Homework 5

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Exercise 1.

I 分布表如下：

Y\X	0	1	2	3
0	$\frac{10}{220}$	$\frac{30}{220}$	$\frac{15}{220}$	$\frac{1}{220}$
1	$\frac{40}{220}$	$\frac{60}{220}$	$\frac{12}{220}$	0
2	$\frac{30}{220}$	$\frac{18}{220}$	0	0
3	$\frac{4}{220}$	0	0	0

II

$$P(X=1)=\sum_{y=0}^{3}P(X=1,Y=y)=\frac{30}{220}+\frac{60}{220}+\frac{18}{220}=\frac{112}{220}$$

Exercise 2.

$$\begin{aligned}P(a<X\le b,c<Y\le d)&=\iint_{[a,b]\times[c,d]}f(x,y)dxdy\\&=\int_a^b\int_c^d\frac{\partial^2F}{\partial x\partial y}(x,y)dydx\\&=\int_a^b\left(\frac{\partial F}{\partial x}(x,d)-\frac{\partial F}{\partial x}(x,c)\right)dx\\&=F(b,d)-F(b,c)-F(a,d)+F(a,c)\end{aligned}$$

Exercise 3.

I

$$f(x,y)=\begin{cases}c & ,\ x^2+y^2\le 1\\ 0 & ,\ x^2+y^2>1\end{cases}\tag{1}$$

又由归一化条件，

$$\iint_{\mathbb{R}^2}f(x,y)dxdy=c\pi=1$$

所以 $c=\dfrac{1}{\pi}$

II

$$f_X(x)=\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\frac{1}{\pi}dy=\begin{cases}\frac{2}{\pi}\sqrt{1-x^2} & ,\ -1\le x\le 1\\ 0 & ,\ |x|>1\end{cases}$$

$$f_Y(y)=\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\frac{1}{\pi}dx=\begin{cases}\frac{2}{\pi}\sqrt{1-y^2} & ,\ -1\le y\le 1\\ 0 & ,\ |y|>1\end{cases}$$

III $P(R\le r)=P(X^2+Y^2\le r^2)=r^2$

IV $f(r)=\dfrac{dP(R\le r)}{dr}=2r$，那么

$$E(R)=\int_0^1 r\cdot 2rdr=\frac{2}{3}$$

Exercise 4.

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f(x,y)\,dy \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right]\right) dy
\end{aligned}
$$

将指数项重新整理：

$$
\begin{aligned}
&\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2} \\
&= \frac{(x-\mu_1)^2}{\sigma_1^2} + \frac{1}{\sigma_2^2}\left[(y-\mu_2)^2 - 2\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)(y-\mu_2)\right] \\
&= \frac{(x-\mu_1)^2}{\sigma_1^2} + \frac{1}{\sigma_2^2}\left[(y-\mu_2) - \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)\right]^2 - \frac{\rho^2}{\sigma_1^2}(x-\mu_1)^2 \\
&= \frac{(x-\mu_1)^2}{\sigma_1^2}(1-\rho^2) + \frac{1}{\sigma_2^2}\left[(y-\mu_2) - \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)\right]^2
\end{aligned}
$$

代入原积分：

$$
f_X(x) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right) \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma_2^2(1-\rho^2)}\left[(y-\mu_2) - \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)\right]^2\right) dy
$$

令 $u = (y-\mu_2) - \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)$，则 $dy = du$，且当 y 从 $-\infty$ 到 ∞ 变化时，u 也从 $-\infty$ 到 ∞ 变化。所以：

$$
\begin{aligned}
f_X(x) &= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right) \int_{-\infty}^{\infty} \exp\left(-\frac{u^2}{2\sigma_2^2(1-\rho^2)}\right) du \\
&= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right) \cdot \sigma_2\sqrt{2\pi(1-\rho^2)} \\
&= \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right)
\end{aligned}
$$

同理，

$$
f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_2} \exp\left(-\frac{(y-\mu_2)^2}{2\sigma_2^2}\right)
$$

因此，二元正态分布的边际分布仍然是正态分布，且：

$$
X \sim N(\mu_1, \sigma_1^2)
$$

$$
Y \sim N(\mu_2, \sigma_2^2)
$$

Exercise 5.

$$f_{Y|X}(y|x) = \frac{\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right]\right)}{\frac{1}{\sqrt{2\pi}\sigma_1} \exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right)}$$

将指数项重新整理：

$$\begin{aligned}
&\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right] - \frac{(x-\mu_1)^2}{2\sigma_1^2} \\
&= \frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right] - \frac{(x-\mu_1)^2(1-\rho^2)}{2\sigma_1^2(1-\rho^2)} \\
&= \frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_1)^2\rho^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right] \\
&= \frac{1}{2(1-\rho^2)}\left[\frac{1}{\sigma_2^2}\left(y-\mu_2-\frac{\rho\sigma_2}{\sigma_1}(x-\mu_1)\right)^2\right]
\end{aligned}$$

代入条件密度表达式：

$$\begin{aligned}
f_{Y|X}(y|x) &= \frac{1}{\sqrt{2\pi}\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)\sigma_2^2}\left(y-\mu_2-\frac{\rho\sigma_2}{\sigma_1}(x-\mu_1)\right)^2\right) \\
&= \frac{1}{\sqrt{2\pi}\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2\sigma_2^2(1-\rho^2)}\left(y-\mu_2-\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)\right)^2\right)
\end{aligned}$$

令 $\mu_{Y|X=x} = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)$ 和 $\sigma_{Y|X}^2 = \sigma_2^2(1-\rho^2)$，则：

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}\sigma_{Y|X}} \exp\left(-\frac{(y-\mu_{Y|X=x})^2}{2\sigma_{Y|X}^2}\right)$$

这表明在给定 $X = x$ 的条件下，Y 的条件分布仍是正态分布：

$$Y|X = x \sim N\left(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1), \sigma_2^2(1-\rho^2)\right)$$

同理可得，在给定 $Y = y$ 的条件下，X 的条件分布也是正态分布：

$$X|Y = y \sim N\left(\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(y-\mu_2), \sigma_1^2(1-\rho^2)\right)$$

Exercise 6.

I $P(X \le x, Y \le y) = 2xy$，所以

$$f(x,y)=\begin{cases}2 & ，0\le x\le 1, 0\le y\le 1\\ 0 & ，其他\end{cases}$$

II

$$f_Y(y)=\int_0^{1-y} 2dx=\begin{cases}2(1-y) & ，0\le y\le 1\\ 0 & ，其他\end{cases}$$

III

$$f_{X|Y}(x|y)=\frac{f(x,y)}{f_Y(y)}=\begin{cases}\dfrac{1}{1-y} & ，0\le x\le 1-y，0<y<1\\ 0 & ，其他\end{cases}$$

Exercise 7.

若 $X_1 \sim P(\lambda_1), X_2 \sim P(\lambda_2)$，
那么

$$\begin{aligned}P(X_1+X_2=k)&=\sum_{i=0}^{k}P(X_1=i)P(X_2=k-i)\\&=\sum_{i=0}^{k}\frac{\lambda_1^i}{i!}\exp(-\lambda_1)\frac{\lambda_2^{k-i}}{(k-i)!}\exp(-\lambda_2)\\&=\exp(-\lambda_1-\lambda_2)\sum_{i=0}^{k}\frac{\lambda_1^i\lambda_2^{k-i}}{i!(k-i)!}\\&=\frac{(\lambda_1+\lambda_2)^k}{k!}\exp(-\lambda_1-\lambda_2)\end{aligned}$$

即 $X_1+X_2 \sim P(\lambda_1+\lambda_2)$.
又由条件概率公式，

$$\begin{aligned}P(X_1=k|X_1+X_2=n)&=\frac{P(X_1=k,X_1+X_2=n)}{P(X_1+X_2=n)}\\&=\frac{P(X_1=k)P(X_2=n-k)}{P(X_1+X_2=n)}\\&=\frac{\dfrac{\lambda_1^k}{k!}\exp(-\lambda_1)\dfrac{\lambda_2^{n-k}}{(n-k)!}\exp(-\lambda_2)}{\dfrac{(\lambda_1+\lambda_2)^n}{n!}\exp(-\lambda_1-\lambda_2)}\\&=\frac{n!}{k!(n-k)!}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}\end{aligned}$$

这表明 $X_1|X_1+X_2=n \sim B(n, \dfrac{\lambda_1}{\lambda_1+\lambda_2})$.

合并泊松过程的每个事件按比例分配给不同来源，这符合二项分布的形式.$(PoissonSplittingProperty)$

Exercise 8.

令 X 和 Y 分别表示甲和乙到达的时间，以分钟为单位，以下午 1 点为起点，那么 X 和 Y 是相互独立的随机变量，且均服从 $(0,60)$ 上的均匀分布.（联合分布即相乘）

所求概率为 $P\{X+10<Y\}+P\{Y+10<X\}$. 根据对称性，它等于 $2P\{X+10<Y\}$，而:

$$\begin{aligned}2P\{X+10<Y\} &= 2\iint\limits_{x+10<y} f(x,y)\,dx\,dy \\ &= 2\iint\limits_{x+10<y} f_X(x)f_Y(y)\,dx\,dy \\ &= 2\int_{10}^{60}\int_{0}^{y-10}\left(\frac{1}{60}\right)^2 dx\,dy \\ &= \frac{2}{60^2}\int_{10}^{60}(y-10)\,dy \\ &= \frac{25}{36}\end{aligned}$$

Exercise 9.

I
$$H_X(x)=\lim_{y\to+\infty}H(x,y)=F(x),\quad H_Y(y)=\lim_{x\to+\infty}H(x,y)=G(y)$$

II X,Y 相互独立表明 $H(x,y)=F(x)G(y)$，即

$$\theta[1-F(x)][1-G(y)]=0，\forall x,y.$$

又 $F(x)$ 和 $G(y)$ 可以取值在 $(0,1)$ 内，$[1-F(x)][1-G(y)]$ 不总为零，故 $\theta=0$.

III 在 F 和 G 分别取 $[0,1]$ 上的均匀分布，并分别选取 $\theta=-1$ 和 $\theta=1$ 即可构造.

Exercise 10.

$$H(X,Y)=C(F(X),G(Y))$$

那么，由 $Copula$ 函数的边际一致性，

$$H_X(x)=\lim_{y\to+\infty}C(F(x),G(y))=C(F(x),1)=F(x)$$

$$H_Y(y)=\lim_{x\to+\infty}C(F(x),G(y))=C(1,G(y))=G(y)$$

Exercise 11.

I 情形 **1**：X 和 Y 均为离散型随机变量

全概率公式：

$$P(X=x)=\sum_{y}P(X=x|Y=y)P(Y=y)$$

贝叶斯公式：

$$P(Y=y|X=x)=\frac{P(X=x|Y=y)P(Y=y)}{P(X=x)}=\frac{P(X=x|Y=y)P(Y=y)}{\sum_t P(X=x|Y=t)P(Y=t)}$$

II 情形 **2**：X 为连续型随机变量，Y 为离散型随机变量

全概率公式：

$$f_X(x)=\sum_{y}f_{X|Y}(x|y)P(Y=y)$$

贝叶斯公式：

$$P(Y=y|X=x)=\frac{f_{X|Y}(x|y)P(Y=y)}{f_X(x)}=\frac{f_{X|Y}(x|y)P(Y=y)}{\sum_t f_{X|Y}(x|t)P(Y=t)}$$

III 情形 **3**：X 为离散型随机变量，Y 为连续型随机变量

全概率公式：

$$P(X=x)=\int_{\mathbb{R}}P(X=x|Y=y)f_Y(y)dy$$

贝叶斯公式：

$$f_{Y|X}(y|x)=\frac{P(X=x|Y=y)f_Y(y)}{P(X=x)}=\frac{P(X=x|Y=y)f_Y(y)}{\int_{\mathbb{R}}P(X=x|Y=t)f_Y(t)dt}$$

IV 情形 **4**：X 和 Y 均为连续型随机变量

全概率公式：

$$f_X(x)=\int_{\mathbb{R}}f_{X|Y}(x|y)f_Y(y)dy$$

贝叶斯公式：

$$f_{Y|X}(y|x)=\frac{f_{X|Y}(x|y)f_Y(y)}{f_X(x)}=\frac{f_{X|Y}(x|y)f_Y(y)}{\int_{\mathbb{R}}f_{X|Y}(x|t)f_Y(t)dt}$$

Exercise 12.

I 由归一化条件，

$$\iint_{\mathbb{R}^2} f(x,y)dxdy = \iint_{x^2+y^2\le 1} \frac{c}{1+x^2+y^2}dxdy = \int_0^{2\pi}\int_0^1 \frac{c}{1+r^2}rdrd\theta = \int_0^{2\pi}\frac{c}{2}\ln 2d\theta = \pi c\ln 2 = 1$$

所以 $c = \frac{1}{\pi \ln 2}$

II $x^2+y^2 \le 1$

$$f_X(x) = \int_{\mathbb{R}} f(x,y)dy = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi\ln 2(1+x^2+y^2)}dy = \frac{2}{\pi\ln 2}\frac{1}{\sqrt{1+x^2}}\arctan\sqrt{\frac{1-x^2}{1+x^2}}$$

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)dx = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi\ln 2(1+x^2+y^2)}dx = \frac{2}{\pi\ln 2}\frac{1}{\sqrt{1+y^2}}\arctan\sqrt{\frac{1-y^2}{1+y^2}}$$

显然

$$f_X(x)f_Y(y) = \frac{4}{\pi^2\ln^2 2}\frac{1}{\sqrt{(1+x^2)(1+y^2)}}\arctan\sqrt{\frac{1-x^2}{1+x^2}}\arctan\sqrt{\frac{1-y^2}{1+y^2}} \neq f(x,y)$$

Exercise 13.

I 首先，$f(x,y) = f_X(x)f_Y(y) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)\cdot\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right) = \frac{1}{2\pi}\exp\left(-\frac{x^2+y^2}{2}\right)$

那么

$$\begin{aligned}
\iint_{\mathbb{R}^2} g(x,y)dxdy &= \iint_{x^2+y^2\le 1}\frac{xy}{100}dxdy + \iint_{\mathbb{R}^2} f(x,y)dxdy \\
&= \iint_{\mathbb{R}^2}\frac{1}{2\pi}\exp\left(-\frac{x^2+y^2}{2}\right)dxdy \\
&= \int_0^{2\pi}\int_0^1 \frac{1}{2\pi}r\exp\left(-\frac{r^2}{2}\right)drd\theta \\
&= \int_0^{+\infty}\exp\left(-\frac{r^2}{2}\right)d\left(\frac{r^2}{2}\right) \\
&= 1
\end{aligned}$$

II

$$f_U(x) = g_X(x) = \int_{\mathbb{R}} g(x,y)dy = \int_{\mathbb{R}} f(x,y)dy + \int_{x^2+y^2\le 1}\frac{xy}{100}dy = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)$$

故 $U \sim N(0,1)$, 同理 $V \sim N(0,1)$

显然 $g(x,y)$ 的形式不符合二重正态分布.

Exercise 14.

```python
import numpy as np
import matplotlib.pyplot as plt

y = np.random.uniform(0, 1, 10000)
x = -np.log(1 - y)

plt.hist(x, bins=50, density=True, alpha=0.6, color='pink')

lam = 1
xmin, xmax = 0, 8
x_pdf = np.linspace(xmin, xmax, 100)
p = lam * np.exp(-lam * x_pdf)
plt.plot(x_pdf, p, 'k', linewidth=2)

plt.xlim(xmin, xmax)
plt.ylim(0, 1)

plt.xlabel('x')
plt.ylabel('Density')
plt.title('Histogram of x and Exponential Distribution PDF')
plt.show()
```