

## Graph Algorithms

### Problems on Graphs

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What kind of problems do we want to solve on/via graphs?

- is the graph fully-connected?
- can we remove an edge and keep it fully-connected?
- is one vertex reachable starting from some other vertex?
- what is the cheapest cost path from  $v$  to  $w$ ?
- which vertices are reachable from  $v$ ? (transitive closure)
- is there a cycle that passes through all vertices? (circuit)
- is there a tree that links all vertices? (spanning tree)
- what is the minimum spanning tree?
- what is the maximal flow through a graph?
- ...
- can a graph be drawn in a plane with no crossing edges? (planar graphs)
- are two graphs "equivalent"? (isomorphism)

## Graph Algorithms

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In this course we examine algorithms for

- connectivity (simple graphs)
- path finding (simple/directed graphs)
- minimum spanning trees (weighted graphs)
- shortest path (weighted graphs)
- maximum flow (weighted graphs)

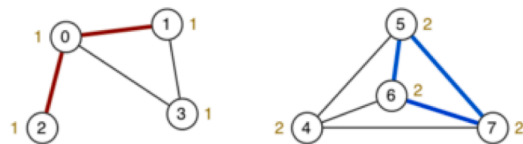
We already looked at *depth-first* (DFS) and *breadth-first* (BFS) traversal ...

## Other DFS Examples

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Other problems to solve via DFS graph search

- checking for the existence of a cycle
- determining which connected component each vertex is in



Graph with two connected components, a path and a cycle

## Exercise #1: Buggy Cycle Check

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A graph has a cycle if

- it has a path of length  $> 1$
- with start vertex  $src$  = end vertex  $dest$
- and without using any edge more than once

We are not required to give the path, just indicate its presence.

The following DFS cycle check has two bugs. Find them.

```
hasCycle(G):
|   Input  graph G
|   Output true if G has a cycle, false otherwise
|
|   choose any vertex v ∈ G
|   return dfsCycleCheck(G, v)
|
dfsCycleCheck(G, v):
|   mark v as visited
|   for each (v, w) ∈ edges(G) do
|       if w has been visited then // found cycle
|           return true
|       else if dfsCycleCheck(G, w) then
|           return true
|   end for
|   return false // no cycle at v
```

1. Only one connected component is checked.
2. The loop

```
for each (v, w) ∈ edges(G) do
```

should exclude the neighbour of  $v$  from which you just came, so as to prevent a single edge  $w-v$  from being classified as a cycle.

## Computing Connected Components

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Problems:

- how many connected subgraphs are there?
- are two vertices in the same connected subgraph?

Both of the above can be solved if we can

- build an array, one element for each vertex  $V$
- indicating which connected component  $V$  is in
- `componentOf [ ] ... array [0..nV-1] of component IDs`

Algorithm to assign vertices to connected components:

```
components(G):
  Input graph G

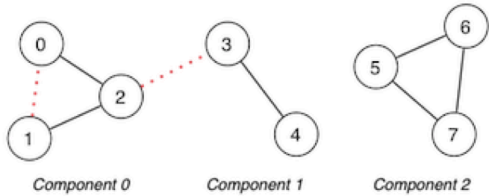
  for all vertices v∈G do
    componentOf[v]=-1
  end for
  compID=0
  for all vertices v∈G do
    if componentOf[v]=-1 then
      dfsComponents(G,v,compID)
      compID=compID+1
    end if
  end for

dfsComponents(G,v,id):
  componentOf[v]=id
  for all vertices w adjacent to v do
    if componentOf[w]=-1 then
      dfsComponents(G,w,id)
    end if
  end for
```

Exercise #2: Connected components

Trace the execution of the algorithm

- 1. on the graph shown below
- 2. on the same graph but with the dotted edges added



Consider neighbours in ascending order

1.

[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
-1	-1	-1	-1	-1	-1	-1	-1
0	-1	-1	-1	-1	-1	-1	-1

2.

[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
-1	-1	-1	-1	-1	-1	-1	-1
0	-1	-1	-1	-1	-1	-1	-1
0	0	-1	-1	-1	-1	-1	-1
0	0	0	-1	-1	-1	-1	-1
0	0	0	0	0	1	1	1

Hamiltonian and Euler Paths

Hamiltonian Path and Circuit

Hamiltonian path problem:

- find a simple path connecting two vertices  $v,w$  in graph  $G$
- such that the path includes each *vertex* exactly once

If  $v = w$ , then we have a *Hamiltonian circuit*

Simple to state, but difficult to solve (*NP*-complete)

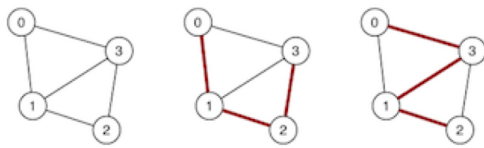
Many real-world applications require you to visit all vertices of a graph:

- Travelling salesman
- Bus routes
- ...

Problem named after Irish mathematician, physicist and astronomer Sir William Rowan Hamilton (1805 - 1865)

... Hamiltonian Path and Circuit

Graph and two possible Hamiltonian paths:



### ... Hamiltonian Path and Circuit

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Approach:

- generate all possible simple paths (using e.g. DFS)
- keep a counter of vertices visited in current path
- stop when find a path containing  $V$  vertices

Can be expressed via a recursive DFS algorithm

- similar to simple path finding approach, except
  - keeps track of path length; succeeds if length =  $v$
  - resets "visited" marker after unsuccessful path

### ... Hamiltonian Path and Circuit

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Algorithm for finding Hamiltonian path:

visited[] // array [0.. $nV-1$ ] to keep track of visited vertices

```
hasHamiltonianPath(G,src,dest):
  for all vertices v∈G do
    visited[v]=false
  end for
  return hamiltonR(G,src,dest,#vertices(G)-1)
```

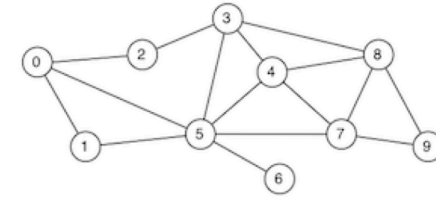
```
hamiltonR(G,v,dest,d):
  Input G      graph
         v      current vertex considered
         dest   destination vertex
         d      distance "remaining" until path found

  if v=dest then
    if d=0 then return true else return false
  else
    visited[v]=true
    for each (v,w)∈edges(G) ∧ ¬visited[w] do
      if hamiltonR(G,w,dest,d-1) then
        return true
      end if
    end for
  end if
  visited[v]=false // reset visited mark
  return false
```

### Exercise #3: Hamiltonian Path

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Trace the execution of the algorithm when searching for a Hamiltonian path from 1 to 6:



Consider neighbours in ascending order

1-0-2-3-4-5-6	$d \neq 0$
1-0-2-3-4-5-7-8-9	no unvisited neighbour
1-0-2-3-4-5-7-9-8	no unvisited neighbour
1-0-2-3-4-7-5-6	$d \neq 0$
1-0-2-3-4-7-8-9	no unvisited neighbour
1-0-2-3-4-7-9-8	no unvisited neighbour
1-0-2-3-4-8-7-5-6	$d \neq 0$
1-0-2-3-4-8-7-9	no unvisited neighbour
1-0-2-3-4-8-9-7-5-6	✓

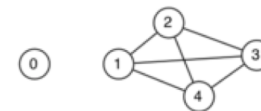
Repeat on your own with  $src=0$  and  $dest=6$

### ... Hamiltonian Path and Circuit

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Analysis: worst case requires  $(V-1)!$  paths to be examined

Consider a graph with isolated vertex and the rest fully-connected



Checking  $hasHamiltonianPath(g,x,0)$  for any  $x$

- requires us to consider every possible path
- e.g 1-2-3-4, 1-2-4-3, 1-3-2-4, 1-3-4-2, 1-4-2-3, ...
- starting from any  $x$ , there are  $3!$  paths  $\Rightarrow 4!$  total paths
- there is no path of length 5 in these  $(V-1)!$  possibilities

There is no known simpler algorithm for this task  $\Rightarrow NP$ -hard.

Note, however, that the above case could be solved in constant time if we had a fast check for 0 and  $x$  being in the same connected component

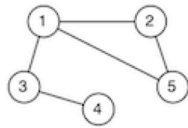
## Euler Path and Circuit

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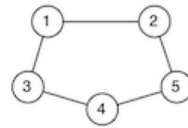
*Euler path* problem:

- find a path connecting two vertices  $v, w$  in graph  $G$
- such that the path includes each *edge* exactly once  
(note: the path does not have to be simple  $\Rightarrow$  can visit vertices more than once)

If  $v = w$ , then we have an *Euler circuit*



Euler Path: 4-3-1-5-2-1



Euler Circuit: 1-2-5-4-3-1

Many real-world applications require you to visit all edges of a graph:

- Postman
- Garbage pickup
- ...

Problem named after Swiss mathematician, physicist, astronomer, logician and engineer Leonhard Euler (1707 - 1783)

### ... Euler Path and Circuit

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One possible "brute-force" approach:

- check for each path if it's an Euler path
- would result in factorial time performance

Can develop a better algorithm by exploiting:

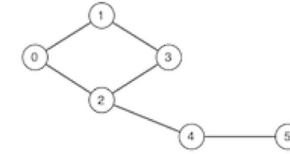
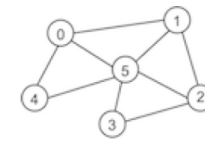
*Theorem.* A graph has an Euler circuit if and only if it is connected and all vertices have even degree

*Theorem.* A graph has a non-circuitous Euler path if and only if it is connected and exactly two vertices have odd degree

## Exercise #4: Euler Paths and Circuits

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Which of these two graphs have an Euler path? an Euler circuit?



No Euler circuit

Only the second graph has an Euler path, e.g. 2-0-1-3-2-4-5

### ... Euler Path and Circuit

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Assume the existence of  $\text{degree}(g, v)$  (degree of a vertex, cf. problem set week 6)

Algorithm to check whether a graph has an Euler path:

```
hasEulerPath(G, src, dest):  
    Input graph G, vertices src, dest  
    Output true if G has Euler path from src to dest  
           false otherwise  
  
    if src != dest then  
        if degree(G, src) or degree(G, dest) is even then  
            return false  
        end if  
    else if degree(G, src) is odd then  
        return false  
    end if  
    for all vertices v in G do  
        if v != src and v != dest and degree(G, v) is odd then  
            return false  
        end if  
    end for  
    return true
```

### ... Euler Path and Circuit

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Analysis of hasEulerPath algorithm:

- assume that connectivity is already checked
- assume that degree is available via  $O(1)$  lookup
- single loop over all vertices  $\Rightarrow O(V)$

If degree requires iteration over vertices

- cost to compute degree of a single vertex is  $O(V)$
- overall cost is  $O(V^2)$

⇒ problem tractable, even for large graphs (unlike Hamiltonian path problem)

For the keen, a linear-time (in the number of edges,  $E$ ) algorithm to compute an Euler path is described in [Sedgewick] Ch.17.7.

## Directed Graphs

### Directed Graphs (Digraphs)

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In our previous discussion of graphs:

- an edge indicates a relationship between two vertices
- an edge indicates nothing more than a relationship

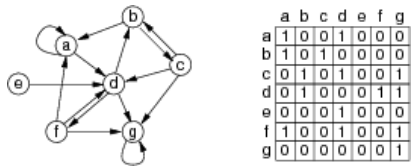
In many real-world applications of graphs:

- edges are directional ( $v \rightarrow w \neq w \rightarrow v$ )
- edges have a *weight* (cost to go from  $v \rightarrow w$ )

### ... Directed Graphs (Digraphs)

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Example digraph and adjacency matrix representation:



	a	b	c	d	e	f	g
a	1	0	0	1	0	0	0
b	1	0	1	0	0	0	0
c	0	1	0	1	0	0	1
d	0	1	0	0	0	1	1
e	0	0	0	1	0	0	0
f	1	0	0	1	0	0	1
g	0	0	0	0	0	0	1

Undirectional ⇒ symmetric matrix

Directional ⇒ non-symmetric matrix

Maximum #edges in a digraph with V vertices:  $V^2$

### ... Directed Graphs (Digraphs)

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Terminology for digraphs ...

*Directed path:* sequence of  $n \geq 2$  vertices  $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_n$

- where  $(v_i, v_{i+1}) \in \text{edges}(G)$  for all  $v_i, v_{i+1}$  in sequence

- if  $v_1 = v_n$ , we have a *directed cycle*

*Reachability:*  $w$  is reachable from  $v$  if  $\exists$  directed path  $v, \dots, w$

## Digraph Applications

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Potential application areas:

Domain	Vertex	Edge
Web	web page	hyperlink
scheduling	task	precedence
chess	board position	legal move
science	journal article	citation
dynamic data	malloc'd object	pointer
programs	function	function call
make	file	dependency

### ... Digraph Applications

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Problems to solve on digraphs:

- is there a directed path from  $s$  to  $t$ ? (transitive closure)
- what is the shortest path from  $s$  to  $t$ ? (shortest path)
- are all vertices mutually reachable? (strong connectivity)
- how to organise a set of tasks? (topological sort)
- which web pages are "important"? (PageRank)
- how to build a web crawler? (graph traversal)

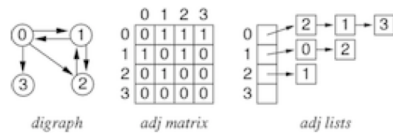
## Digraph Representation

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Similar set of choices as for undirectional graphs:

- array of edges (directed)
- vertex-indexed adjacency matrix (non-symmetric)
- vertex-indexed adjacency lists

$V$  vertices identified by  $0 \dots V-1$



## Reachability

### Transitive Closure

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Given a digraph  $G$  it is potentially useful to know

- is vertex  $t$  reachable from vertex  $s$ ?

Example applications:

- can I complete a schedule from the current state?
- is a malloc'd object being referenced by any pointer?

How to compute transitive closure?

#### ... Transitive Closure

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One possibility:

- implement it via `hasPath(G, s, t)` (itself implemented by DFS or BFS algorithm)
- feasible if `reachable(G, s, t)` is infrequent operation

What if we have an algorithm that frequently needs to check reachability?

Would be very convenient/efficient to have:

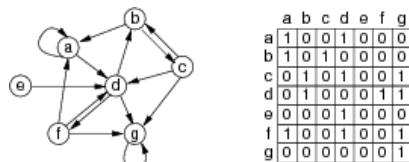
```
reachable(G, s, t):
|   return G.tc[s][t]    // transitive closure matrix
```

Of course, if  $V$  is very large, then this is not feasible.

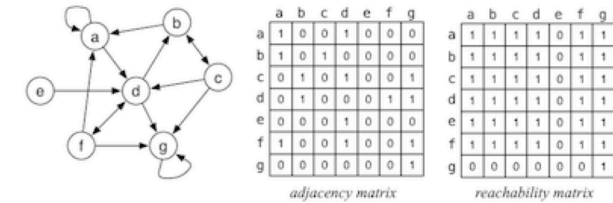
#### Exercise #5: Transitive Closure Matrix

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Which reachable  $s \dots t$  exist in the following graph?



Transitive closure of example graph:



#### ... Transitive Closure

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**Goal:** produce a matrix of reachability values

- if  $tc[s][t]$  is 1, then  $t$  is reachable from  $s$
- if  $tc[s][t]$  is 0, then  $t$  is not reachable from  $s$

So, how to create this matrix?

Observation:

$\forall i, s, t \in \text{vertices}(G)$ :  
 $(s, i) \in \text{edges}(G) \text{ and } (i, t) \in \text{edges}(G) \Rightarrow tc[s][t] = 1$

$tc[s][t] = 1$  if there is a path from  $s$  to  $t$  of length 2 ( $s \rightarrow i \rightarrow t$ )

#### ... Transitive Closure

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If we implement the above as:

```
make tc[][] a copy of edges[][]
for all i vertices(G) do
  for all s vertices(G) do
    for all t vertices(G) do
      if tc[s][i]=1 and tc[i][t]=1 then
        tc[s][t]=1
      end if
    end for
  end for
end for
```

then we get an algorithm to convert `edges` into a `tc`

This is known as *Warshall's algorithm*

#### ... Transitive Closure

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How it works ...

After iteration 1,  $tc[s][t]$  is 1 if

- either  $s \rightarrow t$  exists or  $s \rightarrow 0 \rightarrow t$  exists

After iteration 2,  $tc[s][t]$  is 1 if any of the following exist

- $s \rightarrow t$  or  $s \rightarrow 0 \rightarrow t$  or  $s \rightarrow 1 \rightarrow t$  or  $s \rightarrow 0 \rightarrow 1 \rightarrow t$  or  $s \rightarrow 1 \rightarrow 0 \rightarrow t$

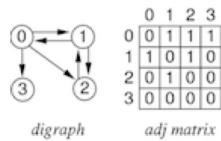
Etc. ... so after the  $V^{th}$  iteration,  $tc[s][t]$  is 1 if

- there is any directed path in the graph from  $s$  to  $t$

Exercise #6: Transitive Closure

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Trace Warshall's algorithm on the following graph:



1<sup>st</sup> iteration  $i=0$ :

tc	[0]	[1]	[2]	[3]
[0]	0	1	1	1
[1]	1	1	1	1
[2]	0	1	0	0
[3]	0	0	0	0

2<sup>nd</sup> iteration  $i=1$ :

tc	[0]	[1]	[2]	[3]
[0]	1	1	1	1
[1]	1	1	1	1
[2]	1	1	1	1
[3]	0	0	0	0

3<sup>rd</sup> iteration  $i=2$ : unchanged

4<sup>th</sup> iteration  $i=3$ : unchanged

... Transitive Closure

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Cost analysis:

- storage: additional  $V^2$  items (each item may be 1 bit)
- computation of transitive closure:  $V^3$
- computation of `reachable()`:  $O(I)$  after having generated `tc[ ][ ]`

Amortisation: would need many calls to `reachable()` to justify other costs

Alternative: use DFS in each call to `reachable()`

Cost analysis:

- storage: cost of queue and set during `reachable`
- computation of `reachable()`: cost of DFS =  $O(V^2)$  (for adjacency matrix)

Digraph Traversal

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Same algorithms as for undirected graphs:

**depthFirst(v):**

1. mark  $v$  as visited
2. for each  $(v,w) \in edges(G)$  do  
    if  $w$  has not been visited then  
        **depthFirst(w)**

**breadth-first(v):**

1. enqueue  $v$
2. while queue not empty do  
    dequeue  $v$   
    if  $v$  not already visited then  
        mark  $v$  as visited  
        enqueue each vertex  $w$  adjacent to  $v$

Example: Web Crawling

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**Goal:** visit every page on the web

**Solution:** breadth-first search with "implicit" graph

```
webCrawl(startingURL):
    mark startingURL as alreadySeen
    enqueue(Q, startingURL)
    while Q is not empty do
        nextPage = dequeue(Q)
        visit nextPage
        for each hyperLink on nextPage do
            if hyperLink not alreadySeen then
                mark hyperLink as alreadySeen
                enqueue(Q, hyperLink)
            end if
        end for
    end for
```

|    **end while**

visit scans page and collects e.g. keywords and links

## PageRank

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**Goal:** determine which Web pages are "important"

**Approach:** ignore page contents; focus on hyperlinks

- treat Web as graph: page = vertex, hyperlink = di-edge
- pages with many incoming hyperlinks are important
- need to computing "incoming degree" for vertices

Problem: the Web is a *very* large graph

- approx.  $10^{14}$  pages,  $10^{15}$  hyperlinks

Assume for the moment that we could build a graph ...

Most frequent operation in algorithm "Does edge (v,w) exist?"

### ... PageRank

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Simple PageRank algorithm:

```
PageRank(myPage):
|   rank=0
|   for each page in the Web do
|   |   if linkExists(page,myPage) then
|   |   |   rank=rank+1
|   |   end if
|   end for
```

Note: requires *inbound* link check (not outbound as assumed above for cost of representation)

### ... PageRank

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$V$  = # pages in Web,  $E$  = # hyperlinks in Web

Costs for computing PageRank for each representation:

Representation	linkExists(v,w)	Cost
Adjacency <i>matrix</i>	edge[v][w]	$I$
Adjacency <i>lists</i>	inLL(list[v],w)	$\cong E/V$

Not feasible ...

- adjacency matrix ...  $V \cong 10^{14} \Rightarrow$  matrix has  $10^{28}$  cells
- adjacency list ...  $V$  lists, each with  $\cong 10$  hyperlinks  $\Rightarrow 10^{15}$  list nodes

So how to really do it?

### ... PageRank

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Approach: the random web surfer

- if we randomly follow links in the web ...
- ... more likely to re-discover pages with many inbound links

```
curr=random page, prev=null
for a long time do
|   if curr not in array ranked[] then
|   |   rank[curr]=0
|   end if
|   rank[curr]=rank[curr]+1
|   if random(0,100)<85 then // with 85% chance ...
|   |   prev=curr
|   |   curr=choose hyperlink from curr // ... crawl on
|   else
|   |   curr=random page // avoid getting stuck
|   |   prev=null
|   end if
end for
```

Could be accomplished while we crawl web to build search index

### Exercise #7: Implementing Facebook

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Facebook could be considered as a giant "social graph"

- what are the vertices?
- what are the edges?
- are edges directional?

What kind of algorithm would ...

- help us find people that you might like to "befriend"?

## Tips for Week 7 Problem Set

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Main theme: *Graph traversal, digraphs*

- Test your understanding of Euler/Hamiltonian paths/cycles
- Test your understanding of directed graphs



- Algorithms:
    - correct the "buggy" cycle check from above
    - maintain "connected component array" as part of graph ADT implementation
  - *Do the online mock test*
  - *Review all concepts, data structures, algorithms from weeks 2-7*
- 

## Summary

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- Graph traversal: cycle check, connected components
  - Hamiltonian paths/circuits, Euler paths/circuits
  - Digraphs: representations, applications
  - Warshall's algorithm to compute reachability
  - Suggested reading (Sedgewick):
    - Hamiltonian/Euler paths ... Ch.17.7
    - Digraphs ... Ch.19.1-19.3
-