第三次习题课参考解答 隐函数微分、多元函数微分学几何应用

1. 计算下列各题:

(1) 已知函数 $z=z(x,y)$ 由方程 $x^2+y^2+z^2=a^2$ 确定，求 $\dfrac{\partial^2 z}{\partial x\partial y}$.

解: 方程 $x^2+y^2+z^2=a^2$ 两边分别对 $x,\ y$ 求偏导，得 $2x+2z\dfrac{\partial z}{\partial x}=0$，$2y+2z\dfrac{\partial z}{\partial y}=0$，

故 $\dfrac{\partial z}{\partial x}=-\dfrac{x}{z},\quad \dfrac{\partial z}{\partial y}=-\dfrac{y}{z}$，这样 $\dfrac{\partial^2 z}{\partial x\partial y}=\dfrac{y}{z^2}\cdot\dfrac{\partial z}{\partial x}=-\dfrac{xy}{z^3}$.

(2) 设函数 $z=z(x,y)$ 由方程 $(z+y)^x=x^2y$ 确定，求 $\left.\dfrac{\partial z}{\partial y}\right|_{(3,3)}$.

解：将 $x=3,\ y=3$ 带入方程 $(z+y)^x=x^2y$，解得 $z=0$.

方程 $(z+y)^x=x^2y$ 两端关于 y 求偏导，得 $x(z+y)^{x-1}(\dfrac{\partial z}{\partial y}+1)=x^2$，

将 $x=3,\ y=3,\ z=0$ 带入上式，得 $\left.\dfrac{\partial z}{\partial y}\right|_{(3,3)}=-\dfrac{2}{3}$.

(3) 设函数 $z=z(x,y)$ 由方程 $xyz+\sqrt{x^2+y^2+z^2}=\sqrt{2}$ 确定，且 $z(1,0)=-1$，求 $\left.dz\right|_{(1,0)}$.

解：方程 $xyz+\sqrt{x^2+y^2+z^2}=\sqrt{2}$ 两边微分，则

$$yzdx+xzdy+xydz+\frac{xdx}{\sqrt{x^2+y^2+z^2}}+\frac{ydy}{\sqrt{x^2+y^2+z^2}}+\frac{zdz}{\sqrt{x^2+y^2+z^2}}=0,$$

将 $(x,y,z)=(1,0,-1)$ 带入上式，有 $\left.dz\right|_{(1,0)}=dx-\sqrt{2}dy.$

2. 设函数 $x=x(z),\ y=y(z)$ 由方程组 $\begin{cases}x^2+y^2+z^2-1=0\\ x^2+2y^2-z^2-1=0\end{cases}$ 确定，求 $\dfrac{dx}{dz},\ \dfrac{dy}{dz}$.

解: 令 $F(x,y,z)=x^2+y^2+z^2-1$，$G(x,y,z)=x^2+2y^2-z^2-1$，则当 $xy\neq 0$ 时，

$\dfrac{\partial(F,G)}{\partial(x,y)}=\begin{pmatrix}2x & 2y\\ 2x & 4y\end{pmatrix}$ 可逆，故方程组 $\begin{cases}x^2+y^2+z^2-1=0\\ x^2+2y^2-z^2-1=0\end{cases}$ 确定了隐函数组

$x=x(z),\ y=y(z)$，且

$$\begin{bmatrix}\dfrac{dx}{dz}\\ \dfrac{dy}{dz}\end{bmatrix}=-\left(\frac{\partial(F,G)}{\partial(x,y)}\right)^{-1}\begin{pmatrix}\dfrac{\partial F}{\partial z}\\ \dfrac{\partial G}{\partial z}\end{pmatrix}=-\frac{1}{4xy}\begin{bmatrix}4y & -2y\\ -2x & 2x\end{bmatrix}\begin{bmatrix}2z\\ -2z\end{bmatrix}=-\frac{1}{4xy}\begin{bmatrix}12yz\\ -8xz\end{bmatrix},$$

所以 $\dfrac{dx}{dz}=-\dfrac{3z}{x},\ \dfrac{dy}{dz}=\dfrac{2z}{y}$.

3. 已知函数 $z=z(x,y)$ 由参数方程 $\begin{cases} x=u\cos v \\ y=u\sin v \\ z=uv \end{cases}$ 给定，试求 $\dfrac{\partial z}{\partial x}$，$\dfrac{\partial z}{\partial y}$.

解: 这个问题涉及到复合函数微分法与隐函数微分法. 因变量 z 以 u,v 为中间变量，u,v 又分别是由方程组 $\begin{cases} x=u\cos v \\ y=u\sin v \end{cases}$ 确定的 x,y 的隐函数，这样 z 是 x,y 的二元复合函数。故由复合函数的链式法则，$z=uv$ 两端分别对 x,y 求偏导，得到

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=v\frac{\partial u}{\partial x}+u\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=v\frac{\partial u}{\partial y}+u\frac{\partial v}{\partial y}$$

由于 u,v 是由方程组 $\begin{cases} x=u\cos v \\ y=u\sin v \end{cases}$ 确定的 x,y 的隐函数，在这两个等式两端分别关于 x,y 求偏导数，得 $\begin{cases} 1=\cos v\dfrac{\partial u}{\partial x}-u\sin v\dfrac{\partial v}{\partial x} \\ 0=\sin v\dfrac{\partial u}{\partial x}+u\cos v\dfrac{\partial v}{\partial x} \end{cases}$ $\qquad\begin{cases} 0=\cos v\dfrac{\partial u}{\partial y}-u\sin v\dfrac{\partial v}{\partial y} \\ 1=\sin v\dfrac{\partial u}{\partial y}+u\cos v\dfrac{\partial v}{\partial y} \end{cases}$

故 $\dfrac{\partial u}{\partial x}=\cos v$，$\dfrac{\partial v}{\partial x}=\dfrac{-\sin v}{u}$，$\dfrac{\partial u}{\partial y}=\sin v$，$\dfrac{\partial v}{\partial y}=\dfrac{\cos v}{u}$.

将这个结果代入前面的式子，得到

$$\frac{\partial z}{\partial x}=v\frac{\partial u}{\partial x}+u\frac{\partial v}{\partial x}=v\cos v-\sin v,$$

$$\frac{\partial z}{\partial y}=v\frac{\partial u}{\partial y}+u\frac{\partial v}{\partial y}=v\sin v+\cos v.$$

4. 设 f，g，$h\in C^1$. 若矩阵 $\dfrac{\partial(g,h)}{\partial(z,t)}$ 可逆，且函数 $u=u(x,y)$ 由方程组 $\begin{cases} u=f(x,y,z,t) \\ g(y,z,t)=0 \\ h(z,t)=0 \end{cases}$ 确定，求 $\dfrac{\partial u}{\partial x},\dfrac{\partial u}{\partial y}$.

解: 解法一、令 $F(x,y,z,t,u)=f(x,y,z,t)-u$. 因为矩阵 $\dfrac{\partial(g,h)}{\partial(z,t)}$ 可逆，因此

$\dfrac{\partial(F,g,h)}{\partial(z,t,u)}=\begin{pmatrix} f'_z & f'_t & -1 \\ g'_z & g'_t & 0 \\ h'_z & h'_t & 0 \end{pmatrix}$ 可逆，从而方程组 $\begin{cases} u=f(x,y,z,t) \\ g(y,z,t)=0 \\ h(z,t)=0 \end{cases}$ 确定了隐函数组

$\begin{cases} z = z(x,y), \\ t = t(x,y), \\ u = u(x,y). \end{cases}$ 故 $\dfrac{\partial(z,t,u)}{\partial(x,y)} = -\left(\dfrac{\partial(F,g,h)}{\partial(z,t,u)}\right)^{-1}\dfrac{\partial(F,g,h)}{\partial(x,y)}$. 其中

$$\left(\frac{\partial(F,g,h)}{\partial(z,t,u)}\right)^{-1} = \frac{1}{g_z'h_t' - g_t'h_z'}\begin{pmatrix} 0 & h_t' & g_t' \\ 0 & h_z' & -g_z' \\ g_z'h_t' - g_t'h' & -(f_z'h_t' - f_t'h_z') & f_z'g_t' - f_t'g_z' \end{pmatrix}$$

且 $\dfrac{\partial(F,g,h)}{\partial(x,y)} = \begin{pmatrix} f_x' & f_y' \\ 0 & g_y' \\ 0 & 0 \end{pmatrix}$. 所以 $\dfrac{\partial u}{\partial y} = \dfrac{\partial f}{\partial y} + \dfrac{\left(\dfrac{\partial f}{\partial t}\dfrac{\partial h}{\partial z} - \dfrac{\partial f}{\partial z}\dfrac{\partial h}{\partial t}\right)\dfrac{\partial g}{\partial y}}{\dfrac{\partial g}{\partial z}\dfrac{\partial h}{\partial t} - \dfrac{\partial g}{\partial t}\dfrac{\partial h}{\partial z}}$.

解法二、 因为矩阵 $\dfrac{\partial(g,h)}{\partial(z,t)}$ 可逆，因此方程组 $\begin{cases} g(y,z,t) = 0 \\ h(z,t) = 0 \end{cases}$ 确定了隐函数组 $\begin{cases} z = z(y), \\ t = t(y). \end{cases}$ 且

$$\begin{pmatrix} \dfrac{dz}{dy} \\ \dfrac{dt}{dy} \end{pmatrix} = -\left(\det\frac{\partial(g,h)}{\partial(z,t)}\right)^{-1}\begin{pmatrix} \dfrac{\partial h}{\partial t} & -\dfrac{\partial g}{\partial t} \\ -\dfrac{\partial h}{\partial z} & \dfrac{\partial g}{\partial z} \end{pmatrix}\begin{pmatrix} \dfrac{\partial g}{\partial y} \\ 0 \end{pmatrix}.$$

对复合函数 $u = f(x,y,z(y),t(y))$ 分别关于 x，y 求偏导，得

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x}, \qquad \frac{\partial u}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{dz}{dy} + \frac{\partial f}{\partial t}\frac{dt}{dy}.$$

故 $\dfrac{\partial u}{\partial y} = \dfrac{\partial f}{\partial y} + \dfrac{\left(\dfrac{\partial f}{\partial t}\dfrac{\partial h}{\partial z} - \dfrac{\partial f}{\partial z}\dfrac{\partial h}{\partial t}\right)\dfrac{\partial g}{\partial y}}{\dfrac{\partial g}{\partial z}\dfrac{\partial h}{\partial t} - \dfrac{\partial g}{\partial t}\dfrac{\partial h}{\partial z}}$.

5. 已知所有二阶实方阵 $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ 构成一个 4 维线性空间 V，定义向量值函数 $\mathbf{f}: V \to V$ 为 $\mathbf{f}(X) = X^2$，求 $\mathbf{f}(X)$ 在 $X_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ 处的全微分。

解：由于 $\mathbf{f}(X) = \begin{pmatrix} x_{11}^2 + x_{12}x_{21} & x_{11}x_{12} + x_{12}x_{22} \\ x_{11}x_{21} + x_{21}x_{22} & x_{22}^2 + x_{12}x_{21} \end{pmatrix}$，因此

$$\frac{\partial \mathbf{f}}{\partial x_{11}} = \begin{pmatrix} 2x_{11} & x_{12} \\ x_{21} & 0 \end{pmatrix}, \quad \frac{\partial \mathbf{f}}{\partial x_{12}} = \begin{pmatrix} x_{21} & x_{11} + x_{22} \\ 0 & x_{21} \end{pmatrix},$$

$$\frac{\partial \mathbf{f}}{\partial x_{21}}=\begin{pmatrix} x_{12} & 0 \\ x_{11}+x_{22} & x_{12} \end{pmatrix},\quad \frac{\partial \mathbf{f}}{\partial x_{22}}=\begin{pmatrix} 0 & x_{12} \\ x_{21} & 2x_{22} \end{pmatrix},$$

故

$$d\mathbf{f}(X_0)=\frac{\partial \mathbf{f}}{\partial x_{11}}(X_0)dx_{11}+\frac{\partial \mathbf{f}}{\partial x_{12}}(X_0)dx_{12}+\frac{\partial \mathbf{f}}{\partial x_{21}}(X_0)dx_{21}+\frac{\partial \mathbf{f}}{\partial x_{22}}(X_0)dx_{22}$$

$$=\begin{pmatrix} 2dx_{11} & dx_{12} \\ dx_{21} & 0 \end{pmatrix}.$$

6. 求解下列各题:

(1) 求螺线 $\begin{cases} x=a\cos t \\ y=a\sin t \\ z=ct \end{cases}$ $(a>0,c>0)$ 在点 $M(\frac{a}{\sqrt{2}},\frac{a}{\sqrt{2}},\frac{\pi c}{4})$ 处的切线与法平面.

解: 由于点 M 对应的参数为 $t_0=\frac{\pi}{4}$, 所以螺线在 M 处的切向量是

$$\vec{v}=(x'(\frac{\pi}{4}),y'(\frac{\pi}{4}),z'(\frac{\pi}{4}))=(-\frac{a}{\sqrt{2}},\frac{a}{\sqrt{2}},c),$$

因而所求切线的参数方程为 $\begin{cases} x=\frac{a}{\sqrt{2}}-\frac{a}{\sqrt{2}}t, \\ y=\frac{a}{\sqrt{2}}+\frac{a}{\sqrt{2}}t, \\ z=\frac{\pi}{4}c+ct, \end{cases}$

法平面方程为 $-\frac{a}{\sqrt{2}}(x-\frac{a}{\sqrt{2}})+\frac{a}{\sqrt{2}}(y-\frac{a}{\sqrt{2}})+c(z-\frac{\pi c}{4})=0$.

(2) 求曲线 $\begin{cases} x^2+y^2+z^2-6=0 \\ z-x^2-y^2=0 \end{cases}$ 在点 $M_0(1,1,2)$ 处的切线方程.

解: 令 $F(x,y,z)=x^2+y^2+z^2-6$, $G(x,y,z)=z-x^2-y^2$, 则

$$\operatorname{grad}F(M_0)=(2,2,4),\qquad \operatorname{grad}G(M_0)=(-2,-2,1)$$

所以曲线在 $M_0(1,1,2)$ 处的切向量为 $\vec{v}=\operatorname{grad}F(M_0)\times\operatorname{grad}G(M_0)=(10,-10,\ 0)$,

于是所求的切线方程为 $\begin{cases} x=1+10t \\ y=1-10t \\ z=2. \end{cases}$

7. 求曲面 $S:2x^2-2y^2+2z=1$ 上切平面与直线 $L:\begin{cases} 3x-2y-z=5 \\ x+y+z=0 \end{cases}$ 平行的切点的轨迹。

解：直线L的方向方向：$\vec{\tau}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 3 & -2 & -1\\ 1 & 1 & 1\end{vmatrix}=-\vec{i}-4\,\vec{j}+5\,\vec{k}$.

切点$P(x,y,z)$处曲面S的法向量：$\vec{n}=4x\,\vec{i}-4y\,\vec{j}+2\,\vec{k}$.

因为$\vec{n}\perp\vec{\tau}\Leftrightarrow \vec{n}\cdot\vec{\tau}=-4x+16y+10=0$，且切点在曲面上，

因此切点的轨迹为空间曲线：$\begin{cases}2x-8y=5\\ 2x^2-2y^2+2z=1,\end{cases}$

该曲线的参数方程：$\begin{cases}x=x\\ y=(2x-5)/8\\ z=(-60x^2-60x+57)/64.\end{cases}$

8．证明球面$S_1: x^2+y^2+z^2=R^2$与锥面$S_2: x^2+y^2=a^2z^2$正交.

证明：所谓两曲面正交是指它们在交点处的法向量互相垂直. 记

$$F(x,y,z)=x^2+y^2+z^2-R^2,\quad G(x,y,z)=x^2+y^2-a^2z^2,$$

设点$M(x,y,z)$是两曲面的公共点。曲面S_1在点$M(x,y,z)$处的法向量是$\vec{v}_1=(x,y,z)^T$，曲面S_2在点$M(x,y,z)$处的法向量为$\vec{v}_2=(x,\ y,\ -a^2z)^T$. 则在点$M(x,y,z)$处有

$$\vec{v}_1\cdot\vec{v}_2=(x,y,z)^T\cdot(x,y,-a^2z)^T=x^2+y^2-a^2z^2=0,$$

即在公共点处两曲面的法向量相互垂直，因此两曲面正交. 证毕

9．已知曲面S的方程$e^z=xy+yz+zx$，求曲面S在$(1,1,0)$处的切平面方程；若曲面S的显式方程为$z=f(x,y)$，求$\operatorname{grad} f(1,1)$.

解：令$F(x,y,z)=e^z-xy-yz-zx$. 则

$$F_x'(1,1,0)=-1,\ \ F_y'(1,1,0)=-1,\ \ F_z'(1,1,0)=-1.$$

所以曲面S在$(1,1,0)$处的法向量为$(-1,-1,-1)$或$(1,1,1)$. 从而曲面S在$(1,1,0)$处的切平面方程$(x-1)+(y-1)+z=0$，即$x+y+z=2$. 因为

$$f_x'(1,1)=-\frac{F_x'(1,1,0)}{F_z'(1,1,0)}=-1,\quad f_y'(1,1)=-\frac{F_y'(1,1,0)}{F_z'(1,1,0)}=-1,$$

所以$\operatorname{grad} f(1,1)=(f_x'(1,1),f_y'(1,1))=(-1,-1).$

10. 求证：通过曲面$S: \mathrm{e}^{xyz}+x-y+z=3$上点$(1,0,1)$的切平面平行于$y$轴.

证明：令$F(x,y,z)=\mathrm{e}^{xyz}+x-y+z-3$. 则曲面$S$在其上的点$(1,0,1)$处的法向量为

$$\left(yz\mathrm{e}^{xyz}+1,xz\mathrm{e}^{xyz}-1,xy\mathrm{e}^{xyz}+1\right)\Big|_{(1,0,1)}=(1,0,1).$$

所以曲面S在点$(1,0,1)$处的切平面方程为$(x-1)+(z-1)=0$，且曲面S在点$(1,0,1)$处的切平面的法向量$(1,0,1)$垂直于y轴. 又知原点不在切平面上，故切平面不含y轴，所以曲面

S 在点 $(1,0,1)$ 处的切平面平行于 y 轴. 证毕

11. 已知 f 可微，证明曲面 $f\left(\dfrac{x-a}{z-c},\dfrac{y-b}{z-c}\right)=0$ 上任意一点处的切平面通过一定点，并求此点位置 .

证明：设 $F(x,y,z)=f\left(\dfrac{x-a}{z-c},\dfrac{y-b}{z-c}\right)$，则

$$\frac{\partial F}{\partial x}=f_1'\cdot\left(\frac{1}{z-c}\right),\quad \frac{\partial F}{\partial y}=f_2'\cdot\left(\frac{1}{z-c}\right),\quad \frac{\partial F}{\partial z}=f_1'\cdot\frac{a-x}{(z-c)^2}+f_2'\cdot\frac{b-y}{(z-c)^2}.$$

故曲面在 $P_0(x_0,y_0,z_0)$ 处的切平面为

$$f_1'(P_0)\frac{x-x_0}{z_0-c}+f_2'(P_0)\frac{y-y_0}{z_0-c}+\left(f_1'(P_0)\frac{a-x_0}{(z_0-c)^2}+f_2'(P_0)\frac{b-y_0}{(z_0-c)^2}\right)(z-z_0)=0\text{ ,}$$

整理得，

$$f_1'(P_0)[(z_0-c)(x-x_0)+(a-x_0)(z-z_0)]+f_2'(P_0)[(z_0-c)(y-y_0)+(b-y_0)(z-z_0)]=0.$$

易见当 $x=a,\ y=b,\ z=c$ 时上式恒等于零。于是曲面 $f\left(\dfrac{x-a}{z-c},\dfrac{y-b}{z-c}\right)=0$ 上任意一点处的切平面通过一定点 (a,b,c) ．证毕

12. 设 G 是可导函数且在自变量取值为零时，导数为零，否则函数的导数都不等于零。曲面 S 由方程 $ax+by+cz=G\left(x^2+y^2+z^2\right)$ 确定，试证明：曲面 S 上任一点的法线与某定直线相交。

证明：在曲面上任取一点 $P(x_0,y_0,z_0)$，则曲面在点 $P(x_0,y_0,z_0)$ 处的法线为

$$\frac{x-x_0}{a-2x_0G'(x_0^2+y_0^2+z_0^2)}=\frac{y-y_0}{b-2y_0G'(x_0^2+y_0^2+z_0^2)}=\frac{z-z_0}{c-2z_0G'(x_0^2+y_0^2+z_0^2)}.$$

设与该法线相交的定直线为 $\dfrac{x-x_1}{\alpha}=\dfrac{y-y_1}{\beta}=\dfrac{z-z_1}{\gamma}$，则这两条相交直线确定一个平面，从而两条相交直线的方向向量的叉积垂直于该平面，因此

$$\left[\left(a-2x_0G'(x_0^2+y_0^2+z_0^2),b-2y_0G'(x_0^2+y_0^2+z_0^2),c-2z_0G'(x_0^2+y_0^2+z_0^2)\right)\times(\alpha,\beta,\gamma)\right]$$
$$\cdot\left(x_1-x_0,y_1-y_0,z_1-z_0\right)=0,$$

即如下三阶矩阵的行列式为零，

$$\begin{vmatrix} a-2x_0G'(x_0^2+y_0^2+z_0^2) & b-2y_0G'(x_0^2+y_0^2+z_0^2) & c-2z_0G'(x_0^2+y_0^2+z_0^2) \\ \alpha & \beta & \gamma \\ x_1-x_0 & y_1-y_0 & z_1-z_0 \end{vmatrix}=0,$$

从而由行列式的运算，得

$$\begin{vmatrix} a & b & c \\ \alpha & \beta & \gamma \\ x_1-x_0 & y_1-y_0 & z_1-z_0 \end{vmatrix}-2G'(x_0^2+y_0^2+z_0^2)\begin{vmatrix} x_0 & y_0 & z_0 \\ \alpha & \beta & \gamma \\ x_1 & y_1 & z_1 \end{vmatrix}=0,$$

所以只要取$(\alpha,\beta,\gamma)=(a,b,c),\quad (x_1,y_1,z_1)=(0,0,0)$，故曲面上任一一点处的法线与定直线$\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}$相交．证毕

13．求过直线$\begin{cases}3x-2y-z=-15\\ x+y+z=10\end{cases}$且与曲面$S:x^2-y^2+z=10$相切的平面方程。

解法一、在曲面$S:x^2-y^2+z=10$上任取一点(x_0,y_0,z_0)，则曲面在(x_0,y_0,z_0)处的切平面方程：$2x_0(x-x_0)-2y_0(y-y_0)+(z-z_0)=0$，即$2x_0x-2y_0y+z=20-z_0$.

将直线方程$\begin{cases}3x-2y-z=-15\\ x+y+z=10\end{cases}$化为$\begin{cases}y=4x+5\\ z=5-5x,\end{cases}$

代入切平面方程，得$(2x_0-8y_0-5)x-10y_0-15+z_0=0$，

故$\begin{cases}2x_0-8y_0-5=0\\ -10y_0-15+z_0=0.\end{cases}$ 又$x_0^2-y_0^2+z_0=10$，解得

$$x_0=\frac{1}{2},\ y_0=-\frac{1}{2},\ z_0=10;\ \text{或}\ x_0=-\frac{7}{2},\ y_0=-\frac{3}{2},\ z_0=0.$$

所以切平面方程为$x+y+z=10$或$7x-3y-z+20=0$.

解法二、设切平面经过曲面$S:x^2-y^2+z=10$上一点(x_0,y_0,z_0)，则切平面的法向量为$(2x_0,-2y_0,1)$．过直线$\begin{cases}3x-2y-z=-15\\ x+y+z=10\end{cases}$的平面束为

$$(x+y+z-10)+\lambda(3x-2y-z+15)=0,$$

其法向量为$(1+3\lambda,1-2\lambda,1-\lambda)$．故

$$\frac{2x_0}{1+3\lambda}=\frac{-2y_0}{1-2\lambda}=\frac{1}{1-\lambda},$$

所以 $x_0=\dfrac{1+3\lambda}{2(1-\lambda)}$，$y_0=\dfrac{2\lambda-1}{2(1-\lambda)}$. 又知 (x_0,y_0,z_0) 既在曲面上，又在平面上，因此

$$\begin{cases} x_0^2-y_0^2+z_0=10 \\ (x_0+y_0+z_0-10)+\lambda(3x_0-2y_0-z_0+15)=0, \end{cases}$$

解得 $\lambda=0$ 或 $\lambda=2$. 故得到切平面方程为 $x+y+z=10$ 或 $7x-3y-z+20=0.$

14. 证明：设 $D\subset\mathbb{R}^2$ 是一个非空区域，且 $z=f(x,y)\in C^2(D)$. 则在旋转变换 $u=x\cos\theta+y\sin\theta,\ v=-x\sin\theta+y\cos\theta$ 下，表达式 $f''_{xx}+f''_{yy}$ 不变.

证明：因为 $\det\dfrac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{vmatrix}=1$，因此存在逆变换 $\begin{cases}x=x(u,v)\\ y=y(u,v)\end{cases}$ 使得通过变量 u,v，f 转化为 x,y 的函数，所以 $f'_x=f'_u u'_x+f'_v v'_x=f'_u\cos\theta-f'_v\sin\theta$，

$$f''_{xx}=f''_{uu}\cos^2\theta-2f''_{uv}\sin\theta\cos\theta+f''_{vv}\sin^2\theta,$$

$$f'_y=f'_u\sin\theta+f'_v\cos\theta,\quad f''_{yy}=f''_{uu}\sin^2\theta+2f''_{uv}\sin\theta\cos\theta+f''_{vv}\cos^2\theta.$$

故 $f''_{xx}+f''_{yy}=f''_{uu}+f''_{vv}$. 证毕