
+& \sum_k(\eta_0-1)\Big(\Psi(\tau_{k0})-\Psi(\tau_{k0}+\tau_{k1})\Big)+\sum_k(\eta_1-1)\Big(\Psi(\tau_{k1})-\Psi(\tau_{k0}+\tau_{k1})\Big)\\
-& \sum_a \sum_{b\in sink(a)} \sum_k \phi_{a\to b,k} log\,\phi_{a\to b,k}-\sum_a \sum_{b\in source(a)} \sum_k \phi_{b\leftarrow a,k} log\,\phi_{b\leftarrow a,k}\\
-& \sum_a \sum_{b\notin sink(a)} \sum_k \phi_{a\to b,k} log\,\phi_{a\to b,k}-\sum_a \sum_{b\notin source(a)} \sum_k \phi_{b\leftarrow a,k} log\,\phi_{b\leftarrow a,k}\\
-& \sum_a log\,\Gamma(\sum_k \gamma_{a,k})+\sum_a\sum_k log\,\Gamma(\gamma_{a,k})-\sum_a\sum_k(\gamma_{a,k}-1)\Big(\Psi(\gamma_{a,k})-\Psi(\sum_h \gamma_{a,h})\Big)\\
-& \sum_k log\,\Gamma(\tau_{k0}+\tau_{k1})+\sum_k log\,\Gamma(\tau_{k0})+\sum log\,\Gamma(\tau_{k1})\\
-& \sum_k(\tau_{k0}-1)\Big(\Psi(\tau_{k0})-\Psi(\tau_{k0}+\tau_{k1})\Big)-\sum_k(\tau_{k1}-1)\Big(\Psi(\tau_{k1})-\Psi(\tau_{k0}+\tau_{k1})\Big)
\end{aligned}
$$

Next we want to find the variational parameters that maximize the variational lower bound:

$$
\begin{aligned}
\mathcal{L}\Big[\underset{a\to b}{\phi_{a\to b,k}}\Big] \;=\;& \phi_{a\to b,k}\phi_{a\leftarrow b,k}\Big(\Psi(\tau_{k0})-\Psi(\tau_{k0}+\tau_{k1})\Big)\\
-\;& \phi_{a\to b,k}\phi_{a\leftarrow b,k} log\,\epsilon\\
+\;& \phi_{a\to b,k}\Big(\Psi(\gamma_{a,k})-\Psi(\sum_h \gamma_{a,h})\Big)\\
-\;& \phi_{a\to b,k} log\,\phi_{a\to b,k}\\
=\;& \phi_{a\to b,k}\Bigg(\phi_{a\leftarrow b,k}\Big(\Psi(\tau_{k0})-\Psi(\tau_{k0}+\tau_{k1})\Big)-\phi_{a\leftarrow b,k} log\,\epsilon\\
& +\Big(\Psi(\gamma_{a,k})-\Psi(\sum_h \gamma_{a,h})\Big)-log\,\phi_{a\to b,k}\Bigg)
\end{aligned}
$$

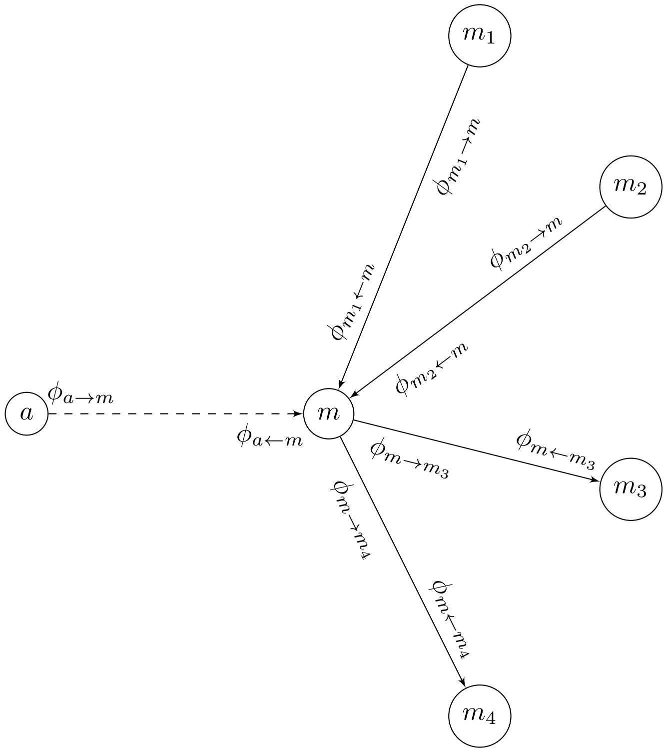
Hence maximizing $\mathcal{L}\Big[\underset{a\to b}{\phi_{a\to b,k}}\Big]$ with respect to $\underset{a\to b}{\phi_{a\to b,k}}$:

$$
\begin{aligned}
\frac{\partial \mathcal{L}\Big[\underset{a\to b}{\phi_{a\to b,k}}\Big]}{\partial \underset{a\to b}{\phi_{a\to b,k}}} & \\
= 0 \quad \Longrightarrow \quad & \Bigg(\phi_{a\leftarrow b,k}\Big(\Psi(\tau_{k0}) - \Psi(\tau_{k0}+\tau_{k1})\Big) - \phi_{a\leftarrow b,k} log\, \epsilon \\
& +\Big(\Psi(\gamma_{a,k}) - \Psi(\sum_h \gamma_{a,h})\Big) - log\, \phi_{a\to b,k}\Bigg) - 1 = 0 \\
\Longrightarrow & \\
\underset{a\to b}{\phi_{a\to b,k}} \quad \propto \quad & exp\Bigg(\phi_{a\leftarrow b,k}\Big(\Psi(\tau_{k0}) - \Psi(\tau_{k0}+\tau_{k1}) - log\, \epsilon\Big) + \Big(\Psi(\gamma_{a,k}) - \Psi(\sum_h \gamma_{a,h})\Big)\Bigg) \\
\propto \quad & \boxed{\epsilon^{-\phi_{a\leftarrow b,k}} \times exp\Bigg(\phi_{a\leftarrow b,k}\Big(\Psi(\tau_{k0}) - \Psi(\tau_{k0}+\tau_{k1})\Big) + \Big(\Psi(\gamma_{a,k}) - \Psi(\sum_h \gamma_{a,h})\Big)\Bigg)}
\end{aligned}
$$

Similarly for $\underset{a\to b}{\phi_{a\leftarrow b,k}}$ we have:

$$
\begin{aligned}
\mathcal{L}\Big[\underset{a\to b}{\phi_{a\leftarrow b,k}}\Big] \quad = \quad & \phi_{a\leftarrow b,k}\Bigg(\phi_{a\to b,k}\Big(\Psi(\tau_{k0}) - \Psi(\tau_{k0}+\tau_{k1})\Big) - \phi_{a\to b,k} log\, \epsilon \\
& +\Psi(\gamma_{b,k}) - \Psi(\sum_h \gamma_{b,h}) - log\, \phi_{a\leftarrow b,k}\Big)\Bigg) \\
\frac{\partial \mathcal{L}\Big[\underset{a\to b}{\phi_{a\leftarrow b,k}}\Big]}{\partial \underset{a\to b}{\phi_{a\leftarrow b,k}}} & \\
= 0 \quad \Longrightarrow \quad & \Bigg(\phi_{a\to b,k}\Big(\Psi(\tau_{k0}) - \Psi(\tau_{k0}+\tau_{k1})\Big) - \phi_{a\to b,k} log\, \epsilon \\
& +\Psi(\gamma_{b,k}) - \Psi(\sum_h \gamma_{b,h}) - log\, \phi_{a\leftarrow b,k}\Big)\Bigg) - 1 = 0 \\
\underset{a\to b}{\phi_{a\leftarrow b,k}} \quad \propto \quad & exp\Bigg(\phi_{a\to b,k}\Big(\Psi(\tau_{k0}) - \Psi(\tau_{k0}+\tau_{k1}) - log\, \epsilon\Big) + \Big(\Psi(\gamma_{b,k}) - \Psi(\sum_h \gamma_{b,h})\Big)\Bigg) \\
\propto \quad & \boxed{\epsilon^{-\phi_{a\to b,k}} \times exp\Bigg(\phi_{a\to b,k}\Big(\Psi(\tau_{k0}) - \Psi(\tau_{k0}+\tau_{k1})\Big) + \Big(\Psi(\gamma_{b,k}) - \Psi(\sum_h \gamma_{b,h})\Big)\Bigg)}
\end{aligned}
$$

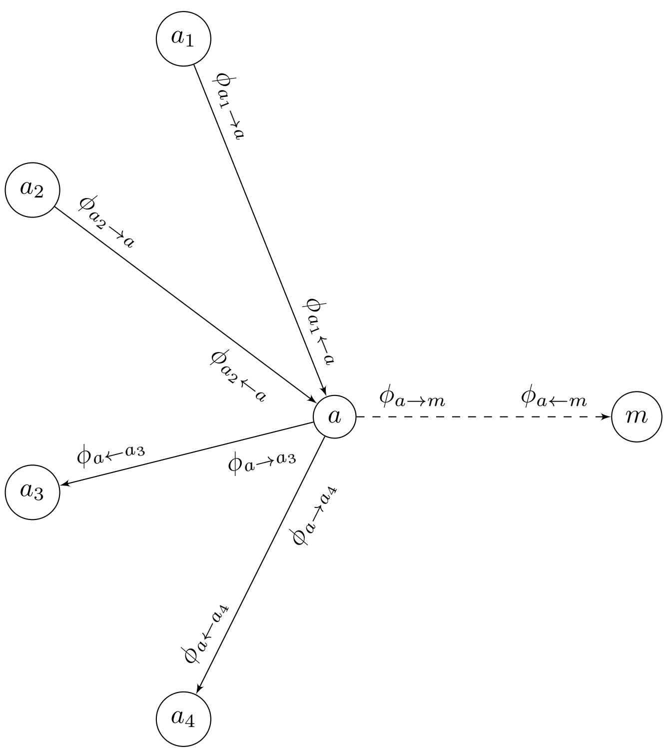
We can now find the same variational parameters for the case where $a \nrightarrow b$ by averaging from the links:



for $m \notin sink(a)$,

$$\phi_{a \leftarrow m,k} = \frac{\sum_{b \in source(m)} \phi_{b \leftarrow m,k} + \sum_{b \in sink(m)} \phi_{m \to b,k}}{outdeg(m) + indeg(m)}$$

The simplifying assumption here is that if there is no directed edge from a to m, then the receptive ϕ for m which is $\phi_{a \leftarrow m}$ is a function of m's attributes. The attributes here are averages of the ϕ's over m's incoming and outgoing links.



for $m \notin sink(a)$,

$$\phi_{a \to m,k} = \frac{\sum_{b \in source(a)} \phi_{b \leftarrow a,k} + \sum_{b \in sink(a)} \phi_{a \to b,k}}{outdeg(a) + indeg(a)}$$

Similarly when there is no directed link from a to m, the variational parameter $\phi_{a\to m}$ is assumed to be averaged over ϕ's of its sources and sinks. Again $\phi_{a\to b}$ here is also only a function of the attributes of node a. it is good to know that

$$\begin{array}{ccc} \phi_{a\to .} & = & \phi_{.\leftarrow a} \\ a \nrightarrow . & & . \nrightarrow a \end{array}$$

Turning into the global parameters:

$$\begin{aligned}
\mathcal{L}\Big[\gamma_{a,k}\Big] &= \Big(\sum_{b\in sink(a)} \phi_{a\to b,k} + \sum_{b\notin sink(a)} \phi_{a\to b,k}\Big)\Big(\Psi(\gamma_{a,k}) - \Psi(\sum_h \gamma_{a,h})\Big) \\
&+ \Big(\sum_{b\in source(a)} \phi_{b\leftarrow a,k} + \sum_{b\notin source(a)} \phi_{b\leftarrow a,k}\Big)\Big(\Psi(\gamma_{a,k}) - \Psi(\sum_h \gamma_{a,h})\Big) \\
&+ (\alpha_k - \gamma_{a,k})\Big(\Psi(\gamma_{a,k}) - \Psi(\sum_h \gamma_{a,h})\Big) + log\,\Gamma(\gamma_{a,k}) - log\,\Gamma(\sum_h \gamma_{a,h}) \\
\frac{\partial \mathcal{L}\Big[\gamma_{a,k}\Big]}{\partial \gamma_{a,k}} &= 0 \\
&\Longrightarrow \Big(\Psi'(\gamma_{a,k}) - \Psi'(\sum_h \gamma_{a,h})\Big)\Bigg(\sum_{b\in sink(a)} \phi_{a\to b,k} + \sum_{b\notin sink(a)} \phi_{a\to b,k} \\
&\quad + \sum_{b\in source(a)} \phi_{b\leftarrow a,k} + \sum_{b\notin source(a)} \phi_{b\leftarrow a,k} + \alpha_k - \gamma_{a,k}\Bigg) = 0 \\
\gamma_{a,k} &= \alpha_k + \sum_{b\in sink(a)} \phi_{a\to b,k} + \sum_{b\notin sink(a)} \phi_{a\to b,k} + \sum_{b\in source(a)} \phi_{b\leftarrow a,k} + \sum_{b\notin source(a)} \phi_{b\leftarrow a,k}
\end{aligned}$$

By replacing for the arguments for the case of nonlinks, we can rewrite this as:

$$\begin{aligned}
\gamma_{a,k} &= \alpha_k + \sum_{b\in sink(a)} \phi_{a\to b,k} + \sum_{b\in source(a)} \phi_{b\leftarrow a,k} + \sum_{b\notin sink(a)} \frac{\sum_{b'\in source(a)} \phi_{b'\leftarrow a,k} + \sum_{b'\in sink(a)} \phi_{a\to b',k}}{indeg(a) + outdeg(a)} \\
&+ \sum_{b\notin source(a)} \frac{\sum_{b'\in source(a)} \phi_{b'\leftarrow a,k} + \sum_{b'\in sink(a)} \phi_{a\to b',k}}{indeg(a) + outdeg(a)}
\end{aligned}$$

simplifying this gives us:

$$\begin{aligned}
\gamma_{a,k} &= \alpha_k + \sum_{b\in sink(a)} \phi_{a\to b,k} + \sum_{b\in source(a)} \phi_{b\leftarrow a,k} + \big(N - 1 - outdeg(a)\big) \times \frac{\sum_{b'\in source(a)} \phi_{b'\leftarrow a,k} + \sum_{b'\in sink(a)} \phi_{a\to b',k}}{indeg(a) + outdeg(a)} \\
&+ \big(N - 1 - indeg(a)\big)\frac{\sum_{b'\in source(a)} \phi_{b'\leftarrow a,k} + \sum_{b'\in sink(a)} \phi_{a\to b',k}}{indeg(a) + outdeg(a)} \\
&= \alpha_k + \sum_{b\in sink(a)} \phi_{a\to b,k} + \sum_{b\in source(a)} \phi_{b\leftarrow a,k} + \frac{\sum_{b\in source(a)} \phi_{b\leftarrow a,k} + \sum_{b\in sink(a)} \phi_{a\to b,k}}{indeg(a) + outdeg(a)} \times \Big(2N - 2 - \big(indeg(a) + outdeg(a)\big)\Big) \\
&= \alpha_k + \Bigg(\sum_{b\in sink(a)} \phi_{a\to b,k} + \sum_{b\in source(a)} \phi_{b\leftarrow a,k}\Bigg) \times \Bigg(1 + \frac{\Big(2N - 2 - \big(indeg(a) + outdeg(a)\big)\Big)}{indeg(a) + outdeg(a)}\Bigg) \\
&= \boxed{\alpha_k + \Bigg(\sum_{b\in sink(a)} \phi_{a\to b,k} + \sum_{b\in source(a)} \phi_{b\leftarrow a,k}\Bigg) \times \Bigg(\frac{2N - 2}{indeg(a) + outdeg(a)}\Bigg)}
\end{aligned}$$

For τ_{k0} and τ_{k1}:

$$
\begin{aligned}
\mathcal{L}\Big[\tau_k\Big] &= \sum_a \sum_{b \in sink(a)} \phi_{a\to b,k}\phi_{a\leftarrow b,k}\Big(\Psi(\tau_{k0}) - \Psi(\tau_{k0}+\tau_{k1})\Big) \\
&+ \sum_a \sum_{b \notin sink(a)} \phi_{a\to b,k}\phi_{a\leftarrow b,k}\Big(\Psi(\tau_{k1}) - \Psi(\tau_{k0}+\tau_{k1})\Big) \\
&+ (\eta_0 - \tau_{k0})\Big(\Psi(\tau_{k0}) - \Psi(\tau_{k0}+\tau_{k1})\Big) \\
&+ (\eta_1 - \tau_{k1})\Big(\Psi(\tau_{k1}) - \Psi(\tau_{k0}+\tau_{k1})\Big) \\
&- log\,\Gamma(\tau_{k0}+\tau_{k1}) + log\,\Gamma(\tau_{k0}) + log\,\Gamma(\tau_{k1}) \\
\frac{\partial\mathcal{L}\Big[\tau_k\Big]}{\partial\tau_k} &= \begin{cases} \dfrac{\partial\mathcal{L}\Big[\tau_{k0}\Big]}{\partial\tau_{k0}} & = 0 \\ \dfrac{\partial\mathcal{L}\Big[\tau_{k1}\Big]}{\partial\tau_{k1}} & = 0 \end{cases} \\
&\Longrightarrow \begin{cases} \Big(\Psi'(\tau_{k0}) - \Psi'(\tau_{k0}+\tau_{k1})\Big)\left(\sum_a \sum_{b \in sink(a)} \phi_{a\to b,k}\phi_{a\leftarrow b,k} + \eta_0 - \tau_{k0}\right) & = 0 \\ \Big(\Psi'(\tau_{k1}) - \Psi'(\tau_{k0}+\tau_{k1})\Big)\left(\sum_a \sum_{b \notin sink(a)} \phi_{a\to b,k}\phi_{a\leftarrow b,k} + \eta_1 - \tau_{k1}\right) & = 0 \end{cases} \\
&\Longrightarrow \\
\tau_{k0} &= \boxed{\eta_0 + \sum_a \sum_{b \in sink(a)} \phi_{a\to b,k}\phi_{a\leftarrow b,k}} \\
\tau_{k1} &= \boxed{\eta_1 + \sum_a \sum_{b \notin sink(a)} \phi_{a\to b,k}\phi_{a\leftarrow b,k}}
\end{aligned}
$$

We can further smplify the expression for τ_{k1}(for the sake of convenience to distinguish between the variational parameter ϕ for links and nonlinks we use the $\bar{\phi}$to represent nonlinks):

$$
\sum_a \sum_{b \notin sink(a)} \bar{\phi}_{a\to b,k}\bar{\phi}_{a\leftarrow b,k} = \sum_a \sum_{b \neq a} \bar{\phi}_{a\to b,k}\bar{\phi}_{a\leftarrow b,k} - \sum_a \sum_{b \in sink(a)} \bar{\phi}_{a\to b,k}\bar{\phi}_{a\leftarrow b,k}
$$

We know that $\bar{\phi}_{a\to b,k}$ only depends on the properties of a, so we denote it by $\bar{\phi}_{a\to b,k} = \bar{\phi}_{a\to .,k} = f(a)$ and similarly $\bar{\phi}_{a\leftarrow b,k} = \bar{\phi}_{.\leftarrow b,k} = f(b)$, hence we can rewrite the $\sum_a \sum_{b\neq a} \bar{\phi}_{a\to .,k}\bar{\phi}_{.\leftarrow b,k}$ as below:

$$
\begin{aligned}
\sum_a \sum_{b\neq a} \bar{\phi}_{a\to .,k}\bar{\phi}_{.\leftarrow b,k} &= \sum_a \sum_{b\neq a} f(a)f(b) \\
&= \sum_a f(a) \sum_{b\neq a} f(b) \\
&= \sum_a \bar{\phi}_{a\to .,k} \sum_{b\neq a} \bar{\phi}_{.\leftarrow b,k} \\
&= \sum_a \bar{\phi}_{a\to .,k} \sum_a \bar{\phi}_{.\leftarrow a,k} - \sum_a (\bar{\phi}_{a\to .,k})^2 \\
&= \Big(\sum_a \bar{\phi}_{a\to .,k}\Big)^2 - \sum_a (\bar{\phi}_{a\to .,k})^2 = \Big(\sum_a \bar{\phi}_{.\leftarrow a,k}\Big)^2 - \sum_a (\bar{\phi}_{.\leftarrow a,k})^2
\end{aligned}
$$

We can apply the same rule to the sum product $\sum_a \sum_{b \in sink(a)} \bar{\phi}_{a\to .,k}\bar{\phi}_{.\leftarrow b,k}$ and rewrite it as:

$$
\begin{aligned}
\sum_a \sum_{b\in sink(a)} \bar{\phi}_{a\to .,k}\bar{\phi}_{.\leftarrow b,k} &= \sum_a \sum_{b\in sink(a)} f(a)f(b) \\
&= \sum_a f(a) \sum_{b\in sink(a)} f(b) \\
&= \sum_a \bar{\phi}_{a\to .,k} \sum_{b\in sink(a)} \bar{\phi}_{.\leftarrow b,k}
\end{aligned}
$$

So using the results above, instead of using the sum over all the non links we can instead write $\sum_a \sum_{b\notin sink(a)} \bar{\phi}_{a\to .,k}\bar{\phi}_{.\leftarrow b,k}$ as follows:

$$
\sum_a \sum_{b\notin sink(a)} \bar{\phi}_{a\to b,k}\bar{\phi}_{a\leftarrow b,k} = \Big(\sum_a \bar{\phi}_{a\to .,k}\Big)^2 - \sum_a (\bar{\phi}_{.\leftarrow a,k})^2 - \sum_a \bar{\phi}_{a\to .,k} \sum_{b\in sink(a)} \bar{\phi}_{.\leftarrow b,k}
$$

Hence, the result for τ_{k1} could be rewritten as:

$$
\tau_{k1} = \boxed{\eta_1 + \Big(\sum_a \bar{\phi}_{a\to .,k}\Big)^2 - \sum_a (\bar{\phi}_{.\leftarrow a,k})^2 - \sum_a \bar{\phi}_{a\to .,k} \sum_{b\in sink(a)} \bar{\phi}_{.\leftarrow b,k}}
$$