矩阵分析作业 6：模和内积

2019.11.01

Exercise.3

> 3. Evaluate the Frobenius matrix norm, 1-norm,2-norm and ∞-norm for each matrix below.
>
> $$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix}.$$

(1). 对于矩阵 A，首先计算 A^*A 的特征值，

$$A^*A - \lambda I = \begin{pmatrix} 2-\lambda & -4 \\ -4 & 8-\lambda \end{pmatrix}$$

由 $(2-\lambda)(8-\lambda) - 16 = 0$ 可得 $\lambda_1 = 10,\ \lambda_2 = 0$，因此可得，

$$||A||_2 = \sqrt{\lambda_{max}} = \sqrt{10}$$

$$||A||_F = \left(\sum_{i,j} |a_{ij}^2|\right)^{\frac{1}{2}} = \sqrt{10}$$

$$||A||_1 = \underset{j}{max} \sum_i |a_{ij}| = 4$$

$$||A||_\infty = \underset{i}{max} \sum_j |a_{ij}| = 3$$

(2). 对于矩阵 B，首先计算 B^*B 的特征值，

$$B^*B - \lambda I = \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix}$$

由 $(1-\lambda)^3 = 0$ 可得 $\lambda = 1$，因此可得，

$$||B||_2 = \sqrt{\lambda_{max}} = 1$$

$$||B||_F = \left(\sum_{i,j} |b_{ij}^2|\right)^{\frac{1}{2}} = \sqrt{3}$$

$$||B||_1 = \underset{j}{max} \sum_i |b_{ij}| = 1$$

$$||B||_\infty = \underset{i}{max} \sum_j |b_{ij}| = 1$$

(3). 对于矩阵 A，首先计算 C^*C 的特征值，

$$C^*C - \lambda I = \begin{pmatrix} 36-\lambda & -18 & 36 \\ -18 & 9-\lambda & -18 \\ 36 & -18 & 36-\lambda \end{pmatrix}$$

由 $|C^*C - \lambda I| = 0$ 可得 $\lambda_1 = 81$, $\lambda_2 = \lambda_3 = 0$，因此可得，

$$||C||_2 = \sqrt{\lambda_{max}} = \sqrt{81} = 9$$

$$||C||_F = \left(\sum_{i,j} |c_{ij}^2|\right)^{\frac{1}{2}} = \sqrt{81} = 9$$

$$||C||_1 = \max_j \sum_i |c_{ij}| = 10$$

$$||C||_\infty = \max_i \sum_j |c_{ij}| = 10$$

Exercise.10

10. Let $\mathcal{S} = span\{\mathbf{x}_1 = (1,1,1,-1)^T, \quad \mathbf{x}_2 = (2,-1,-1,1)^T, \quad \mathbf{x}_3 = (-1,2,2,1)^T\}$.
   (a) Use the classical GramSchmidt algorithm (with exact arithmetic) to determine an orthonormal basis for $\mathcal{S}$.
   (b) Repeat part (a) using the modified GramSchmidt algorithm, and compare the results.

(a) 施密特正正交化过程如下：

$$k = 1: u_1 \leftarrow \frac{x_1}{||x_1||} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$$

$$k = 2: u_2 \leftarrow x_2 - (u_1^T x_2)u_1 = x_2 + \frac{1}{2}u_1 = \frac{1}{4}\begin{pmatrix} 9 \\ -3 \\ -3 \\ 3 \end{pmatrix}, \; u_2 \leftarrow \frac{u_2}{||u_2||} = \frac{\sqrt{3}}{6}\begin{pmatrix} 3 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$$k = 3: u_3 \leftarrow x_3 - (u_1^T x_3)u_1 - (u_2^T x_3)u_2 = x_3 - u_1 + \sqrt{3}u_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$$

$$u_3 \leftarrow \frac{u_3}{||u_3||} = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$$

$thus:$

$$u_1=\frac{1}{2}\begin{pmatrix}1\\1\\1\\-1\end{pmatrix},u_2=\frac{\sqrt{3}}{6}\begin{pmatrix}3\\-1\\-1\\1\end{pmatrix},u_3=\frac{1}{\sqrt{6}}\begin{pmatrix}0\\1\\1\\2\end{pmatrix}$$

所以 $\{u_1,u_2,u_3\}$ 构成了 $\mathcal{S}$ 空间的一组标准正交基。

(b) 使用改进的施密特正交化过程如下：

$$k=1:u_1\leftarrow\frac{x_1}{||x_1||}=\frac{1}{2}\begin{pmatrix}1\\1\\1\\-1\end{pmatrix},so\{u_1,u_2,u_3\}\leftarrow\{u_1,x_2,x_3\}$$

$$k=2:(u_1^Tu_2)=-\frac{1}{2},(u_1^Tu_3)=1,so$$

$$u_2\leftarrow u_2-(u_1^Tu_2)u_1=u_2+\frac{1}{2}u_1=\frac{3}{4}\begin{pmatrix}3\\-1\\-1\\1\end{pmatrix},$$

$$u_3\leftarrow u_3-(u_1^Tu_3)u_1=u_3-u_1=\frac{3}{2}\begin{pmatrix}-1\\1\\1\\1\end{pmatrix},$$

$$and$$

$$u_2\leftarrow\frac{u_2}{||u_2||}=\frac{\sqrt{3}}{6}\begin{pmatrix}3\\-1\\-1\\1\end{pmatrix}$$

$$k=3:(u_2^Tu_3)=-\sqrt{3}$$

$$u_3\leftarrow u_3(u_2^Tu_3)u_2=u_3+\sqrt{3}u_2=\begin{pmatrix}0\\1\\1\\2\end{pmatrix}$$

$$u_3\leftarrow\frac{u_3}{||u_3||}=\frac{1}{\sqrt{6}}\begin{pmatrix}0\\1\\1\\2\end{pmatrix},$$

$$thus:$$

$$u_1=\frac{1}{2}\begin{pmatrix}1\\1\\1\\-1\end{pmatrix},u_2=\frac{\sqrt{3}}{6}\begin{pmatrix}3\\-1\\-1\\1\end{pmatrix},u_3=\frac{1}{\sqrt{6}}\begin{pmatrix}0\\1\\1\\2\end{pmatrix}$$

此时 $\{u_1,u_2,u_3\}$ 构成了 $\mathcal{S}$ 空间的一组标准正交基。
由过程 (a) 和 (b) 的结果对比来看，在精确表示时，改进后算法的结果与改进前的结果一样。

Exercise.12

12. Let $\mathbf{A}=\begin{pmatrix}1&0&-1\\1&2&1\\1&1&-3\\0&1&1\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}1&1&1&1\end{pmatrix}$

(a) Determine the rectangular QR factorization of $\mathbf{A}$.
(b) Use the QR factor from part (a) to determine the least squares solution to $\mathbf{Ax}=\mathbf{b}$.

(a)QR 分解过程如下：

$$k=1:\ r_{11}\leftarrow||a_1||=\sqrt{3},\qquad q_1\leftarrow\frac{a_1}{r_{11}}=\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\\0\end{pmatrix}$$

$$k=2:r_{12}\leftarrow q_1^Ta_2=\sqrt{3},\qquad q_2\leftarrow a_2-r_{12}q_1=a_2-\sqrt{3}q_1=\begin{pmatrix}-1\\1\\0\\1\end{pmatrix},$$

$$r_{22}=||q_2||=\sqrt{3},\qquad q_2\leftarrow\frac{q_2}{r_{22}}=\frac{1}{\sqrt{3}}\begin{pmatrix}-1\\1\\0\\1\end{pmatrix}$$

$$k=3:r_{13}\leftarrow q_1^Ta_3=-\sqrt{3},\qquad r_{23}\leftarrow q_2^Ta_3=\sqrt{3},$$

$$q_3\leftarrow a_3-r_{13}q_1-r_{23}q_2=a_3+\sqrt{3}q_1-\sqrt{3}q_2=\begin{pmatrix}1\\1\\-2\\0\end{pmatrix},$$

$$r_{33}=||q_3||=\sqrt{6},\qquad q_3\leftarrow\frac{q_3}{r_{33}}=\frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\-2\\0\end{pmatrix}$$

所以，可以得到，

$$Q=\begin{pmatrix}\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{6}\\ \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{6}\\ \frac{\sqrt{3}}{3} & 0 & -\frac{\sqrt{6}}{3}\\ 0 & \frac{\sqrt{3}}{3} & 0\end{pmatrix},\qquad R=\begin{pmatrix}\sqrt{3} & \sqrt{3} & -\sqrt{3}\\ 0 & \sqrt{3} & \sqrt{3}\\ 0 & 0 & \sqrt{6}\end{pmatrix}$$

(b) 在上一题中已经求得 A 的 QR 分解形式 $A=QR$，则等式 $Ax=b$ 可以有如下形式

$$Ax=b\Leftrightarrow QRx=b\Leftrightarrow Q^TQRx=Q^Tb\Leftrightarrow Rx=Q^Tb$$

即

$$\begin{pmatrix}\sqrt{3} & \sqrt{3} & -\sqrt{3}\\ 0 & \sqrt{3} & \sqrt{3}\\ 0 & 0 & \sqrt{6}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & 0\\ -\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3}\\ \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{3} & 0\end{pmatrix}\begin{pmatrix}1\\ 1\\ 1\\ 1\end{pmatrix}=\begin{pmatrix}\sqrt{3}\\ \frac{\sqrt{3}}{3}\\ 0\end{pmatrix}$$

通过回代法可以求得

$$x=\begin{pmatrix}\frac{2}{3}\\ \frac{1}{3}\\ 0\end{pmatrix}$$

Exercise.16

16. Let $\mathbf{u}=(-2\ 1\ 3\ \ -1)^T$ and $\mathbf{v}=(1\ 4\ 0\ \ -1)^T$.
  (a) Determine the orthogonal projection of $\mathbf{u}$ onto $span\{\mathbf{v}\}$.
  (b) Determine the orthogonal projection of $\mathbf{v}$ onto $span\{\mathbf{u}\}$.
  (c) Determine the orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}^{\perp}$.
  (d) Determine the orthogonal projection of $\mathbf{v}$ onto $\mathbf{u}^{\perp}$.

(a) 由于 $v=(1\ \ 4\ \ 0\ \ -1)^T$，则

$$P_v=\frac{vv^*}{v^*v}=\frac{1}{18}\begin{pmatrix}1 & 4 & 0 & -1\\ 4 & 16 & 0 & -4\\ 0 & 0 & 0 & 0\\ -1 & -4 & 0 & 1\end{pmatrix}$$

因此，u 在 v 张成空间上的正交投影为

$$P_vu=\frac{1}{18}\begin{pmatrix}1 & 4 & 0 & -1\\ 4 & 16 & 0 & -4\\ 0 & 0 & 0 & 0\\ -1 & -4 & 0 & 1\end{pmatrix}\begin{pmatrix}-2\\ 1\\ 3\\ -1\end{pmatrix}=\frac{1}{6}\begin{pmatrix}1\\ 4\\ 0\\ -1\end{pmatrix}$$

(b) 由于 $u=(-2\ \ 1\ \ 3\ \ -1)^T$，则

$$P_u=\frac{uu^*}{u^*u}=\frac{1}{15}\begin{pmatrix} 4 & -2 & -6 & 2 \\ -2 & 1 & 3 & -1 \\ -6 & 3 & 9 & -3 \\ 2 & -1 & -3 & 1 \end{pmatrix}$$

因此，v 在 u 张成空间上的正交投影为

$$P_uv=\frac{1}{15}\begin{pmatrix} 4 & -2 & -6 & 2 \\ -2 & 1 & 3 & -1 \\ -6 & 3 & 9 & -3 \\ 2 & -1 & -3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \\ 0 \\ -1 \end{pmatrix}=\frac{1}{5}\begin{pmatrix} -2 \\ 1 \\ 3 \\ -1 \end{pmatrix}$$

(c) 由于 $v=(1\ \ 4\ \ 0\ \ -1)^T$，则

$$P_{v^\perp}=I-\frac{vv^*}{v^*v}=I-\frac{1}{18}\begin{pmatrix} 1 & 4 & 0 & -1 \\ 4 & 16 & 0 & -4 \\ 0 & 0 & 0 & 0 \\ -1 & -4 & 0 & 1 \end{pmatrix}=\frac{1}{18}\begin{pmatrix} 17 & -4 & 0 & 1 \\ -4 & 2 & 0 & 4 \\ 0 & 0 & 18 & 0 \\ 1 & 4 & 0 & 17 \end{pmatrix}$$

因此，u 在 $v^\perp$ 张成空间上的正交投影为

$$P_{v^\perp}u=\frac{1}{18}\begin{pmatrix} 17 & -4 & 0 & 1 \\ -4 & 2 & 0 & 4 \\ 0 & 0 & 18 & 0 \\ 1 & 4 & 0 & 17 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \\ 3 \\ -1 \end{pmatrix}=\begin{pmatrix} -\frac{13}{6} \\ \frac{1}{3} \\ 3 \\ -\frac{5}{6} \end{pmatrix}$$

(d) 由于 $u=(-2\ \ 1\ \ 3\ \ -1)^T$，则

$$P_{u^\perp}=I-\frac{uu^*}{u^*u}=I-\frac{1}{15}\begin{pmatrix} 11 & 2 & 6 & -2 \\ 2 & 14 & -3 & 1 \\ 6 & -3 & 6 & 3 \\ -2 & 1 & 3 & 14 \end{pmatrix}$$

因此，v 在 $u^\perp$ 张成空间上的正交投影为

$$P_{u^\perp}v=\frac{1}{15}\begin{pmatrix} 11 & 2 & 6 & -2 \\ 2 & 14 & -3 & 1 \\ 6 & -3 & 6 & 3 \\ -2 & 1 & 3 & 14 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \\ 0 \\ -1 \end{pmatrix}=\frac{1}{5}\begin{pmatrix} 7 \\ 19 \\ -3 \\ -4 \end{pmatrix}$$