T_1

① 掷10次硬币，正面向上的次数 X_1，

离散型 $\Omega=\{0,1\}^{10}$

② pku中抽取一名同学，其bmi X_2.

连续型 $\Omega=\{$所有pkuer$\}$.

③ pku中抽取100名同学，支持早八取消的人数 X_3.

离散型 $\Omega=\{100$名pkuer意见所有可能$\}$ $(\{0,1\}^{100})$

④ 随机抽取某天某站的公交车到站情况 X_4

离散型 $\Omega=\{$所有车的到站情况$\}$.

⑤ 随机抽取某天网络学堂的响应时间 X_5

连续型 $\Omega=\{$所有响应时间$\}$.

T_2. proof:

<1> $\forall\ y > x \quad F(y) - F(x) = P(x < X \leq y) \geq 0$

可 $F(x)$ 单增.

又 $F(x)$ 有界

那么由单调收敛定理知

$\lim\limits_{x\to+\infty} F(x)$, $\lim\limits_{x\to-\infty} F(x)$ 存在.

又 $P(\Omega) = 1 \quad P(\phi) = 0$

$\Rightarrow \lim\limits_{x\to-\infty} F(x) = 0 \qquad \lim\limits_{x\to+\infty} F(x) = 1$.

<2> 取 $\{x_n\}$ 单调递减且收敛于 x.

此时有 $\{X \leq x_n\}$ 为递减事件序列

$(F(x_n) \downarrow)$

$\Rightarrow \lim\limits_{n\to\infty} F(x_n) = F(x)$

(<1> 取 $\{x_n\}$ 单增, $\bigcup\limits_{n=1}^{\infty} [X \leq x_n] = X$ 也可证).

<3> $P(a \leq X \leq b)$

$= P(a < X \leq b) + P(X = a)$

$= F(b) - F(a) + F(a) - \lim\limits_{x\to a^-} F(x)$

$= F(b) - \lim\limits_{x\to a^-} F(x)$

T3

<1>

$$P(\omega_i) = \frac{1}{3} \quad (i = 1, 2, 3)$$

显然, $P(X \in I) = P(Y \in I)$ 12.

<2> $(X+Y)(\omega_1) = 3$

$(X+Y)(\omega_2) = 5$

$(X+Y)(\omega_3) = 4$

$\Rightarrow P(X+Y=3) = P(X+Y=5) = P(X+Y=4) = \frac{1}{3}$.

$(Y-X)(\omega_1) = 1$

$(Y-X)(\omega_2) = 1$

$(Y-X)(\omega_3) = -2$.

$\Rightarrow P(Y-X=1) = \frac{2}{3} \qquad P(Y-X=-2) = \frac{1}{3}$.

T4 Proof:

$$Var(X) = E[(X - E(X))^2].$$

$$= \sum (x_i - E(X))^2 f(x_i)$$

$$= \sum x_i^2 f(x_i) - 2E(X) \sum x_i f(x_i) + E^2(X) \sum f(x_i)$$

$$= E(X^2) - E^2(X)$$

一致.

T_5

<1> $X \in \{1, 2, \cdots, a+1\}$.

$$P(X=k) = \frac{\binom{a}{k-1}\binom{b}{1}}{\binom{a+b}{k}}$$

<2> $X \in \mathbb{N}^*$

$$P(X=k) = \left(\frac{a}{a+b}\right)^{k-1} \cdot \frac{b}{a+b}$$

$$\overset{0<p=\frac{a}{a+b}<1}{=\!=} p^{k-1}(1-p)$$

$$E(X) = \sum_{k=1}^{\infty} k p^{k-1}(1-p)$$

$$\overset{\text{Abel}}{=\!=} \lim_{n\to\infty} (1-p)\left[n \cdot \frac{1-p^n}{1-p} \cdot 1 - \sum_{i=1}^{n-1} \frac{1-p^i}{1-p} \cdot 1\right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{1-p} - np^n - \frac{p^n}{1-p}\right]$$

$$= \frac{1}{1-p}$$

$$= \frac{a+b}{b}$$

T_6

$P(X=0) = P(Y=1) = 0.999$

$P(X=114514) = P(Y=0) = 0.001$

$\Rightarrow P(Y>X) = 0.998001 > 0.99$

而 $E(X) = 114.514 > 100E(Y) = 99.9$

T_7

<1> $X \in \mathbb{N}^*$

$$P(X=k) = (1-p)^{k-1}p$$

<2>

$$E(X) = \sum_{k=1}^{\infty} kp(1-p)^{k-1}$$

$$= \frac{1}{p} \quad (\text{用 } T_5 \text{ <2>})$$

$$E(X^2) = \sum_{k=1}^{\infty} k^2 p(1-p)^{k-1}$$

实际上，我们有：

$$\sum_{i=1}^{\infty} i^n x^i = \frac{1}{(1-x)^{n+1}} \sum_{k=0}^{n} A(n,k)x^{k+1} = \frac{x}{(1-x)^{n+1}} A_n(x)$$

其中 $A(n,m) = \sum_{k=0}^{m} \binom{n+1}{k}(m+1-k)^n(-1)^k$ 为欧拉数。

$A_n(e) = \sum_{k=0}^{n} A(n,k)e^k$ 为欧拉多项式。

Proof: 不难注意到

$$\sum_{i=1}^{\infty} i^n x^i = \sum_{i=1}^{\infty} \sum_{j=0}^{n} {n \brace j} (i)_{\underline{j}}\, x^i$$

$$= \sum_{j=0}^{n} {n \brace j} \sum_{i=1}^{\infty} (i)_{\underline{j}}\, x^i$$

$$= \frac{1}{x} \sum_{j=0}^{n} {n \brace j} j! \left(\frac{x}{1-x}\right)^{j+1}$$

$$= \left(\frac{x}{1-x}\right)^{n+1} \cdot \frac{1}{x} \sum_{j=0}^{n} {n \brace j} j! \left(\frac{1}{x}-1\right)^{n-j}$$

$$= \left(\frac{x}{1-x}\right)^{n+1} \cdot \frac{1}{x} A_n\left(\frac{1}{x}\right)$$

$$= \frac{x}{(1-x)^{n+1}} A_n(x) \quad \square$$

代入即知

$$E(X^2)=\frac{2-p}{p^2}$$

$$\Rightarrow Var(X)=E(X^2)-E^2(X)$$

$$=\frac{1-p}{p^2}$$

T8

由中心极限定理. $(X\sim B(25,0.6))$

$$P(X\geqslant 15)=P\left\{\frac{X-25\times0.6}{\sqrt{25\times0.6\times0.4}}\geqslant\frac{15-25\times0.6}{\sqrt{25\times0.6\times0.4}}\right\}$$

$$\approx 1-\Phi(0)\approx 0.5000\ (\approx 0.5858)$$

$$P(X>20)=P(X\geqslant 20.5)$$

$$=P\left\{\frac{X-25\times0.6}{\sqrt{25\times0.6\times0.4}}\geqslant\frac{20.5-25\times0.6}{\sqrt{25\times0.6\times0.4}}\right\}$$

$$\approx 1-\Phi(2.25)$$

$$\approx 0.0122\ (\approx 0.0095)$$

$$P(X<10)=P(X\leqslant 9.5)$$

$$=\cdots\approx 0.0122\ (\approx 0.0132)$$

(括号中为计算机模拟结果，显然n还不够多大导致误差较大)

T9

对于 $X \sim B(n,p)$

直接考虑

$$E(X^k) = \sum_{i=0}^{n} i^k \binom{n}{i} p^i (1-p)^{n-i}$$

$$= \sum_{i=1}^{n} i^k \binom{n}{i} p^i (1-p)^{n-i}$$

$$= np \sum_{i=1}^{n} i^{k-1} \binom{n-1}{i-1} p^{i-1} (1-p)^{n-i}$$

$$= np \sum_{j=0}^{n-1} (j+1)^{k-1} \binom{n-1}{j} p^j (1-p)^{n-1-j}$$

$$= np\, E[(Y+1)^{k-1}]$$

其中 $Y \sim B(n-1, p)$

那么 令 $k=1$ 即有 $E(X) = np$

令 $k=2$ 即有 $E(X^2) = np\, E(Y+1)$

$$= np[(n-1)p+1]$$

$$\Rightarrow \mathrm{Var}(X) = E(X^2) - E^2(X)$$

$$= np(1-p)$$

T_{10}

<1> $$P(X=m) = \frac{\binom{M}{m}\binom{N-M}{n-m}}{\binom{N}{n}}$$

<2> $$\frac{\hat{N}}{M} = \frac{n}{m} \Rightarrow \hat{N} = \frac{nM}{m}. \quad \left(\hat{N} = \left[\frac{nM}{m}\right]\right)$$

<3> 记 $P(X=m) = P_m(N)$

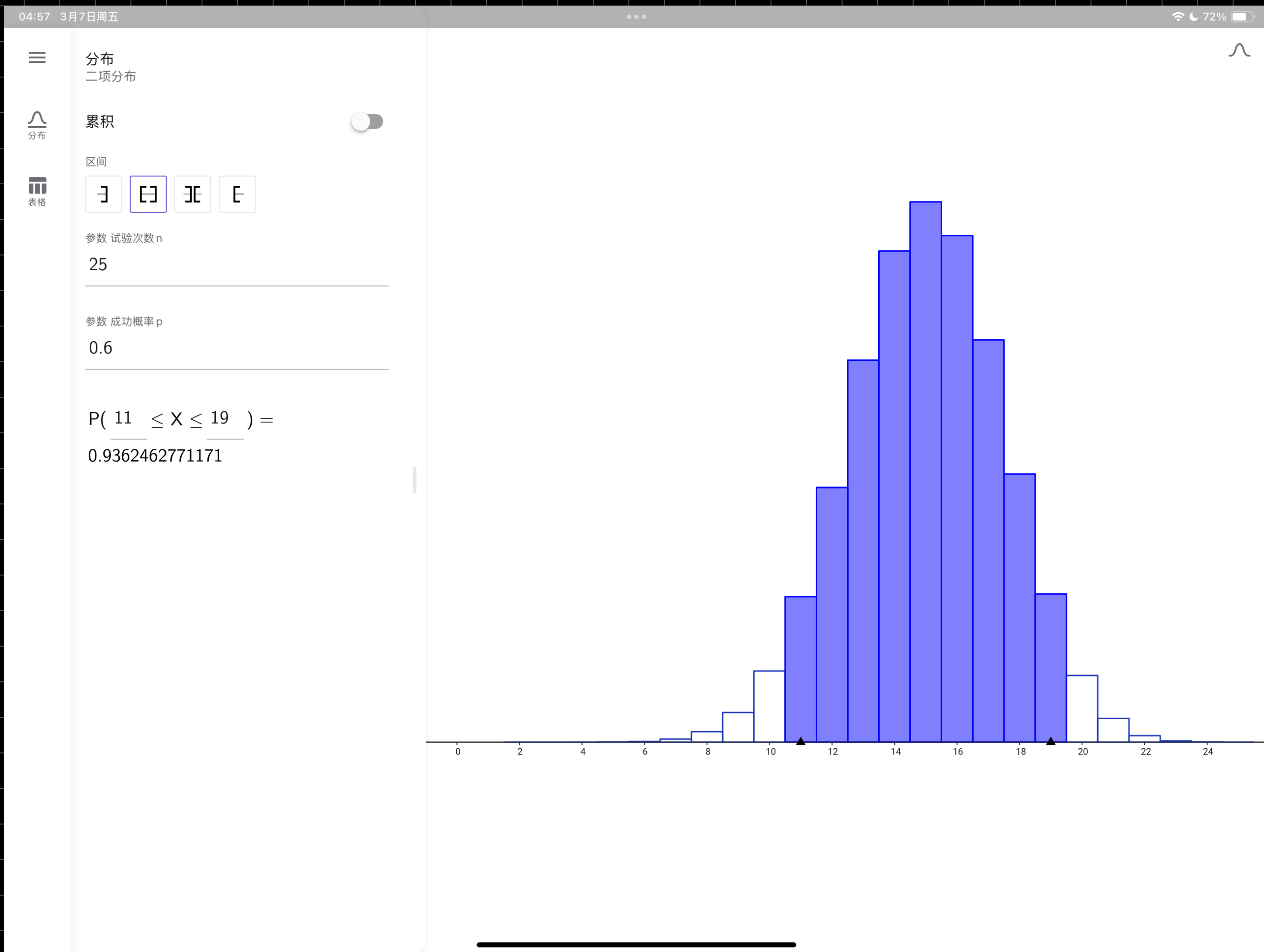
则 $$\frac{P_m(N)}{P_m(N-1)} = \frac{(N-M)(N-n)}{N(N-M-n+m)} \geq 1$$

$$\Rightarrow \quad N \leq \frac{nM}{m}.$$

即 $\hat{N} = \left[\frac{nM}{m}\right]$ 与<2>一致.

<4> 极大似然估计即取最可能使 n 中带 m 个标记的 N.

T_{11} 使用 CAS 计算器辅助分析.



<1> $x = 15$ 时.

<2> $\mu = X$

<3> $\sigma^2 = 5.978$

<4> $P(\mu - 2\sigma < X \leq \mu + 2\sigma) = P(11 \leq X \leq 19)$

$= 0.9362$.