① <1> $S(\beta_0, \beta_1) = \sum [y_i - (\beta_1 x_i + \beta_0)]^2$

$\frac{\partial S}{\partial \beta_0} = \frac{\partial S}{\partial \beta_1} = 0 \Rightarrow \hat{\beta}_1 = \frac{S_{xy}}{S_{xx}}$ $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$ 过 $(\bar{x}, \bar{y})$

<2> $Cov(\hat{\beta}_0, \hat{\beta}_1) = cov(\bar{y} - \hat{\beta}_1 \bar{x}, \hat{\beta}_1)$

$= -\bar{x}\, Var(\hat{\beta}_1) \rightarrow = \sum \left(\frac{x_i - \bar{x}}{S_{xx}}\right)^2 Var(y_i)$

$= -\frac{\bar{x}\sigma^2}{S_{xx}}$ $= \frac{\sigma^2}{S_{xx}}$

$\bar{x} = 0$ 不相关.

<3> $Var(\hat{\beta}_1) = \frac{\sigma^2}{S_{xx}}$ 最小 $\Rightarrow$ s^2 最大

可取 $\lfloor \frac{n}{2} \rfloor$ 个 -1 $\lceil \frac{n}{2} \rceil$ 个 1.

<4> $\beta_0 = 0 \Rightarrow \beta_1 = \frac{\sum x_i y_i}{\sum x_i^2}$

2. <1> $\varepsilon_i \sim N(0, \sigma^2) \Rightarrow y_i \sim N(\beta_0 + \beta_1 x_i, \sigma^2)$

$\Rightarrow L(\beta_0, \beta_1, \sigma^2) = \prod \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{[y_i - (\beta_0 + \beta_1 x_i)]^2}{2\sigma^2}\right)$

$\Rightarrow l(\beta_0, \beta_1, \sigma^2) = -\frac{n}{2}\ln 2\pi - \frac{n}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum (y_i - \beta_0 - \beta_1 x_i)^2.$

$\frac{\partial l}{\partial \sigma^2} = 0 \Rightarrow \hat{\sigma}^2 = \frac{\sum (y_i - \beta_0 - \beta_1 x_i)^2}{n} = \frac{RSS}{n}.$

(2) $RSS = \sum (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2 = \sum (y_i - (\bar{y} - \beta_1 \bar{x}) - \hat{\beta}_1 x_i)^2$

$$= \sum [(y_i - \bar{y}) - \hat{\beta}_1 (x_i - \bar{x})]^2$$

$$= S_{yy} - 2\hat{\beta}_1 S_{xy} + \hat{\beta}_1 S_{xx}$$

$$= S_{yy} - \frac{S_{xy}^2}{S_{xx}}$$

$\Rightarrow E(RSS) = E(S_{yy}) - \frac{E(S_{xy}^2)}{S_{xx}} = \beta_1^2 S_{xx} + (n-1)\sigma^2 - \frac{\beta_1^2 S_{xx}^2 + \sigma^2 S_{xx}}{S_{xx}}$

$$= (n-2)\sigma^2$$

可 $\hat{\sigma}^2 = \frac{RSS}{n-2}$

(3) $\beta_0 = 0$ 时 $E(RSS) = (n-1)\sigma^2$ 可 $\hat{\sigma}^2 = \frac{RSS}{n-1}$

又 $x = x_0$ 时 $y_0 = \beta_1 x_0 + \varepsilon_0$ $\hat{y}_0 = \hat{\beta}_1 x_0$

$$\Rightarrow y_0 \sim N(\beta_1 x_0, \sigma^2)$$

$$\hat{y}_0 \sim N(\beta_1 x_0, \frac{\sigma^2}{S_{xx}} \cdot x_0^2)$$

$$\Rightarrow y_0 - \hat{y}_0 \sim N(0, (1+\frac{x_0^2}{S_{xx}})\sigma^2)$$

又 $\frac{(n-1)\hat{\sigma}^2}{\sigma^2} \sim \chi^2(n-1)$ $\Rightarrow \frac{y_0 - \hat{y}_0}{\hat{\sigma}\sqrt{1+\frac{x_0^2}{S_{xx}}}} \sim t(n-1)$

$\Rightarrow$ $\hat{\beta}_1 x_0 \pm t_{\frac{\alpha}{2}}(n-1) \cdot \hat{\sigma}\sqrt{1+\frac{x_0^2}{S_{xx}}}$