AN EXPLORATION OF SHELL SORT GAP INTERVALS

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Abstract. The search for an optimal gap sequence to be used in Shell's sort algorithm continues. This paper evaluates twenty-one uniformly distributed gap sequence factors on a closed interval $[2, 3]$ to test the hypothesis that $\frac{5}{2}$ is the optimal gap factor.

Contents

1. Introduction

A search of the internet for Shell sort gap sequences yields an interesting discussion on Stack Overflow. The URL is `https://stackoverflow.com/questions/2539545/fastest-gap-sequence-for-shell-sort`. That discussion states that the optimum gap sequence is 1, 4, 10, 23, 57, 132, 301, 701, 1750. This sequence can be found in The Online Encyclopedia of Integer Sequences (`https://oeis.org/A102549`), and is credited to Marcin Ciura. After 1750, the Stack Overflow page asserts that the next number in the sequence is the product of $\frac{5}{2}$ and the previous number ($n_i = \lfloor \frac{5n_{i-1}}{2} \rfloor$). Unfortunately, no literature was cited to support this assertion.

Many sources on the Internet offer a fine explanation of Shell's sorting algorithm and its history. There is no compelling reason to duplicate that information here. The focus will be on the effect of those twenty-one gap factors on sort performance. Suffice it to say that, when recursion is not appropriate, Shell's algorithm remains a popular alternative. This is particularly true when the data are partially sequenced. Another advantage of this algorithm is that it is easily implemented.

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Theoretical analysis of Shell's algorithm is extremely challenging. Therefore, the empirical method was employed. A Github repository `https://github.com/56phil/Shell.git` contains the raw data and source code.

2. Procedure

The two Python programs written for this experiment are named shell and plot. All sorting and data collection procedures are performed in shell. Plot organizes the raw data for Gnuplot. The raw data are saved in a pickle file named `observations.pickle`. As an alternative, the data are also available as a JSON file named `observations.txt`.

The six array lengths tested include a random element called m, to insure that powers of two are not used. The size values have the form:

$$2^n \pm m \ \ where\ 18 \leq n \leq 22\ and\ 0 \leq m \leq 2^{n-4}$$

Every combination of array size and gap factor is tested seven times, using the same set of seven unique, randomly generated arrays. The time used to execute each sort is saved in the observations structure and presented in the observations section. Except for the box and whiskers plot, only the median time for each combination of array size and gap factor are used.

All gap sequences begin with 1, 4 10, 23, 57, 132, 301, 701, 1750. A gap factor becomes active when gap values greater than 1750 are required. Each value after 1750 is the product of the current gap factor and the previous gap value. The process continues until $n_i > \lfloor \frac{N}{3} \rfloor$.

All observations were made using a MacBook Pro, model 11,5 with a 2.8 GHz i7 processor and 16 GB of memory. The operating system used was MacOS 10.12.6. Programs were written for Python 2.7.10 with NumPy imported. The graphics were generated with Gnuplot.

3. Observations

3.1. **Summary.** Table 1, on page 3, summarizes the relationship between gap factor and median time for each combination of array size and gap factor. The bottom row displays a composite score for each gap factor. Each factor has a variable that contains the sum of its position score squared from the preceding six rows. Last place gets a point and the fastest factor gets 441 points. The structure is sorted using that score as the key, and the quicker gap factors are placed in the right.

3.2. **Run Times.** Tables 2 through 7, beginning on page 5, display the median run time, of seven sorts, performed using each gap factor and array size. Note that whole numbers make poor gap factors.

3.3. **Statistics.** Tables 8 through 13, starting on page 8, contain the usual statistics (minimum, mean, maximum, standard deviation, and median) for run times of each gap factor and array size.

3.4. **Time v. Size.** Figure 2 on page 11 is a visualization of data contained in tables 2 through 7. The scale for the x-axis is logarithmic, base two.

3.5. **Visual Statistics.** Figure 1 on page 4 displays six plots. Each one includes a histogran, an ogive, and at the bottom, the distribution of all run times for that array size.

3.6. **Distribution of Times.** Figure 3 on page 12 provides six box and whiskers plots indicating the distribution of the 147 run times for each array size. The number of outliers increase as the array lengths. This is due to the remarkably slow times for factors 2.0 and 3.0.

4. Conclusion

$\frac{5}{2}$ is a poor good choice. Referring to the composite score row in table 1, on page 3, one can see that $\frac{23}{10}$ is a better gap factor.

Clearly, a recursive algorithm such as Hoare's Quicksort will be substantially faster. However, there are times when resources are limited, or recursion is inappropriate. In such times, Shell's algorithm, with the right set of gaps, is a worthy alternative.

Finally, it is important to consider the environment in which this experiment was performed. Using a multitasking multiuser operating system, MacOS Sierra in this case, adds a random element to the measurements. The execution of other tasks in background may have had an unknown effect on the observations.

A. Tables and Figures

N	← Slower						Gap Factor									Faster →					
247,959	2.00	3.00	2.90	2.75	2.70	2.25	2.95	2.30	2.85	2.60	2.20	2.65	2.50	2.45	2.55	2.80	2.35	2.10	2.15	2.05	2.40
556,034	3.00	2.00	2.90	2.65	2.75	2.85	2.95	2.80	2.50	2.55	2.70	2.60	2.45	2.40	2.35	2.05	2.15	2.20	2.10	2.30	2.25
1,037,375	2.00	3.00	2.50	2.35	2.60	2.65	2.10	2.20	2.15	2.75	2.55	2.25	2.45	2.40	2.05	2.70	2.80	2.30	2.85	2.90	2.95
1,970,626	2.00	3.00	2.15	2.05	2.10	2.35	2.50	2.90	2.85	2.25	2.70	2.80	2.95	2.40	2.20	2.60	2.75	2.65	2.30	2.45	2.55
4,365,249	2.00	3.00	2.50	2.95	2.90	2.85	2.60	2.75	2.05	2.80	2.10	2.35	2.65	2.15	2.20	2.70	2.55	2.45	2.25	2.40	2.30
8,895,124	3.00	2.00	2.95	2.85	2.90	2.80	2.50	2.75	2.70	2.65	2.05	2.35	2.10	2.60	2.55	2.25	2.45	2.30	2.40	2.15	2.20
Composite	2.00	3.00	2.50	2.90	2.75	2.85	2.95	2.60	2.65	2.35	2.70	2.80	2.10	2.05	2.15	2.25	2.20	2.55	2.45	2.40	2.30

Table 1. Summary of effect of gap factors on performance

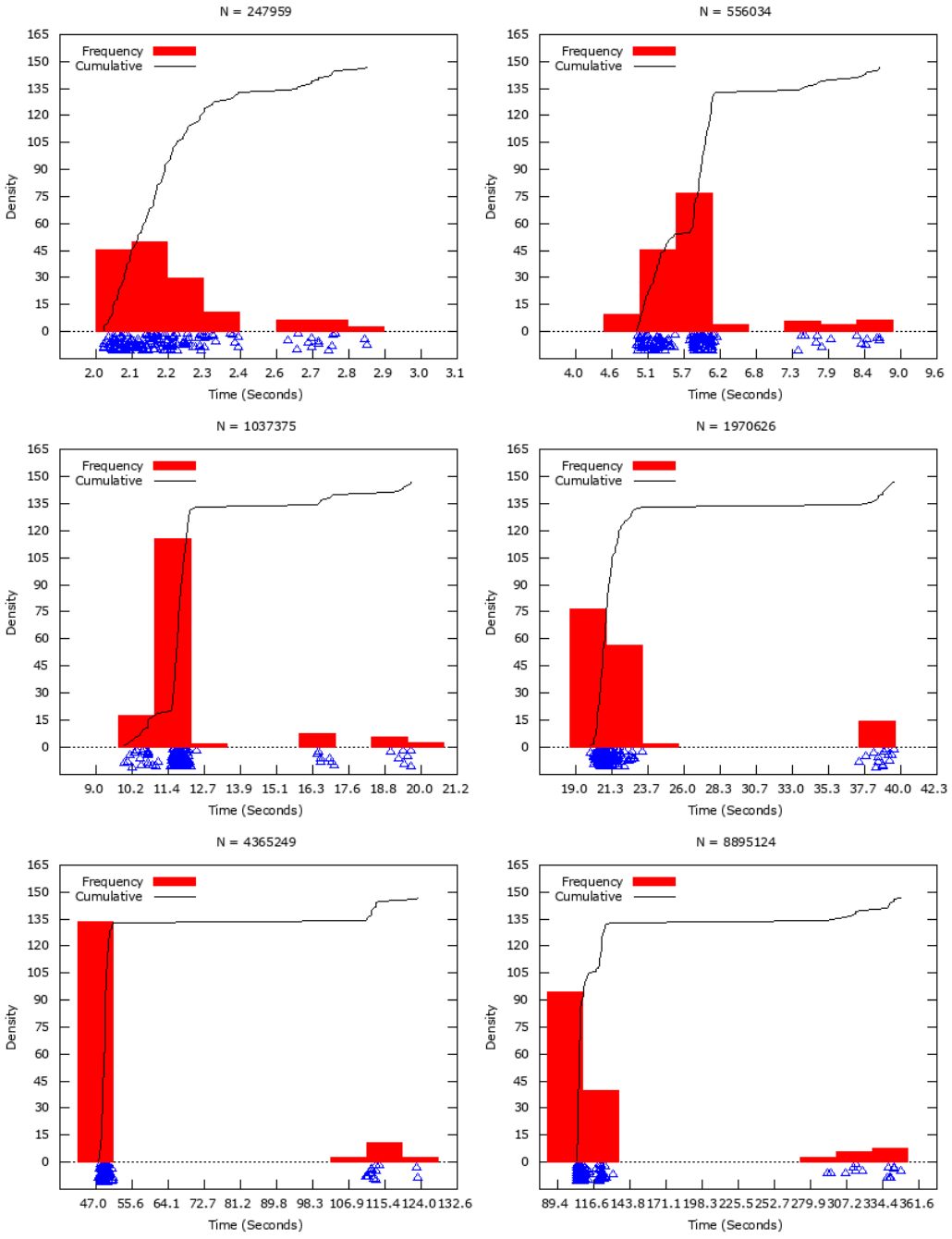

FIGURE 1. Visualization of Statistics

Factor	Run Times						
2.00	2.68574	2.71395	2.73002	2.68532	2.66285	2.82671	2.83372
2.05	2.14485	2.11681	2.07856	2.08778	2.05813	2.06436	2.09055
2.10	2.23106	2.10665	2.05284	2.05151	2.10409	2.09397	2.06858
2.15	2.06857	2.02425	2.06975	2.09290	2.20183	2.30444	2.17650
2.20	2.08509	2.23728	2.18001	2.02439	2.18626	2.26697	2.05178
2.25	2.20886	2.11937	2.24816	2.12811	2.11259	2.36742	2.26957
2.30	2.34268	2.17862	2.19476	2.08336	2.28546	2.28047	2.12963
2.35	2.10314	2.04073	2.14190	2.27573	2.12640	2.05471	2.41203
2.40	2.18451	2.02269	2.05272	2.02996	2.07559	2.10244	2.17960
2.45	2.17818	2.03982	2.08720	2.04546	2.15322	2.25170	2.21349
2.50	2.22600	2.15497	2.23638	2.18710	2.15533	2.06655	2.14430
2.55	2.06594	2.14081	2.16365	2.43760	2.08137	2.09745	2.21133
2.60	2.20744	2.18442	2.24938	2.23883	2.13735	2.13231	2.09162
2.65	2.10890	2.20646	2.09475	2.07390	2.15950	2.15877	2.27769
2.70	2.35628	2.27995	2.21042	2.26755	2.21305	2.05303	2.16457
2.75	2.32501	2.36089	2.32083	2.22904	2.15113	2.21845	2.12865
2.80	2.10534	2.18438	2.13816	2.20430	2.12364	2.03756	2.23047
2.85	2.23518	2.17657	2.05709	2.08795	2.18910	2.29708	2.23115
2.90	2.33271	2.32185	2.21564	2.33365	2.42005	2.42767	2.33211
2.95	2.10896	2.43606	2.16967	2.32097	2.31502	2.04939	2.20214
3.00	2.73554	2.66280	2.65252	2.62105	2.59155	2.72525	2.61967

TABLE 2. Run times for array of size 247,959

Factor	Run Times						
2.00	7.79135	7.71495	7.41761	7.69446	7.47680	7.93016	7.51997
2.05	5.20382	5.23691	5.03664	4.96797	5.39322	5.29131	5.30473
2.10	5.50401	5.04045	5.39545	5.25581	5.12448	5.06233	4.99809
2.15	5.11093	5.21499	4.99968	5.25030	5.40855	5.07231	5.16582
2.20	4.99233	5.28418	4.95351	5.24665	5.42649	5.14088	5.06683
2.25	5.07708	5.32013	5.00786	5.01964	4.94207	4.99033	5.19524
2.30	5.04015	5.06492	4.95139	5.10565	5.21063	5.15182	5.44010
2.35	5.37846	4.97270	5.30499	5.26574	5.28107	5.19064	5.08518
2.40	5.22291	5.46424	5.37024	5.52293	5.29908	5.83137	5.81076
2.45	5.81886	5.81837	5.91566	5.75613	5.80013	5.77348	5.82701
2.50	5.80861	5.88267	6.06548	5.88612	5.94178	6.08083	5.95163
2.55	5.80747	5.88977	5.83085	6.04934	6.01179	6.05029	5.83795
2.60	5.83192	5.81093	5.94105	5.99832	5.88493	5.83051	6.08939
2.65	5.88611	5.99720	6.13819	6.04551	5.95294	6.01881	6.07407
2.70	5.92044	5.84288	5.77496	5.88631	5.82683	5.89663	5.98294
2.75	6.00033	6.09861	5.94959	6.10477	6.09099	5.82821	5.88548
2.80	5.96940	5.92026	6.06647	5.81812	6.03530	5.89868	6.07985
2.85	5.93021	5.99041	5.98989	5.87517	6.09013	5.96831	6.02706
2.90	5.86767	5.96920	5.97720	6.08090	6.08919	6.10272	6.07145
2.95	5.91651	5.97769	6.16951	5.90526	6.11460	5.92391	6.01786
3.00	8.67795	8.39369	8.63829	8.66954	8.45638	8.31761	8.48605

TABLE 3. Run times for array of size 556,034

Factor	Run Times						
2.00	19.25139	19.65175	19.52589	19.64955	19.37781	18.95809	19.43052
2.05	12.14401	11.70850	11.71947	11.69107	11.89917	11.65555	11.92322
2.10	11.63359	12.04306	11.75428	12.19470	12.07531	11.63234	11.84752
2.15	11.52209	11.97701	12.02158	11.74337	12.10203	11.77296	11.81839
2.20	12.13422	12.15373	12.11596	11.78236	11.76390	11.83543	11.64703
2.25	11.71330	11.79674	11.65092	11.78172	11.84441	11.94086	11.86124
2.30	11.86347	11.61136	11.66668	11.67831	11.81954	11.66532	11.87382
2.35	11.91868	12.09375	11.94043	11.96807	11.98186	11.95723	11.75366
2.40	11.57587	11.79629	11.79204	11.54087	11.63059	12.14921	11.74813
2.45	11.74038	11.75287	11.88554	11.59726	12.01966	11.78440	11.72355
2.50	12.00859	12.16508	12.39272	12.04476	11.85589	12.09896	12.04231
2.55	11.58087	11.80286	11.67923	11.78076	12.15391	11.93920	12.03251
2.60	11.97912	11.99108	11.84020	11.94484	11.78317	11.93254	11.55707
2.65	11.82312	11.87325	11.68572	11.89168	12.08694	11.65552	12.06704
2.70	11.58909	12.04215	11.67001	11.62067	11.86686	11.82390	11.71642
2.75	11.71276	12.03750	11.72266	11.81090	11.78605	12.10167	11.91265
2.80	11.62655	11.74929	12.00928	11.60990	11.70931	11.68706	11.76097
2.85	11.79259	11.10819	11.58860	11.02489	10.90184	10.31296	10.14671
2.90	10.57930	10.59111	10.74636	10.76371	10.77598	10.08741	10.37403
2.95	10.79915	10.75926	9.96541	10.76486	10.23883	10.50539	10.53382
3.00	16.90298	16.47198	16.58811	16.98387	16.54193	16.59710	17.05697

TABLE 4. Run times for array of size 1,037,375

Factor	Run Times						
2.00	39.38279	39.19163	38.53710	39.56146	38.93386	38.26565	38.54424
2.05	22.62315	21.28924	23.33690	21.55685	21.15504	22.13422	21.75070
2.10	21.30959	21.36436	21.36594	21.57960	21.89666	22.76869	20.60501
2.15	22.00835	21.70452	22.54162	21.75448	21.30550	21.06463	21.76741
2.20	20.41813	20.49716	20.76557	21.94115	20.15832	20.78437	22.45846
2.25	20.96729	20.97200	21.20307	21.78421	20.75965	20.56422	21.03291
2.30	21.01918	21.05601	20.41713	20.61532	19.96620	20.33012	20.55570
2.35	20.61453	21.20023	21.64370	20.49107	21.73384	20.86344	22.72654
2.40	21.47436	22.10195	21.27226	20.79331	20.62477	20.66876	20.40139
2.45	20.76231	20.90324	20.32024	20.46415	20.34982	20.97203	20.20983
2.50	21.13097	20.77314	20.66091	21.72695	21.08990	22.58229	22.39002
2.55	20.30675	21.11277	20.38649	20.38114	21.15756	20.43833	20.19302
2.60	20.89036	20.73147	20.17433	20.81219	20.78200	20.50383	20.74070
2.65	20.61010	20.37160	21.07381	20.72036	21.30277	20.73007	20.55452
2.70	20.66481	20.31767	20.90581	20.41565	21.41485	21.05447	20.98307
2.75	20.72573	20.88316	20.79213	20.65476	20.89425	20.60679	20.56156
2.80	20.48807	21.00047	21.54523	20.94379	20.89038	20.54535	20.63577
2.85	21.24749	20.35109	21.06426	20.84699	20.62513	20.98661	21.53956
2.90	20.97037	21.33573	20.98479	21.25388	20.90527	21.08087	21.07078
2.95	21.23222	20.88506	20.66464	20.88943	21.14017	20.15037	21.73718
3.00	38.85648	38.37497	39.27818	38.80821	37.33777	37.71689	39.01771

TABLE 5. Run times for array of size 1,970,626

Factor	Run Times						
2.00	113.21041	113.26105	112.12870	112.79090	114.12121	113.34831	112.11675
2.05	49.30749	48.23878	49.09939	49.83975	49.11896	48.94008	50.18229
2.10	48.82444	49.64399	49.31216	48.46250	49.27243	49.09054	49.05045
2.15	48.66179	48.78452	49.24244	49.24228	48.53146	48.34859	48.86433
2.20	48.33168	47.89452	48.78042	49.50495	48.96384	48.70316	48.26087
2.25	48.84767	48.41086	49.15341	49.25571	47.86379	48.60021	48.18536
2.30	48.62687	48.62473	47.81690	48.32565	48.26525	47.47081	48.48945
2.35	48.62491	49.17226	49.24081	48.43357	49.03645	48.44276	49.81583
2.40	49.12705	49.31038	48.56904	48.83559	48.00164	48.28379	48.10859
2.45	47.97090	48.73544	49.36356	48.27642	48.63115	49.21518	47.87525
2.50	50.67344	49.91113	50.78838	50.19264	49.40856	50.92242	50.24060
2.55	48.37801	48.67117	49.18117	48.72960	47.78437	48.19296	49.20774
2.60	49.20860	49.22058	49.00645	48.19100	49.23894	48.71932	49.30174
2.65	49.17013	48.87964	48.83620	48.76230	49.48472	48.52606	49.73881
2.70	48.99466	48.67768	49.10948	48.63164	49.12793	48.64291	48.46528
2.75	49.16035	49.64286	49.17422	49.17446	49.14777	49.93551	49.36334
2.80	49.16746	49.66479	48.89939	49.03565	48.49939	49.11505	49.46878
2.85	48.99367	49.52959	49.85554	49.88556	49.26784	50.50302	49.19876
2.90	49.61247	48.58484	49.82016	49.26752	50.88134	49.05363	50.02417
2.95	49.47462	49.62340	49.66697	49.78688	49.23448	49.68674	50.16653
3.00	111.55887	111.02547	112.34665	111.03842	122.93085	123.20042	112.14226

TABLE 6. Run times for array of size 4,365,249

Factor	Run Times						
2.00	305.93703	319.23091	312.54303	314.06352	313.54919	293.10959	295.07074
2.05	104.85004	105.45605	106.67733	106.31431	105.82800	105.38121	106.24027
2.10	105.65020	105.10275	105.55068	105.52013	105.26608	106.23839	106.80159
2.15	104.34457	106.24505	104.21439	103.90708	104.45233	104.22972	104.66902
2.20	103.98292	104.33804	105.72297	104.90463	104.29058	103.93918	104.80185
2.25	105.36901	105.46172	104.90363	104.64076	105.69683	104.80580	105.14536
2.30	104.83249	104.73758	103.66936	104.59665	104.53835	105.02008	104.08301
2.35	105.88235	105.68679	105.24014	106.65356	105.69155	105.28519	106.63165
2.40	104.22180	104.48496	104.45291	104.69152	104.48235	104.08573	104.45229
2.45	104.67460	106.18788	106.13354	104.97895	104.30085	104.71662	106.19882
2.50	108.58484	109.42854	108.93333	109.69946	108.99018	108.54078	108.90152
2.55	105.50062	106.47568	105.17529	105.60771	104.88492	104.84362	105.03737
2.60	105.00628	105.60328	105.74064	105.54010	105.99068	105.05108	105.48596
2.65	105.46267	106.37486	105.16566	106.22479	111.22261	105.39907	106.43420
2.70	105.98075	106.58304	106.83846	106.20708	111.19709	113.28633	107.68087
2.75	110.51783	112.62984	108.67548	107.83499	106.96033	107.01907	109.66880
2.80	118.33894	121.06802	123.47471	122.57159	120.72714	120.71937	120.81627
2.85	122.28119	122.39143	126.31724	123.20967	122.73775	121.54424	122.25013
2.90	121.96453	122.86133	122.80897	121.24396	122.20591	121.53710	118.42910
2.95	120.28669	123.92856	130.90757	124.74606	122.50761	124.93325	124.61616
3.00	342.53367	347.98769	343.21231	339.58075	340.18518	339.00139	337.73770

TABLE 7. Run times for array of size 8,895,124

Factor	Minimum	Mean	Maximum	Std. Dev.	Median
2.00	2.6628e+00	2.7340e+00	2.8337e+00	6.4053e-02	2.7139e+00
2.05	2.0581e+00	2.0916e+00	2.1448e+00	2.8076e-02	2.0878e+00
2.10	2.0515e+00	2.1012e+00	2.2311e+00	5.7103e-02	2.0940e+00
2.15	2.0242e+00	2.1340e+00	2.3044e+00	9.0712e-02	2.0929e+00
2.20	2.0244e+00	2.1474e+00	2.2670e+00	8.7102e-02	2.1800e+00
2.25	2.1126e+00	2.2077e+00	2.3674e+00	8.7959e-02	2.2089e+00
2.30	2.0834e+00	2.2136e+00	2.3427e+00	8.6098e-02	2.1948e+00
2.35	2.0407e+00	2.1649e+00	2.4120e+00	1.2349e-01	2.1264e+00
2.40	2.0227e+00	2.0925e+00	2.1845e+00	6.1880e-02	2.0756e+00
2.45	2.0398e+00	2.1384e+00	2.2517e+00	7.6768e-02	2.1532e+00
2.50	2.0665e+00	2.1672e+00	2.2364e+00	5.2895e-02	2.1553e+00
2.55	2.0659e+00	2.1712e+00	2.4376e+00	1.1841e-01	2.1408e+00
2.60	2.0916e+00	2.1773e+00	2.2494e+00	5.4643e-02	2.1844e+00
2.65	2.0739e+00	2.1543e+00	2.2777e+00	6.5536e-02	2.1588e+00
2.70	2.0530e+00	2.2207e+00	2.3563e+00	8.9052e-02	2.2130e+00
2.75	2.1287e+00	2.2477e+00	2.3609e+00	8.3530e-02	2.2290e+00
2.80	2.0376e+00	2.1463e+00	2.2305e+00	6.0922e-02	2.1382e+00
2.85	2.0571e+00	2.1820e+00	2.2971e+00	7.8392e-02	2.1891e+00
2.90	2.2156e+00	2.3405e+00	2.4277e+00	6.5512e-02	2.3327e+00
2.95	2.0494e+00	2.2289e+00	2.4361e+00	1.2515e-01	2.2021e+00
3.00	2.5915e+00	2.6583e+00	2.7355e+00	5.0480e-02	2.6525e+00

TABLE 8. Statistics for array of size 247,959

Factor	Minimum	Mean	Maximum	Std. Dev.	Median
2.00	7.4176e+00	7.6493e+00	7.9302e+00	1.7137e-01	7.6945e+00
2.05	4.9680e+00	5.2049e+00	5.3932e+00	1.4060e-01	5.2369e+00
2.10	4.9981e+00	5.1972e+00	5.5040e+00	1.7911e-01	5.1245e+00
2.15	4.9997e+00	5.1747e+00	5.4086e+00	1.2380e-01	5.1658e+00
2.20	4.9535e+00	5.1587e+00	5.4265e+00	1.5754e-01	5.1409e+00
2.25	4.9421e+00	5.0789e+00	5.3201e+00	1.2355e-01	5.0196e+00
2.30	4.9514e+00	5.1378e+00	5.4401e+00	1.4517e-01	5.1057e+00
2.35	4.9727e+00	5.2113e+00	5.3785e+00	1.2973e-01	5.2657e+00
2.40	5.2229e+00	5.5031e+00	5.8314e+00	2.2107e-01	5.4642e+00
2.45	5.7561e+00	5.8157e+00	5.9157e+00	4.7373e-02	5.8184e+00
2.50	5.8086e+00	5.9453e+00	6.0808e+00	9.1786e-02	5.9418e+00
2.55	5.8075e+00	5.9254e+00	6.0503e+00	1.0013e-01	5.8898e+00
2.60	5.8109e+00	5.9124e+00	6.0894e+00	9.5441e-02	5.8849e+00
2.65	5.8861e+00	6.0161e+00	6.1382e+00	7.5954e-02	6.0188e+00
2.70	5.7750e+00	5.8759e+00	5.9829e+00	6.2867e-02	5.8863e+00
2.75	5.8282e+00	5.9940e+00	6.1048e+00	1.0275e-01	6.0003e+00
2.80	5.8181e+00	5.9697e+00	6.0798e+00	8.9654e-02	5.9694e+00
2.85	5.8752e+00	5.9816e+00	6.0901e+00	6.3400e-02	5.9899e+00
2.90	5.8677e+00	6.0226e+00	6.1027e+00	8.0670e-02	6.0714e+00
2.95	5.9053e+00	6.0036e+00	6.1695e+00	9.5862e-02	5.9777e+00
3.00	8.3176e+00	8.5199e+00	8.6780e+00	1.3278e-01	8.4860e+00

TABLE 9. Statistics for array of size 556,034

Factor	Minimum	Mean	Maximum	Std. Dev.	Median
2.00	1.8958e+01	1.9406e+01	1.9652e+01	2.2678e-01	1.9431e+01
2.05	1.1656e+01	1.1820e+01	1.2144e+01	1.6393e-01	1.1719e+01
2.10	1.1632e+01	1.1883e+01	1.2195e+01	2.0792e-01	1.1848e+01
2.15	1.1522e+01	1.1851e+01	1.2102e+01	1.8319e-01	1.1818e+01
2.20	1.1647e+01	1.1919e+01	1.2154e+01	1.9416e-01	1.1835e+01
2.25	1.1651e+01	1.1798e+01	1.1941e+01	8.9150e-02	1.1797e+01
2.30	1.1611e+01	1.1740e+01	1.1874e+01	1.0056e-01	1.1678e+01
2.35	1.1754e+01	1.1945e+01	1.2094e+01	9.3726e-02	1.1957e+01
2.40	1.1541e+01	1.1748e+01	1.2149e+01	1.8957e-01	1.1748e+01
2.45	1.1597e+01	1.1786e+01	1.2020e+01	1.2376e-01	1.1753e+01
2.50	1.1856e+01	1.2087e+01	1.2393e+01	1.5262e-01	1.2045e+01
2.55	1.1581e+01	1.1853e+01	1.2154e+01	1.8601e-01	1.1803e+01
2.60	1.1557e+01	1.1861e+01	1.1991e+01	1.4237e-01	1.1933e+01
2.65	1.1656e+01	1.1869e+01	1.2087e+01	1.5516e-01	1.1873e+01
2.70	1.1589e+01	1.1761e+01	1.2042e+01	1.4819e-01	1.1716e+01
2.75	1.1713e+01	1.1869e+01	1.2102e+01	1.4168e-01	1.1811e+01
2.80	1.1610e+01	1.1736e+01	1.2009e+01	1.2333e-01	1.1709e+01
2.85	1.0147e+01	1.0982e+01	1.1793e+01	5.5992e-01	1.1025e+01
2.90	1.0087e+01	1.0560e+01	1.0776e+01	2.3365e-01	1.0591e+01
2.95	9.9654e+00	1.0510e+01	1.0799e+01	2.8819e-01	1.0534e+01
3.00	1.6472e+01	1.6735e+01	1.7057e+01	2.2067e-01	1.6597e+01

TABLE 10. Statistics for array of size 1,037,375

Factor	Minimum	Mean	Maximum	Std. Dev.	Median
2.00	3.8266e+01	3.8917e+01	3.9561e+01	4.4969e-01	3.8934e+01
2.05	2.1155e+01	2.1978e+01	2.3337e+01	7.2387e-01	2.1751e+01
2.10	2.0605e+01	2.1556e+01	2.2769e+01	6.1239e-01	2.1366e+01
2.15	2.1065e+01	2.1735e+01	2.2542e+01	4.4104e-01	2.1754e+01
2.20	2.0158e+01	2.1003e+01	2.2458e+01	7.9409e-01	2.0766e+01
2.25	2.0564e+01	2.1040e+01	2.1784e+01	3.5764e-01	2.0972e+01
2.30	1.9966e+01	2.0566e+01	2.1056e+01	3.5569e-01	2.0556e+01
2.35	2.0491e+01	2.1325e+01	2.2727e+01	7.2327e-01	2.1200e+01
2.40	2.0401e+01	2.1048e+01	2.2102e+01	5.5405e-01	2.0793e+01
2.45	2.0210e+01	2.0569e+01	2.0972e+01	2.8320e-01	2.0464e+01
2.50	2.0661e+01	2.1479e+01	2.2582e+01	7.1201e-01	2.1131e+01
2.55	2.0193e+01	2.0568e+01	2.1158e+01	3.6604e-01	2.0386e+01
2.60	2.0174e+01	2.0662e+01	2.0890e+01	2.2771e-01	2.0741e+01
2.65	2.0372e+01	2.0766e+01	2.1303e+01	2.9540e-01	2.0720e+01
2.70	2.0318e+01	2.0822e+01	2.1415e+01	3.5510e-01	2.0906e+01
2.75	2.0562e+01	2.0731e+01	2.0894e+01	1.2160e-01	2.0726e+01
2.80	2.0488e+01	2.0864e+01	2.1545e+01	3.3482e-01	2.0890e+01
2.85	2.0351e+01	2.0952e+01	2.1540e+01	3.6358e-01	2.0987e+01
2.90	2.0905e+01	2.1086e+01	2.1336e+01	1.4495e-01	2.1071e+01
2.95	2.0150e+01	2.0957e+01	2.1737e+01	4.5730e-01	2.0889e+01
3.00	3.7338e+01	3.8484e+01	3.9278e+01	6.6263e-01	3.8808e+01

TABLE 11. Statistics for array of size 1,970,626

Factor	Minimum	Mean	Maximum	Std. Dev.	Median
2.00	1.1212e+02	1.1300e+02	1.1412e+02	6.6266e-01	1.1321e+02
2.05	4.8239e+01	4.9247e+01	5.0182e+01	5.8293e-01	4.9119e+01
2.10	4.8462e+01	4.9094e+01	4.9644e+01	3.4955e-01	4.9091e+01
2.15	4.8349e+01	4.8811e+01	4.9242e+01	3.1394e-01	4.8785e+01
2.20	4.7895e+01	4.8634e+01	4.9505e+01	4.8859e-01	4.8703e+01
2.25	4.7864e+01	4.8617e+01	4.9256e+01	4.6985e-01	4.8600e+01
2.30	4.7471e+01	4.8231e+01	4.8627e+01	4.0321e-01	4.8326e+01
2.35	4.8434e+01	4.8967e+01	4.9816e+01	4.6581e-01	4.9036e+01
2.40	4.8002e+01	4.8605e+01	4.9310e+01	4.6877e-01	4.8569e+01
2.45	4.7875e+01	4.8581e+01	4.9364e+01	5.3506e-01	4.8631e+01
2.50	4.9409e+01	5.0305e+01	5.0922e+01	4.9658e-01	5.0241e+01
2.55	4.7784e+01	4.8592e+01	4.9208e+01	4.7940e-01	4.8671e+01
2.60	4.8191e+01	4.8984e+01	4.9302e+01	3.7259e-01	4.9209e+01
2.65	4.8526e+01	4.9057e+01	4.9739e+01	3.9813e-01	4.8880e+01
2.70	4.8465e+01	4.8807e+01	4.9128e+01	2.4521e-01	4.8678e+01
2.75	4.9148e+01	4.9371e+01	4.9936e+01	2.8390e-01	4.9174e+01
2.80	4.8499e+01	4.9122e+01	4.9665e+01	3.5084e-01	4.9115e+01
2.85	4.8994e+01	4.9605e+01	5.0503e+01	4.7899e-01	4.9530e+01
2.90	4.8585e+01	4.9606e+01	5.0881e+01	6.8762e-01	4.9612e+01
2.95	4.9234e+01	4.9663e+01	5.0167e+01	2.6462e-01	4.9667e+01
3.00	1.1103e+02	1.1489e+02	1.2320e+02	5.1907e+00	1.1214e+02

TABLE 12. Statistics for array of size 4,365,249

Factor	Minimum	Mean	Maximum	Std. Dev.	Median
2.00	2.9311e+02	3.0764e+02	3.1923e+02	9.3089e+00	3.1254e+02
2.05	1.0485e+02	1.0582e+02	1.0668e+02	5.8820e-01	1.0583e+02
2.10	1.0510e+02	1.0573e+02	1.0680e+02	5.4746e-01	1.0555e+02
2.15	1.0391e+02	1.0458e+02	1.0625e+02	7.1327e-01	1.0434e+02
2.20	1.0394e+02	1.0457e+02	1.0572e+02	5.8181e-01	1.0434e+02
2.25	1.0464e+02	1.0515e+02	1.0570e+02	3.5508e-01	1.0515e+02
2.30	1.0367e+02	1.0450e+02	1.0502e+02	4.3294e-01	1.0460e+02
2.35	1.0524e+02	1.0587e+02	1.0665e+02	5.3430e-01	1.0569e+02
2.40	1.0409e+02	1.0441e+02	1.0469e+02	1.8295e-01	1.0445e+02
2.45	1.0430e+02	1.0531e+02	1.0620e+02	7.6743e-01	1.0498e+02
2.50	1.0854e+02	1.0901e+02	1.0970e+02	3.9088e-01	1.0893e+02
2.55	1.0484e+02	1.0536e+02	1.0648e+02	5.2867e-01	1.0518e+02
2.60	1.0501e+02	1.0549e+02	1.0599e+02	3.2852e-01	1.0554e+02
2.65	1.0517e+02	1.0661e+02	1.1122e+02	1.9413e+00	1.0622e+02
2.70	1.0598e+02	1.0825e+02	1.1329e+02	2.6314e+00	1.0684e+02
2.75	1.0696e+02	1.0904e+02	1.1263e+02	1.9071e+00	1.0868e+02
2.80	1.1834e+02	1.2110e+02	1.2347e+02	1.5021e+00	1.2082e+02
2.85	1.2154e+02	1.2296e+02	1.2632e+02	1.4479e+00	1.2239e+02
2.90	1.1843e+02	1.2158e+02	1.2286e+02	1.4007e+00	1.2196e+02
2.95	1.2029e+02	1.2456e+02	1.3091e+02	3.0081e+00	1.2462e+02
3.00	3.3774e+02	3.4146e+02	3.4799e+02	3.2047e+00	3.4019e+02

TABLE 13. Statistics for array of size 8,895,124

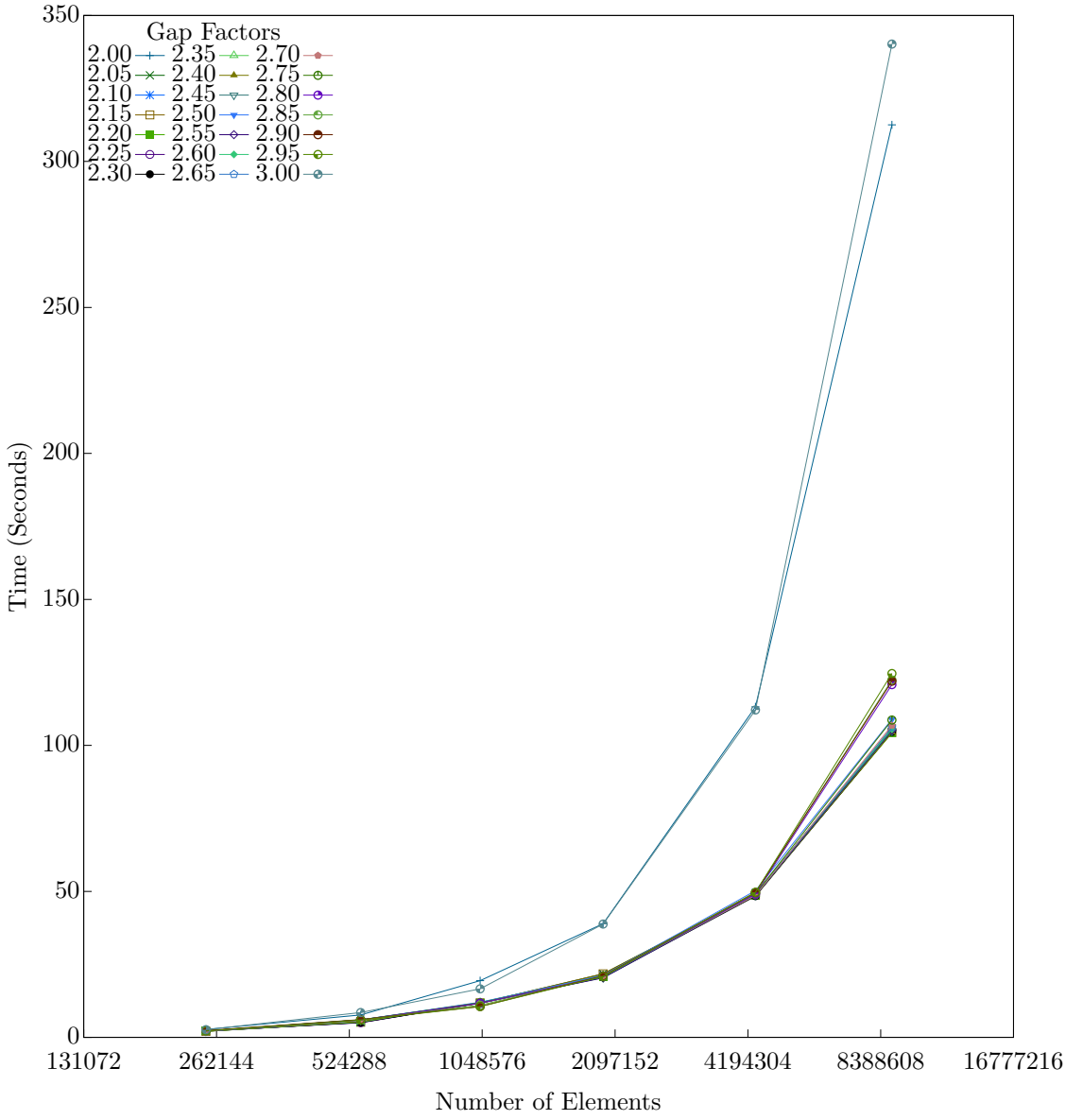

FIGURE 2. Time v. Size

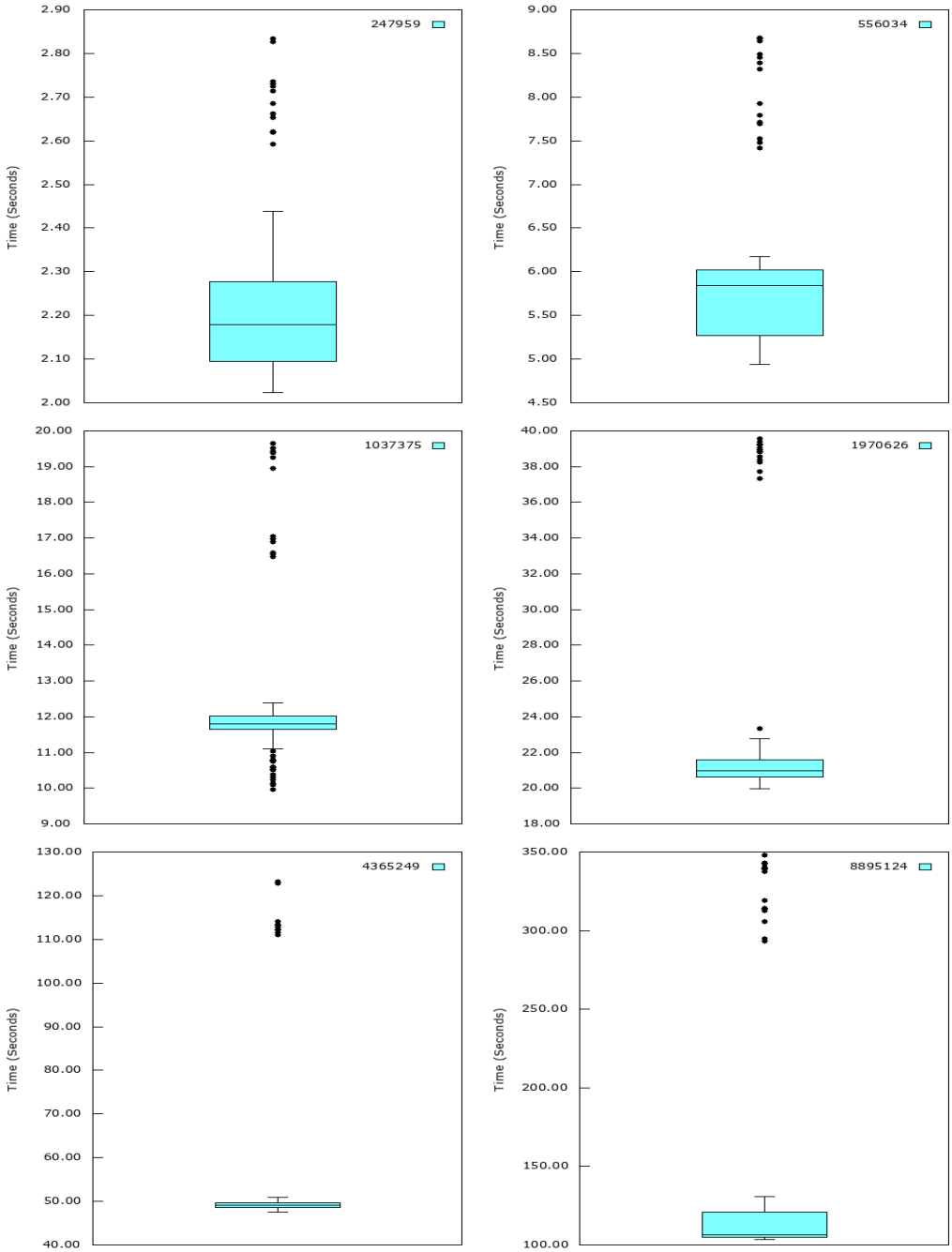

FIGURE 3. Box & Whiskers