Some computational results for $\mathbf{P}_1 - P_1$ and $\mathbf{P}_2 - P_1$ TraceFEM for the surface Stokes problem

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Contents

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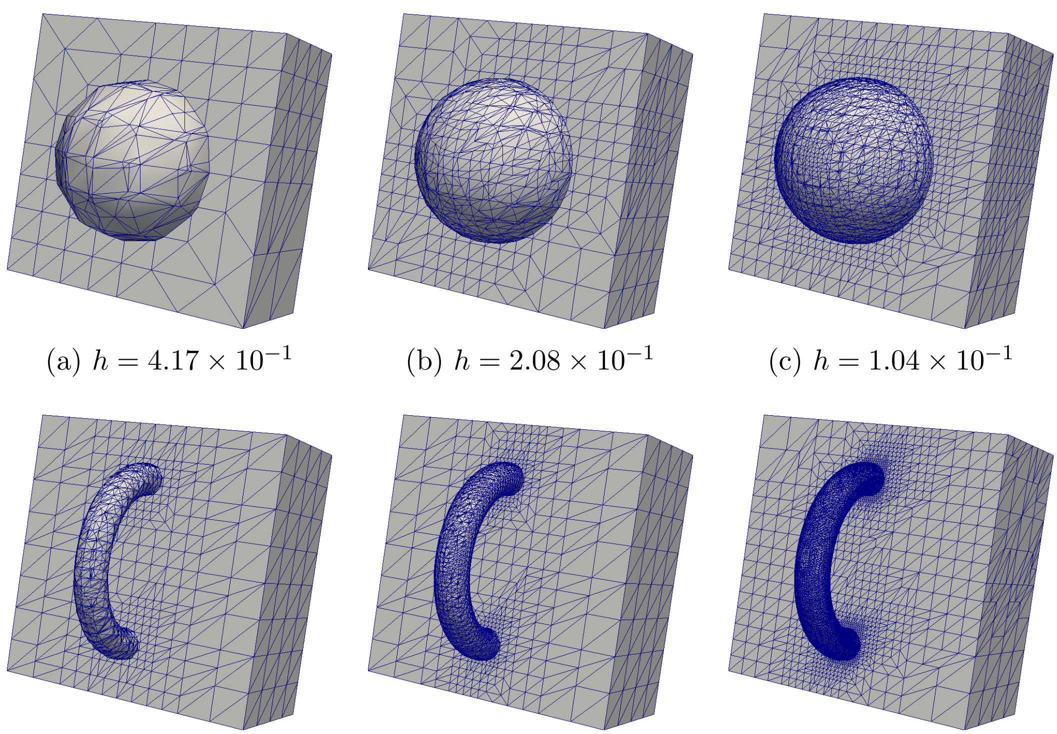
1 Preliminaries

1.1 Bilinear forms and matrices. We set $n_{\mathbf{A}}$ to be the number of velocity d.o.f. and $n_{\mathbf{S}}$ to be the number of pressure d.o.f. Vector stiffness, divergence, pressure mass, normal stabilization, and full stabilization matrices resulting from Trace FEM discretization of the surface Stokes problem [2] are defined via

$$
\begin{aligned}
\langle \mathbf{A}\,\vec{\mathbf{u}}, \vec{\mathbf{v}} \rangle &\approx \int_\Gamma \left(2\, E_{s,\Gamma}(\mathbf{u}) : E_{s,\Gamma}(\mathbf{v}) + \mathbf{u}\cdot\mathbf{v} + \tau\, (\mathbf{u}\cdot\mathbf{n}_\Gamma)\,(\mathbf{v}\cdot\mathbf{n}_\Gamma)\right) \mathrm{d}s \\
&\quad + \rho_u \int_{\Omega_h^\Gamma} \frac{\partial \mathbf{u}}{\partial \mathbf{n}_\Gamma}\cdot\frac{\partial \mathbf{v}}{\partial \mathbf{n}_\Gamma}\,\mathrm{d}\mathbf{x}, \quad \mathbf{A} \in \mathbb{R}^{n_{\mathbf{A}}\times n_{\mathbf{A}}}, \\
\langle \mathbf{B}\,\vec{\mathbf{u}}, \vec{\mathbf{q}} \rangle &\approx \int_\Gamma \nabla_\Gamma q \cdot \mathbf{u}\,\mathrm{d}s, \quad \mathbf{B} \in \mathbb{R}^{n_{\mathbf{S}}\times n_{\mathbf{A}}}, \\
\langle \mathbf{M}_0\,\vec{\mathbf{p}}, \vec{\mathbf{q}} \rangle &\approx \int_\Gamma p\, q\,\mathrm{d}s, \quad \mathbf{M}_0 \in \mathbb{R}^{n_{\mathbf{S}}\times n_{\mathbf{S}}}, \\
\langle \mathbf{C}_n\,\vec{\mathbf{p}}, \vec{\mathbf{q}} \rangle &\approx \rho_p \int_{\Omega_h^\Gamma} \frac{\partial p}{\partial \mathbf{n}_\Gamma}\frac{\partial q}{\partial \mathbf{n}_\Gamma}\,\mathrm{d}\mathbf{x}, \quad \mathbf{C}_n \in \mathbb{R}^{n_{\mathbf{S}}\times n_{\mathbf{S}}}, \\
\langle \mathbf{C}_{\text{full}}\,\vec{\mathbf{p}}, \vec{\mathbf{q}} \rangle &\approx \rho_p \int_{\Omega_h^\Gamma} \nabla p\cdot\nabla q\,\mathrm{d}\mathbf{x}, \quad \mathbf{C}_{\text{full}} \in \mathbb{R}^{n_{\mathbf{S}}\times n_{\mathbf{S}}},
\end{aligned}
\tag{1}
$$

respectively. We use notations as in [2], in particular, Ω_h^Γ is the domain consisting of tetrahedra cut by the surface $\Gamma \coloneqq \{\mathbf{x}\in\mathbb{R}^3 \,:\, \phi(\mathbf{x}) = 0\}$. Here $\vec{\mathbf{u}}$ denotes a vector of d.o.f. corresponding to a FE interpolant $\mathbf{u}$ (analogously for $\vec{\mathbf{p}}$ and p). See (5) and (6) for the computational details. Mesh-dependent parameters τ, ρ_u, and ρ_p are chosen to be proportional to some power of $h \coloneqq$ the typical mesh size for tetrahedra from Ω_h^Γ. Γ is chosen either as the unit sphere or torus, $\Gamma = \Gamma_{\text{sph}}$ or $\Gamma = \Gamma_{\text{tor}}$ (see Figure 1). The background domain is chosen as a cube $\Omega \coloneqq (-5/3,\, 5/3)^3$.



(a) $h = 4.17\times10^{-1}$ (b) $h = 2.08\times10^{-1}$ (c) $h = 1.04\times10^{-1}$

(d) $h = 1.04\times10^{-1}$ (e) $h = 5.21\times10^{-2}$ (f) $h = 2.6\times10^{-2}$

Figure 1: Cutaway of the bulk mesh and Γ_h for $\ell = 1, 2, 3$ for Γ_{sph} (top) and Γ_{tor} (bottom)

To build the initial triangulation, we divide Ω into 2^3 cubes and further tessellate each cube into 6 tetrahedra. Then the mesh is gradually refined towards the surface, and $\ell \in \mathbb{N}$ denotes the level of refinement, with the mesh size $h_\ell = \frac{5}{3} 2^{-\ell}$; see Figure 1 for the illustration of the bulk meshes and the induced mesh on the embedded surface for three consecutive refinement levels.

1.2 Quadratures for bilinear forms. We denote by $P_h^n \subset \bar{P}_h^n$ spaces of continuous and discontinuous nodal P_n interpolants defined on Ω_h^Γ, respectively. For a function f, $I_h^n(f) \in P_h^n$ is the corresponding interpolant; we will use the notation f_h^n to emphasize that $f_h^n \in P_h^n$ and f_h^n approximates f in some sense, but $I_h^n(f) \neq f_h^n$.

We set

$$\Gamma_h^n := \{\mathbf{x} \in \mathbb{R}^3 : \big(I_h^n(\phi)\big)(\mathbf{x}) = 0\}, \tag{2}$$

$$\mathbf{n}_{\Gamma_h^n} = \frac{\nabla I_h^n(\phi)}{\|\nabla I_h^n(\phi)\|} \notin \bar{P}_h^m \text{ for any } m \text{ if } n > 1. \tag{3}$$

Note that Γ_h^n is a continuous piecewise P_n surface in Ω_h^Γ, and $\Gamma_h^n \neq I_h^n(\Gamma)$. The unit normal $\mathbf{n}_{\Gamma_h^n}$ is not a rational function; it is continuous in $T \in \Omega_h^\Gamma$ and discontinuous on faces. We also define

$$\Gamma_{h/m}^{2\to 1} := \{\mathbf{x} \in \mathbb{R}^3 : \Big(I_{h/m}^1\big(I_h^2(\phi)\big)\Big)(\mathbf{x}) = 0\}. \tag{4}$$

Note that $I_{h/2}^1\big(I_h^2(\phi)\big) = I_{h/2}^1(\phi)$ (since in order to build both $I_{h/2}^1$ and I_h^2 the same values of ϕ are used), and $I_{h/m}^1\big(I_h^2(\phi)\big) \neq I_{h/m}^1(\phi)$ for $m > 2$. Thus we have $\Gamma_{h/2}^{2\to 1} = \Gamma_{h/2}^1$, and $\Gamma_{h/m}^{2\to 1} \neq \Gamma_{h/m}^1$ for $m > 2$. We refer to Figures 2 and 3.

We implemented two options for the matrix assembly (1). The first one is

$$\begin{aligned}
\langle \mathbf{A}\,\vec{\mathbf{u}}, \vec{\mathbf{v}}\rangle &= \int^5_{\Gamma_{h/m}^{2\to 1}} \big(2\, E_{s,\Gamma_h^2}(\mathbf{u}) : E_{s,\Gamma_h^2}(\mathbf{v}) + \mathbf{u}\cdot\mathbf{v} + \tau\,(\mathbf{u}\cdot\mathbf{n}_{\Gamma_h^2})\,(\mathbf{v}\cdot\mathbf{n}_{\Gamma_h^2})\big)\,\mathrm{d}s \\
&\quad + \rho_u \int^5_{\Omega_h^\Gamma} \frac{\partial \mathbf{u}}{\partial \mathbf{n}_{\Gamma_h^2}} \cdot \frac{\partial \mathbf{v}}{\partial \mathbf{n}_{\Gamma_h^2}}\,\mathrm{d}\mathbf{x}, \quad \mathbf{A} \in \mathbb{R}^{n_\mathbf{A} \times n_\mathbf{A}}, \\
\langle \mathbf{B}\,\vec{\mathbf{u}}, \vec{\mathbf{q}}\rangle &= \int^5_{\Gamma_{h/m}^{2\to 1}} \nabla_{\Gamma_h^2} q \cdot \mathbf{u}\,\mathrm{d}s, \quad \mathbf{B} \in \mathbb{R}^{n_\mathbf{S} \times n_\mathbf{A}}, \\
\langle \mathbf{M}_0\,\vec{\mathbf{p}}, \vec{\mathbf{q}}\rangle &= \int^5_{\Gamma_{h/m}^{2\to 1}} p\,q\,\mathrm{d}s, \quad \mathbf{M}_0 \in \mathbb{R}^{n_\mathbf{S} \times n_\mathbf{S}}, \\
\langle \mathbf{C}_n\,\vec{\mathbf{p}}, \vec{\mathbf{q}}\rangle &= \rho_p \int^5_{\Omega_h^\Gamma} \frac{\partial p}{\partial \mathbf{n}_{\Gamma_h^2}} \frac{\partial q}{\partial \mathbf{n}_{\Gamma_h^2}}\,\mathrm{d}\mathbf{x}, \quad \mathbf{C}_n \in \mathbb{R}^{n_\mathbf{S} \times n_\mathbf{S}}, \\
\langle \mathbf{C}_{\text{full}}\,\vec{\mathbf{p}}, \vec{\mathbf{q}}\rangle &= \rho_p \int^5_{\Omega_h^\Gamma} \nabla p \cdot \nabla q\,\mathrm{d}\mathbf{x}, \quad \mathbf{C}_{\text{full}} \in \mathbb{R}^{n_\mathbf{S} \times n_\mathbf{S}}.
\end{aligned} \tag{5}$$

- $\int^5_{\Gamma_{h/m}^{2\to 1}} \cdot\,\mathrm{d}s$ denotes a composite quadrature rule that is exact for $\bar{P}_h^5(\Gamma_{h/m}^{2\to 1})$, i.e. this quadrature is exact for piecewise polynomials up to degree 5 on each triangular patch $\gamma \in \Gamma_{h/m}^{2\to 1}$,
- $\int^5_{\Omega_h^\Gamma} \cdot\,\mathrm{d}\mathbf{x}$ denotes a composite quadrature rule that is exact for $\bar{P}_h^5(\Omega_h^\Gamma)$, i.e. this quadrature is exact for piecewise polynomials up to degree 5 on each tetrahedron $T \in \Omega_h^\Gamma$,
- E_{s,Γ_h^2} and $\nabla_{\Gamma_h^2}$ are defined as their continuous analogues with $\mathbf{n}_\Gamma$ in $\mathbf{P}_\Gamma$ replaced with $\mathbf{n}_{\Gamma_h^2}$.

The second option is

$$
\begin{aligned}
\langle \mathbf{A}\,\vec{\mathbf{u}}, \vec{\mathbf{v}}\rangle &= \int^{5}_{\Gamma^{2\to 1}_{h/m}} \left(2\,E_{s,\Gamma^{2\to1}_{h/m}}(\mathbf{u}) : E_{s,\Gamma^{2\to1}_{h/m}}(\mathbf{v}) + \mathbf{u}\cdot\mathbf{v} + \tau\,(\mathbf{u}\cdot\mathbf{n}_{\Gamma^2_h})\,(\mathbf{v}\cdot\mathbf{n}_{\Gamma^2_h})\right)\mathrm{d}s \\
&\quad + \rho_u \int^{5}_{\Omega^{\Gamma}_h} \frac{\partial \mathbf{u}}{\partial \mathbf{n}_{\Gamma^2_h}} \cdot \frac{\partial \mathbf{v}}{\partial \mathbf{n}_{\Gamma^2_h}}\,\mathrm{d}\mathbf{x}, \quad \mathbf{A} \in \mathbb{R}^{n_{\mathbf{A}}\times n_{\mathbf{A}}}, \\
\langle \mathbf{B}\,\vec{\mathbf{u}}, \vec{\mathbf{q}}\rangle &= \int^{5}_{\Gamma^{2\to 1}_{h/m}} \nabla_{\Gamma^{2\to1}_{h/m}} q \cdot \mathbf{u}\,\mathrm{d}s, \quad \mathbf{B} \in \mathbb{R}^{n_{\mathbf{S}}\times n_{\mathbf{A}}}, \\
\langle \mathbf{M}_0\,\vec{\mathbf{p}}, \vec{\mathbf{q}}\rangle &= \int^{5}_{\Gamma^{2\to 1}_{h/m}} p\,q\,\mathrm{d}s, \quad \mathbf{M}_0 \in \mathbb{R}^{n_{\mathbf{S}}\times n_{\mathbf{S}}}, \\
\langle \mathbf{C}_n\,\vec{\mathbf{p}}, \vec{\mathbf{q}}\rangle &= \rho_p \int^{5}_{\Omega^{\Gamma}_h} \frac{\partial p}{\partial \mathbf{n}_{\Gamma^2_h}} \frac{\partial q}{\partial \mathbf{n}_{\Gamma^2_h}}\,\mathrm{d}\mathbf{x}, \quad \mathbf{C}_n \in \mathbb{R}^{n_{\mathbf{S}}\times n_{\mathbf{S}}}, \\
\langle \mathbf{C}_{\text{full}}\,\vec{\mathbf{p}}, \vec{\mathbf{q}}\rangle &= \rho_p \int^{5}_{\Omega^{\Gamma}_h} \nabla p \cdot \nabla q\,\mathrm{d}\mathbf{x}, \quad \mathbf{C}_{\text{full}} \in \mathbb{R}^{n_{\mathbf{S}}\times n_{\mathbf{S}}}.
\end{aligned}
\tag{6}
$$

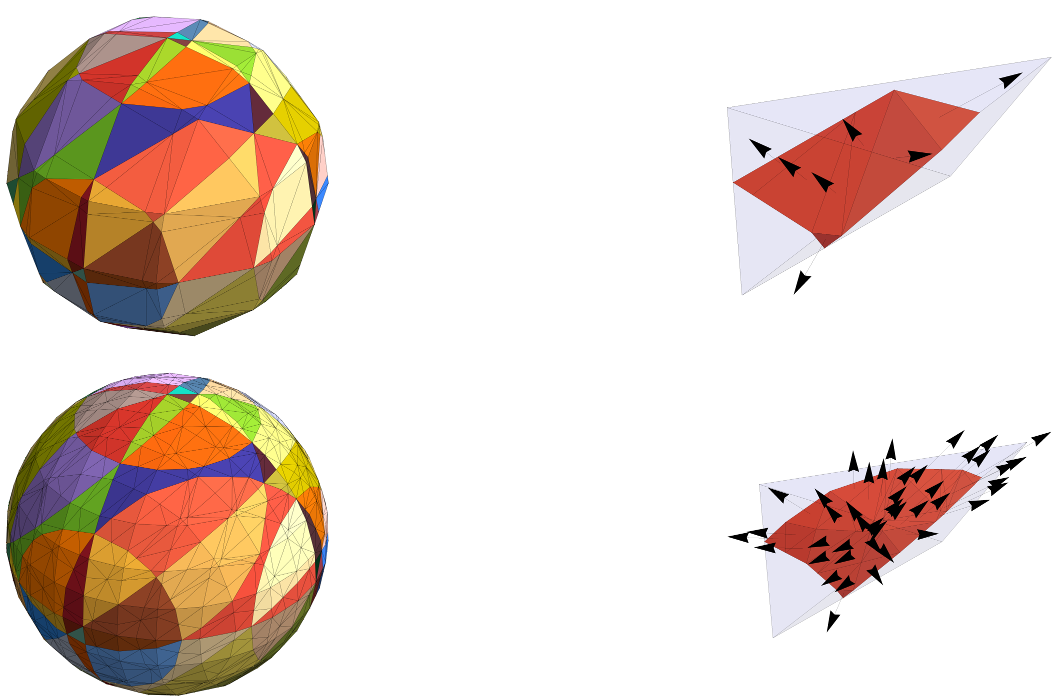


Figure 2: $\Gamma = \Gamma_{\text{sph}}$, $\phi(\mathbf{x}) = \|\mathbf{x}\|^2 - 1$, $h = 8.33 \times 10^{-1}$. Top-left: triangular patches $\gamma \in \Gamma^{2\to1}_{h/2} = \Gamma^1_{h/2}$ (different color corresponds to a different tetrahedron $T \in \Omega^{\Gamma}_h$). Top-right: a patch γ and its normals. Bottom-left and bottom-right: same for $\Gamma^{2\to1}_{h/4} = \Gamma^1_{h/4}$. **Note that since $\phi \in P^2$, we have that $\Gamma^{2\to1}_{h/m} = \Gamma^1_{h/m} \to \Gamma$ as $m \to \infty$ even for fixed h**

For both formulations (5) and (6) the loading vectors for moments and continuity equations are approximated as

$$
\begin{aligned}
\mathbf{f}_i &= \int_{\Gamma^{2\to 1}_{h/m}}^{5} \mathbf{f}\cdot\boldsymbol{\phi}_i \,\mathrm{d}s, \quad i=1,2,\ldots,n_{\mathbf{A}},\\
\mathbf{g}_i &= -\int_{\Gamma^{2\to 1}_{h/m}}^{5} g\,\phi_i \,\mathrm{d}s, \quad i=1,2,\ldots,n_{\mathbf{S}},
\end{aligned}
\tag{7}
$$

respectively. Here $\boldsymbol{\phi}_i$ and ϕ_i are vector and scalar Lagrange basis functions defined on Ω_h^Γ.

Figure 3: $\Gamma = \Gamma_{\mathrm{sph}}$, $\phi(\mathbf{x}) = \|\mathbf{x}\|^{1/2} - 1$, $h = 8.33\times 10^{-1}$. Left: triangular patches $\gamma \in \Gamma^1_{h/2}$ (different color corresponds to a different tetrahedron $T \in \Omega_h^\Gamma$). Right: same for $\Gamma^{2\to 1}_{h/4} \neq \Gamma^1_{h/4}$. **Note that since $\phi \notin \bar{P}_h^2$, we have that $\Gamma^{2\to 1}_{h/m} \neq \Gamma^1_{h/m}$ for $m > 2$, and $\Gamma^{2\to 1}_{h/m} \to \Gamma^2_h \neq \Gamma$ as $m \to \infty$ for fixed h**

As in [1], we refer to (1) as **inconsistent formulation**. We also consider the same formulation as in (1) but with the first term $\mathbf{A}_s$ in the definition of $\mathbf{A}$ changed as

$$
\langle \mathbf{A}_s\,\vec{\mathbf{u}}, \vec{\mathbf{v}}\rangle \approx \int_\Gamma 2\left(E_{s,\Gamma}(\mathbf{u}) - (\mathbf{u}\cdot\mathbf{n}_\Gamma)\,\mathbf{H}_\Gamma\right) : \left(E_{s,\Gamma}(\mathbf{v}) - (\mathbf{v}\cdot\mathbf{n}_\Gamma)\,\mathbf{H}_\Gamma\right) \mathrm{d}s, \tag{8}
$$

where the shape operator is defined as $\mathbf{H}_\Gamma \coloneqq \nabla_\Gamma \mathbf{n}_\Gamma \coloneqq \mathbf{P}_\Gamma\,\nabla\mathbf{n}^e_\Gamma\,\mathbf{P}_\Gamma$, $\mathbf{H}_\Gamma : \mathcal{O}(\Gamma) \to \mathbb{R}^3$. We refer to (8) as **consistent formulation**.

Similarly to (5) and (6), we consider two discretizations of (8):

$$
\langle \mathbf{A}_s\,\vec{\mathbf{u}}, \vec{\mathbf{v}}\rangle = \int_{\Gamma^{2\to 1}_{h/m}}^{5} 2\left(E_{s,\Gamma^2_h}(\mathbf{u}) - (\mathbf{u}\cdot\mathbf{n}_{\Gamma^2_h})\,\mathbf{H}_{\Gamma^2_h}\right) : \left(E_{s,\Gamma^2_h}(\mathbf{v}) - (\mathbf{v}\cdot\mathbf{n}_{\Gamma^2_h})\,\mathbf{H}_{\Gamma^2_h}\right) \mathrm{d}s \tag{9}
$$

and

$$
\langle \mathbf{A}_s\,\vec{\mathbf{u}}, \vec{\mathbf{v}}\rangle = \int_{\Gamma^{2\to 1}_{h/m}}^{5} 2\left(E_{s,\Gamma^{2\to 1}_{h/m}}(\mathbf{u}) - (\mathbf{u}\cdot\mathbf{n}_{\Gamma^{2\to 1}_{h/m}})\,\mathbf{H}_{\Gamma^2_h}\right) : \left(E_{s,\Gamma^{2\to 1}_{h/m}}(\mathbf{v}) - (\mathbf{v}\cdot\mathbf{n}_{\Gamma^{2\to 1}_{h/m}})\,\mathbf{H}_{\Gamma^2_h}\right) \mathrm{d}s. \tag{10}
$$

Note that $\mathbf{n}_\Gamma = \nabla\phi/\|\nabla\phi\|$ is defined in $\mathcal{O}(\Gamma)$, so $\nabla\mathbf{n}_\Gamma$ makes sense and

$$
\nabla\mathbf{n}_\Gamma = \left(\mathbf{I} - \frac{\nabla\phi\,\nabla\phi^T}{\|\nabla\phi\|^2}\right)\frac{\nabla^2\phi}{\|\nabla\phi\|} = \mathbf{P}_\Gamma\,\frac{\nabla^2\phi}{\|\nabla\phi\|}.
$$

If $\mathbf{n}_\Gamma = \mathbf{n}^e_\Gamma$, one gets

$$
\mathbf{H}_\Gamma = \mathbf{P}_\Gamma\,\frac{\nabla^2\phi}{\|\nabla\phi\|}\,\mathbf{P}_\Gamma. \tag{11}
$$

Thus we define $\mathbf{H}_{\Gamma_h^2}$ to be as in (11) but with ϕ replaced with $I_h^2(\phi)$, i.e.

$$\mathbf{H}_{\Gamma_h^2} := \mathbf{P}_{\Gamma_h^2} \frac{\nabla^2 I_h^2(\phi)}{\|\nabla I_h^2(\phi)\|} \mathbf{P}_{\Gamma_h^2}. \tag{12}$$

Indeed, computation of $\mathbf{H}_{\Gamma_h^2}$ requires Hessians of shape functions.

Depending on the choice of ϕ, we may or may not have $\mathbf{n}_\Gamma = \mathbf{n}_\Gamma^e$. Note that the choice $\phi = d$ is sufficient for this, but not necessary. Consider this choices of ϕ for Γ_{sph}:

1. $\phi_1(\mathbf{x}) = \|\mathbf{x}\| - 1 = d(\mathbf{x})$, $\nabla\phi_1/\|\nabla\phi_1\| = \mathbf{n}_\Gamma^e$,
2. $\phi_2(\mathbf{x}) = \|\mathbf{x}\|^2 - 1 \in P^2$, $\nabla\phi_2/\|\nabla\phi_2\| = \nabla\phi_1/\|\nabla\phi_1\| = \mathbf{n}_\Gamma^e$,
3. $\phi_3(\mathbf{x}) = e^{\phi_2(\mathbf{x})}\, x^2 + y^2 + z^2 - 1$, $\nabla\phi_3/\|\nabla\phi_3\| \neq \mathbf{n}_\Gamma^e$, i.e. $\nabla\phi_3/\|\nabla\phi_3\| = \mathbf{n}_\Gamma$ only on Γ_{sph}.

As for the case 2: note that if ϕ is piecewise quadratic in Ω_h^Γ and defines a normal that is equal to its extension, then $\mathbf{H}_{\Gamma_h^2} = \mathbf{H}_\Gamma$, i.e. the approximation is **exact**.

For the approach (6), there is also an option to approximate $\mathbf{H}$ as $\mathbf{P}_{\Gamma_{h/m}^{2\to1}} \frac{\nabla^2 I_h^2(\phi)}{\|\nabla I_h^2(\phi)\|} \mathbf{P}_{\Gamma_{h/m}^{2\to1}}$ since we build $\mathbf{P}_{\Gamma_{h/m}^{2\to1}}$ anyway. We chose to use (12) for both (5) and (6).

1.3 Error computation. Note that $\mathbb{H}^1(\Gamma)$-error (for e.g. $\mathbf{P}_2 - P_1$ FE) can be cheaply approximated as $\langle \mathbf{w}, \mathbf{A}_s \mathbf{w}\rangle^{1/2}$, $\mathbf{w}$:= vector of d.o.f. corresponding to $\mathbf{P}_h^2$ interpolant $I_h^2(\mathbf{u}^e) - \mathbf{u}_h$, $\mathbf{A}_s$:= matrix corresponding to the first term of $\mathbf{A}$ in (6). Thus the errors are approximated as

$$\begin{aligned}
\|\mathbf{u} - \mathbf{u}_h\|_{\mathbb{H}^1(\Gamma)} &= \|I_h^k(\mathbf{u}^e) - \mathbf{u}_h\|_{\mathbb{H}^1\left(\Gamma_{h/m}^{2\to1}\right)} + O(h^k),\\
\|\mathbf{u} - \mathbf{u}_h\|_{\mathbb{L}^2(\Gamma)} &= \|I_h^k(\mathbf{u}^e) - \mathbf{u}_h\|_{\mathbb{L}^2\left(\Gamma_{h/m}^{2\to1}\right)} + O(h^{k+1}),\\
\|p - p_h\|_{\mathbb{L}^2(\Gamma)} &= \|I_h^1(p^e) - p_h\|_{\mathbb{L}^2\left(\Gamma_{h/m}^{2\to1}\right)} + O(h^2)
\end{aligned} \tag{13}$$

for $m > 1$. Here $k = 1$ for $\mathbf{P}_1 - P_1$ FEM and $k = 2$ for $\mathbf{P}_2 - P_1$. For consistent penalty approach matrix $\mathbf{A}_s$ is computed as in (9) or (10).

2 Convergence results

2.1 Manufactured solution. We solve model problem from [2, p. 20], $\Gamma = \Gamma_{\text{sph}}$[1]. We set

$$\tilde{\mathbf{u}}(x,y,z) := (-z^2, y, x)^T, \quad \tilde{p}(x,y,z) := x\,y^2 + z, \quad \phi(\mathbf{x}) := \|\mathbf{x}\|^2 - 1. \tag{14}$$

The exact solution on the unit sphere is chosen as

$$\mathbf{u} := \mathbf{P}_\Gamma\, \tilde{\mathbf{u}}^e, \quad p := \tilde{p}^e. \tag{15}$$

Thus we have $\int_\Gamma p\,\mathrm{d}\mathbf{x} = 0$, $p \equiv p^e$, $\mathbf{u} \equiv \mathbf{u}^e$ in $\mathcal{O}(\Gamma)$, and $\mathbf{u}$ is a tangential field. Note that for our choice of ϕ in (14) we have

$$\mathbf{n}_{\Gamma_h^2} = \mathbf{n}_\Gamma^e \text{ in } \mathcal{O}(\Gamma), \quad \Gamma_{h/m}^{2\to1} = \Gamma_{h/m}^1, \quad \mathbf{n}_{\Gamma_{h/m}^{2\to1}} = \mathbf{n}_{\Gamma_{h/m}^1} \text{ on } \Gamma, \tag{16}$$

and

$$\mathbf{n}_{\Gamma_{h/m}^1} \to \mathbf{n}_\Gamma, \quad \Gamma_{h/m}^1 \to \Gamma \tag{17}$$

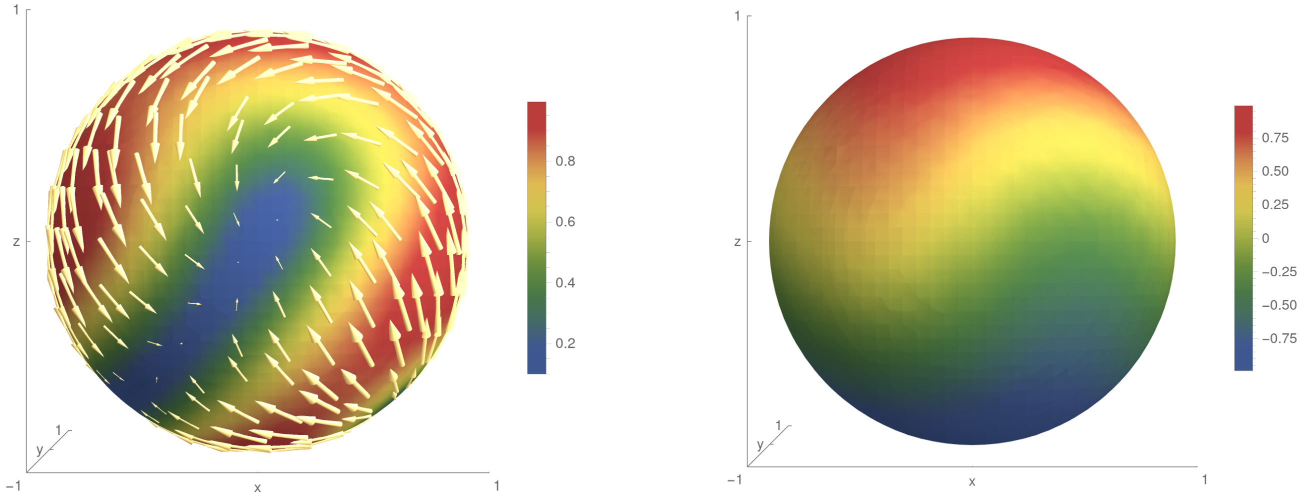


Figure 4: Exact velocity solution (Left) and pressure solution (Right) as in (15)

as one increases m **even for fixed** h.

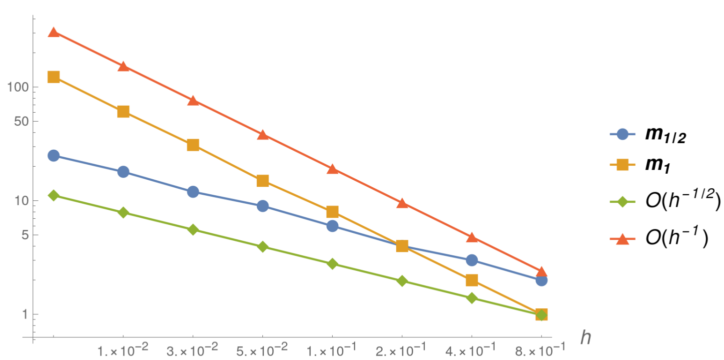
We have

$$\mathbf{n}_{\Gamma_{\rm sph}}(\mathbf{x}) = \|\mathbf{x}\|^{-1}\,\mathbf{x} = \mathbf{n}^e_{\Gamma_{\rm sph}}(\mathbf{x}), \tag{18}$$

$$\mathbf{P}_{\Gamma_{\rm sph}}(\mathbf{x}) = \|\mathbf{x}\|^{-2}\begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix} = \mathbf{P}^e_{\Gamma_{\rm sph}}(\mathbf{x}), \tag{19}$$

$$\mathbf{H}_{\Gamma_{\rm sph}}(\mathbf{x}) = \|\mathbf{x}\|^{-3}\begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix} \neq \mathbf{H}^e_{\Gamma_{\rm sph}}(\mathbf{x}) = \mathbf{P}_{\Gamma_{\rm sph}}(\mathbf{x}). \tag{20}$$

We consider two choices for virtual refinement: $m \propto h^{-1/2}$ and $m \propto h^{-1}$. The first choice assures h^3-accurate approximation of Γ and $h^{3/2}$ accurate approximation of the normal vector, whereas the second choice assures h^4- and h^2-approximations. We refer to Figure 5.



$\mathbf{m}_{1/2} \coloneqq \{2, 3, 4, 6, 9, 12, 18, 25\}$,
$\mathbf{m}_1 \coloneqq \{1, 2, 4, 8, 15, 31, 61, 123\}$.

$m \in \mathbf{m}_{1/2}$ behaves as $h^{-1/2}$, and $m \in \mathbf{m}_1$ behaves as h^{-1}

Figure 5: Virtual refinement parameter m for $\Gamma^{2\to 1}_{h/m}$

[1] In [2] they use $\mathbf{u} \coloneqq \mathbf{P}\,\tilde{\mathbf{u}}$, i.e. $\mathbf{u} \neq \mathbf{u}^e$. I prefer $\mathbf{u} \equiv \mathbf{u}^e$ as in [1].

Table 1: Errors for normals and shape operator, $\Gamma = \Gamma_{\text{sph}}$. Please see (16) and (17)

$m \in \mathbf{m}_{1/2}$ as in Figure 5					
h	$\|\mathbf{n}^e_\Gamma - \mathbf{n}_{\Gamma^2_h}\|_{\mathbb{L}^2(\Gamma^1_{h/m})}$	$\|\mathbf{n}^e_\Gamma - \mathbf{n}_{\Gamma^1_{h/m}}\|_{\mathbb{L}^2(\Gamma^1_{h/m})}$	Order	$\|\mathbf{H}^e_\Gamma - \mathbf{H}_{\Gamma^2_h}\|_{\mathbb{L}^2(\Gamma^1_{h/m})}$	Order
8.3×10^{-1}	8.1×10^{-16}	6.4×10^{-1}		2.3×10^{-1}	
4.2×10^{-1}	1.1×10^{-15}	$2. \times 10^{-1}$	1.7	2.5×10^{-2}	3.2
2.1×10^{-1}	2.3×10^{-15}	7.6×10^{-2}	1.4	3.5×10^{-3}	2.8
$1. \times 10^{-1}$	5.2×10^{-15}	2.5×10^{-2}	1.6	3.9×10^{-4}	3.2
5.2×10^{-2}	9.3×10^{-15}	8.4×10^{-3}	1.6	4.3×10^{-5}	3.2
2.6×10^{-2}	1.9×10^{-14}	3.1×10^{-3}	1.4	6.1×10^{-6}	2.8
1.3×10^{-2}	3.6×10^{-14}	$1. \times 10^{-3}$	1.6	6.8×10^{-7}	3.2

2.2 $\mathbf{P}_2 - P_1$ Trace FEM.

Next we compare inconsistent and consistent Trace FEM penalty formulations.

2.2.1 Inconsistent penalty formulation.

We use the normal stabilization matrix $\mathbf{C}_n$. We stick to the approach (5), so with (16) we have

$$
\begin{aligned}
\langle \mathbf{A}\,\vec{\mathbf{u}}, \vec{\mathbf{v}} \rangle &= \int^5_{\Gamma^1_{h/2}} \big(2\, E_{s,\Gamma}(\mathbf{u}) : E_{s,\Gamma}(\mathbf{v}) + \mathbf{u}\cdot\mathbf{v} + \tau\,(\mathbf{u}\cdot\mathbf{n}_\Gamma)\,(\mathbf{v}\cdot\mathbf{n}_\Gamma)\big)\,\mathrm{d}s \\
&\quad + \rho_u \int^5_{\Omega^\Gamma_h} \frac{\partial \mathbf{u}}{\partial \mathbf{n}_\Gamma} \cdot \frac{\partial \mathbf{v}}{\partial \mathbf{n}_\Gamma}\,\mathrm{d}\mathbf{x}, \quad \mathbf{A} \in \mathbb{R}^{n_\mathbf{A} \times n_\mathbf{A}}, \\
\langle \mathbf{B}\,\vec{\mathbf{u}}, \vec{\mathbf{q}} \rangle &= \int^5_{\Gamma^1_{h/2}} \nabla_\Gamma q \cdot \mathbf{u}\,\mathrm{d}s, \quad \mathbf{B} \in \mathbb{R}^{n_\mathbf{S} \times n_\mathbf{A}}, \\
\langle \mathbf{M}_0\,\vec{\mathbf{p}}, \vec{\mathbf{q}} \rangle &= \int^5_{\Gamma^1_{h/2}} p\,q\,\mathrm{d}s, \quad \mathbf{M}_0 \in \mathbb{R}^{n_\mathbf{S} \times n_\mathbf{S}}, \\
\langle \mathbf{C}_n\,\vec{\mathbf{p}}, \vec{\mathbf{q}} \rangle &= \rho_p \int^5_{\Omega^\Gamma_h} \frac{\partial p}{\partial \mathbf{n}_\Gamma} \frac{\partial q}{\partial \mathbf{n}_\Gamma}\,\mathrm{d}\mathbf{x}, \quad \mathbf{C}_n \in \mathbb{R}^{n_\mathbf{S} \times n_\mathbf{S}}.
\end{aligned}
\tag{21}
$$

Table 2: Convergence results. $\tau = h^{-2}$, $\rho_u = h$, $\rho_p = h$. Matrices are assembled as in (21)

$m \in \mathbf{m}_{1/2}$ as in Figure 5						
h	$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbb{H}^1}$	Order	$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbb{L}^2}$	Order	$\|p - p_h\|_{\mathbb{L}^2}$	Order
8.3×10^{-1}	3.		1.8		2.1	
4.2×10^{-1}	1.8	7.4×10^{-1}	9.1×10^{-1}	9.7×10^{-1}	1.7	2.7×10^{-1}
2.1×10^{-1}	$7. \times 10^{-1}$	1.4	3.4×10^{-1}	1.4	6.9×10^{-1}	1.3
$1. \times 10^{-1}$	$2. \times 10^{-1}$	1.8	9.9×10^{-2}	1.8	$2. \times 10^{-1}$	1.8
5.2×10^{-2}	5.2×10^{-2}	1.9	2.6×10^{-2}	1.9	5.2×10^{-2}	1.9
2.6×10^{-2}	1.3×10^{-2}	2.	6.5×10^{-3}	2.	1.3×10^{-2}	2.
1.3×10^{-2}	3.3×10^{-3}	2.	1.6×10^{-3}	2.	3.3×10^{-3}	2.

h	$\|\mathbf{u}_h \cdot \mathbf{n}\|_{\mathbb{L}^2}$	Order	Outer iterations	Residual norm
8.33×10^{-1}	1.8		24	6.2×10^{-9}
4.17×10^{-1}	9.2×10^{-1}	9.4×10^{-1}	31	5.4×10^{-9}
2.08×10^{-1}	3.5×10^{-1}	1.4	30	9.8×10^{-9}
1.04×10^{-1}	9.9×10^{-2}	1.8	27	7.8×10^{-9}
5.21×10^{-2}	2.6×10^{-2}	1.9	26	8.3×10^{-9}
2.6×10^{-2}	6.5×10^{-3}	2.	26	9.6×10^{-9}
1.3×10^{-2}	1.6×10^{-3}	2.	35	$7. \times 10^{-9}$

For statistics: using 64 CPUs, computation of the meshlevel 3 ($h = 2.08 \times 10^{-1}$) takes ∼1 minute, meshlevel 4 takes ∼7 minutes, meshlevel 5 takes ∼50 minutes, meshlevel 6 takes 4.8 hours, and meshlevel 7 takes ∼21.3 hours.

2.2.2 Consistent penalty formulation. We consider the same formulation as in (21), but with the first term $\mathbf{A}_s$ in the definition of $\mathbf{A}$ changed according to (9). Thus with (16) we have

$$\langle \mathbf{A}_s\, \vec{\mathbf{u}}, \vec{\mathbf{v}} \rangle = \int_{\Gamma^1_{h/m}}^{5} 2\left(E_{s,\Gamma}(\mathbf{u}) - (\mathbf{u}\cdot\mathbf{n}_\Gamma)\,\mathbf{H}_\Gamma\right) : \left(E_{s,\Gamma}(\mathbf{v}) - (\mathbf{v}\cdot\mathbf{n}_\Gamma)\,\mathbf{H}_\Gamma\right) \mathrm{d}s. \tag{22}$$

Table 3: Convergence results. $\tau = h^{-2}$, $\rho_u = h^{-1}$, $\rho_p = h$, $\mathbf{C}_\star = \mathbf{C}_{\text{full}}$. Matrices are assembled as in (21)–(22)

$m \in \mathbf{m}_{1/2}$ as in Figure 5						
h	$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbb{H}^1}$	Order	$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbb{L}^2}$	Order	$\|p - p_h\|_{\mathbb{L}^2}$	Order
8.3×10^{-1}	1.6		7.8×10^{-1}		1.3	
4.2×10^{-1}	6.9×10^{-1}	1.2	3.9×10^{-1}	1.	8.1×10^{-1}	6.3×10^{-1}
2.1×10^{-1}	2.4×10^{-1}	1.5	1.3×10^{-1}	1.6	3.1×10^{-1}	1.4
$1. \times 10^{-1}$	8.1×10^{-2}	1.6	3.6×10^{-2}	1.8	1.1×10^{-1}	1.5
5.2×10^{-2}	2.4×10^{-2}	1.8	9.5×10^{-3}	1.9	3.2×10^{-2}	1.7
2.6×10^{-2}	6.5×10^{-3}	1.9	2.4×10^{-3}	2.	8.8×10^{-3}	1.9
1.3×10^{-2}	1.8×10^{-3}	1.8	6.1×10^{-4}	2.	2.5×10^{-3}	1.8

h	$\|\mathbf{u}_h \cdot \mathbf{n}\|_{\mathbb{L}^2}$	Order	Outer iterations	Residual norm
8.33×10^{-1}	3.5×10^{-1}		15	1.3×10^{-9}
4.17×10^{-1}	5.4×10^{-2}	2.7	21	3.5×10^{-9}
2.08×10^{-1}	4.9×10^{-3}	3.4	27	5.5×10^{-9}
1.04×10^{-1}	$5. \times 10^{-4}$	3.3	29	5.8×10^{-9}
5.21×10^{-2}	4.9×10^{-5}	3.4	29	5.1×10^{-9}
2.6×10^{-2}	$5. \times 10^{-6}$	3.3	28	8.4×10^{-9}
1.3×10^{-2}	5.9×10^{-7}	3.1	34	9.1×10^{-9}

Table 4: Convergence results. $\tau = h^{-2}$, $\rho_u = h^{-1}$, $\rho_p = h$, $\mathbf{C}_\star = \mathbf{C}_n$. Matrices are assembled as in (21)−(22)

$m \in \mathbf{m}_{1/2}$ as in Figure 5						
h	$\lVert\mathbf{u}-\mathbf{u}_h\rVert_{\mathbb{H}^1}$	Order	$\lVert\mathbf{u}-\mathbf{u}_h\rVert_{\mathbb{L}^2}$	Order	$\lVert p-p_h\rVert_{\mathbb{L}^2}$	Order
8.3×10^{-1}	1.2		4.8×10^{-1}		4.2×10^{-1}	
4.2×10^{-1}	3.7×10^{-1}	1.8	6.1×10^{-2}	3.	1.1×10^{-1}	2.
2.1×10^{-1}	9.2×10^{-2}	2.	5.8×10^{-3}	3.4	2.5×10^{-2}	2.1
$1. \times 10^{-1}$	2.2×10^{-2}	2.1	5.6×10^{-4}	3.4	6.3×10^{-3}	2.
5.2×10^{-2}	5.4×10^{-3}	2.	5.2×10^{-5}	3.4	1.7×10^{-3}	1.8
2.6×10^{-2}	1.4×10^{-3}	2.	5.4×10^{-6}	3.3	5.3×10^{-4}	1.7
1.3×10^{-2}	3.5×10^{-4}	2.	6.9×10^{-7}	3.	1.8×10^{-4}	1.6

h	$\lVert\mathbf{u}_h \cdot \mathbf{n}\rVert_{\mathbb{L}^2}$	Order	Outer iterations	Residual norm
8.33×10^{-1}	3.4×10^{-1}		26	7.3×10^{-9}
4.17×10^{-1}	5.3×10^{-2}	2.7	33	4.8×10^{-9}
2.08×10^{-1}	4.9×10^{-3}	3.4	31	$6. \times 10^{-9}$
1.04×10^{-1}	$5. \times 10^{-4}$	3.3	27	8.3×10^{-9}
5.21×10^{-2}	4.9×10^{-5}	3.4	26	8.6×10^{-9}
2.6×10^{-2}	$5. \times 10^{-6}$	3.3	26	7.5×10^{-9}
1.3×10^{-2}	5.9×10^{-7}	3.1	34	$8. \times 10^{-9}$

$m \in \mathbf{m}_1$ as in Figure 5						
h	$\lVert\mathbf{u}-\mathbf{u}_h\rVert_{\mathbb{H}^1}$	Order	$\lVert\mathbf{u}-\mathbf{u}_h\rVert_{\mathbb{L}^2}$	Order	$\lVert p-p_h\rVert_{\mathbb{L}^2}$	Order
8.3×10^{-1}	2.2		6.4×10^{-1}		7.4×10^{-1}	
4.2×10^{-1}	3.8×10^{-1}	2.5	6.1×10^{-2}	3.4	1.2×10^{-1}	2.6
2.1×10^{-1}	9.2×10^{-2}	2.1	5.8×10^{-3}	3.4	2.5×10^{-2}	2.2
$1. \times 10^{-1}$	2.2×10^{-2}	2.1	5.6×10^{-4}	3.4	6.1×10^{-3}	2.1
5.2×10^{-2}	5.3×10^{-3}	2.	5.2×10^{-5}	3.4	1.6×10^{-3}	1.9
2.6×10^{-2}	1.3×10^{-3}	2.	5.2×10^{-6}	3.3	4.1×10^{-4}	2.
1.3×10^{-2}	3.4×10^{-4}	2.	$6. \times 10^{-7}$	3.1	$1. \times 10^{-4}$	2.

h	$\lVert\mathbf{u}_h \cdot \mathbf{n}\rVert_{\mathbb{L}^2}$	Order	Outer iterations	Residual norm
8.33×10^{-1}	4.5×10^{-1}		26	2.6×10^{-9}
4.17×10^{-1}	5.3×10^{-2}	3.1	33	5.1×10^{-9}
2.08×10^{-1}	4.9×10^{-3}	3.4	31	$6. \times 10^{-9}$
1.04×10^{-1}	$5. \times 10^{-4}$	3.3	27	7.3×10^{-9}
5.21×10^{-2}	4.9×10^{-5}	3.4	25	6.4×10^{-9}
2.6×10^{-2}	$5. \times 10^{-6}$	3.3	26	4.3×10^{-9}
1.3×10^{-2}	5.8×10^{-7}	3.1	34	7.8×10^{-9}

For statistics: using 80 CPUs, computation of the meshlevel 7 ($h = 1.3 \times 10^{-2}$) takes ∼27 hours with $m = 18$ (computation also involves errors for normals and the shape operator).

Table 6: Convergence results. $\tau = h^{-2}$, $\rho_u = h^{-1}$, $\rho_p = 1$, $\mathbf{C}_\star = \mathbf{C}_n$. Matrices are assembled as in (21)–(22)

$m \in \mathbf{m}_{1/2}$ as in Figure 5						
h	$\|\mathbf{u}-\mathbf{u}_h\|_{\mathbb{H}^1}$	Order	$\|\mathbf{u}-\mathbf{u}_h\|_{\mathbb{L}^2}$	Order	$\|p-p_h\|_{\mathbb{L}^2}$	Order
8.3×10^{-1}	1.3		$5. \times 10^{-1}$		4.9×10^{-1}	
4.2×10^{-1}	3.8×10^{-1}	1.7	6.7×10^{-2}	2.9	1.5×10^{-1}	1.7
2.1×10^{-1}	$1. \times 10^{-1}$	1.9	8.6×10^{-3}	3.	$5. \times 10^{-2}$	1.5
$1. \times 10^{-1}$	2.4×10^{-2}	2.1	1.3×10^{-3}	2.7	1.2×10^{-2}	2.
5.2×10^{-2}	5.6×10^{-3}	2.1	1.8×10^{-4}	2.9	2.1×10^{-3}	2.6
2.6×10^{-2}	1.4×10^{-3}	2.	2.3×10^{-5}	3.	4.1×10^{-4}	2.4
1.3×10^{-2}	3.4×10^{-4}	2.	2.9×10^{-6}	3.	$1. \times 10^{-4}$	2.

h	$\|\mathbf{u}_h \cdot \mathbf{n}\|_{\mathbb{L}^2}$	Order	Outer iterations	Residual norm
8.33×10^{-1}	3.4×10^{-1}		25	5.1×10^{-9}
4.17×10^{-1}	5.3×10^{-2}	2.7	33	3.3×10^{-9}
2.08×10^{-1}	4.9×10^{-3}	3.4	32	$1. \times 10^{-8}$
1.04×10^{-1}	$5. \times 10^{-4}$	3.3	29	8.3×10^{-9}
5.21×10^{-2}	4.9×10^{-5}	3.4	27	4.7×10^{-9}
2.6×10^{-2}	$5. \times 10^{-6}$	3.3	26	5.1×10^{-9}
1.3×10^{-2}	5.9×10^{-7}	3.1	34	8.5×10^{-9}

Table 7: Convergence results. $\tau = h^{-2}$, $\rho_u = h^{-1}$, $\rho_p = h^{-1}$, $\mathbf{C}_\star = \mathbf{C}_n$. Matrices are assembled as in (21)–(22)

$m \in \mathbf{m}_{1/2}$ as in Figure 5						
h	$\|\mathbf{u}-\mathbf{u}_h\|_{\mathbb{H}^1}$	Order	$\|\mathbf{u}-\mathbf{u}_h\|_{\mathbb{L}^2}$	Order	$\|p-p_h\|_{\mathbb{L}^2}$	Order
8.3×10^{-1}	1.3		5.2×10^{-1}		5.6×10^{-1}	
4.2×10^{-1}	$4. \times 10^{-1}$	1.7	8.7×10^{-2}	2.6	$2. \times 10^{-1}$	1.5
2.1×10^{-1}	1.3×10^{-1}	1.6	2.1×10^{-2}	2.1	1.2×10^{-1}	7.6×10^{-1}
$1. \times 10^{-1}$	5.2×10^{-2}	1.3	7.5×10^{-3}	1.5	6.4×10^{-2}	9.1×10^{-1}
5.2×10^{-2}	$2. \times 10^{-2}$	1.4	2.6×10^{-3}	1.5	2.6×10^{-2}	1.3
2.6×10^{-2}	6.3×10^{-3}	1.7	7.7×10^{-4}	1.7	8.5×10^{-3}	1.6
1.3×10^{-2}	1.8×10^{-3}	1.8	2.1×10^{-4}	1.9	2.4×10^{-3}	1.8

h	$\|\mathbf{u}_h \cdot \mathbf{n}\|_{\mathbb{L}^2}$	Order	Outer iterations	Residual norm
8.33×10^{-1}	3.4×10^{-1}		25	3.1×10^{-9}
4.17×10^{-1}	5.3×10^{-2}	2.7	31	5.5×10^{-9}
2.08×10^{-1}	4.9×10^{-3}	3.4	33	4.4×10^{-9}
1.04×10^{-1}	$5. \times 10^{-4}$	3.3	31	$1. \times 10^{-8}$
5.21×10^{-2}	4.9×10^{-5}	3.4	31	3.7×10^{-9}
2.6×10^{-2}	$5. \times 10^{-6}$	3.3	28	9.8×10^{-9}
1.3×10^{-2}	5.9×10^{-7}	3.1	34	9.6×10^{-9}

3 Inf-sup stability: pressure Schur complement generalized eigenvalues

3.1 Solution description. We define matrices

$$\mathbf{C}_0 := \mathbf{0}, \quad \mathbf{M}_n := \mathbf{M}_0 + \mathbf{C}_n, \quad \mathbf{M}_{\text{full}} := \mathbf{M}_0 + \mathbf{C}_{\text{full}}. \tag{23}$$

We are interested in (generalized) extreme eigenvalues of the pressure Schur complement matrices

$$\mathbf{S}_0 \coloneqq \mathbf{B}\,\mathbf{A}^{-1}\,\mathbf{B}^T, \quad \mathbf{S}_n \coloneqq \mathbf{S}_0 + \mathbf{C}_n, \quad \mathbf{S}_{\text{full}} \coloneqq \mathbf{S}_0 + \mathbf{C}_{\text{full}}, \tag{24}$$

i.e. in solving

$$\mathbf{S}_\star\,\mathbf{x} = \lambda\,\mathbf{M}_\star\,\mathbf{x}, \tag{25}$$

where "$\star$" stands for "0," "n," or "full." We denote by $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_{n\mathbf{s}} = O(1)$ the spectrum of (25).

Computing $\mathbf{A}^{-1}$ in (24) becomes troublesome already for $h = 5.21{\times}10^{-2}$ ($n_{\mathbf{A}} = 32736$ for $\mathbf{u} \in \mathbf{P}_1$ FE space): although $\mathbf{A}$ is sparse, $\mathbf{A}^{-1}$ is dense and consumes 8.5+ GB in double-precision arithmetic. A quick research showed that `Mathematica` has no built-in matrix-free eigenvalue routines. `Intel MKL`'s FEAST algorithm for computing (generalized) eigenvalues in an interval is suitable for matrix-free implementations; however, it requires some expensive operations to be implemented (e.g. matrix-matrix multiplications $\mathbf{Y} \leftarrow \mathbf{S}_\star\,\mathbf{X}$, $\mathbf{Y} \leftarrow \mathbf{M}_\star\,\mathbf{X}$ and approximating the action of inverses in the form $\mathbf{y} \leftarrow (\sigma\,\mathbf{M}_\star - \mathbf{S}_\star)^{-1}\,\mathbf{x}$).

Taking this into account, instead of (25) we consider a perturbed[2] problem

$$\underbrace{\begin{bmatrix} \mathbf{A} & \mathbf{B}^T \\ \mathbf{B} & -\mathbf{C}_\star \end{bmatrix}}_{\mathcal{A}_\star \coloneqq} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \mu\,\underbrace{\begin{bmatrix} \epsilon\,\mathbf{A} & \\ & \mathbf{M}_\star \end{bmatrix}}_{\mathcal{M}_\star^\epsilon \coloneqq} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \tag{26}$$

with $0 < \epsilon \ll 1$. For $\mathcal{A}_0$ and $\mathcal{M}_0^\epsilon$ we have

$$\mu = -\lambda + o(1) \quad \text{or} \quad \epsilon^{-1} + \lambda + o(1), \qquad \epsilon \to 0. \tag{27}$$

This makes it easy to pick only "correct" eigenvalues. To ease the computation further we replace the $(1,1)$-block of $\mathcal{M}_\star^\epsilon$ with $\epsilon\,\mathbf{I}$.

To make sure that results are consistent we solve (26) for $\epsilon = 10^{-5}$ and $\epsilon = 10^{-6}$; for the coarse mesh levels we also check that the dense solver for (25) and the iterative one for (26) give solutions that coincide in finite precision. See Table 8.

Table 8: Eigenvalues differences: dense solver for (25) and the iterative solver for (26). Corresponding eigenvalues are denoted as $\{\lambda_i\}$ and $\{\lambda_i^\mu\}$. Results are shown for consistent $\mathbf{P}_2{-}P_1$, $\tau = h^{-2}$, $\rho_u = h^{-1}$, $\rho_p = h$, $m \in \mathbf{m}_{1/2}$ as in Figure 5

$\Gamma = \Gamma_{\text{sph}}$						
h	$\mathbf{S}_0$		$\mathbf{S}_n$		$\mathbf{S}_{\text{full}}$	
	$\lvert\lambda_2 - \lambda_2^\mu\rvert$	$\lvert\lambda_{n\mathbf{s}} - \lambda_{n\mathbf{s}}^\mu\rvert$	$\lvert\lambda_2 - \lambda_2^\mu\rvert$	$\lvert\lambda_{n\mathbf{s}} - \lambda_{n\mathbf{s}}^\mu\rvert$	$\lvert\lambda_2 - \lambda_2^\mu\rvert$	$\lvert\lambda_{n\mathbf{s}} - \lambda_{n\mathbf{s}}^\mu\rvert$
8.33×10^{-1}	1.22×10^{-7}	5.97×10^{-6}	4.28×10^{-7}	2.09×10^{-7}	1.05×10^{-6}	1.5×10^{-7}
4.17×10^{-1}	5.14×10^{-9}	1.2×10^{-5}	8.74×10^{-7}	4.72×10^{-6}	9.25×10^{-6}	3.06×10^{-6}
2.08×10^{-1}	1.06×10^{-8}	4.15×10^{-5}	1.05×10^{-6}	4.34×10^{-5}	7.1×10^{-6}	2.8×10^{-5}

3.2 Dependency of the spectrum on the mesh size. Here we test inconsistent $\mathbf{P}_1{-}P_1$ and consistent $\mathbf{P}_2{-}P_1$ approaches.

[2] The majority of generalized eigenvalue solvers require left-hand-side matrix to be Hermitian and right-hand-side matrix to be Hermitian **positive definite**; that's why we need to introduce $\epsilon > 0$.

Table 9: Spectrum of (25) for inconsistent $\mathbf{P}_1 - P_1$, $\tau = h^{-2}$, $\rho_u = h$, $\rho_p = h$, $m \equiv 2$

$\Gamma = \Gamma_{\text{sph}}$								
h	$n_{\mathbf{A}}$	$n_{\mathbf{S}}$	$\mathbf{S}_0$		$\mathbf{S}_n$		$\mathbf{S}_{\text{full}}$	
			λ_2	$\lambda_{n_{\mathbf{S}}}$	λ_2	$\lambda_{n_{\mathbf{S}}}$	λ_2	$\lambda_{n_{\mathbf{S}}}$
8.33×10^{-1}	153	51	1.32×10^{-2}	1.42	7.48×10^{-1}	1.13	9.58×10^{-1}	1.06
4.17×10^{-1}	570	190	5.12×10^{-3}	1.04	5.77×10^{-1}	1.	8.54×10^{-1}	1.
2.08×10^{-1}	1992	664	4.4×10^{-3}	7.93×10^{-1}	3.87×10^{-1}	1.	6.71×10^{-1}	1.
1.04×10^{-1}	8292	2764	2.01×10^{-3}	7.79×10^{-1}	2.19×10^{-1}	1.	5.82×10^{-1}	1.
5.21×10^{-2}	32736	10912	6.04×10^{-5}	9.81×10^{-1}	1.17×10^{-1}	1.	5.37×10^{-1}	1.
2.6×10^{-2}	131592	43864	3.53×10^{-5}	8.67×10^{-1}	5.72×10^{-2}	1.	5.16×10^{-1}	1.
1.3×10^{-2}	525864	175288	2.16×10^{-6}	7.34×10^{-1}	2.84×10^{-2}	1.	5.04×10^{-1}	1.

$\Gamma = \Gamma_{\text{tor}}$								
h	$n_{\mathbf{A}}$	$n_{\mathbf{S}}$	$\mathbf{S}_0$		$\mathbf{S}_n$		$\mathbf{S}_{\text{full}}$	
			λ_2	$\lambda_{n_{\mathbf{S}}}$	λ_2	$\lambda_{n_{\mathbf{S}}}$	λ_2	$\lambda_{n_{\mathbf{S}}}$
2.08×10^{-1}	972	324	5.04×10^{-2}	4.93	2.84×10^{-1}	1.35	3.64×10^{-1}	1.19
1.04×10^{-1}	4740	1580	2.99×10^{-3}	3.83	1.58×10^{-1}	1.02	3.35×10^{-1}	1.01
5.21×10^{-2}	19704	6568	1.11×10^{-3}	5.45	7.73×10^{-2}	1.01	3.25×10^{-1}	1.
2.6×10^{-2}	80808	26936	1.2×10^{-4}	5.42	3.07×10^{-2}	1.01	3.21×10^{-1}	1.
1.3×10^{-2}	327036	109012	1.77×10^{-5}	5.23	1.18×10^{-2}	1.01	3.16×10^{-1}	1.

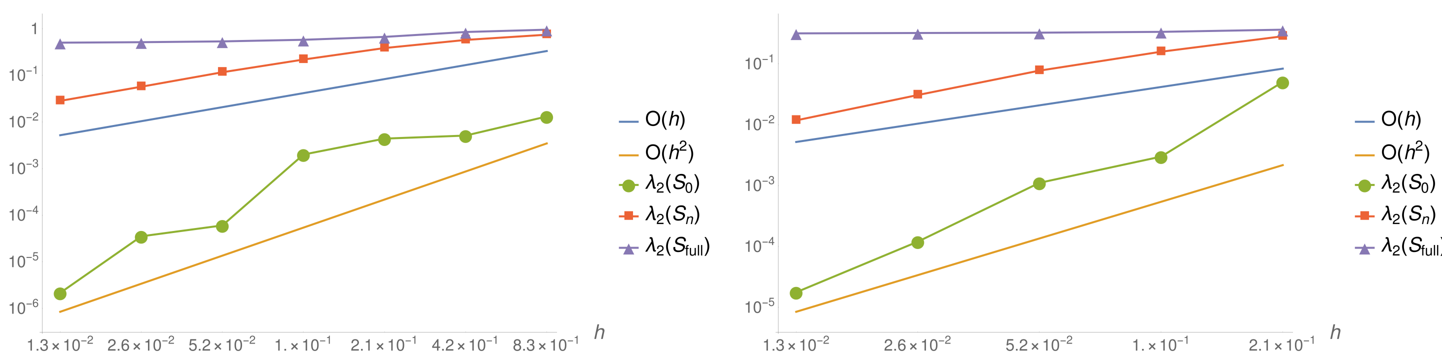


Figure 6: Log-log plot of λ_2 for Table 9. Left: $\Gamma = \Gamma_{\text{sph}}$, right: $\Gamma = \Gamma_{\text{tor}}$

Table 10: Spectrum of (25) for consistent $\mathbf{P}_2 - P_1$, $\tau = h^{-2}$, $\rho_u = h^{-1}$, $\rho_p = h$, $m \in \mathbf{m}_{1/2}$ as in Figure 5

$\Gamma = \Gamma_{\text{sph}}$								
h	$n_{\mathbf{A}}$	$n_{\mathbf{S}}$	$\mathbf{S}_0$		$\mathbf{S}_n$		$\mathbf{S}_{\text{full}}$	
			λ_2	$\lambda_{n_{\mathbf{S}}}$	λ_2	$\lambda_{n_{\mathbf{S}}}$	λ_2	$\lambda_{n_{\mathbf{S}}}$
8.33×10^{-1}	789	51	2.33×10^{-1}	1.07	6.3×10^{-1}	1.	8.81×10^{-1}	1.
4.17×10^{-1}	3276	190	4.72×10^{-2}	6.97×10^{-1}	5.29×10^{-1}	1.	7.64×10^{-1}	1.
2.08×10^{-1}	11718	664	7.93×10^{-2}	6.7×10^{-1}	5.09×10^{-1}	1.	6.39×10^{-1}	1.
1.04×10^{-1}	48762	2764	3.71×10^{-2}	6.69×10^{-1}	5.03×10^{-1}	1.	5.73×10^{-1}	1.
5.21×10^{-2}	193086	10912	1.81×10^{-3}	6.68×10^{-1}	4.98×10^{-1}	1.	5.36×10^{-1}	1.
2.6×10^{-2}	775998	43864	6.65×10^{-4}	6.65×10^{-1}	4.92×10^{-1}	1.	5.17×10^{-1}	1.

$\Gamma = \Gamma_{\text{tor}}$								
h	$n_{\mathbf{A}}$	$n_{\mathbf{S}}$	$\mathbf{S}_0$		$\mathbf{S}_n$		$\mathbf{S}_{\text{full}}$	
			λ_2	$\lambda_{n_{\mathbf{S}}}$	λ_2	$\lambda_{n_{\mathbf{S}}}$	λ_2	$\lambda_{n_{\mathbf{S}}}$
2.08×10^{-1}	5580	324	2.15×10^{-1}	9.56×10^{-1}	3.12×10^{-1}	1.	3.4×10^{-1}	1.
1.04×10^{-1}	28116	1580	1.59×10^{-2}	7.6×10^{-1}	3.21×10^{-1}	1.	3.35×10^{-1}	1.
5.21×10^{-2}	116592	6568	1.31×10^{-3}	7.48×10^{-1}	3.21×10^{-1}	1.	3.26×10^{-1}	1.
2.6×10^{-2}	477708	26936	1.9×10^{-4}	7.42×10^{-1}	3.2×10^{-1}	1.	3.22×10^{-1}	1.

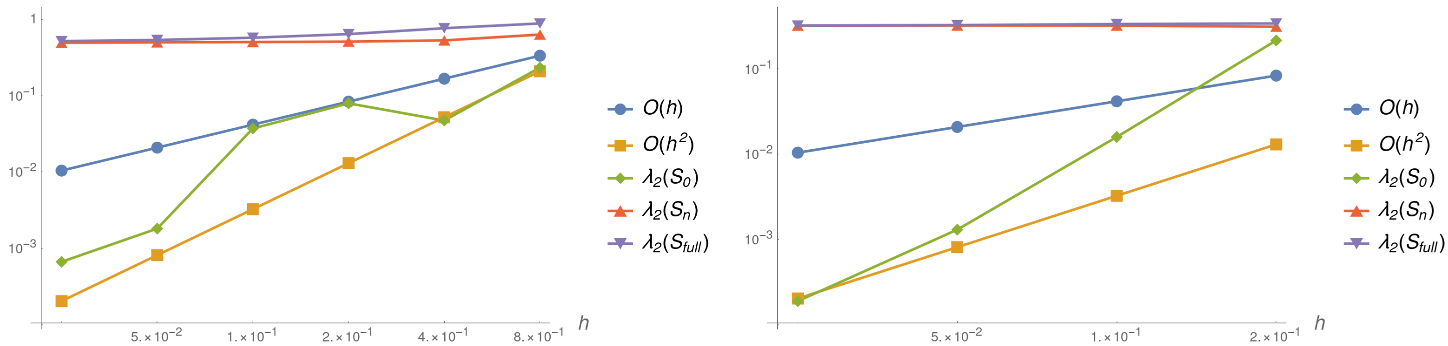
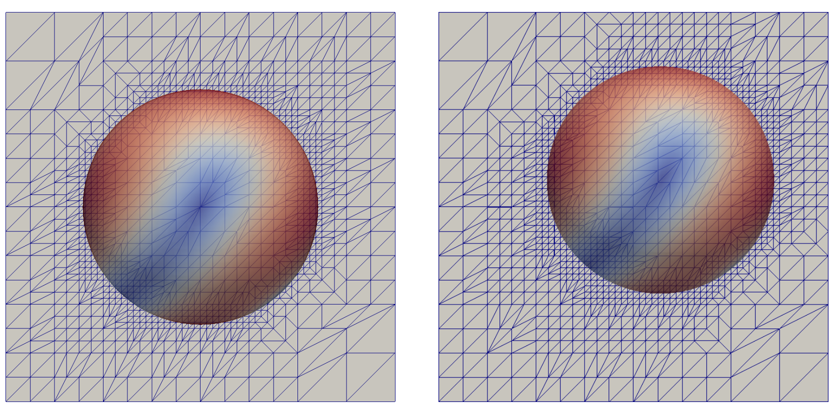


Figure 7: Log-log plot of λ_2 for Table 10. Left: $\Gamma = \Gamma_{\text{sph}}$, right: $\Gamma = \Gamma_{\text{tor}}$

3.3 Sensitivity of the spectrum to levelset shifts. In this section we investigate the sensitivity of the spectrum to levelset shifts

$$\Gamma \mapsto \Gamma + \alpha\,\mathbf{s} \tag{28}$$

for some $\alpha \in \mathbb{R}$ and $\mathbf{s} \in \mathbb{R}^3$, $\|\mathbf{s}\| = 1$. We refer to Figure 8.



(a) Γ_{sph} (b) $\Gamma_{\text{sph}} + \alpha\,\mathbf{s}$

Figure 8: $\|\mathbf{u}_h\|$ on the unit sphere (left) and the shifted unit sphere (right). Here $\mathbf{s} = (1,1,1)^T/\sqrt{3}$, $\alpha = 0.4$, and $h = 5.21 \times 10^{-2}$

Table 11: Spectrum of (25) for perturbed levelset $\Gamma + \alpha\,\mathbf{s}$ for consistent $\mathbf{P}_2{-}P_1$, $\tau = h^{-2}$, $\rho_u = h^{-1}$, $\rho_p = h$, $m \in \mathbf{m}_{1/2}$ as in Figure 5. Here $\mathbf{s} = (1,1,1)^T/\sqrt{3}$, $h = 1.04 \times 10^{-1}$

Surface	$\mathbf{S}_0$		$\mathbf{S}_n$		$\mathbf{S}_{\text{full}}$	
	λ_2	$\lambda_{n\mathbf{s}}$	λ_2	$\lambda_{n\mathbf{s}}$	λ_2	$\lambda_{n\mathbf{s}}$
$\Gamma_{\text{sph}} + 0.0\,\mathbf{s}$	3.714×10^{-2}	6.69×10^{-1}	5.03×10^{-1}	1.	5.731×10^{-1}	1.
$\Gamma_{\text{sph}} + 0.1\,\mathbf{s}$	1.313×10^{-3}	6.87×10^{-1}	5.03×10^{-1}	1.	5.733×10^{-1}	1.
$\Gamma_{\text{sph}} + 0.2\,\mathbf{s}$	1.248×10^{-3}	6.7×10^{-1}	5.03×10^{-1}	1.	5.73×10^{-1}	1.
$\Gamma_{\text{sph}} + 0.3\,\mathbf{s}$	1.036×10^{-2}	6.72×10^{-1}	5.031×10^{-1}	1.	5.73×10^{-1}	1.
$\Gamma_{\text{sph}} + 0.4\,\mathbf{s}$	5.315×10^{-4}	6.72×10^{-1}	5.031×10^{-1}	1.	5.731×10^{-1}	1.

Surface	$\mathbf{S}_0$		$\mathbf{S}_n$		$\mathbf{S}_{\text{full}}$	
	λ_2	$\lambda_{n\mathbf{s}}$	λ_2	$\lambda_{n\mathbf{s}}$	λ_2	$\lambda_{n\mathbf{s}}$
$\Gamma_{\text{tor}} + 0.00\,\mathbf{s}$	1.591×10^{-2}	7.6×10^{-1}	3.208×10^{-1}	1.	3.348×10^{-1}	1.
$\Gamma_{\text{tor}} + 0.05\,\mathbf{s}$	9.204×10^{-3}	1.14	3.207×10^{-1}	1.	3.353×10^{-1}	1.
$\Gamma_{\text{tor}} + 0.10\,\mathbf{s}$	$3. \times 10^{-3}$	1.91	3.189×10^{-1}	1.	3.349×10^{-1}	1.
$\Gamma_{\text{tor}} + 0.15\,\mathbf{s}$	8.67×10^{-3}	1.02	3.208×10^{-1}	1.	3.354×10^{-1}	1.
$\Gamma_{\text{tor}} + 0.20\,\mathbf{s}$	6.683×10^{-3}	3.04	3.208×10^{-1}	1.	3.353×10^{-1}	1.

References

[1] T. Jankuhn and A. Reusken. Trace Finite Element Methods for Surface Vector-Laplace Equations. *arXiv e-prints*, April 2019.

[2] M. Olshanskii, A. Quaini, A. Reusken, and V. Yushutin. A finite element method for the surface stokes problem. *SIAM Journal on Scientific Computing*, 40(4):A2492–A2518, 2018.