

Week 09: Search Tree Algorithms 1

Searching

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An extremely common application in computing

- given a (large) collection of *items* and a *key* value
- find the item(s) in the collection containing that key
 - item = (key, val₁, val₂, ...) (i.e. a structured data type)
 - key = value used to distinguish items (e.g. student ID)

Applications: Google, databases,

... Searching

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Since searching is a very important/frequent operation, many approaches have been developed to do it

Linear structures: arrays, linked lists, files

Arrays = random access. Lists, files = sequential access.

Cost of searching:

	Array	List	File
Unsorted	O(n) (linear scan)	O(n) (linear scan)	O(n) (linear scan)
Sorted	O(log n) (binary search)	O(n) (linear scan)	O(log n) (seek, seek>, ...)

- $O(n)$... linear scan (search technique of last resort)
- $O(\log n)$... binary search, *search trees* (trees also have other uses)

Also (cf. COMP9021): hash tables ($O(1)$), but only under optimal conditions)

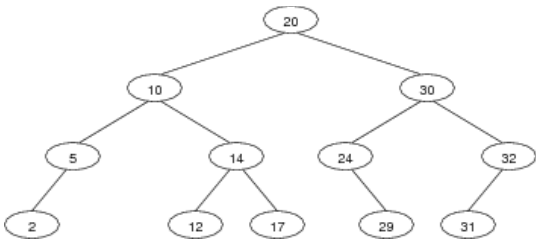
... Searching

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Maintaining the order in sorted arrays and files is a costly operation.

Search trees are as efficient to search but more efficient to maintain.

Example: the following tree corresponds to the sorted array [2, 5, 10, 12, 14, 17, 20, 24, 29, 30, 31, 32]:



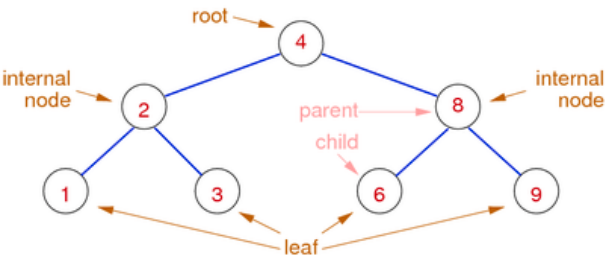
Tree Data Structures

Trees

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Trees are connected graphs

- consisting of nodes and edges (called *links*), with no cycles (no "up-links")
- each node contains a **data** value (or key+data)
- each node has **links** to $\leq k$ other child nodes ($k=2$ below)

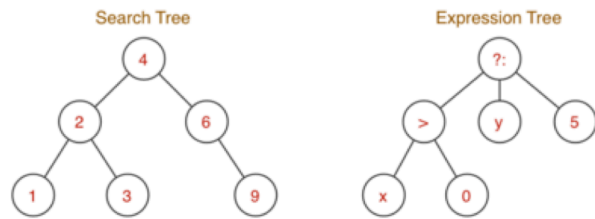


... Trees

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Trees are used in many contexts, e.g.

- representing hierarchical data structures (e.g. expressions)
- efficient searching (e.g. sets, symbol tables, ...)

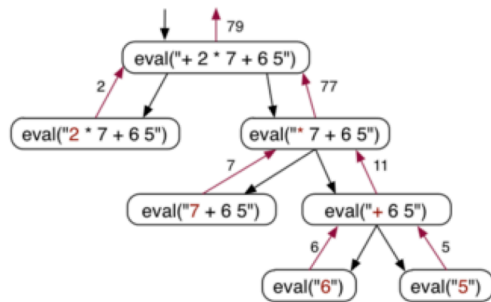


... Trees

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Trees can be used as a data structure, but also for *illustration*.

E.g. showing evaluation of a prefix arithmetic expression



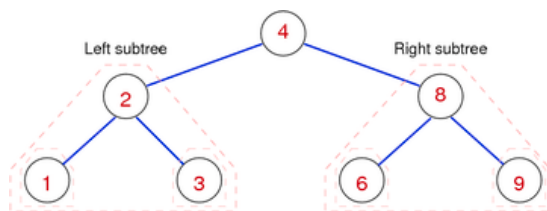
... Trees

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Binary trees ($k=2$ children per node) can be defined recursively, as follows:

A *binary tree* is either

- empty (contains no nodes)
- consists of a *node*, with two *subtrees*
 - node contains a value
 - left and right subtrees are *binary trees*



... Trees

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Other special kinds of tree

- *m-ary tree*: each internal node has exactly m children
- *Ordered tree*: all left values $<$ root, all right values $>$ root
- *Balanced tree*: has \approx minimal height for a given number of nodes
- *Degenerate tree*: has \approx maximal height for a given number of nodes

Search Trees

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Binary Search Trees

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Binary search trees (or *BSTs*) have the characteristic properties

- each node is the root of 0, 1 or 2 subtrees
- all values in any left subtree are less than root
- all values in any right subtree are greater than root
- these properties applies over all nodes in the tree

(perfectly) *balanced trees* have the properties

- #nodes in left subtree = #nodes in right subtree
- this property applies over all nodes in the tree



... Binary Search Trees

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Operations on BSTs:

- *insert*(Tree,Item) ... add new item to tree via key
- *delete*(Tree,Key) ... remove item with specified key from tree
- *search*(Tree,Key) ... find item containing key in tree
- plus, "bookkeeping" ... *new()*, *free()*, *show()*, ...

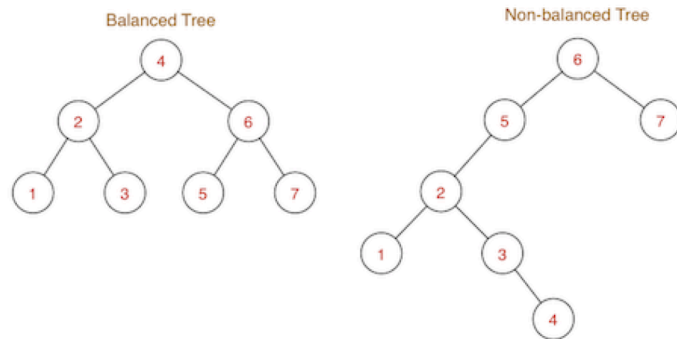
Notes:

- nodes contain *Items*; we just show *Item.key*
- keys are unique (not technically necessary)

... Binary Search Trees

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Examples of binary search trees:



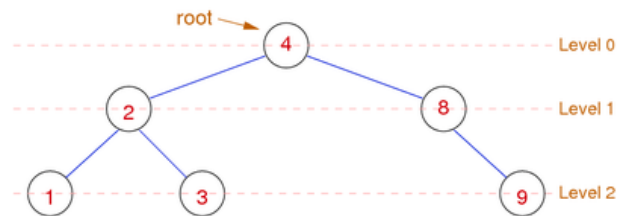
Shape of tree is determined by order of insertion.

... Binary Search Trees

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Level of node = path length from root to node

Height (or: depth) of tree = max path length from root to leaf



Height-balanced tree: \forall nodes: height(left subtree) = height(right subtree)

Time complexity of tree algorithms is typically $O(\text{height})$

Exercise #1: Insertion into BSTs

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For each of the sequences below

- start from an initially empty binary search tree
- show tree resulting from inserting values in order given

(a) 4 2 6 5 1 7 3

(b) 6 5 2 3 4 7 1

(c) 1 2 3 4 5 6 7

Assume new values are always inserted as new leaf nodes

(a) the balanced tree from 3 slides ago (height = 2)

(b) the non-balanced tree from 3 slides ago (height = 4)

(c) a fully degenerate tree of height 6

Representing BSTs

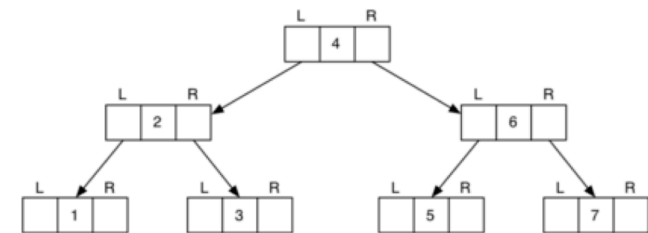
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Binary trees are typically represented by node structures

- containing a value, and pointers to child nodes

Most tree algorithms move *down* the tree.

If upward movement needed, add a pointer to parent.



... Representing BSTs

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Typical data structures for trees ...

```
// a Tree is represented by a pointer to its root node
typedef struct Node *Tree;
```

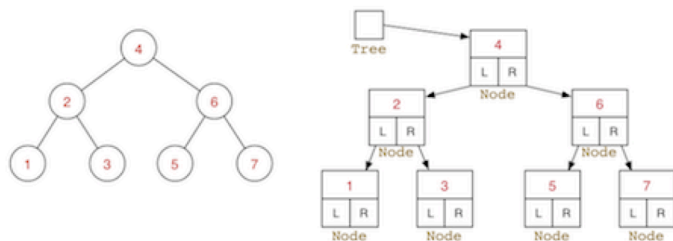
```
// a Node contains its data, plus left and right subtrees
typedef struct Node {
    int data;
    Tree left, right;
} Node;
```

```
// some macros that we will use frequently
#define data(tree) ((tree)->data)
#define left(tree) ((tree)->left)
#define right(tree) ((tree)->right)
```

We ignore items \Rightarrow data in Node is just a key

... Representing BSTs

Abstract data vs concrete data ...



Tree Algorithms

Searching in BSTs

Most tree algorithms are best described recursively:

```
TreeSearch(tree,item):
  Input  tree, item
  Output true if item found in tree, false otherwise

  if tree is empty then
    return false
  else if item < data(tree) then
    return TreeSearch(left(tree),item)
  else if item > data(tree) then
    return TreeSearch(right(tree),item)
  else // found
    return true
  end if
```

Insertion into BSTs

Insert an item into appropriate subtree:

```
insertAtLeaf(tree,item):
  Input  tree, item
  Output tree with item inserted

  if tree is empty then
    return new node containing item
  else if item < data(tree) then
    return insertAtLeaf(left(tree),item)
  else if item > data(tree) then
    return insertAtLeaf(right(tree),item)
```

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```
else
  return tree // avoid duplicates
end if
```

Tree Traversal

Iteration (traversal) on ...

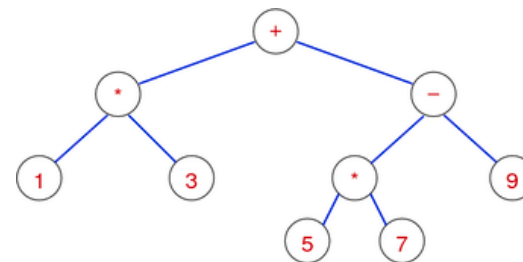
- Lists ... visit each value, from first to last
- Graphs ... visit each vertex, order determined by DFS/BFS/...

For binary Trees, several well-defined visiting orders exist:

- *preorder* (NLR) ... visit root, then left subtree, then right subtree
- *inorder* (LNR) ... visit left subtree, then root, then right subtree
- *postorder* (LRN) ... visit left subtree, then right subtree, then root
- *level-order* ... visit root, then all its children, then all their children

... Tree Traversal

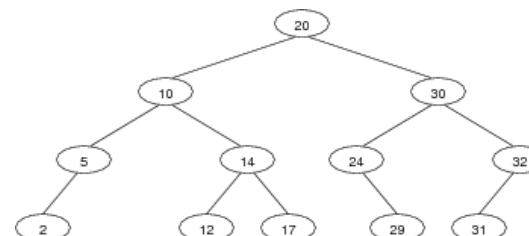
Consider "visiting" an expression tree like:



NLR: + * 1 3 - * 5 7 9 (prefix-order: useful for building tree)
LNR: 1 * 3 + 5 * 7 - 9 (infix-order: "natural" order)
LRN: 1 3 * 5 7 * 9 - + (postfix-order: useful for evaluation)
Level: + * - 1 3 * 9 5 7 (level-order: useful for printing tree)

Exercise #2: Tree Traversal

Show NLR, LNR, LRN traversals for the following tree:



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NLR (preorder): 20 10 5 2 14 12 17 30 24 29 32 31

LNR (inorder): 2 5 10 12 14 17 20 24 29 30 31 32

LRN (postorder): 2 5 12 17 14 10 29 24 31 32 30 20

Exercise #3: Non-recursive traversals

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Write a non-recursive *preorder* traversal algorithm.

Assume that you have a stack ADT available.

```
showBSTreePreorder(t):
  Input tree t

  push t onto new stack S
  while stack is not empty do
    t=pop(S)
    print data(t)
    if right(t) is not empty then
      push right(t) onto S
    end if
    if left(t) is not empty then
      push left(t) onto S
    end if
  end while
```

Joining Two Trees

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An auxiliary tree operation ...

Tree operations so far have involved just one tree.

An operation on two trees: $t = \text{joinTrees}(t_1, t_2)$

- Pre-conditions:
 - takes two BSTs; returns a single BST
 - $\max(\text{key}(t_1)) < \min(\text{key}(t_2))$
 - Post-conditions:
 - result is a BST (i.e. fully ordered)
 - containing all items from t_1 and t_2
-

... Joining Two Trees

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Method for performing tree-join:

- find the min node in the right subtree (t_2)
- replace min node by its right subtree
- elevate min node to be new root of both trees

Advantage: doesn't increase height of tree significantly

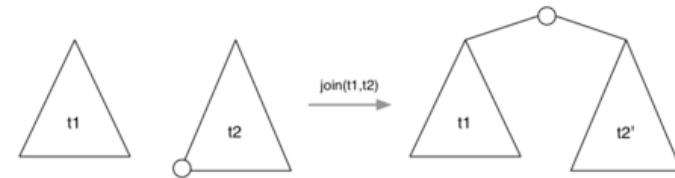
$x \leq \text{height}(t) \leq x+1$, where $x = \max(\text{height}(t_1), \text{height}(t_2))$

Variation: choose deeper subtree; take root from there.

... Joining Two Trees

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Joining two trees:



Note: t_2' may be less deep than t_2

... Joining Two Trees

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Implementation of tree-join:

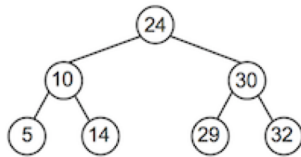
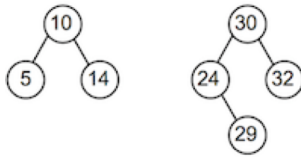
```
joinTrees( $t_1, t_2$ ):
  Input trees  $t_1, t_2$ 
  Output  $t_1$  and  $t_2$  joined together

  if  $t_1$  is empty then return  $t_2$ 
  else if  $t_2$  is empty then return  $t_1$ 
  else
    curr= $t_2$ , parent=NULL
    while left(curr) is not empty do      // find min element in  $t_2$ 
      parent=curr
      curr=left(curr)
    end while
    if parent≠NULL then
      left(parent)=right(curr) // unlink min element from parent
      right(curr)= $t_2$ 
    end if
    left(curr)= $t_1$ 
    return curr                      // curr is new root
  end if
```

Exercise #4: Joining Two Trees

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Join the trees



Deletion from BSTs

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Insertion into a binary search tree is easy.

Deletion from a binary search tree is harder.

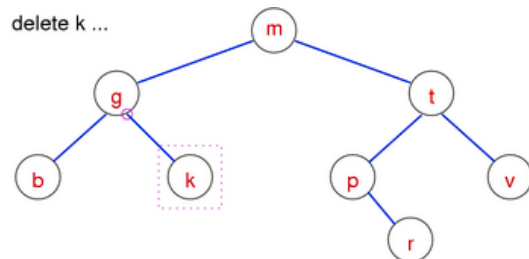
Four cases to consider ...

- empty tree ... new tree is also empty
- zero subtrees ... unlink node from parent
- one subtree ... replace by child
- two subtrees ... replace by successor, join two subtrees

... Deletion from BSTs

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Case 2: item to be deleted is a leaf (zero subtrees)

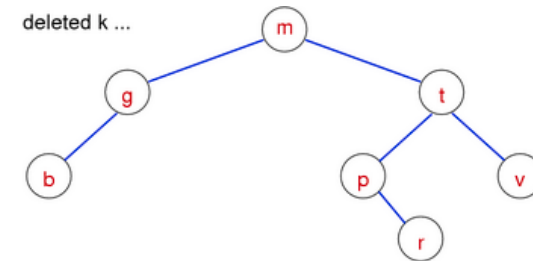


Just delete the item

... Deletion from BSTs

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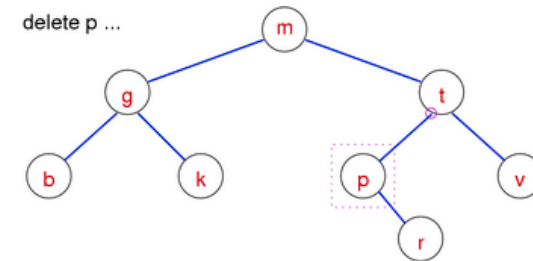
Case 2: item to be deleted is a leaf (zero subtrees)



... Deletion from BSTs

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Case 3: item to be deleted has one subtree

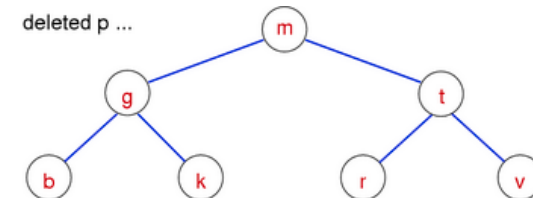


Replace the item by its only subtree

... Deletion from BSTs

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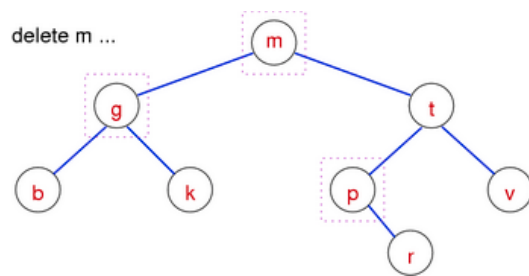
Case 3: item to be deleted has one subtree



... Deletion from BSTs

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Case 4: item to be deleted has two subtrees

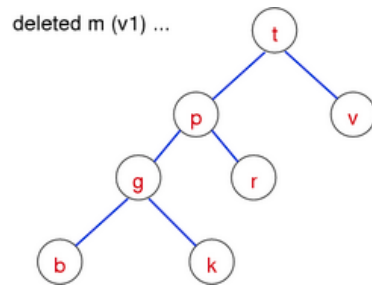


Version 1: right child becomes new root, attach left subtree to min element of right subtree

... Deletion from BSTs

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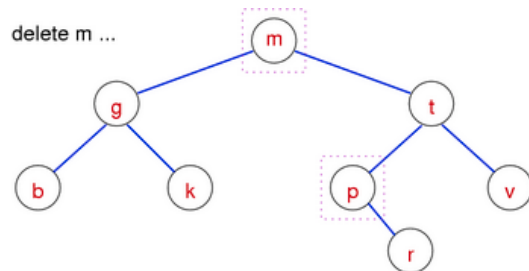
Case 4: item to be deleted has two subtrees



... Deletion from BSTs

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Case 4: item to be deleted has two subtrees

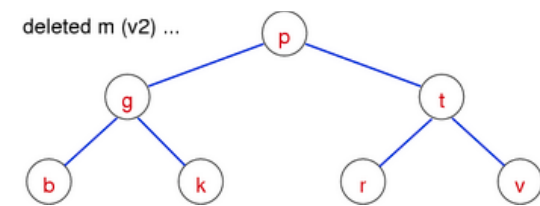


Version 2: *join* left and right subtree

... Deletion from BSTs

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Case 4: item to be deleted has two subtrees



... Deletion from BSTs

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Pseudocode (version 2):

TreeDelete(t, item):

Input tree t, item

Output t with item deleted

```

if t is not empty then           // nothing to do if tree is empty
  if item < data(t) then           // delete item in left subtree
    left(t)=TreeDelete(left(t),item)
  else if item > data(t) then     // delete item in right subtree
    right(t)=TreeDelete(right(t),item)
  else                           // node 't' must be deleted
    if left(t) and right(t) are empty then // 0 children
      new=empty tree
    else if left(t) is empty then // 1 child
      new=right(t)
    else if right(t) is empty then // 1 child
      new=left(t)
    else                          // 2 children
      new=joinTrees(left(t),right(t))
    end if
    free memory allocated for t
    t=new
  end if
end if
return t

```

Balanced BSTs

Balanced Binary Search Trees

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Goal: build binary search trees which have

- minimum height \Rightarrow minimum worst case search cost

Perfectly balanced tree with N nodes has

- $\text{abs}(\text{\#nodes}(\text{LeftSubtree}) - \text{\#nodes}(\text{RightSubtree})) < 2$, for every node
- height of $\log_2 N \Rightarrow$ worst case search $O(\log N)$

Three *strategies* to improving worst case search in BSTs:

- *randomise* — reduce chance of worst-case scenario occurring
- *amortise* — do more work at insertion to make search faster
- *optimise* — implement all operations with performance bounds

Operations for Rebalancing

To assist with rebalancing, we consider new operations:

Left rotation

- move right child to root; rearrange links to retain order

Right rotation

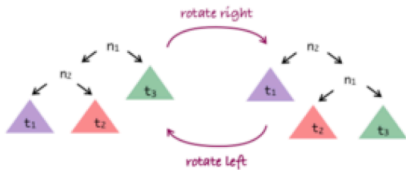
- move left child to root; rearrange links to retain order

Insertion at root

- each new item is added as the new root node

Tree Rotation

In tree below: $t_1 < n_2 < t_2 < n_1 < t_3$



... Tree Rotation

Method for rotating tree T right:

- N_1 is current root; N_2 is root of N_1 's left subtree
- N_1 gets new left subtree, which is N_2 's right subtree
- N_1 becomes root of N_2 's new right subtree
- N_2 becomes new root

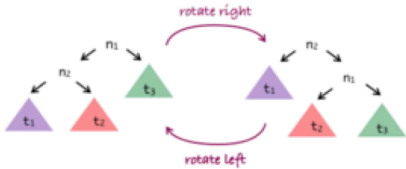
Left rotation: swap left/right in the above.

Cost of tree rotation: $O(1)$

... Tree Rotation

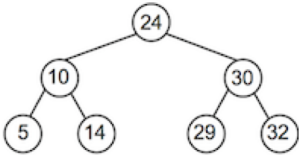
Algorithm for right rotation:

```
rotateRight( $n_1$ ):
|   Input   tree  $n_1$ 
|   Output  $n_1$  rotated to the right
|
|   if  $n_1$  is empty or left( $n_1$ ) is empty then
|       return  $n_1$ 
|   end if
|    $n_2 = \text{left}(n_1)$ 
|   left( $n_1$ ) = right( $n_2$ )
|   right( $n_2$ ) =  $n_1$ 
|   return  $n_2$ 
```

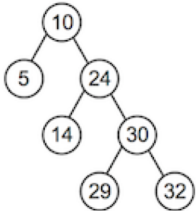


Exercise #5: Tree Rotation

Consider the tree t :

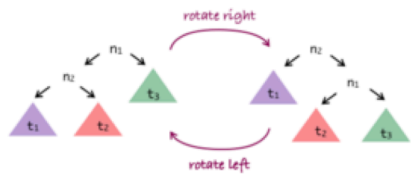


Show the result of rotateRight(t)



Exercise #6: Tree Rotation

Write the algorithm for left rotation



```
rotateLeft(n2):
|   Input  tree n2
|   Output n2 rotated to the left
|
|   if n2 is empty or right(n2) is empty then
|       return n2
|   end if
|   n1=right(n2)
|   right(n2)=left(n1)
|   left(n1)=n2
|   return n1
```

Insertion at Root

Previous description of BSTs inserted at leaves.

Different approach: insert new item at root.

Potential disadvantages:

- large-scale rearrangement of tree for each insert

Potential advantages:

- recently-inserted items are close to root
- low cost if recent items more likely to be searched

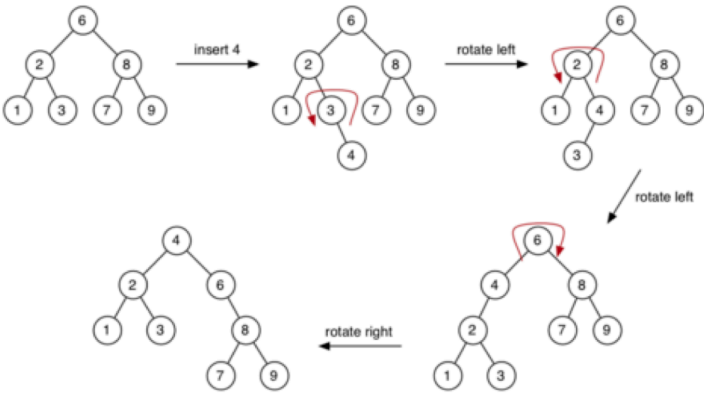
... Insertion at Root

Method for inserting at root:

- base case:
 - tree is empty; make new node and make it root
- recursive case:

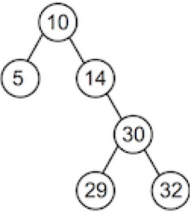
- insert new node as root of appropriate subtree
- lift new node to root by rotation

... Insertion at Root

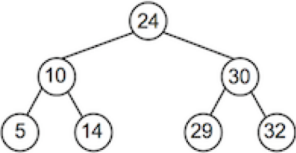


Exercise #7: Insertion at Root

Consider the tree t:



Show the result of insertAtRoot(t, 24)



... Insertion at Root

Analysis of insertion-at-root:

- same complexity as for insertion-at-leaf: $O(\text{height})$
- tendency to be balanced, but no balance guarantee
- benefit comes in searching
 - for some applications, search favours recently-added items
 - insertion-at-root ensures these are close to root
- could even consider "move to root when found"
 - effectively provides "self-tuning" search tree

Rebalancing Trees

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An approach to balanced trees:

- insert into leaves as for simple BST
- periodically, rebalance the tree

Question: how frequently/when/how to rebalance?

NewTreeInsert(tree,item):

```

Input  tree, item
Output tree with item randomly inserted

t=insertAtLeaf(tree,item)
if #nodes(t) mod  $k = 0$  then
    t=rebalance(t)
end if
return t
  
```

E.g. rebalance after every 20 insertions \Rightarrow choose $k=20$

Note: To do this efficiently we would need to change tree data structure and basic operations:

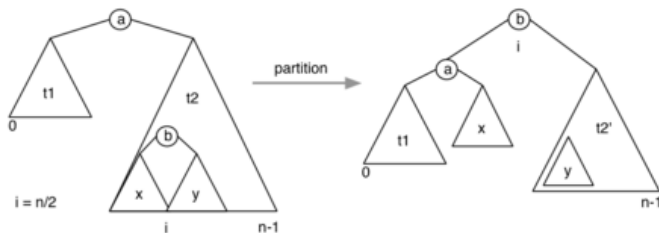
```

typedef struct Node {
    int data;
    int nnodes; // #nodes in my tree
    Tree left, right; // subtrees
} Node;
  
```

... Rebalancing Trees

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How to rebalance a BST? Move median item to root.



... Rebalancing Trees

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Implementation of rebalance:

rebalance(t):

```

Input  tree t with n nodes
Output t rebalanced

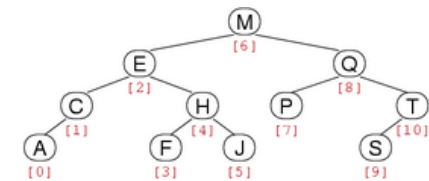
if n ≥ 3 then
    t=partition(t, ⌊n/2⌋) // put node with median key at root
    left(t)=rebalance(left(t)) // then rebalance each subtree
    right(t)=rebalance(right(t))
end if
return t
  
```

... Rebalancing Trees

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New operation on trees:

- **partition(tree,i)**: re-arrange tree so that element with index i becomes root

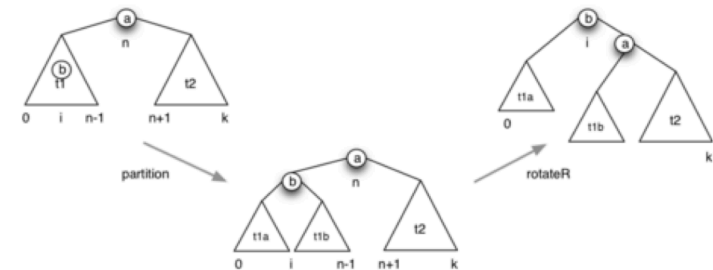


For tree with N nodes, indices are $0 \dots N-1$

... Rebalancing Trees

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Partition: moves i^{th} node to root



... Rebalancing Trees

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Implementation of partition operation:

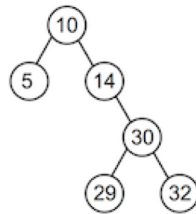
```
partition(tree, i):  
    Input  tree with n nodes, index i  
    Output tree with  $i^{\text{th}}$  item moved to the root  
  
    m=#nodes(left(tree))  
    if i < m then  
        left(tree)=partition(left(tree), i)  
        tree=rotateRight(tree)  
    else if i > m then  
        right(tree)=partition(right(tree), i-m-1)  
        tree=rotateLeft(tree)  
    end if  
    return tree
```

Note: $\text{size}(\text{tree}) = n$, $\text{size}(\text{left}(\text{tree})) = m$, $\text{size}(\text{right}(\text{tree})) = n-m-1$ (why -1?)

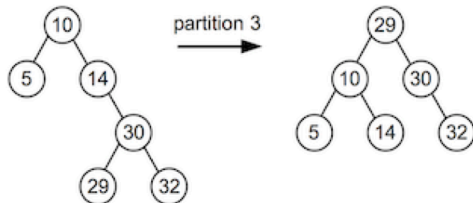
Exercise #8: Partition

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Consider the tree t:



Show the result of `partition(t, 3)`



... Rebalancing Trees

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Analysis of rebalancing: visits every node $\Rightarrow O(N)$

Cost means not feasible to rebalance after each insertion.

When to rebalance? ... Some possibilities:

- after every k insertions
- whenever "imbalance" exceeds threshold

Either way, we tolerate worse search performance for periods of time.

Does it solve the problem? ... Not completely \Rightarrow Solution: real balanced trees (next week)

Application of BSTs: Sets

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Trees provide efficient search.

Sets require efficient search

- to find where to insert/delete
- to test for set membership

Logical to implement a set ADT via BSTree

... Application of BSTs: Sets

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Assuming we have `Tree` implementation

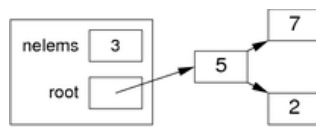
- which precludes duplicate key values
- which implements insertion, search, deletion

then `Set` implementation is

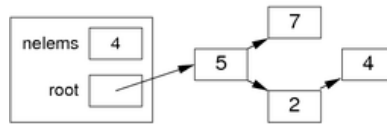
- `SetInsert(Set, Item) = TreeInsert(Tree, Item)`
- `SetDelete(Set, Item) = TreeDelete(Tree, Item.Key)`
- `SetMember(Set, Item) = TreeSearch(Tree, Item.Key)`

... Application of BSTs: Sets

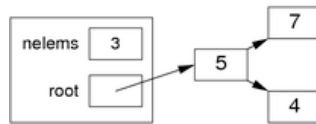
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After SetInsert(s,4):



After SetDelete(s,2):



- Suggested reading:
 - Sedgewick, Ch.12.5-12.6,12.8-12.9

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... Application of BSTs: Sets

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Concrete representation:

```
#include <BSTree.h>
```

```
typedef struct SetRep {
    int    nelems;
    Tree   root;
} SetRep;
```

```
Set newSet() {
    Set S = malloc(sizeof(SetRep));
    assert(S != NULL);
    S->nelems = 0;
    S->root = newTree();
    return S;
}
```

Summary

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- Binary search tree (BST) data structure
- BST insertion and deletion
- Other tree operations
 - tree rotation
 - tree partition
 - joining trees