Some computational results for generalized pressure Schur complement eigenvalues of the surface Stokes problem

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1 Bilinear forms and matrices

We set $n_{\mathbf{A}}$ to be the number of velocity d.o.f. and $n_{\mathbf{S}}$ to be the number of pressure d.o.f. Vector stiffness, divergence, pressure mass, normal stabilization, and full stabilization matrices resulting from TraceFEM discretization of the surface Stokes problem [1] are defined via

$$\langle \mathbf{A}\,\bar{\mathbf{u}}, \bar{\mathbf{v}}\rangle = \int_\Gamma \big(E_s(\mathbf{u}) : E_s(\mathbf{v}) + \mathbf{u}\cdot\mathbf{v} + \tau\, u_N\, v_N\big)\,\mathrm{d}s + \rho_u \int_{\Omega_h^\Gamma} ([\nabla\mathbf{u}]\,\hat{\mathbf{n}})\cdot([\nabla\mathbf{v}]\,\hat{\mathbf{n}})\,\mathrm{d}\mathbf{x}, \quad \mathbf{A}\in\mathbb{R}^{n_{\mathbf{A}}\times n_{\mathbf{A}}}, \tag{1}$$

$$\langle \mathbf{B}\,\bar{\mathbf{u}}, \bar{\mathbf{q}}\rangle = -\int_\Gamma q\,\mathrm{div}_\Gamma\,\mathbf{u}\,\mathrm{d}s, \quad \mathbf{B}\in\mathbb{R}^{n_{\mathbf{S}}\times n_{\mathbf{A}}}, \tag{2}$$

$$\langle \mathbf{M}_0\,\bar{\mathbf{p}}, \bar{\mathbf{q}}\rangle = \int_\Gamma p\,q\,\mathrm{d}s, \quad \mathbf{M}_0\in\mathbb{R}^{n_{\mathbf{S}}\times n_{\mathbf{S}}}, \tag{3}$$

$$\langle \mathbf{C}_n\,\bar{\mathbf{p}}, \bar{\mathbf{q}}\rangle = \rho_p \int_{\Omega_h^\Gamma} \frac{\partial p}{\partial\hat{\mathbf{n}}}\frac{\partial q}{\partial\hat{\mathbf{n}}}\,\mathrm{d}\mathbf{x}, \quad \mathbf{C}_n\in\mathbb{R}^{n_{\mathbf{S}}\times n_{\mathbf{S}}}, \tag{4}$$

$$\langle \mathbf{C}_{\text{full}}\,\bar{\mathbf{p}}, \bar{\mathbf{q}}\rangle = \rho_p \int_{\Omega_h^\Gamma} \nabla p\cdot\nabla q\,\mathrm{d}\mathbf{x}, \quad \mathbf{C}_{\text{full}}\in\mathbb{R}^{n_{\mathbf{S}}\times n_{\mathbf{S}}}, \tag{5}$$

respectively. Here $\bar{\mathbf{u}}$ denotes a vector of d.o.f. corresponding to a FE interpolant $\mathbf{u}$ (analogously for $\bar{\mathbf{p}}$ and p). Mesh-dependent parameters are set as

$$\tau = h^{-2}, \quad \rho_u = \rho_p = h, \tag{6}$$

and h is the mesh size of Ω_h^Γ. Γ is chosen either as the unit sphere or torus, $\Gamma = \Gamma_{\text{sph}}$ or $\Gamma = \Gamma_{\text{tor}}$ (see Figure 1).

We also define matrices

$$\mathbf{M}_n \coloneqq \mathbf{M}_0 + \mathbf{C}_n, \tag{7}$$

$$\mathbf{M}_{\text{full}} \coloneqq \mathbf{M}_0 + \mathbf{C}_{\text{full}}. \tag{8}$$

We are interested in (generalized) extreme eigenvalues of the pressure Schur complement matrices

$$\mathbf{S}_0 \coloneqq \mathbf{B}\,\mathbf{A}^{-1}\,\mathbf{B}^T, \tag{9}$$

$$\mathbf{S}_n \coloneqq \mathbf{S}_0 + \mathbf{C}_n, \tag{10}$$

$$\mathbf{S}_{\text{full}} \coloneqq \mathbf{S}_0 + \mathbf{C}_{\text{full}}, \tag{11}$$

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i.e. in solving

$$\mathbf{S}_\star \, \mathbf{x} = \lambda \, \mathbf{M}_\star \, \mathbf{x}, \tag{12}$$

where "$\star$" stands for "0", "n" or "full." We denote by $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_{n_\mathbf{S}} = O(1)$ the spectrum of (12).

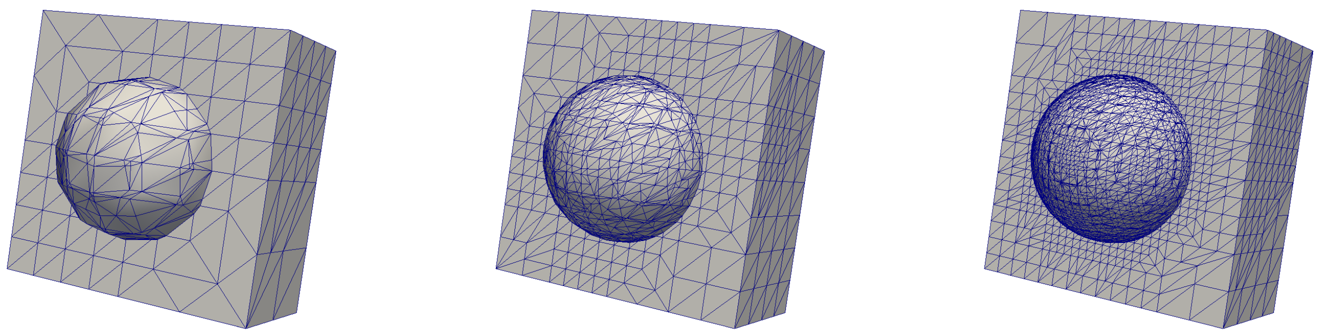
2 Solution description

(TBA)

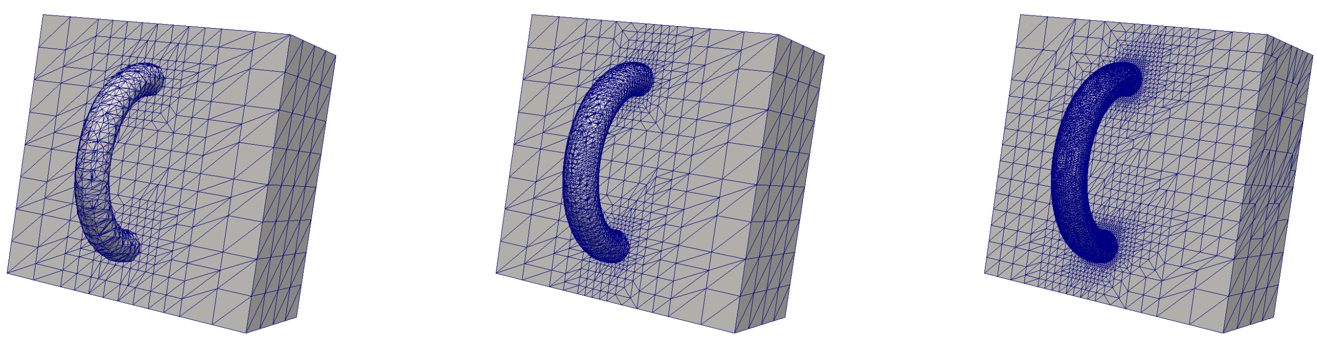
$n_\mathbf{A}$ is the number of velocity d.o.f, $n_\mathbf{S}$ is the number of pressure d.o.f., $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_{n_\mathbf{S}}$ is the (approximate) spectrum of $\mathbf{S}$, and r_i are the residual norm, i.e.

$$r_i \coloneqq \| \mathbf{S} \, \mathbf{x} - \lambda_i \, \mathbf{M} \, \mathbf{x} \|_2$$

Note: $h = 2.6 \times 10^{-2}$ for $\mathbf{P}_2 - P_1$ (the last mesh level) was computed w/ $\epsilon = 10^{-5}$, and everything else w/ $\epsilon = 10^{-4}$. Apparently $\epsilon = 10^{-4}$ did not work for the finest mesh level: λ_2 turned out to be negative for full and normal stabs.



(a) $h = 8.33 \times 10^{-1}$ (b) $h = 4.17 \times 10^{-1}$ (c) $h = 2.08 \times 10^{-1}$



(d) $h = 2.08 \times 10^{-1}$ (e) $h = 1.04 \times 10^{-1}$ (f) $h = 5.21 \times 10^{-2}$

Figure 1: First three mesh levels for Γ_{sph} (top) and Γ_{tor} (bottom)

Table 1: $\mathbf{P}_1 - P_1$ for Γ_{sph}

h	$n_{\mathbf{A}}$	$n_{\mathbf{S}}$	$\mathbf{S}_0$		$\mathbf{S}_n$		$\mathbf{S}_{\text{full}}$	
			λ_2	$\lambda_{n_{\mathbf{S}}}$	λ_2	$\lambda_{n_{\mathbf{S}}}$	λ_2	$\lambda_{n_{\mathbf{S}}}$
8.33×10^{-1}	153	51	1.32×10^{-2}	1.42	7.48×10^{-1}	1.13	9.58×10^{-1}	1.06
4.17×10^{-1}	570	190	5.12×10^{-3}	1.04	5.77×10^{-1}	1.	8.54×10^{-1}	1.
2.08×10^{-1}	1992	664	4.4×10^{-3}	7.93×10^{-1}	3.87×10^{-1}	1.	6.71×10^{-1}	1.
1.04×10^{-1}	8292	2764	2.01×10^{-3}	7.75×10^{-1}	2.19×10^{-1}	1.	5.82×10^{-1}	1.
5.21×10^{-2}	32736	10912	6.04×10^{-5}	9.81×10^{-1}	1.17×10^{-1}	1.	5.37×10^{-1}	1.
2.6×10^{-2}	131592	43864	3.53×10^{-5}	8.67×10^{-1}	5.72×10^{-2}	1.	5.16×10^{-1}	1.
1.3×10^{-2}	525864	175288	2.16×10^{-6}	7.34×10^{-1}	2.84×10^{-2}	1.	5.04×10^{-1}	1.
h	$n_{\mathbf{A}}$	$n_{\mathbf{S}}$	$\mathbf{S}_0$		$\mathbf{S}_n$		$\mathbf{S}_{\text{full}}$	
			r_2	$r_{n_{\mathbf{S}}}$	r_2	$r_{n_{\mathbf{S}}}$	r_2	$r_{n_{\mathbf{S}}}$
8.33×10^{-1}	153	51	$2. \times 10^{-17}$	$8. \times 10^{-10}$	$1. \times 10^{-7}$	$4. \times 10^{-8}$	$3. \times 10^{-7}$	$1. \times 10^{-7}$
4.17×10^{-1}	570	190	$3. \times 10^{-18}$	$3. \times 10^{-10}$	$6. \times 10^{-7}$	$1. \times 10^{-3}$	$1. \times 10^{-7}$	$8. \times 10^{-4}$
2.08×10^{-1}	1992	664	$2. \times 10^{-17}$	$6. \times 10^{-9}$	$6. \times 10^{-8}$	$9. \times 10^{-4}$	$2. \times 10^{-10}$	$8. \times 10^{-3}$
1.04×10^{-1}	8292	2764	$6. \times 10^{-16}$	$9. \times 10^{-10}$	$2. \times 10^{-8}$	$2. \times 10^{-3}$	$2. \times 10^{-8}$	$3. \times 10^{-3}$
5.21×10^{-2}	32736	10912	$8. \times 10^{-19}$	$1. \times 10^{-11}$	$1. \times 10^{-5}$	$7. \times 10^{-4}$	$1. \times 10^{-3}$	$7. \times 10^{-4}$
2.6×10^{-2}	131592	43864	$5. \times 10^{-18}$	$2. \times 10^{-12}$	$8. \times 10^{-9}$	$7. \times 10^{-4}$	$3. \times 10^{-8}$	$9. \times 10^{-4}$
1.3×10^{-2}	525864	175288	$5. \times 10^{-22}$	$8. \times 10^{-14}$	$8. \times 10^{-12}$	$2. \times 10^{-4}$	$3. \times 10^{-5}$	$4. \times 10^{-4}$

Table 2: $\mathbf{P}_2 - P_1$ for Γ_{sph}

h	$n_{\mathbf{A}}$	$n_{\mathbf{S}}$	$\mathbf{S}_0$		$\mathbf{S}_n$		$\mathbf{S}_{\text{full}}$	
			λ_2	$\lambda_{n_{\mathbf{S}}}$	λ_2	$\lambda_{n_{\mathbf{S}}}$	λ_2	$\lambda_{n_{\mathbf{S}}}$
8.33×10^{-1}	789	51	3.22×10^{-1}	1.73	8.27×10^{-1}	1.17	9.68×10^{-1}	1.07
4.17×10^{-1}	3240	190	9.17×10^{-2}	1.08	6.45×10^{-1}	1.	8.56×10^{-1}	1.
2.08×10^{-1}	11718	664	1.78×10^{-1}	8.31×10^{-1}	5.49×10^{-1}	1.	6.75×10^{-1}	1.
1.04×10^{-1}	48762	2764	1.04×10^{-1}	8.13×10^{-1}	5.14×10^{-1}	1.	5.82×10^{-1}	1.
5.21×10^{-2}	193014	10912	2.99×10^{-3}	9.89×10^{-1}	5.02×10^{-1}	1.	5.34×10^{-1}	1.
2.6×10^{-2}	775998	43864	1.17×10^{-3}	7.9×10^{-1}	4.96×10^{-1}	1.	5.17×10^{-1}	1.
h	$n_{\mathbf{A}}$	$n_{\mathbf{S}}$	$\mathbf{S}_0$		$\mathbf{S}_n$		$\mathbf{S}_{\text{full}}$	
			r_2	$r_{n_{\mathbf{S}}}$	r_2	$r_{n_{\mathbf{S}}}$	r_2	$r_{n_{\mathbf{S}}}$
8.33×10^{-1}	789	51	$4. \times 10^{-9}$	$4. \times 10^{-10}$	$2. \times 10^{-8}$	$2. \times 10^{-7}$	$2. \times 10^{-7}$	$3. \times 10^{-7}$
4.17×10^{-1}	3240	190	$6. \times 10^{-12}$	$4. \times 10^{-9}$	$7. \times 10^{-10}$	$4. \times 10^{-2}$	$3. \times 10^{-10}$	$4. \times 10^{-2}$
2.08×10^{-1}	11718	664	$1. \times 10^{-10}$	$3. \times 10^{-9}$	$2. \times 10^{-6}$	$7. \times 10^{-3}$	$2. \times 10^{-9}$	$1. \times 10^{-2}$
1.04×10^{-1}	48762	2764	$1. \times 10^{-11}$	$9. \times 10^{-10}$	$2. \times 10^{-5}$	$2. \times 10^{-3}$	$2. \times 10^{-7}$	$2. \times 10^{-3}$
5.21×10^{-2}	193014	10912	$2. \times 10^{-16}$	$3. \times 10^{-12}$	$7. \times 10^{-5}$	$1. \times 10^{-3}$	$5. \times 10^{-7}$	$2. \times 10^{-3}$
2.6×10^{-2}	775998	43864	$7. \times 10^{-18}$	$5. \times 10^{-12}$	$4. \times 10^{-8}$	$3. \times 10^{-4}$	$2. \times 10^{-8}$	$6. \times 10^{-4}$

Table 3: $\mathbf{P}_1 - P_1$ for Γ_{tor}

h	$n_{\mathbf{A}}$	$n_{\mathbf{S}}$	$\mathbf{S}_0$		$\mathbf{S}_n$		$\mathbf{S}_{\text{full}}$	
			λ_2	$\lambda_{n_{\mathbf{S}}}$	λ_2	$\lambda_{n_{\mathbf{S}}}$	λ_2	$\lambda_{n_{\mathbf{S}}}$
2.08×10^{-1}	972	324	5.04×10^{-2}	4.93	2.84×10^{-1}	1.35	3.64×10^{-1}	1.19
1.04×10^{-1}	4740	1580	2.99×10^{-3}	3.83	1.58×10^{-1}	1.02	3.35×10^{-1}	1.01
h	$n_{\mathbf{A}}$	$n_{\mathbf{S}}$	$\mathbf{S}_0$		$\mathbf{S}_n$		$\mathbf{S}_{\text{full}}$	
			r_2	$r_{n_{\mathbf{S}}}$	r_2	$r_{n_{\mathbf{S}}}$	r_2	$r_{n_{\mathbf{S}}}$
2.08×10^{-1}	972	324	$3. \times 10^{-10}$	$3. \times 10^{-17}$	$9. \times 10^{-13}$	$5. \times 10^{-8}$	$1. \times 10^{-13}$	$3. \times 10^{-7}$
1.04×10^{-1}	4740	1580	$1. \times 10^{-15}$	$2. \times 10^{-18}$	$5. \times 10^{-11}$	$5. \times 10^{-8}$	$4. \times 10^{-10}$	$8. \times 10^{-8}$

Table 4: $\mathbf{P}_2 - P_1$ for Γ_{tor}

h	$n_{\mathbf{A}}$	$n_{\mathbf{S}}$	$\mathbf{S}_0$		$\mathbf{S}_n$		$\mathbf{S}_{\text{full}}$	
			λ_2	$\lambda_{n_{\mathbf{S}}}$	λ_2	$\lambda_{n_{\mathbf{S}}}$	λ_2	$\lambda_{n_{\mathbf{S}}}$
2.08×10^{-1}	5184	324	9.92×10^{-2}	3.89	1.33×10^{-1}	1.37	1.75×10^{-1}	1.19
1.04×10^{-1}	27906	1580	1.46×10^{-2}	4.35	2.84×10^{-1}	1.04	2.99×10^{-1}	1.02
h	$n_{\mathbf{A}}$	$n_{\mathbf{S}}$	$\mathbf{S}_0$		$\mathbf{S}_n$		$\mathbf{S}_{\text{full}}$	
			r_2	$r_{n_{\mathbf{S}}}$	r_2	$r_{n_{\mathbf{S}}}$	r_2	$r_{n_{\mathbf{S}}}$
2.08×10^{-1}	5184	324	$1. \times 10^{-16}$	$7. \times 10^{-13}$	$5. \times 10^{-16}$	$1. \times 10^{-8}$	$7. \times 10^{-16}$	$4. \times 10^{-8}$
1.04×10^{-1}	27906	1580	$2. \times 10^{-19}$	$4. \times 10^{-18}$	$1. \times 10^{-14}$	$7. \times 10^{-8}$	$1. \times 10^{-15}$	$2. \times 10^{-7}$

References

[1] M. Olshanskii, A. Quaini, A. Reusken, and V. Yushutin. A finite element method for the surface stokes problem. *SIAM Journal on Scientific Computing*, 40(4):A2492–A2518, 2018.