PROBABILITY

Set Theory, Operations of Sets, Some Basic Terms and Concepts, Mathematical Definition of Probability, Odds Against and Odds in favour of an event, Addition Theorem, Multiplication Theorem, Conditional Probability, Boole's Inequality, Baye's Theorem, Binomial Distribution for Repeated Trails, Some Important Results

SET THEORY

The collection of well defined things is called set. Well defined means a law by which we are able to find whether a given thing is contained in the given set or not.

Example : A = { x ; x , an odd number less than 15 } means

A = { 1, 3, 5, 7, 9 , 11, 13 }

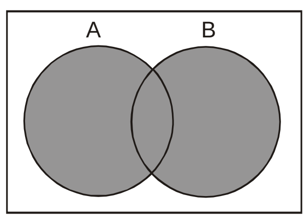
Example : W = { x ; x is a whole number } means

W = { 0 , 1, 2, 3, 4, 5, ... }

OPERATIONS OF SETS

The basic operations of sets and their related results are as follows.

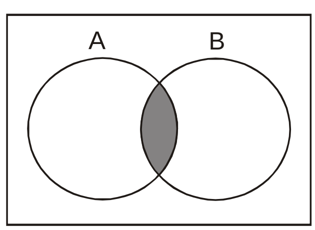
Union of sets : The union of two sets is represented by $A \cup B$ or A + B. This set contains those elements which are in A or in B or in A & B both. So



$A \cup B$

$$A \cup B = \{ x ; x \in A \text{ or } x \in B \}$$

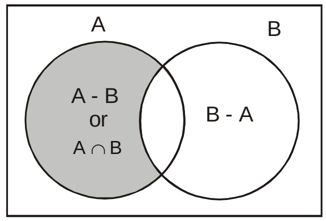
Intersection of two sets : The intersection of two sets is represented by $A \cap B$ or AB. It contains those all elements which are contained in both sets A & B both so



$A \cap B$

$$A \cap B = \{ x ; x \in A \text{ and } x \in B \}$$

Difference of two sets : If A and B are two sets then A–B represents the set of those elements which are in A and not in B. In the same manner B–A represents the set of those elements which are in B and not in A. So



$$A - B = A \cap \bar{B} = \{ x \,;\, x \in A \text{ and } x \notin B \}$$

SOME BASIC TERMS AND CONCEPTS

An Experiment : An action or operation resulting in two or more outcomes is called an experiment.

Example : (i) Tossing of a coin is an experiment. There are two possible outcomes head or tail.

Example : (ii) Drawing a card from a pack of 52 cards is an experiment. There are 52 possible outcomes.

Sample Space : The set of all possible outcomes of an experiment is called the sample space, denoted by S. An element of S is called a sample point.

Example : (i) In the experiment of tossing of a coin, the sample space has two points corresponding to head (H) and Tail (T) i.e. S {H, T}

Example : (ii) When we throw a die then any one of the numbers 1, 2, 3, 4, 5 and 6 will come up. So the sample space

S = {1, 2, 3, 4, 5, 6}.

Event : Any subset of sample space is an event.

Example : (i) If the experiment is done throwing a die which has faces numbered 1 to 6, then

S = {1, 2, 3, 4, 5, 6}.

Then A = {1, 3, 5}, B {2, 4, 6}, the null set ϕ and S itself are some events with respect to S.

The null set ϕ is called the impossible event or null event.

Example : (ii) Getting 7 when a die is thrown is called a null event.

The entire sample space is called the certain event.

Simple Event : An event is called a simple event if it is a singleton subset of the sample space S.

Example : (i) When a coin is tossed, sample space S = {H, T}

Let A = {H} = the event of occurrence of head.

and B = {T} = the event of occurrence of tail.

Here A and B are simple events.

Example : (ii) When a die is thrown, sample space S = {1, 2, 3, 4, 5, 6},

Let A = {5} = the event of occurrence of 5.

B = {2} = the event of occurrences of 2.

Here A and B are simple events.

Compound Events : It is the joint occurrence of two or more simple events.

Example : The event of at least one head appears when two fair coins are tossed is a compound event

i.e. A = {H T, T H, H H}.

Equally Likely Events : A number of simple events are said to be equally likely if there is no reason for one event to occur in preference to any other event.

Example : In drawing a card from a well shuffled pack, there are 52 outcomes which are equally likely.

Exhaustive Events : All the possible outcomes taken together in which an experiment can result are said to be exhaustive.

Example : A card is drawn from well shuffled pack.

- The following events are exhaustive.

  (i) The card is black (ii) The card is red.

- The following events are not exhaustive.

  (i) The card is heart (ii) The card is diamond

If A and B are exhaustive events of the sample space S, then $A \cup B = S$

In general if $E_1, E_2, E_3, \ldots\ldots\ldots E_n$ are the exhaustive events of the sample space then

$E_1 \cup E_2 \cup E_3, \ldots\ldots\ldots \cup E_n = S$

Mutually Exclusive Events : If two events cannot occur simultaneously, then they are mutually exclusive.

If A and B are mutually exclusive, then $A \cap B = \phi$.

Example : In drawing a card from a well shuffled pack, the following events.

A = the card is a spade; B = the card is a heart are mutually exclusive.

(i) In a single throw of a coin either the head or the tail will appear and not both.

(ii) In a throw of a cubic die, either an odd number or an even number will turn up and not both.

Following events are not mutually exclusive :

(a) The card is a heart (b) The card is a king

The card can be king of heart.

Complement of an Event : The complement of an event A, denoted by $\overline{A}$, A' or A^C, is the set of all sample points of the space other then the sample points in A.

Example : In the experiment of casting a fair die, S = {1, 2, 3, 4, 5, 6}

If A = {1, 3, 5, 6}, then A^C = {2, 4}

Note **:** (a) $A \cup A^C = S$ (b) $A \cap A^C = \phi$.

MATHEMATICAL DEFINITION OF PROBABILITY

Let the outcomes of an experiment consists of *n* exhaustive mutually exclusive and equally likely cases. Then the sample spaces S has *n* sample points. If an event A consists of m sample points, $(0 \le m \le n)$, then the probability of event A, denoted by P(A) is defined to be m/n i.e. P(A) = m/n.

Let S = $a_1, a_2, \ldots\ldots, a_n$ be the sample space

- $P(S) = \frac{n}{n} = 1$ corresponding to the certain event.
- $P(\phi) = \frac{0}{n} = 0$ corresponding to the null event ϕ or impossible event.
- If $A_i = \{a_i\}$, i = 1, ..., n then A_i is the event corresponding to a single sample point a_i. Then $P(A_i) = \frac{1}{n}$.

- $0 \le P(A) \le 1$

  $\because \quad 0 \le m \le n \Rightarrow \quad 0 \le \frac{m}{n} \le 1 \quad \Rightarrow 0 \le P(A) \le 1$

- $P(A') = 1 - P(A)$

  If the event A has m elements, then A' has (n – m) elements in S.

  $\therefore\ P(A') = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - P(A)$

SOLVED EXAMPLES

Example. If three cards are drawn from a pack of 52 cards , what is the chance that all will be queen ?

Solution. If the sample space be s then n(s) = the total number of ways of drawing 3 out of

52 cards = ${}^{52}C_3$

Now, if A = the event of drawing three queens then n(A) = ${}^{4}c_3$

$$\therefore\ P(E) = \frac{n(A)}{n(s)} = \frac{{}^{4}C_3}{{}^{52}C_3} = \frac{4}{\frac{52 \times 51 \times 50}{3 \times 2}} = \frac{1}{5525}$$

Example. Words are formed with the letters of the word PEACE. Find the probability that 2 E's come together

Solution. Total number of words which can be formed with the letters P, E, A, C, E $= \frac{\underline{|5}}{\underline{|2}} = 60$

Number of words in which 2 E's come together $= \underline{|4} = 24$

$$\therefore \text{reqd. prob.} = \frac{24}{60} = \frac{2}{5}$$

Example. A bag contains 5 red and 4 green balls. For balls are drawn at random then find the probability that two balls are of red and two balls are of green colour.

Solution. n(s) = the total number of ways of drawing 4 balls out of total 9 balls : ${}^{9}C_4$

If A_1 = the event of drawing 2 red balls out of 5 red balls then n (A_1) = ${}^{5}C_2$

A_2 = the event of drawing 2 green balls out of 4 green balls then n (A_2) = ${}^{4}C_2$

$\therefore\ n(A) = n(A_1) \cdot n(A_2) = {}^{5}C_2 \times {}^{4}C_2$

$$\therefore\ P(A) = \frac{n(A)}{n(s)} = \frac{{}^{5}C_2 \times {}^{4}C_2}{{}^{9}C_4}$$

$$= \frac{\frac{5 \times 4 \times 4 \times 3}{2 \times 2}}{\frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2}} = \frac{10}{21}$$

Example. Two dice are thrown at a time. Find the probability of the following

(i) these numbers shown are equal

(ii) the difference of numbers shown is 1

Solution. The sample space in a throw of two dice

$s = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$

$\therefore$ total no. of cases n (s) = $6 \times 6 = 36$.

(i) Here E_1 = the event of showing equal number on both dice

$= \{(1, 1)\ (2, 2)\ (3, 3)\ (4, 4)\ (5, 5)\ (6, 6)\}$

$$\therefore\ n(E_1) = 6 \quad \therefore\ P(E_1) = \frac{n(E_1)}{n(s)} = \frac{6}{36} = \frac{1}{6}$$

(ii) Here E_2 = the event of showing numbers whose difference is 1.

$= \{(1, 2)\ (2, 1)\ (2, 3)\ (3, 2)\ (3, 4)\ (4, 3)\ (4, 5)\ (5, 4)\ (5, 6)\ (6, 5)\}$

$$\therefore\ n(E_2) = 10 \quad \therefore\ p(E_2) = \frac{n(E_2)}{n(s)} = \frac{10}{36} = \frac{5}{18}$$

ODDS AGAINST AND ODDS IN FAVOUR OF AN EVENT

Let there be m + n equally likely, mutually exclusive and exhaustive cases out of which an event A can occur in m cases and does not occur in n cases. Then by definition of probability of occurrences

$$= \frac{m}{m+n}$$

The probability of non-occurrence $= \dfrac{n}{m+n}$

$\therefore\ P(A) : P(A') = m : n$

Thus the odd in favour of occurrences of the event A are defined by m : n i.e. P(A) : P(A'); and the odds against the occurrence of the event A are defined by n : m i.e. P(A') : P(A).

SOLVED EXAMPLES

Example. Find the odds in favors of getting a king when a card is drawn from a well shuffled pack of 52 cards

Solution. $$\frac{{}^{4}C_1}{{}^{48}C_1} = \frac{4}{48} = \frac{1}{12}$$

Example. In a single cast with two dice find the odds against drawing 7

Solution. E = {(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)}.

$\therefore$ P(E) = $\frac{6}{6\times 6}=\frac{1}{6}$. So, the odds against drawing 7 = $\frac{P(\bar{E})}{P(E)}=\frac{1-\frac{1}{6}}{\frac{1}{6}}=\frac{5}{1}$ = 5 : 1.

ADDITION THEOREM

- If A and B are any events in S, then

  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

  Since the probability of an event is a nonnegative number, it follows that

  $P(A \cup B) \le P(A) + P(B)$

  For three events A, B and C in S we have

  $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$.

General form of addition theorem

For n events $A_1, A_2, A_3, \ldots\ldots A_n$ in S, we have

$P(A_1 \cup A_2 \cup A_3 \cup A_4 \ldots\ldots\ldots \cup A_n)$

$$= \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) \ldots\ldots\ldots\ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap A_3 \ldots\ldots \cap A_n)$$

- If A and B are mutually exclusive, then $P(A \cap B) = 0$ so that $P(A \cup B) = P(A) + P(B)$ **[*Special Addition rule*]**

  Generalizing if $A_1, A_2, A_3, \ldots\ldots, A_n$ are n mutually exclusive events,

  then $P(A_1 \cup A_2 \cup \ldots\ldots \cup A_n) = P(A_1) + P(A_2) + \ldots\ldots. + P(A_n)$.

- If A is any event in S, then

  $P(A') = 1 - P(A) \because A \cup A' = s$ and $A \cap A' = \phi$.

SOLVED EXAMPLES

Example. One digit is selected from 20 positive integers. What is the probability that it is divisible by 3 or 4.

Solution. Let A = event that selected number is divisible by 3

B = event that selected number is divisible by 4.

Here, the events are not mutually exclusive then

$$P(A)=\frac{6}{20},\ P(B)=\frac{5}{20},P(AB)=\frac{1}{20}$$

$$\therefore P(A+B)=P(A)+P(B)-P(AB)$$

$$=\frac{6}{20}+\frac{5}{20}-\frac{1}{20}=\frac{10}{20}=\frac{1}{2}$$

Example. A bag contains 6 white, 5 black and 4 red balls. Find the probability of getting either a white or a black ball in a single draw.

Solution. Let A = event that we get a white ball,

B = event that we get a black ball

So, the events are mutually exclusive

$$P(A)=\frac{{}^{6}C_1}{{}^{15}C_1},P(B)=\frac{{}^{5}C_1}{{}^{15}C_1}$$

So, $P(A+B)=P(A)+P(B)=\frac{6}{15}+\frac{5}{15}=\frac{11}{15}$

MULTIPLICATION THEOREM

When events are independent :

If A_1 , A_2 , ... , A_n are independent events, then P (A_1 , A_2 , ... A_n) = P (A_1) P(A_2) .... P (A_n) .

So if A and B are two independent events then happening of B will have no effect on A. So

P (A/B) = P(A) and P (B/A) = P (B), then

P $(A\cap B)$ = P (A). P (B)

OR

P (AB) = P (A) . P (B)

When events are not independent

The probability of simultaneous happening of two events A and B is equal to the probability of A multiplied by the conditional probability of B with respect to A (or probability of B multiplied by the conditional probability of A with respect of B) i.e

P $(A\cap B)$ = P (A) . P (B/A) or P (B) . P (A/B)

OR

P (AB) = P(A) . P (B/A) or P (B) . P (A/B)

SOLVED EXAMPLES

Example. Let A, B, C be 3 independent events such that $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{2}$, $P(C) = \frac{1}{4}$. Then find probability of exactly 2 events occuring out of 3 events-

Solution. P (exactly two of A, B, C occur)

$$= P(B \cap C) + P(C \cap A) + P(A \cap B) - 3P(A \cap B \cap C)$$

$$= P(B) \cdot P(C) + P(C) \cdot P(A) + P(A) \cdot P(B) - 3P(A) \cdot P(B) \cdot P(C)$$

$$= \frac{1}{2}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{2} - 3\cdot\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{4} = \frac{1}{4}$$

Example. A bag contains 3 red, 6 white and 7 blue balls. Two balls are drawn one by one. What is the probability that first ball is white and second ball is blue when first drawn ball is not replaced in the bag?

Solution. Let A be the event of drawing first ball white and B be the event of drawing second ball blue.

Here A and B are dependent events.

$$P(A) = \frac{6}{16},\ P\left(\frac{B}{A}\right) = \frac{7}{15}$$

$$P(AB) = P(A).P\left(\frac{B}{A}\right) = \frac{6}{16} \times \frac{7}{15} = \frac{7}{40}$$

Example. Three coins are tossed together. What is the probability that first shows head, second shows tail and third shows head?

Solution. Let A, B, C denote three given component events which are mutually independent.

So, $P(ABC) = P(A).P(B).P(C) = \frac{1}{2}.\frac{1}{2}.\frac{1}{2} = \frac{1}{8}$

Example. A bag contains 4 red and 4 blue balls. Four balls are drawn one by one from the bag, then find the probability that the drawn balls are in alternate colour.

Solution. E_1 : Event that first drawn ball is red, second is blue and so on.

E_2 : Event that first drawn ball is blue, second is red and so on.

$\therefore P(E_1) = \frac{4}{8} \times \frac{4}{7} \times \frac{3}{6} \times \frac{3}{5}$ and $P(E_2) = \frac{4}{8} \times \frac{4}{7} \times \frac{3}{6} \times \frac{3}{5}$

$$P(E) = P(E_1) + P(E_2) = 2 \times \frac{4}{8}.\frac{4}{7}.\frac{3}{6}.\frac{3}{5} = \frac{6}{35}$$

Probability of at least one of the n Independent events

If p_1, p_2, p_3, ... p_n are the probabilites of n independent events A_1, A_2, A_3 ..... A_n then the probability of happening of at least one of these event is

$$1 - [(1 - p_1)(1 - p_2) \ldots\ldots (1 - p_n)]$$

$$P(A_1 + A_2 + A_3 + \ldots\ldots + A_n) = 1 - P(\bar{A}_1)\, P(\bar{A}_2)\, P(\bar{A}_3 \ \ldots\ldots\ P(\bar{A}_n)$$

SOLVED EXAMPLES

Example. A probalem of mathematic is given to three students A, B and C. Whose chances of solving it are
1/2, 1/3 , 1/4 respectively. Then probability that the problem is solved is

Solution. Obviously the events of solving the problem by A, B and C are independent. Therefore required probability

$$=1-\left[\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\right] = 1 - \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{2} = \frac{3}{4}$$

CONDITIONAL PROBABILITY

Conditional Probability : If A and B are any events in S then the conditional probability of B relative to A is given by

$$P(B/A) = \frac{P(B \cap A)}{P(A)}, \quad \text{If } P(A) \neq 0$$

Independent Events : If the occurrence of the event A does not depend on the occurrence or the non-occurrence of the event B then A and B are said to be independent events.

Clearly P(B/A) = P(B) and P(A/B) = P(A).

SOLVED EXAMPLES

Example. A coin is tossed thrice. If E be the event of showing atleast two heads and F the event of showing head in the first throw, then find $P\left(\frac{E}{F}\right)$

Solution. There are following 8 outcomes of three throws: HHH, HHT, HTH, HTT THH, THT, TTH, TTT

Also $P(E \cap F) = \frac{3}{8}$ and $P(F) = \frac{4}{8}$

$$\therefore \text{ reqd. prob. } = P\left(\frac{E}{F}\right) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{3}{8}}{\frac{4}{8}} = \frac{3}{4}$$

Example. Two dice are thrown. Find the probability that the numbers appeared has a sum of 8 if it is known that the second dice always exhibits 4

Solution. Let A be the event of occurence of 4 always on the second die

$$= \{(1,4),(2,4),(3,4),(4,4),(5,4),(6,4)\} \quad ; \quad \therefore n(A) = 6$$

and B be the event of occurence of such numbers on both dice whose sum is $8 = \{(4,4)\}$.

Thus, $A \cap B = A \cap \{(4,4)\} = \{(4,4)\}$

$\therefore n(A \cap B) = 1$

$\therefore P\left(\frac{B}{A}\right) = \frac{n(A \cap B)}{n(A)} = \frac{1}{6}$

BOOLE'S INEQUALITY

- For any two events A and B.

  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

  $\therefore \quad P(A \cup B) \le P(A) + P(B) \qquad \{\because P(A \cap B) \ge 0\}$

- For any three events A, B, C

  $P(A \cup B \cup C) \le P(A) + P(B) + P(C)$

- In general for any n events $A_1, A_2, \ldots A_n$

  $P(A_1 \cup A_2 \cup \ldots \cup A_n) \le P(A_1) + P(A_2) + \ldots + P(A_n)$

SOLVED EXAMPLES

Example. If two events such that $P(A) = \frac{1}{3}$ and $P(B) = \frac{2}{3}$, then find $P(A \cup B) \ge$

Solution. $P(A \cup B) = P(A) + P(B) - P(AB)$

and clearly $P(AB) \le \frac{1}{3}$

Hence, $P(A \cup B) \ge P(B) \Rightarrow P(A \cup B) \ge \frac{2}{3}$

BAYE'S THEOREM

Let $A_1, A_2, \ldots, A_n$ be n mutually exclusive and exhaustive events of the sample space S and A is event which can occur with any of the events then $P\left(\frac{A_i}{A}\right) = \frac{P(A_i)\ P(A/A_i)}{\sum_{i=1}^{n} P(A_i)\ P(A/A_i)}$

SOLVED EXAMPLES

Example. A bag A contains 2 white and 3 red balls and a bag B contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and it is found to be red. Find the probability that it was drawn from the bag B.

Solution. Let $E_1 =$ The event of ball being drawn from bag A

$E_2 =$ The event of ball being drawn from bag B.

$E =$ The event of ball being red.

Since, both the bags are equally likely to be selected, therefore

$$P(E_1) = P(E_2) = \frac{1}{2} \text{ and } P\left(\frac{E}{E_1}\right) = \frac{3}{5} \text{ and } P\left(\frac{E}{E_2}\right) = \frac{5}{9}$$

$\therefore$ Required probability

$$P\left(\frac{E_2}{E}\right) = \frac{P(E_2)P\left(\frac{E}{E_2}\right)}{P(E_1).P\left(\frac{E}{E_1}\right) + P(E_2)P\left(\frac{E}{E_2}\right)} = \frac{\frac{1}{2} \times \frac{5}{9}}{\frac{1}{2} \times \frac{3}{5} + \frac{1}{2} \times \frac{5}{9}} = \frac{25}{52}$$

BINOMIAL DISTRIBUTION FOR REPEATED TRIALS

Binomial Experiment : Any experiment which has only two outcomes is known as binomial experiment.

Outcomes of such an experiment are known as success and failure.

Probability of success is denoted by p and probability of failure by q.

$\therefore$ p + q = 1

If binomial experiment is repeated n times, then

$(p + q)^n = {}^nC_0\, q^n + {}^nC_1\, p\, q^{n-1} + {}^nC_2\, p^2\, q^{n-2} + \ldots.. + {}^nC_r\, p^r\, q^{n-r} + \ldots\ldots.. + {}^nC_n\, p^n = 1$

Probability of exactly r successes in n trials = ${}^nC_r p^r q^{n-r}$

Probability of at most r successes in n trails = $\sum_{\lambda=0}^{r} {}^nC_\lambda p^\lambda q^{n-\lambda}$

Probability of atleast r successes in n trails = $\sum_{\lambda=r}^{n} {}^nC_\lambda p^\lambda q^{n-\lambda}$

Probability of having I^{st} success at the r^{th} trials = $p\, q^{r-1}$.

The mean the variance and the standard deviation of binomial distribution are np, npq, $\sqrt{npq}$.

SOLVED EXAMPLES

Example. Two dice are tossed four times find the probability of getting

(i) equal digits exactly two times

(ii) equal digits at least two times

(iii) equal digits at the most two times

Solution. Let A be the event of getting equal digits on the dice. Since number of exhaustive cases is 36 and favourable cases is 6.

$$\therefore P(A) = p = \frac{6}{36} = \frac{1}{6},\ P(\bar{A}) = q = 1 - \frac{1}{6} = \frac{5}{6}$$

Hence by Binomial theorem, we have

$$\left(\frac{5}{6}+\frac{1}{6}\right)^4 = \left(\frac{5}{6}\right)^4 + {}^4C_1\left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right) + {}^4C_2\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right)^2 + {}^4C_3\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)^3 + \left(\frac{1}{6}\right)^4$$

Thus from above result, we have

(i) Probability of getting equal digits exactly two times

$$= {}^4C_2\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right)^2 = \frac{25}{216}$$

(ii) Probability of getting equal digits at least two times

$$= {}^4C_2\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right)^2 + {}^4C_3\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)^3 + \left(\frac{1}{6}\right)^4$$

$$= \frac{25}{216} + \frac{20}{1296} + \frac{1}{1296} = \frac{171}{1296}$$

(iii) Probability of getting equal digits at the most two times

$$= \left(\frac{5}{6}\right)^4 + {}^4C_1\left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right) + {}^4C_2\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right)^2$$

$$= \frac{625}{1296} + \frac{500}{1296} + \frac{150}{1296} = \frac{1275}{1296}$$

SOME IMPORTANT RESULTS

(i) Let A and B be two events, then

- $P(A) + P(\bar{A}) = 1$
- $P(A + B) = 1 - (\bar{A}\,\bar{B})$
- $P(A/B) = \dfrac{P(AB)}{P(B)}$
- $P(A + B) = P(AB) + P(\bar{A}\,B) + P(A\,\bar{B})$
- $A \subset B \Rightarrow P(A) \le P(B)$

- $P(\overline{A}\ B) = P(B) - P(AB)$
- $P(AB) \leq P(A)\ P(B) \leq P(A + B) \leq P(A) + P(B)$
- $P(AB) = P(A) + P(B) - P(A + B)$
- P(Exactly one event) $= P(A\ \overline{B}) + P(\overline{A}\ B)$

$$= P(A) + P(B) - 2P(AB)$$

$$= P(A + B) - P(AB)$$

- P(neither A nor B) $= P(\overline{A}\ \overline{B}) = 1 - P(A + B)$
- $P(\overline{A} + \overline{B}) = 1 - P(AB)$

(ii) Number of exhaustive cases of tossing n coins simultaneously (or of tossing a coin n times) = 2^n

(iii) Number of exhaustive cases of throwing n dice simultaneously (or throwing one dice n times) = 6^n

(iv) Playing Cards :

- Total Cards : 52(26 red, 26 black)
- Four suits : Heart, Diamond, Spade, Club - 13 cards each
- Court Cards : 12 (4 Kings, 4 queens, 4 jacks)
- Honour Cards : 16 (4 aces, 4 kings, queens, 4 jacks)

(v) Probability regarding n letters and their envelopes :

If n letters corresponding to n envelopes are placed in the envelopes at random, then

- Probability that all letters are in right envelopes $= \frac{1}{n!}$
- Probability that all letters are not in right envelopes $= 1 - \frac{1}{n!}$
- Probability that no letters is in right envelopes $= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots.. + (-1)^n \frac{1}{n!}$
- Probability that exactly r letters are in right envelopes $= \frac{1}{r!}\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots.. + (-1)^{n-r} \frac{1}{(n-r)!}\right]$

SOLVED EXAMPLES

Example. There are four letters and four envelopes, the letters are placed into the envelopes at random, find the probability that all letters are placed in the wrong envelope.

Solution. We know from the above given formula that probability that no letter is in right envelope out of n letters and n envelopes is given by

$$= \left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} ..... + (-1)^n \frac{1}{n!}\right]$$

Since all 4 letters are to be placed in wrong envelopes then required probability

$$= \left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right] = \frac{1}{2} - \frac{1}{6} + \frac{1}{24} = \frac{3}{8}$$

CBSE SECTION

SECTION - I

1. If P(A) = 0.8, P(B) = 0.5 and P(B/A) = 0.4 find (i) $P(A \cap B)$ (ii) $P(A/B)$ (iii) $P(A \cup B)$.

2. Mother, father and son line up at random for a family picture. If E is the event 'son on one end' and F is the event 'Father in middle', find P(F/E).

3. Let A and B be independent events with P(A) = 0.3 and P(B) = 0.4. Find

(i) $P(A \cap B)$ (ii) $P(A \cup B)$ (iii) P(A/B) (iv) P(B/A)

4. A random variable X has a probability distribution P(X) of the following form where k is some number:

$$P(X) = \begin{cases} k, & \text{if } x = 0 \\ 2k, & \text{if } x = 1 \\ 3k, & \text{if } x = 2 \\ 0, & \text{otherwise} \end{cases}$$

Determine (i) k (ii) $P(X < 2)$ (iii) $P(X \le 2)$ (iv) $P(x \ge 2)$

SECTION - II

1. Events E and F are known to be independent. Examine, if the following are independent

(i) $\overline{E}, F$, (ii) $E, \overline{F}$, (iii) $\overline{E}, \overline{F}$

2. A random variable X has the following probability distribution

X	0	1	2	3	4	5	6	7
P(X)	0	K	2k	2k	3k	k^2	$2k^2$	$7k^2+k$

3. A doctor is to visit a partient. From the past experience, it is known that the probabilities that he will come by train, bus, scooter or by other means of transport are respectively $\frac{3}{10}, \frac{1}{5}, \frac{1}{10}$ and $\frac{2}{5}$. The probabilities that he will be late are $\frac{1}{4}, \frac{1}{3}, \frac{1}{12}$ if he comes by train, bus and scooter respectively, but if he comes by other means of transport, then he well not be late. When he arrives, he is late. What is the probability that he comes by train?

4. Two groups are competing for the position on Board of Directors of a corporation. The probabilities that the first and the second groups will win are 0.6 and 0.4 respectively. Further, if the first group wins, the probability of introducting a new product is 0.7 and corresponding probability is 0.3 if second group wins. Find the probability that the new product introduced was by second group.

5. Two cards are drawn simultaneously (or successively without replacement) from a well-shuffled pack of 52 cards. Find the mean, variance and standard diviation of the number of kings.

6. An unbiased die is thrown again and again until three sixes are obtained. Find the probability of obtaining third six is sixth throw of a dice.

7. Ten cards numbered 1 through 10 are placed in a box, mixed up thoroughly and then one card is drawn randomly. If it is known that the number on the drawn card is more than 3, what is the probability that it is an even number?

8. In a school there are 1000 students, out of which 430 are girls. It is known that out of 430, 10% of the girls study in class XII. What is the probability that a student chosen randomly studies in class XII, given that the chosen student is a girl?

9. Consider the experiment of tossing a coin. If the coin shows head, toss it again but if it shows tail then throw a die. Find the conditional probability of the event 'the die shows a number greater than 4' given that 'there is at least one tail'.

10. A fair die is rolled. Consider the events $E = \{1,3,5\}$, $F = \{2,3\}$ and $G = \{2,3,4,5\}$. Find

(i) P(E/F) and P(F/E) (ii) P(E/G) and P(G/E) (iii) $P(E \cup F/G)$ (iv) $P(E \cap F/G)$

11. Given that the two numbers appearing on two disc are different. Find the probability of the event 'the sum of numbers on the dice is 4'

12. A box of oranges is inspected by examining three randomly selected oranges drawn without replacement. If all the three oragnes are good, the box is approved for sale, otherwise it is rejected. Find the probability that a box containing 15 organges out of which 12 are good and 3 are bad one will be approved for sale.

13. Events A and B are such that $P(A) = \frac{1}{2}$, $P(B) = \frac{7}{12}$ and P(not A or not B) = $\frac{1}{4}$. State whether A and B are independent?

14. Given three identical boxes I, II and III each containing two coins. In box I, both coins are gold coins, in box II both are silver coins and in box III, there is one gold and one silver coin. A person choses the box at random and takes out a coin. If the selected coin is of gold, what is the probability that the other coin in the box is also of gold?

15. Suppose that the reliability of a HIV test is specified as follows: of people having HIV, 90% of the test detect the disease but 10% go undetected. Of the people free of HIV, 99% of the tests are judged HIV-ve but 1% are diagnosed as showing HIV + ve. From a large population of which only 0.1% have HIV, one person is selected at random, given the HIV test, and the pathologist reports him/her as HIV + ve. What is the probability that the person actually has HIV?

16. There are three coins, one is a two headed coin, another is a baised coin that comes up head 75% of the time and third is an unbiased coin. One of the three coins is chosen at random and tossed, it shows a head. What is the probability that it was a two headed coin?

17. Suppose a girl throws a die. If she gets 5 or 6, she tosses a coin three times and notes the number of heads, If shw gets 1, 2, 3 or 4 she tosses a coin once and notes whether a head or tail is obatined. If she obtained exactly one head, what is the probability that shw threw 1, 2, 3 or 4 with a die?

18. Ten eggs are drawn successively, with replacement, from a lot containing 10% defective eggs. Find the probability that there is at least one defective egg?

19. Five cards are drawn successively, with replacement, from a well shuffled deck of 52 cards. What is the probability that : (i) all the five cards are spades (ii) only 3 cards are spades (iii) none of spade?

20. A bag consists of 10 balls each marked with one of the digits 0 to 9. If four balls are drawn successively with replacement from the bag, what is the probability that none is marked with digit 0?

21. Suppose X has a binomial distribution $B\left(6, \frac{1}{2}\right)$, Show that X = 3 is the most likely outcome.

22. Find the probability of throwing at most 2 sixes in 6 throws of a single dice.

23. The probability of a shooter hitting a target is $\frac{3}{4}$. How many maximum number of must he/she fire so that the probability of hitting the target at least one is more than 0.99?

24. Suppose that 5% of men and 0.25% of women have grey hair. A grey haired person is selected at random. What is the probability of this person being male? Assume that there are equal number of males and females.

CBSE SECTION (ANSWER KEY)

SECTION - I

1. (i) 0.32 (ii) 0.64 (iii) 0.98

2. $\frac{1}{2}$

3. (i) 0.12 (ii) 0.58 (iii) 0.3 (iv) 0.4

4. (i) $\frac{1}{6}$ (ii) $\frac{1}{2}$ (iii) 1 (iv) $\frac{1}{2}$

SECTION - II

2. (i) $\frac{1}{10}$ (ii) $\frac{3}{10}$ (iii) $\frac{19}{100}$ (iv) $\frac{3}{10}$

3. $\frac{1}{2}$

4. $\frac{2}{9}$

5. $\mu = \frac{2}{13}; \sigma^2 = \frac{400}{2873}; \sigma = 0.37$

6. $\frac{625}{23328}$

7. $\frac{4}{7}$

8. 0.1

9. $\frac{2}{9}$

10. (i) $\frac{1}{2}, \frac{1}{3}$ (ii) $\frac{1}{2}, \frac{2}{3}$ (iii) $\frac{3}{4}, \frac{1}{4}$

11. $\frac{1}{15}$

12. $\frac{44}{91}$

13. Not independent

14. $\frac{2}{3}$

15. 0.083

16. $\frac{4}{9}$

17. $\frac{8}{11}$

18. $1-\left(\frac{9}{10}\right)^{10}$

19. (i) $\left(\frac{1}{4}\right)^5$ (ii) $\frac{90}{(4)^5}$ (iii) $\left(\frac{3}{4}\right)^5$

20. $\left(\frac{9}{10}\right)^4$

22. $\frac{35}{18}\left(\frac{5}{6}\right)^4$

23. 4

24. $\frac{20}{21}$