
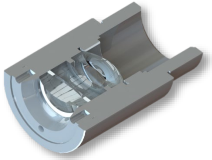



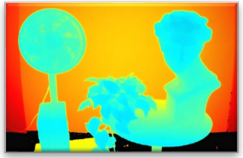
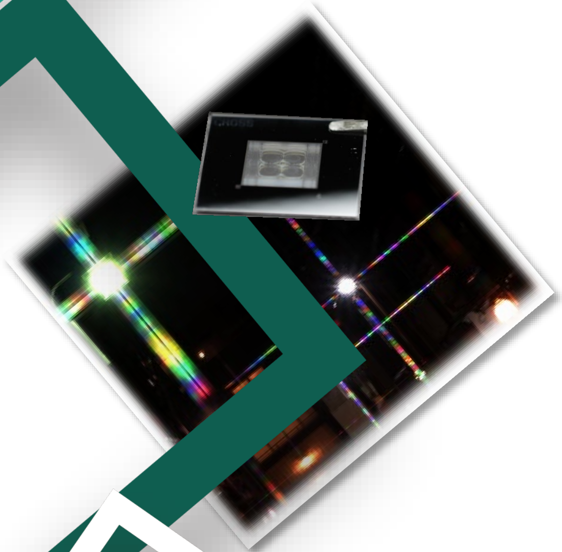

Computational Imaging

Lecture 15: Computing Toolbox: The Alternating Direction Method of Multipliers (ADMM) and Applications

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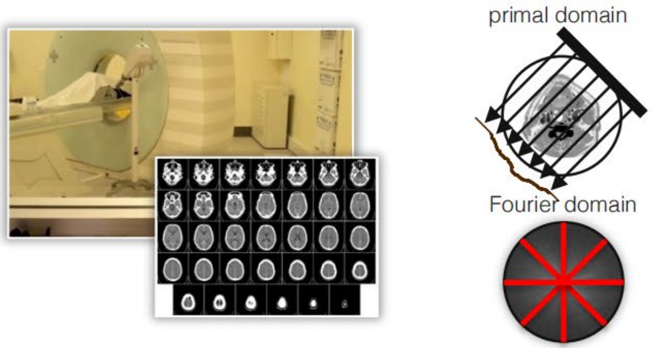
Overview

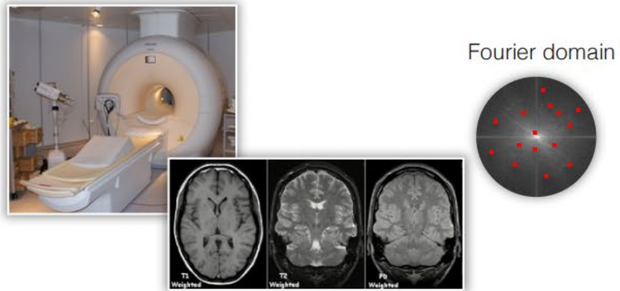
- Inverse Problem
- Single-pixel Camera and ADMM
- Bayesian Perspective of Inverse Problems
- High-dimensional Inverse Problems

Inverse Problem

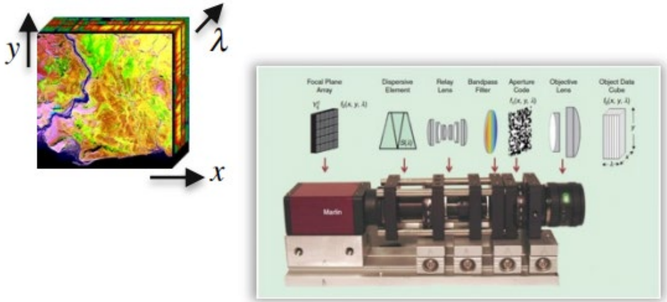
Inverse Problems in Imaging



Computed Tomography (CT)



Magnetic Resonance Imaging (MRI)



Hyper-spectrum Imaging

- Computational Photography
- Light-field Imaging
- Thermal Imaging
- …

Linear Inverse Problem + Regularization

- Write imaging problem as optimization problem:

$$\hat{\mathbf{x}} = \operatorname{argmin}_{\mathbf{x}} \frac{\mu}{2} \left\| \mathbf{b} - \mathbf{A} \cdot \mathbf{x} \right\|_2^2 + \Gamma(\mathbf{x})$$

Where

- $\mathbf{x}$ is intrinsic/latent image, $\hat{\mathbf{x}}$ is its estimate
- $\mathbf{b}$ is the measurement / observation
- $\mathbf{A}$ describes imaging system
- Γ is an image prior or regularizer

- Examples:
  - Deblurring: $\mathbf{A}$ = image blur, $\mathbf{b}$ = blurred image, $\mathbf{x}$ = sharp image
  - Tomography: $\mathbf{A}$ = projection matrix, $\mathbf{b}$ = projected images, $\mathbf{x}$ = volume

Slides Credit: Wolfgang Heidrich

Under-determined Inverse Problems

- Image formation model: $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\eta}, \quad \boldsymbol{b} \in \mathbb{R}^{M}, \boldsymbol{x} \in \mathbb{R}^{N}, \boldsymbol{A} \in \mathbb{R}^{M\times N}$

- What makes it under-determined (or a compressive imaging problem): $M<N$

- Problem: infinitely many solutions satisfy the observations!
  Same problem as ill-posed problems! $\longrightarrow$ need image priors

Under-determined Inverse Problems

- Image formation model: $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\eta}, \quad \boldsymbol{b} \in \mathbb{R}^M, \boldsymbol{x} \in \mathbb{R}^N, \boldsymbol{A} \in \mathbb{R}^{M \times N}$

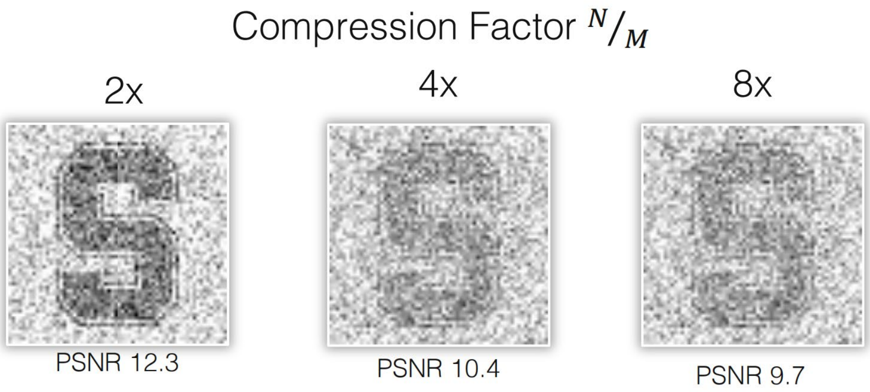
- Standard approach – the least-norm solution: $\tilde{\boldsymbol{x}}_{\mathrm{ln}} = \boldsymbol{A}^{\boldsymbol{T}}\left(\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{T}}\right)\boldsymbol{b}$

- This is the solution of optimization problem
$$\begin{aligned} &\mathrm{minimize}_x \ \|\boldsymbol{x}\|_2 \\ &\text{subject to} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{aligned}$$

Note: among the infinitely many solutions satisfying the observations, the least-norm solution is the one with the smallest L2 norm, thus equivalent to ‖·‖2 regularizer

Under-determined Inverse Problems

- Image formation model: $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\eta}, \quad \boldsymbol{b} \in \mathbb{R}^M, \boldsymbol{x} \in \mathbb{R}^N, \boldsymbol{A} \in \mathbb{R}^{M \times N}$
- Standard approach – the least-norm solution: $\tilde{\boldsymbol{x}}_{\text{ln}} = \boldsymbol{A}^{\boldsymbol{T}}\left(\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{T}}\right)\boldsymbol{b}$
- Results (not great):



Other Inverse Problems in Imaging

- All these inverse problems have important applications and are very different

- Yet, they all boil down to the same inverse problem, each with a different matrix *A*:

$$\text{minimize}_{\boldsymbol{x}} \ \frac{1}{2}\|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}\|_2^2 + \lambda\Psi(\boldsymbol{x})$$

- The methods derived here also apply to all those problems and applications; single-pixel imaging is a great example problem

  →"if you can solve this, you can solve almost anything"

Review of HQS for General Inverse Problems

- Objective or "loss" function of general inverse problem:

$$\text{minimize}_{\boldsymbol{x}} \ \frac{1}{2}\|\boldsymbol{b}-\boldsymbol{A}\boldsymbol{x}\|_2^2 + \underset{\uparrow \atop \text{weight of regularizer}}{\lambda}\Psi(\boldsymbol{x})$$

- Reformulate as:

$$\text{minimize}_{\{x,z\}} \ \underbrace{\frac{1}{2}\|\boldsymbol{b}-\boldsymbol{A}\boldsymbol{x}\|_2^2}_{f(\boldsymbol{x})} + \underbrace{\lambda\Psi(\boldsymbol{z})}_{g(\boldsymbol{z})}$$

$$\text{subject to} \ \ \boldsymbol{D}\boldsymbol{x}-\boldsymbol{z}=0$$

- Remove constraints using penalty term (equivalent for large ρ):

$$L_\rho(\boldsymbol{x},\boldsymbol{z}) = f(\boldsymbol{x}) + g(\boldsymbol{z}) + \underbrace{\frac{\rho}{2}\|\boldsymbol{D}\boldsymbol{x}-\boldsymbol{z}\|_2^2}_{\text{penalty term}}$$

Review of HQS for General Inverse Problems

$$L_\rho(\boldsymbol{x}, \boldsymbol{z}) = f(\boldsymbol{x}) + g(\boldsymbol{z}) + \frac{\rho}{2}\|\boldsymbol{D}\boldsymbol{x} - \boldsymbol{z}\|_2^2$$

- Alternating gradient descent approach to solving penalty formulation leads to following iterative algorithm:

while not converged:

$$\boldsymbol{x} \leftarrow \mathrm{prox}_{f,\rho}(\boldsymbol{z}) = \arg\min_{\boldsymbol{x}} L_\rho(\boldsymbol{x}, \boldsymbol{z}) = \arg\min_{\boldsymbol{x}} f(\boldsymbol{x}) + \frac{\rho}{2}\|\boldsymbol{D}\boldsymbol{x} - \boldsymbol{z}\|_2^2$$

$$\boldsymbol{z} \leftarrow \mathrm{prox}_{g,\rho}(\boldsymbol{D}\boldsymbol{x}) = \arg\min_{\boldsymbol{z}} L_\rho(\boldsymbol{x}, \boldsymbol{z}) = \arg\min_{\boldsymbol{z}} g(\boldsymbol{z}) + \frac{\rho}{2}\|\boldsymbol{D}\boldsymbol{x} - \boldsymbol{z}\|_2^2$$

Review of HQS for General Inverse Problems

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$$L_\rho(\boldsymbol{x},\boldsymbol{z}) = f(\boldsymbol{x}) + g(\boldsymbol{z}) + \frac{\rho}{2}\|\boldsymbol{D}\boldsymbol{x}-\boldsymbol{z}\|_2^2$$

$$L_\rho(\boldsymbol{x},\boldsymbol{z}) = \frac{1}{2}\|\boldsymbol{A}\boldsymbol{x}-\boldsymbol{b}\|_2^2 + \lambda\Psi(\boldsymbol{x}) + \frac{\rho}{2}\|\boldsymbol{D}\boldsymbol{x}-\boldsymbol{z}\|_2^2$$

$\boldsymbol{x} \in \mathbb{R}^N$ unknown image

$\boldsymbol{A} \in \mathbb{R}^{M\times N}$ matrix describing image formation model

$\boldsymbol{z} \in \mathbb{R}^{2N}, \boldsymbol{D} = \begin{bmatrix}\boldsymbol{D}_x \\ \boldsymbol{D}_y\end{bmatrix} \in \mathbb{R}^{2N\times N}$ for TV regularizer

$\boldsymbol{z} \in \mathbb{R}^{N}, \boldsymbol{D} = \boldsymbol{I} \in \mathbb{R}^{N\times N}$ for denoising or other regularizers

Review of HQS for General Inverse Problems

x - update:

$$\boldsymbol{x} \leftarrow \mathrm{prox}_{\|\cdot\|_2,\rho}(\boldsymbol{z}) = \arg\min_{\boldsymbol{x}} \frac{1}{2}\|\boldsymbol{A}\boldsymbol{x}-\boldsymbol{b}\|_2^2 + \frac{\rho}{2}\|\boldsymbol{D}\boldsymbol{x}-\boldsymbol{z}\|_2^2$$

$$\boldsymbol{x} \leftarrow \underbrace{\left(\boldsymbol{A}^T\boldsymbol{A}+\rho\boldsymbol{D}^T\boldsymbol{D}\right)}_{\widetilde{\boldsymbol{A}}}{}^{-1}\underbrace{\left(\boldsymbol{A}^T\boldsymbol{b}+\rho\boldsymbol{D}^T\boldsymbol{z}\right)}_{\widetilde{\boldsymbol{b}}}$$

- For general inverse problems, we don't necessarily have an efficient closed-form solution for this problem, like we did for the deconvolution problem
- Use matrix-free iterative solver, such as the conjugate gradient method, to solve $\widetilde{\boldsymbol{A}}\boldsymbol{x} = \widetilde{\boldsymbol{b}}$ (e.g., scipy.sparse.linalg.cg)

Review of HQS for General Inverse Problems

- **z** – update for TV regularizer in closed form:

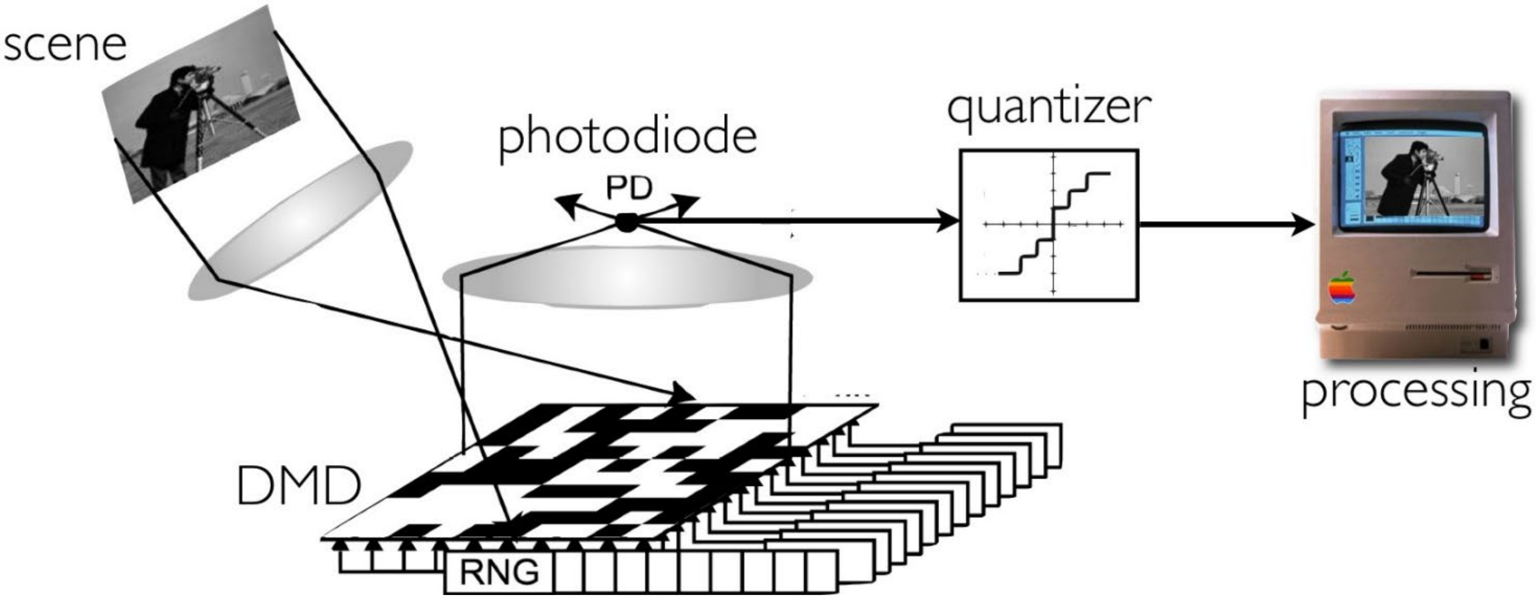
$$\boldsymbol{z} \leftarrow \operatorname{prox}_{\|\cdot\|_1,\rho}(\boldsymbol{D}\boldsymbol{x}) = \arg\min_{\boldsymbol{z}} \lambda\|\boldsymbol{z}\|_1 + \frac{\rho}{2}\|\boldsymbol{D}\boldsymbol{x} - \boldsymbol{z}\|_2^2 = \mathcal{S}_\kappa(\boldsymbol{v})$$

- **z** – update for denoising-based regularizer in closed form:

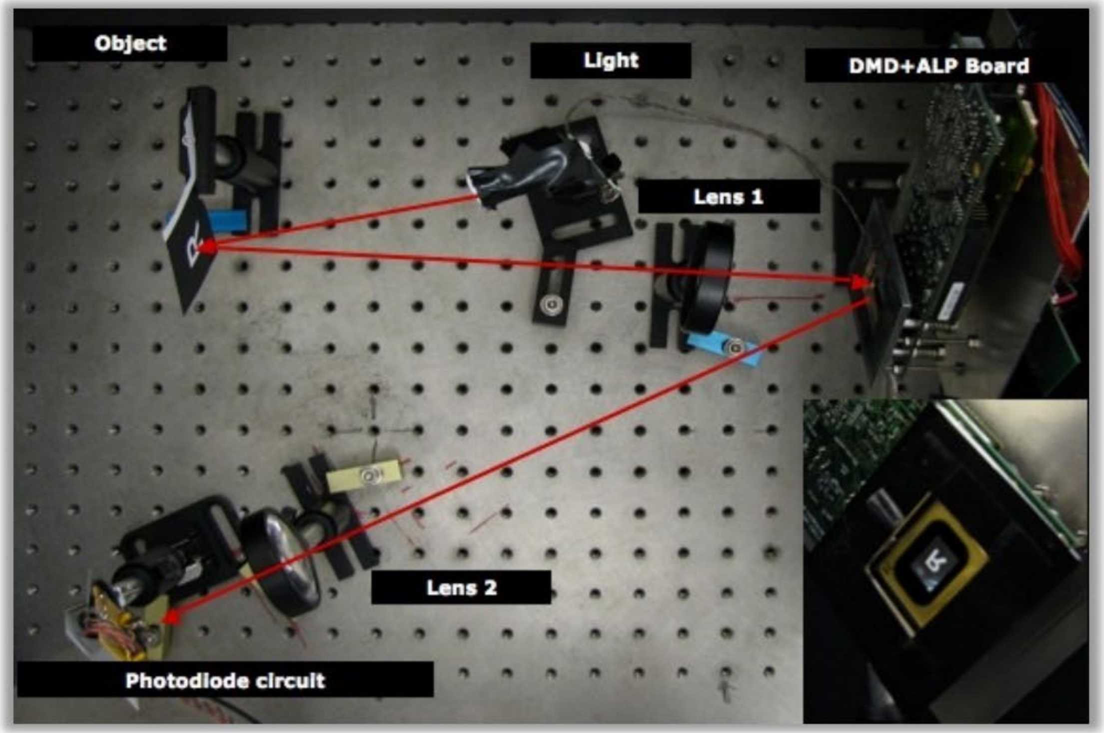
$$\boldsymbol{z} \leftarrow \operatorname{prox}_{\mathcal{D},\rho}(\boldsymbol{x}) = \arg\min_{\boldsymbol{z}} \lambda\Psi(\boldsymbol{z}) + \frac{\rho}{2}\|\boldsymbol{x} - \boldsymbol{z}\|_2^2 = \mathcal{D}\left(\boldsymbol{x}, \sigma^2 = \frac{\lambda}{\rho}\right)$$

Single-pixel Camera and ADMM

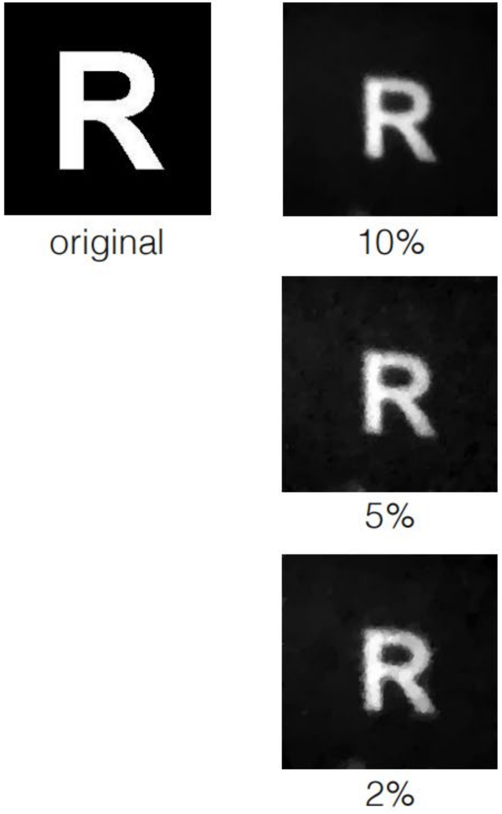
Single-pixel Imaging



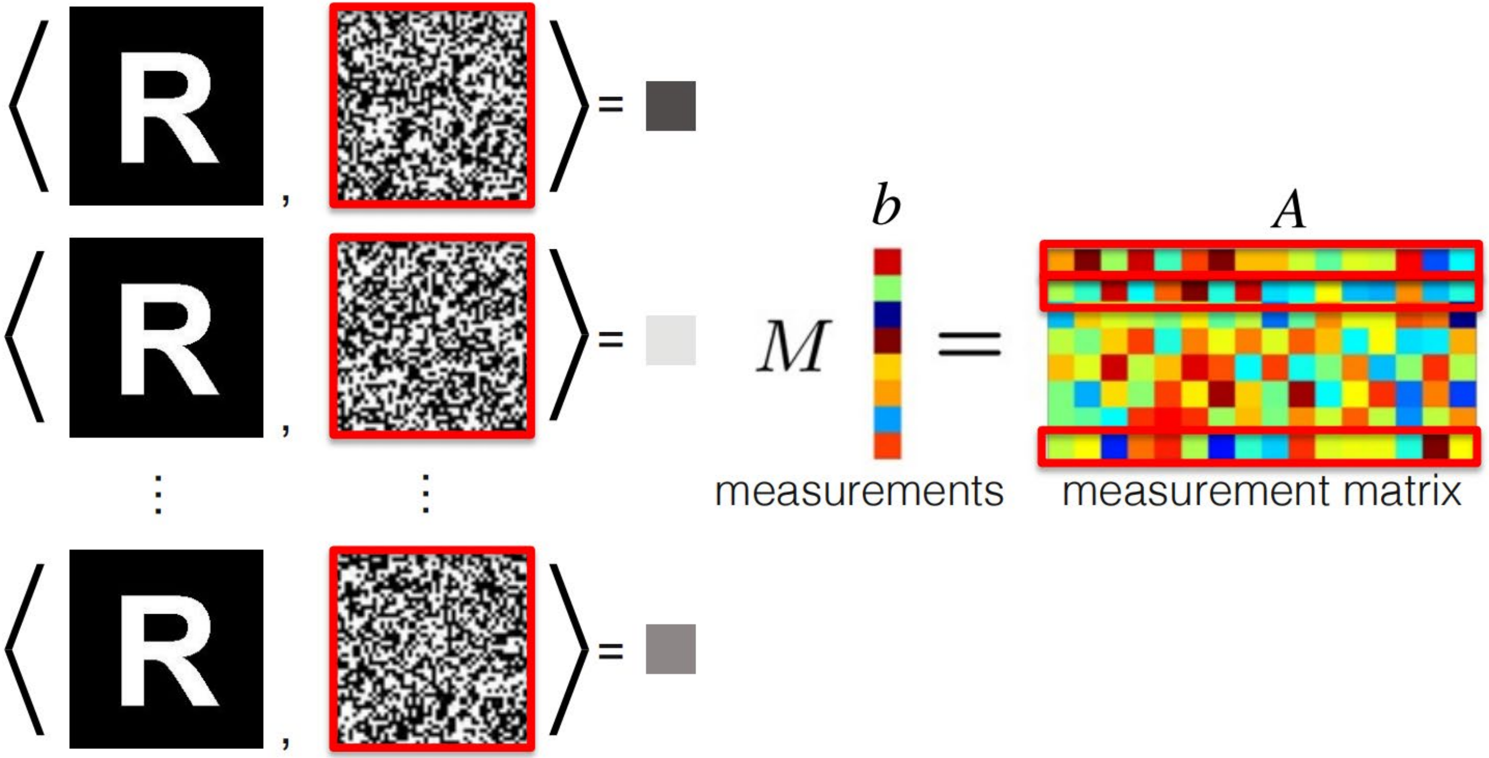
Single-pixel Imaging



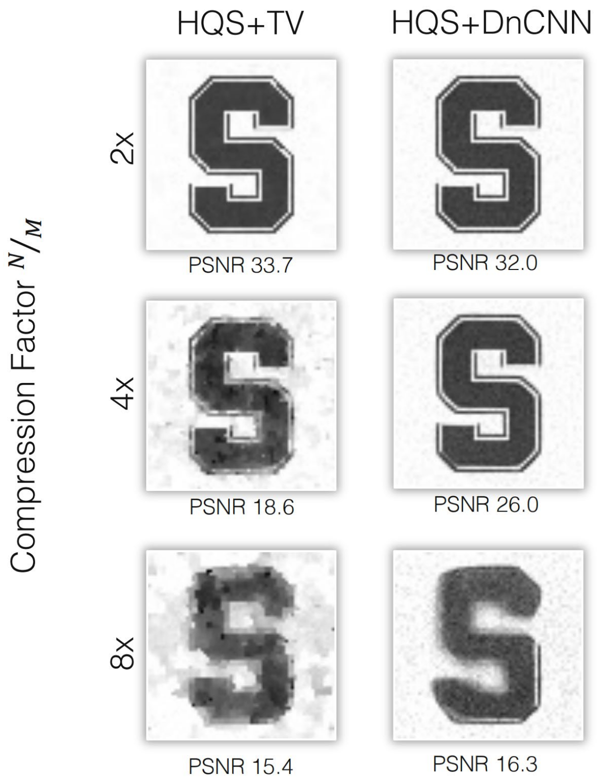
Duarte et al. 2008



Single-pixel Imaging



HQS for Single-pixel Imaging



- Works okay for low compression factor, i.e., when M is close to N
- Not very robust for larger compression factors
- Formulation using penalty term is not adequate → need something more robust

HQS vs. ADMM

- Objective function: $$\text{minimize}_{\boldsymbol{x}} \ \frac{1}{2}\|\boldsymbol{b}-\boldsymbol{A}\boldsymbol{x}\|_2^2 + \lambda\Psi(\boldsymbol{x})$$

- Reformulate as: $$\text{minimize}_{\{x,z\}} \ \underbrace{\frac{1}{2}\|\boldsymbol{b}-\boldsymbol{A}\boldsymbol{x}\|_2^2}_{f(\boldsymbol{x})} + \underbrace{\lambda\Psi(\boldsymbol{z})}_{g(\boldsymbol{z})}$$ $$\text{subject to} \ \ \boldsymbol{D}\boldsymbol{x}-\boldsymbol{z}=0$$

- Penalty Method of HQS: $$L_\rho^{(\mathrm{HQS})}(\boldsymbol{x},\boldsymbol{z}) = f(\boldsymbol{x}) + g(\boldsymbol{z}) + \frac{\rho}{2}\|\boldsymbol{D}\boldsymbol{x}-\boldsymbol{z}\|_2^2$$

- Augmented Lagrangian: $$L_\rho^{(\mathrm{ADMM})}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{y}) = f(\boldsymbol{x}) + g(\boldsymbol{z}) + \boldsymbol{y}^T(\boldsymbol{D}\boldsymbol{x}-\boldsymbol{z}) + \frac{\rho}{2}\|\boldsymbol{D}\boldsymbol{x}-\boldsymbol{z}\|_2^2$$ $$\overset{\boldsymbol{u}=(1/\rho)\boldsymbol{y}}{=} f(\boldsymbol{x}) + g(\boldsymbol{z}) + \frac{\rho}{2}\|\boldsymbol{D}\boldsymbol{x}-\boldsymbol{z}+\boldsymbol{u}\|_2^2 - \frac{\rho}{2}\|\boldsymbol{u}\|_2^2$$

ADMM

$$L_\rho^{(\mathrm{ADMM})}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) = f(\boldsymbol{x}) + g(\boldsymbol{z}) + \frac{\rho}{2}\|\boldsymbol{D}\boldsymbol{x} - \boldsymbol{z} + \boldsymbol{u}\|_2^2 - \frac{\rho}{2}\|\boldsymbol{u}\|_2^2$$

$\boldsymbol{x} \in \mathbb{R}^N$ unknown image

$\boldsymbol{A} \in \mathbb{R}^{M\times N}$ matrix describing image formation model

$\boldsymbol{z},\boldsymbol{u} \in \mathbb{R}^{2N}, \boldsymbol{D} = \begin{bmatrix}\boldsymbol{D}_x \\ \boldsymbol{D}_y\end{bmatrix} \in \mathbb{R}^{2N\times N}$ for TV regularizer

$\boldsymbol{z},\boldsymbol{u} \in \mathbb{R}^{N}, \boldsymbol{D} = \boldsymbol{I} \in \mathbb{R}^{N\times N}$ for denoising or other regularizers

ADMM

$$L_\rho^{(\mathrm{ADMM})}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) = f(\boldsymbol{x}) + g(\boldsymbol{z}) + \frac{\rho}{2}\|\boldsymbol{D}\boldsymbol{x} - \boldsymbol{z} + \boldsymbol{u}\|_2^2 - \frac{\rho}{2}\|\boldsymbol{u}\|_2^2$$

- Alternating gradient descent approach to solving Augmented Lagrangian:

while not converged:

$$\boldsymbol{x} \leftarrow \mathrm{prox}_{f,\rho}(\boldsymbol{z}) = \arg\min_x L_\rho^{(\mathrm{ADMM})}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) = \arg\min_{\boldsymbol{x}} f(\boldsymbol{x}) + \frac{\rho}{2}\|\boldsymbol{D}\boldsymbol{x} - \boldsymbol{z} + \boldsymbol{u}\|_2^2$$

$$\boldsymbol{z} \leftarrow \mathrm{prox}_{g,\rho}(\boldsymbol{D}\boldsymbol{x}) = \arg\min_z L_\rho^{(\mathrm{ADMM})}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) = \arg\min_{\boldsymbol{z}} g(\boldsymbol{z}) + \frac{\rho}{2}\|\boldsymbol{D}\boldsymbol{x} - \boldsymbol{z} + \boldsymbol{u}\|_2^2$$

$$\boldsymbol{u} \leftarrow \boldsymbol{u} + \boldsymbol{D}\boldsymbol{x} - \boldsymbol{z}$$

ADMM

x - update:

$$\boldsymbol{x} \leftarrow \mathrm{prox}_{\|\cdot\|_2,\rho}(\boldsymbol{z}) = \arg\min_{\boldsymbol{x}} \frac{1}{2}\|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|_2^2 + \frac{\rho}{2}\|\boldsymbol{D}\boldsymbol{x} - \boldsymbol{z} + \boldsymbol{u}\|_2^2,$$

$$\boldsymbol{x} \leftarrow \underbrace{\left(\boldsymbol{A}^T\boldsymbol{A} + \rho\boldsymbol{D}^T\boldsymbol{D}\right)^{-1}}_{\widetilde{\boldsymbol{A}}}\underbrace{\left(\boldsymbol{A}^T\boldsymbol{b} + \rho\boldsymbol{D}^T(\boldsymbol{z} - \boldsymbol{u})\right)}_{\widetilde{\boldsymbol{b}}}$$

- Same general x-update as HQS, use matrix-free iterative solver, such as the conjugate gradient method, to solve $\widetilde{\boldsymbol{A}}\boldsymbol{x} = \widetilde{\boldsymbol{b}}$ (e.g., scipy.sparse.linalg.cg)

ADMM

- **z** – update for TV regularizer in closed form:

$$\boldsymbol{z} \leftarrow \mathrm{prox}_{\|\cdot\|_1,\rho}(\boldsymbol{v}) = \arg\min_z \lambda\|\boldsymbol{z}\|_1 + \frac{\rho}{2}\|\boldsymbol{v}-\boldsymbol{z}\|_2^2 = \mathcal{S}_\kappa(\boldsymbol{v}),\ \boldsymbol{v} = \boldsymbol{Dx}+\boldsymbol{u}$$

- **z** – update for denoising-based regularizer in closed form:

$$\boldsymbol{z} \leftarrow \mathrm{prox}_{\mathcal{D},\rho}(\boldsymbol{x}+\boldsymbol{u}) = \arg\min_z \lambda\Psi(\boldsymbol{z}) + \frac{\rho}{2}\|\boldsymbol{x}-\boldsymbol{z}+\boldsymbol{u}\|_2^2 = \mathcal{D}\left(\boldsymbol{x}+\boldsymbol{u}, \sigma^2 = \frac{\lambda}{\rho}\right)$$

→Same z-update rules as HQS!

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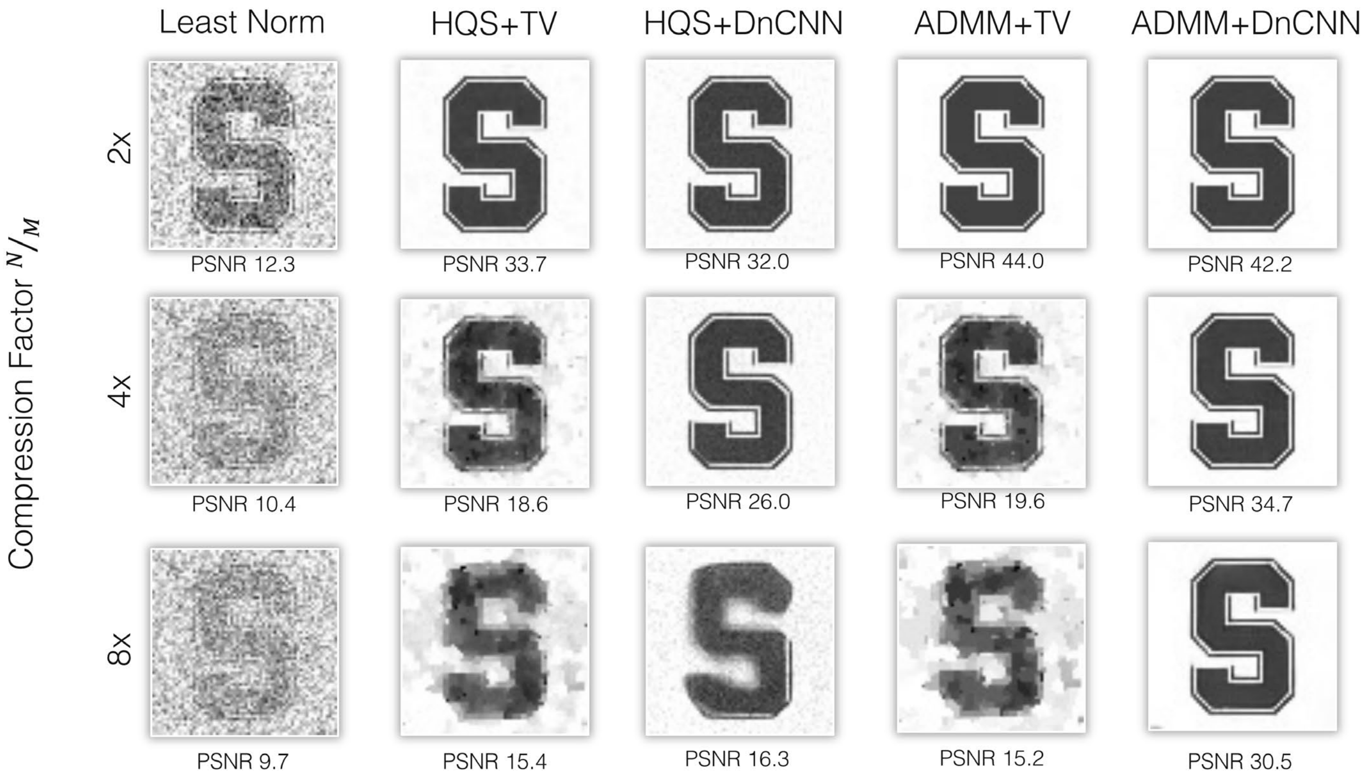
ADMM for inverse problem with denoiser

initialize ρ and λ
$\mathbf{x} = zeros\,(W, H)\,;$
$\mathbf{z} = zeros\,(W, H)\,;$
$\mathbf{u} = zeros\,(W, H)\,;$
for $k = 1$ **to** max_iters **do**
    $\mathbf{x} = \mathbf{prox}_{\|\cdot\|_2,\rho}\,(\mathbf{v}) = \text{cg\_solve}\left(\mathbf{A}^T\mathbf{A} + \rho\mathbf{I}, \mathbf{A}^T\mathbf{b} + \rho\,(\mathbf{z} - \mathbf{u})\right)$
    $\mathbf{prox}_{\mathcal{D},\rho}\,(\mathbf{x} + \mathbf{u}) = \mathcal{D}\left(\mathbf{x} + \mathbf{u}, \sigma^2 = \frac{\lambda}{\rho}\right)$
    $\mathbf{u} = \mathbf{u} + \mathbf{x} - \mathbf{z}$
end for

ADMM for inverse problem with TV

initialize ρ and λ
$\mathbf{x} = zeros\,(W, H)\,;$
$\mathbf{z} = zeros\,(W, H, 2)\,;$
$\mathbf{u} = zeros\,(W, H, 2)\,;$
for $k = 1$ **to** max_iters **do**
    $\mathbf{x} = \mathbf{prox}_{\|\cdot\|_2,\rho}\,(\mathbf{z} - \mathbf{u}) = \text{cg\_solve}\left(\mathbf{A}^T\mathbf{A} + \rho\mathbf{D}^T\mathbf{D}, \mathbf{A}^T\mathbf{b} + \rho\mathbf{D}^T\,(\mathbf{z} - \mathbf{u})\right)$
    $\mathbf{z} = \mathbf{prox}_{\|\cdot\|_1,\rho}\,(\mathbf{D}\mathbf{x} + \mathbf{u}) = \mathcal{S}_{\lambda/\rho}\,(\mathbf{D}\mathbf{x} + \mathbf{u})$
    $\mathbf{u} = \mathbf{u} + \mathbf{D}\mathbf{x} - \mathbf{z}$
end for

ADMM – Results



Bayesian Perspective of Inverse Problems

Bayesian Perspective of Gaussian Noise

- Image formation model: $\boldsymbol{b} = \boldsymbol{Ax} + \boldsymbol{\eta}, \quad \boldsymbol{b} \in \mathbb{R}^M, \boldsymbol{x} \in \mathbb{R}^N, \boldsymbol{A} \in \mathbb{R}^{M\times N}$

- Joint probability of all observations: $$p(\boldsymbol{b}|\boldsymbol{x},\sigma) = \prod_{i=1}^{M} p(\boldsymbol{b}_i|\boldsymbol{x}_i,\sigma) \propto \boldsymbol{e}^{-\frac{\|\boldsymbol{b}-\boldsymbol{Ax}\|_2^2}{2\sigma^2}}$$

- Bayes' rule: $$p(\boldsymbol{x}|\boldsymbol{b},\sigma) = \frac{p(\boldsymbol{b}|\boldsymbol{x},\sigma)p(\boldsymbol{x})}{p(\boldsymbol{b})} \propto p(\boldsymbol{b}|\boldsymbol{x},\sigma)\boldsymbol{p}(\boldsymbol{x})$$

- Maximum-a-posterior (MAP) solution:

$$\begin{aligned}\boldsymbol{x}_{MAP} &= \arg\min_{\boldsymbol{x}} \; -\log\big(p(\boldsymbol{x}|\boldsymbol{b},\sigma)\big) \\ &= \arg\min_{\boldsymbol{x}} \frac{1}{2\sigma^2}\|\boldsymbol{b}-\boldsymbol{Ax}\|_2^2 + \Psi(\boldsymbol{x})\end{aligned}$$

Bayesian Perspective of Poisson Noise

- Image formation model: $\boldsymbol{b} = \mathcal{P}(\boldsymbol{Ax}), \quad \boldsymbol{b} \in \mathbb{R}^M, \boldsymbol{x} \in \mathbb{R}^N, \boldsymbol{A} \in \mathbb{R}^{M \times N}$

- Probability of observation i:

$$p(\boldsymbol{b}_i|\boldsymbol{x}) = \frac{(\boldsymbol{Ax})_i^{\boldsymbol{b}_i} \, e^{-(\boldsymbol{Ax})_i}}{\boldsymbol{b}_i!}$$

- Joint probability of all observations:

$$p(\boldsymbol{b}|\boldsymbol{x}) = \prod_{i=1}^{M} p(\boldsymbol{b}_i|\boldsymbol{x})$$

$$= \prod_{i=1}^{M} e^{\log((\boldsymbol{Ax})_i)\boldsymbol{b}_i} \cdot e^{-(\boldsymbol{Ax})_i} \cdot \frac{1}{\boldsymbol{b}_i!}$$

Bayesian Perspective of Poisson Noise

- Image formation model: $\boldsymbol{b} = \mathcal{P}(\boldsymbol{A}\boldsymbol{x}), \quad \boldsymbol{b} \in \mathbb{R}^M, \boldsymbol{x} \in \mathbb{R}^N, \boldsymbol{A} \in \mathbb{R}^{M \times N}$

- Bayes' rule: $p(\boldsymbol{x}|\boldsymbol{b}, \sigma) = \dfrac{p(\boldsymbol{b}|\boldsymbol{x}, \sigma)p(\boldsymbol{x})}{p(\boldsymbol{b})} \propto p(\boldsymbol{b}|\boldsymbol{x}, \sigma)p(\boldsymbol{x})$

- Maximum-a-posterior (MAP) solution:

$$\boldsymbol{x}_{MAP} = \arg\min_x \; -\log\big(p(\boldsymbol{x}|\boldsymbol{b}, \sigma)\big) = -\log\big(p(\boldsymbol{b}|\boldsymbol{x})\big) - \log\big(p(\boldsymbol{x})\big)$$

$$= \arg\min_x \; -\log\big(p(\boldsymbol{b}|\boldsymbol{x})\big) + \lambda\Psi(\boldsymbol{x})$$

ADMM+TV for Poisson Noise & Nonnegativity

- Objective function:

$$\text{minimize}_x \; \underbrace{-\log\big(p(\boldsymbol{b}|\boldsymbol{x})\big)}_{\text{includes } \boldsymbol{A}} + \underbrace{\lambda\Psi(\boldsymbol{x})}_{\text{does not include } \boldsymbol{A}}$$

- Reformulate as:

$$\text{minimize}_{\{x,z\}} \; \underbrace{-\log\big(p(\boldsymbol{b}|\boldsymbol{z}_1)\big)}_{g_1(\boldsymbol{z}_1)} + \underbrace{\lambda_1\|\boldsymbol{z}_2\|_1}_{g_2(\boldsymbol{z}_2)} + \underbrace{\mathcal{J}_{\mathbb{R}_+}(\boldsymbol{z}_3)}_{g_3(\boldsymbol{z}_3)}$$

- Indicator function:

$$\mathcal{J}_{\mathbb{R}_+}(v) = \begin{cases} 0 & v > 0 \\ \infty & \text{otherwise} \end{cases} \qquad \text{subject to} \; \underbrace{\begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{D} \\ \boldsymbol{I} \end{bmatrix}}_{\boldsymbol{K}} \boldsymbol{x} - \underbrace{\begin{bmatrix} \boldsymbol{z}_1 \\ \boldsymbol{z}_2 \\ \boldsymbol{z}_3 \end{bmatrix}}_{\boldsymbol{z}} = 0$$

- Scaled Augmented Lagrangian:

$$L_\rho^{(\text{ADMM})}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) = \sum_i g_i(\boldsymbol{z}_i) + \frac{\rho}{2}\|\boldsymbol{K}\boldsymbol{x} - \boldsymbol{z} + \boldsymbol{u}\|_2^2 - \frac{\rho}{2}\|\boldsymbol{u}\|_2^2$$

ADMM+TV for Poisson Noise & Nonnegativity

$$L_\rho^{(\mathrm{ADMM})}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) = \sum_i g_i(\boldsymbol{z}_i) + \frac{\rho}{2}\|\boldsymbol{K}\boldsymbol{x} - \boldsymbol{z} + \boldsymbol{u}\|_2^2 - \frac{\rho}{2}\|\boldsymbol{u}\|_2^2$$

- Alternating gradient descent approach to solving **Augmented Lagrangian**:
- Derivation of all these proximal operators in the course notes on Noise, Denoising, and Image Reconstruction with Noise!

while not converged:

$$\boldsymbol{x} \leftarrow \mathrm{prox}_{\|\cdot\|_2,\rho}(\boldsymbol{z}) = \arg\min_x L_\rho^{(\mathrm{ADMM})}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) = \arg\min_{\boldsymbol{x}} \frac{1}{2}\|\boldsymbol{K}\boldsymbol{x} - \boldsymbol{z} + \boldsymbol{u}\|_2^2$$

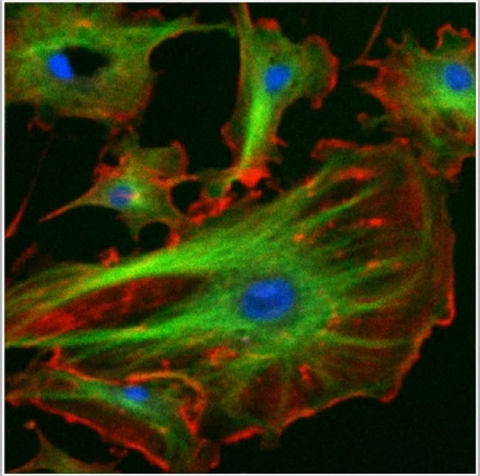
for all i:

$$\boldsymbol{z}_{\boldsymbol{i}} \leftarrow \mathrm{prox}_{g_i,\rho}(\boldsymbol{x}) = \arg\min_{z_i} L_\rho^{(\mathrm{ADMM})}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) = \arg\min_{z_i} g_i(\boldsymbol{z}_i) + \frac{\rho}{2}\|\boldsymbol{K}\boldsymbol{x} - \boldsymbol{z} + \boldsymbol{u}\|_2^2$$

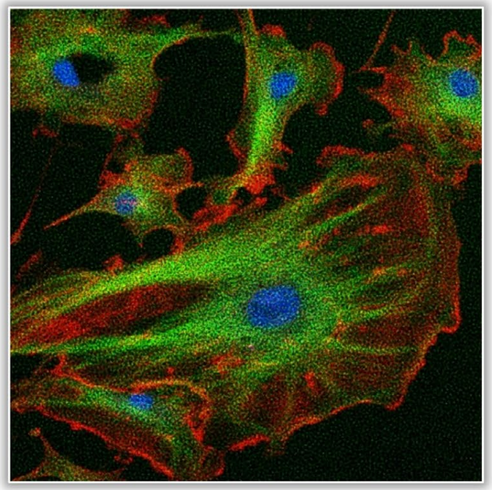
$$\boldsymbol{u} \leftarrow \boldsymbol{u} + \boldsymbol{K}\boldsymbol{x} - \boldsymbol{z}$$

ADMM+TV for Poisson Noise & Nonnegativity

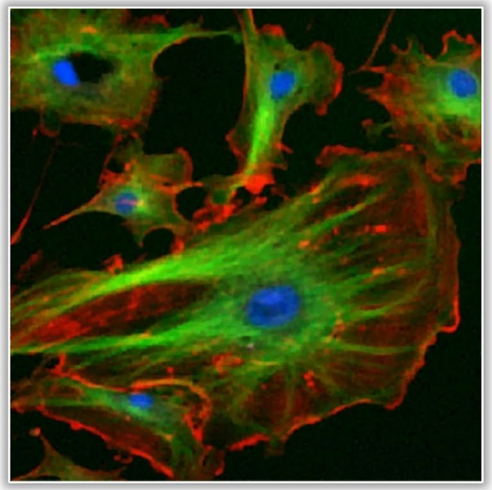
Blurry & Noisy Measurements



Richardson-Lucy Method
(maximum likelihood solution)



ADMM+TV+Nonnegativity
(maximum-a-posteriori solution)



High-dimensional Inverse Problems

Problem Description

SPAD Array: **Advantage**

- Picosecond level resolution
- Single-photon level sensitivity
- Compact and relative costs compared with PMT and Streak Camera

SPAD Array: **Disadvantage**

- Low spatial resolution → Solution: Compressive Sensing
- Low fill-factor → Solution: Optics

Improving Fill-factor of SPAD

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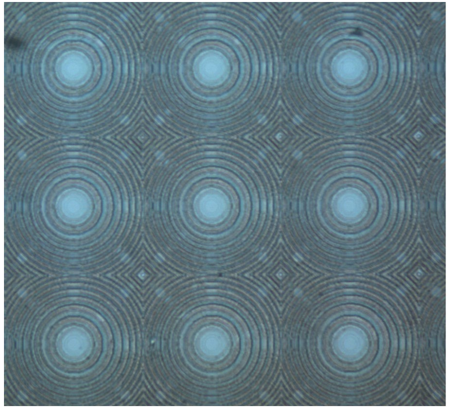
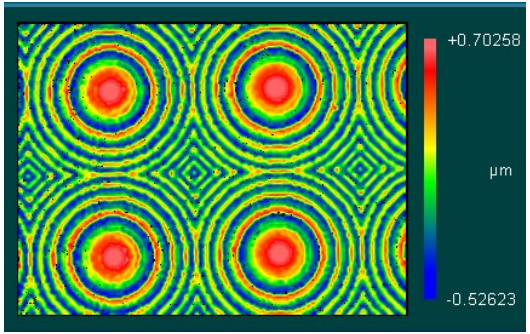
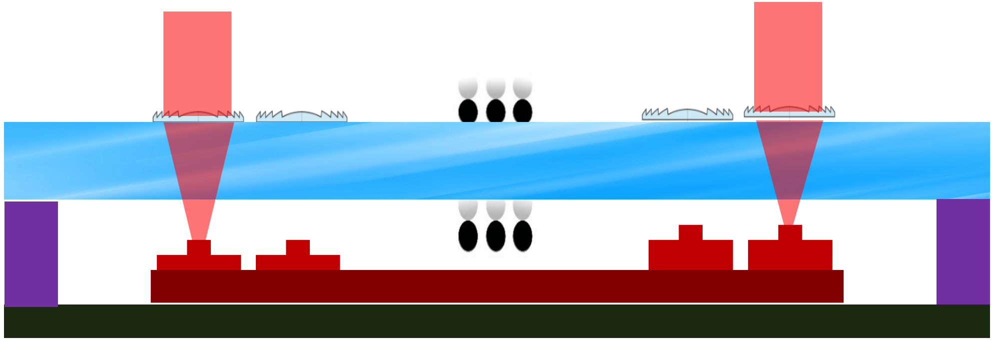
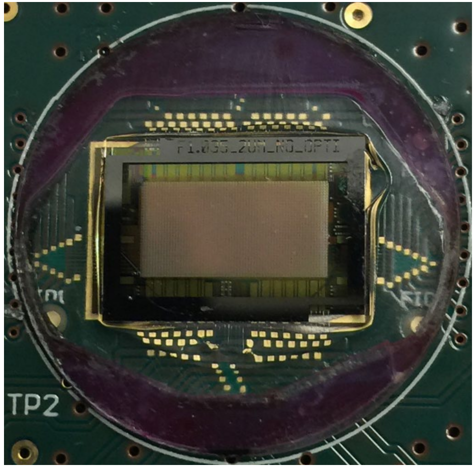
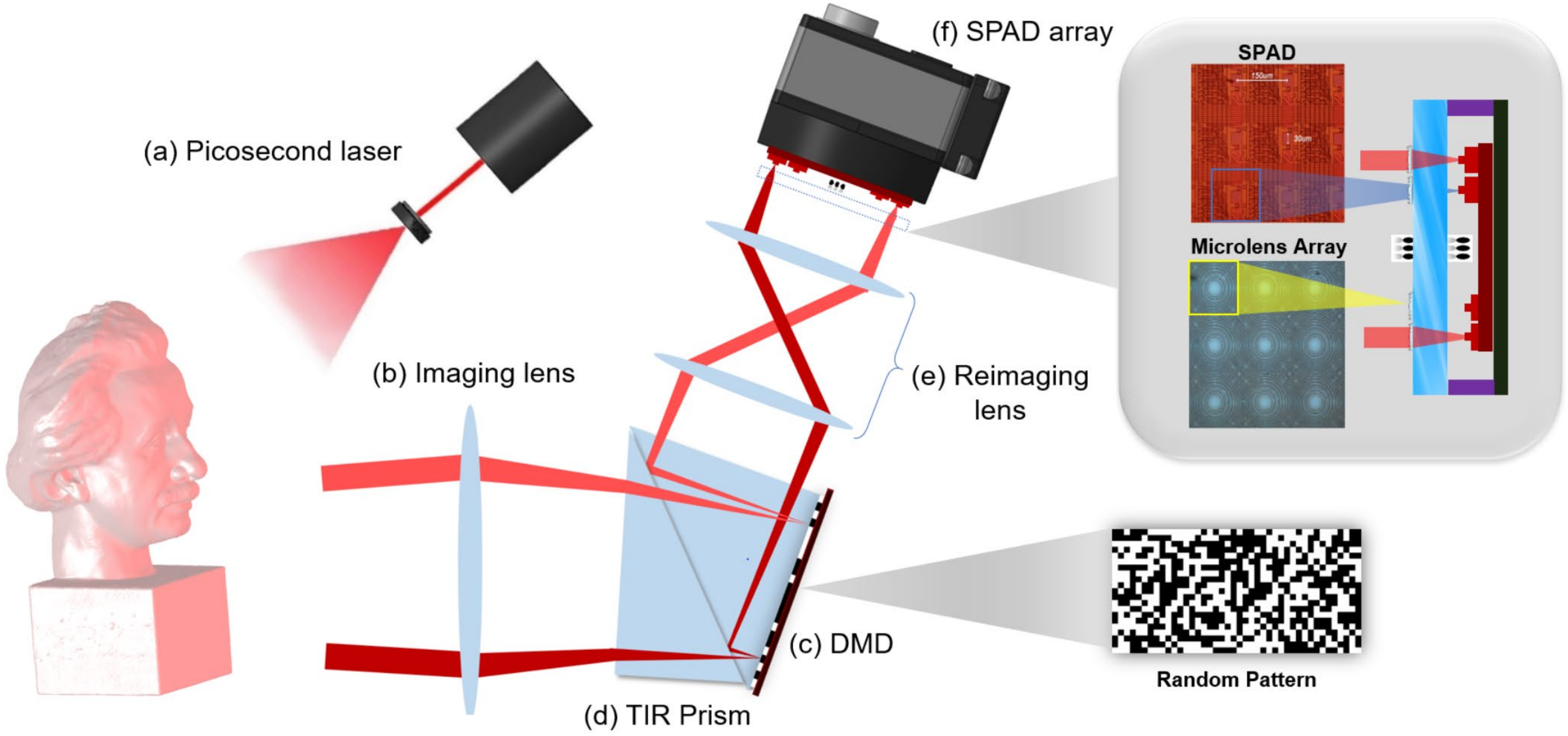






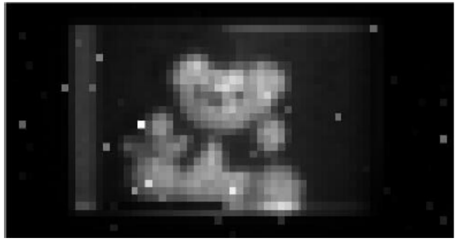


CS SPAD Camera Design

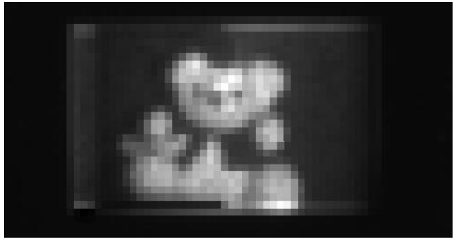


Calibration

Hot pixels $\mathcal{H}$ and background light

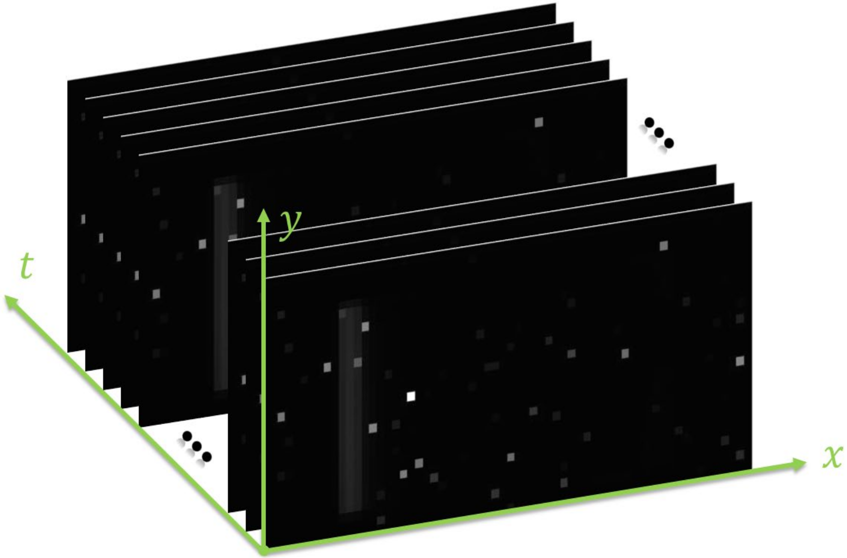


Original data modulated by DMD

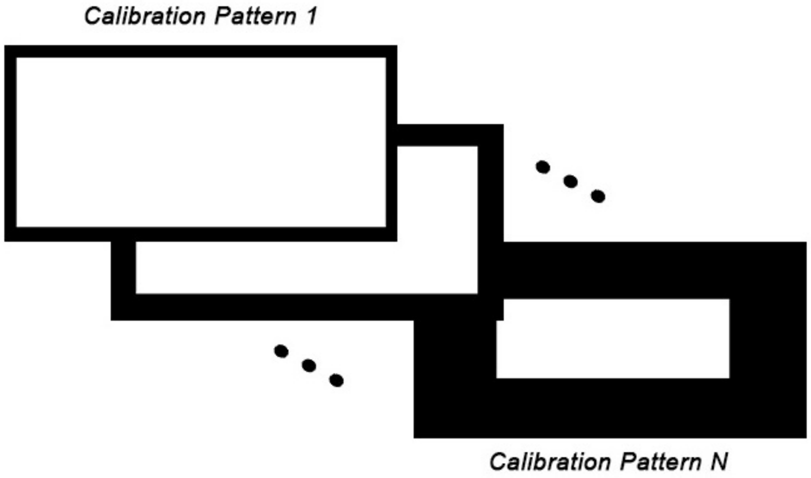


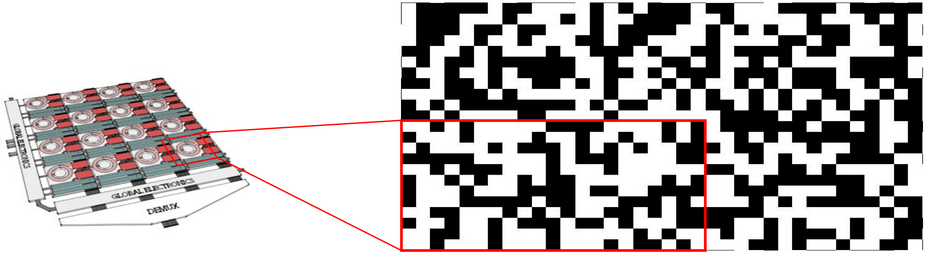
Hot pixels and background light removed

Hot pixels and background light 'Cube'



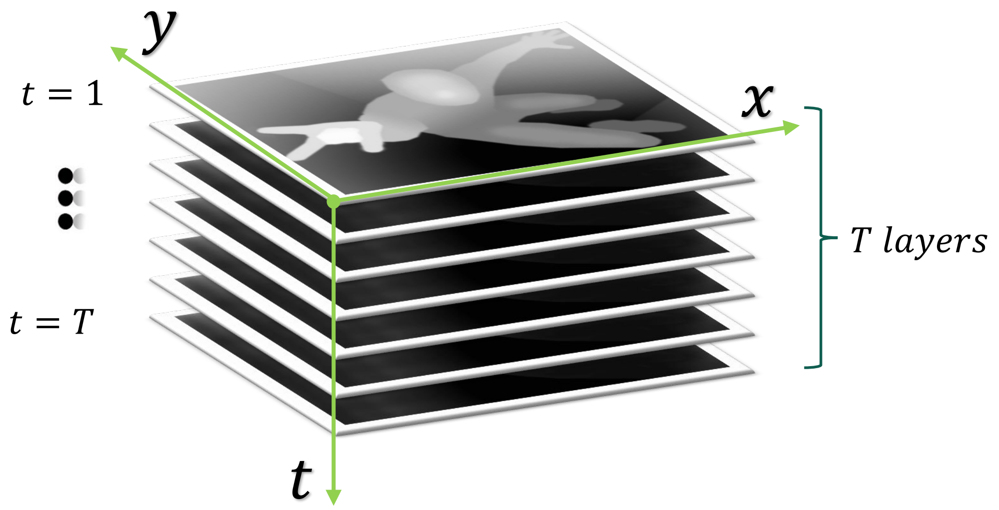
Calibration

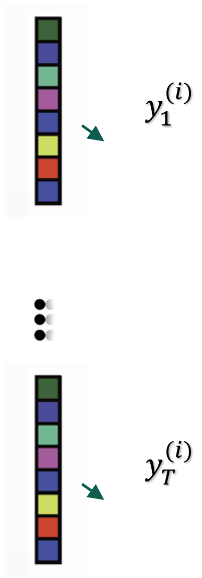




Calibration between DMD and SPAD pixels

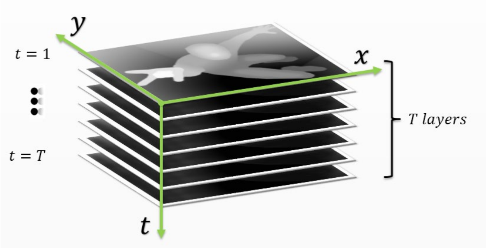
Image Formation

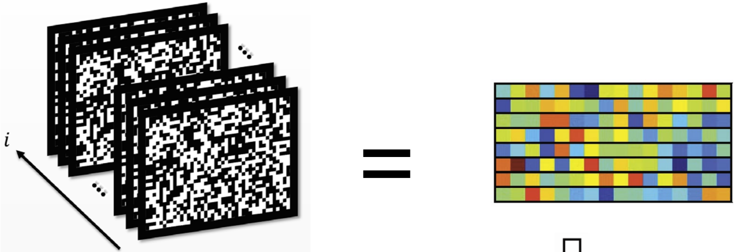


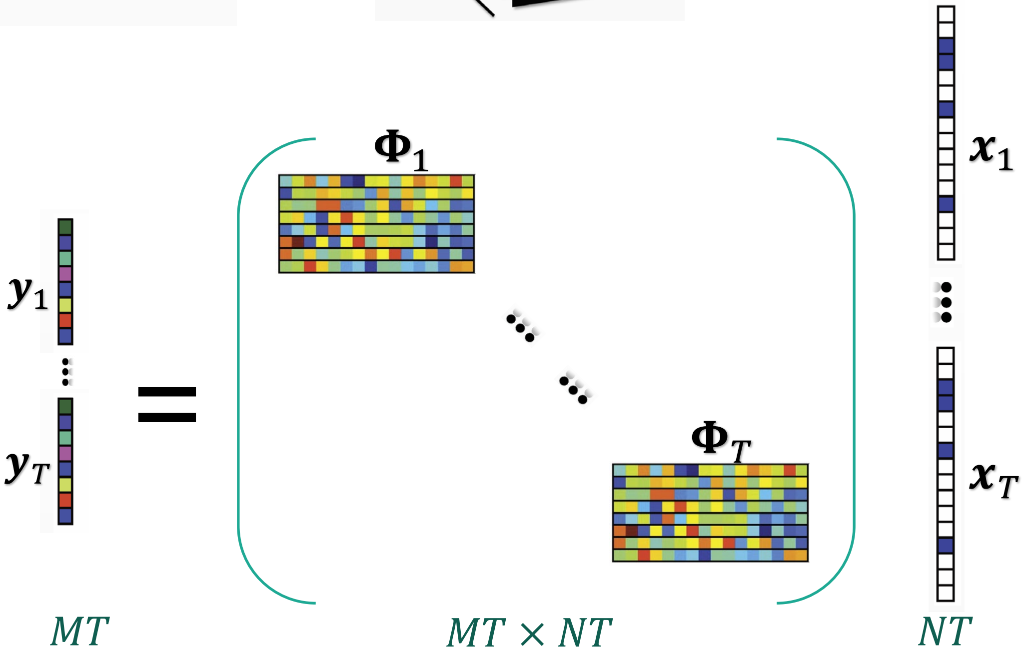


$$y_{int}^{(i)} = \Sigma_t y_t^{(i)}$$

Image Formation in 4D







Build the Objective Function

- Objective function: $\hat{\boldsymbol{X}} = \arg\min_{\boldsymbol{X}} \frac{1}{2}\|\boldsymbol{\Psi}(\boldsymbol{X}) - \boldsymbol{Y}\|_2^2 + \sum_i \lambda_i \boldsymbol{D}_i(\boldsymbol{X})$

subject to: $\begin{cases} \boldsymbol{D}_{1,2}(\boldsymbol{X}) = \|\nabla_s \boldsymbol{X}\|_1 \\ \boldsymbol{D}_3(\boldsymbol{X}) = \|\nabla_\tau \boldsymbol{X}\|_1 \end{cases}$

Where $Y \in R^{K\times T\times n\times m}$, $X \in R^{T\times n\times m}$

- Reformulate as

$$\hat{\boldsymbol{X}} = \arg\min_{\boldsymbol{X},\boldsymbol{w}} \Sigma_i \|\boldsymbol{w}_i\|_1$$

$$s.t. \begin{cases} \boldsymbol{D}_i(\boldsymbol{X}) = \|\boldsymbol{w}_i\|_1 \\ \frac{1}{2}\|\boldsymbol{\Psi}(\boldsymbol{X}) - \boldsymbol{Y}\|_2^2 < \epsilon, \end{cases}$$

Solve the Objective Function

➢ Augmented Lagrangian”

$$
\begin{aligned}
\mathcal{L}\{\boldsymbol{w}, \boldsymbol{X}, \boldsymbol{\sigma}, \boldsymbol{\delta}\} &= \Sigma_i \|\boldsymbol{w}_i\|_1 \\
&- \boldsymbol{\sigma}^T(\boldsymbol{D}(\boldsymbol{X}) - \boldsymbol{w}) - \boldsymbol{\delta}^T(\boldsymbol{\Psi}((\boldsymbol{X}) - \boldsymbol{Y})) \\
&+ \frac{\beta}{2}\|\boldsymbol{D}(\boldsymbol{X}) - \boldsymbol{w}\|_2^2 + \frac{\zeta}{2}\|\boldsymbol{\Psi}((\boldsymbol{X}) - \boldsymbol{Y})\|_2^2.
\end{aligned}
$$

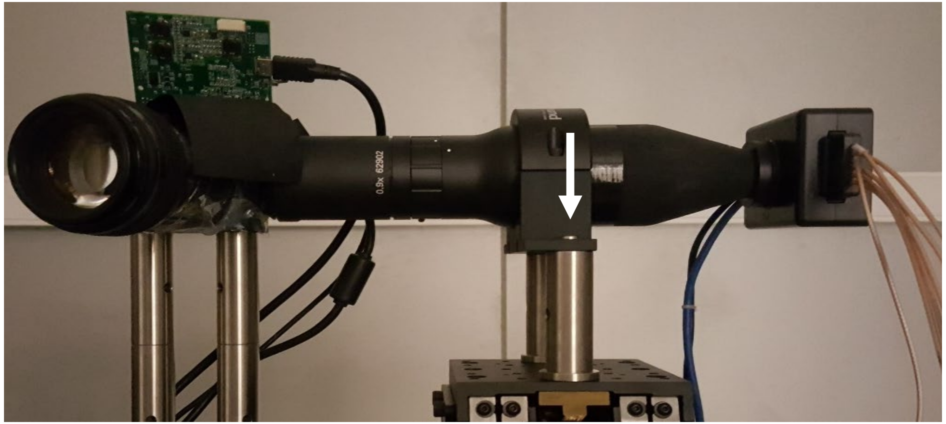
➢ Solving with TVAL3 in 4D

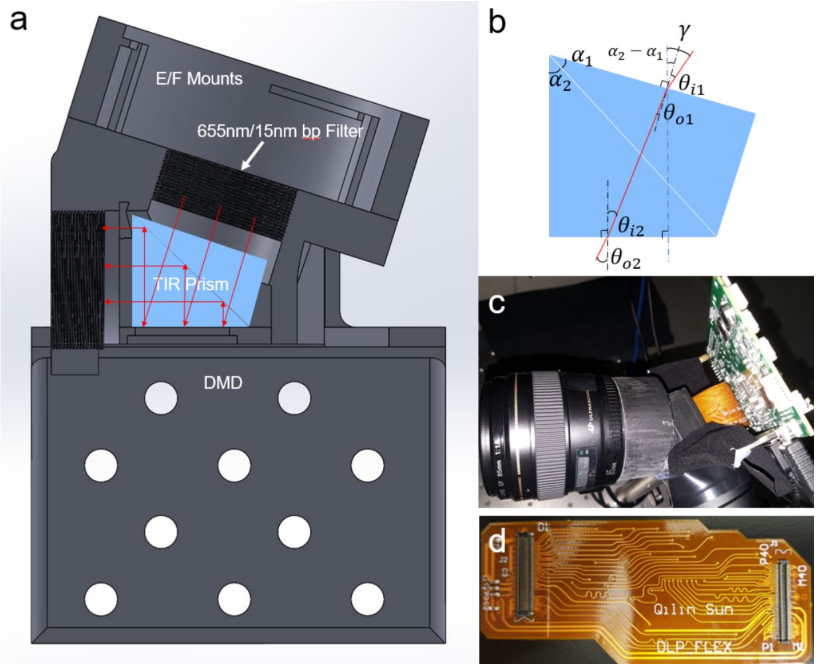
Input: Ψ, $\boldsymbol{Y}$, opts
Result: $\hat{\boldsymbol{X}}$

while $\|\boldsymbol{X}_p - \boldsymbol{X}\|_2 > tol$ **do**
- **Step 1**. $\boldsymbol{X}_p = \boldsymbol{X}^k$
- **Step 2**. Fix $\boldsymbol{w}^k$, do Gradient Descent to $\mathcal{L}\{\boldsymbol{w}^k, \boldsymbol{X}, \boldsymbol{\sigma}, \boldsymbol{\delta}\}$
  - a) compute step length $\tau > 0$ by **BB** rules
  - b) determine $\boldsymbol{X}^{k+1}$ by
    $\boldsymbol{X}^{k+1} = \boldsymbol{X}^k - \alpha\tau\partial_{\boldsymbol{X}}\mathcal{L}\{\boldsymbol{w}^k, \boldsymbol{X}^k, \boldsymbol{\sigma}^k, \boldsymbol{\delta}^k\}$
- **Step 3**. compute $\boldsymbol{w}^{k+1}$ by shrinkage
  $\boldsymbol{w}^{k+1} = \text{shrink}(\boldsymbol{D}(\boldsymbol{X}^{k+1}) - \boldsymbol{\sigma}/\beta, 1/\beta)$
- **Step 4**. update Lagrangian Multipliers by
  $\boldsymbol{\sigma}^{k+1} = \boldsymbol{\sigma}^k - \beta(\boldsymbol{D}(\boldsymbol{X}^{k+1}) - \boldsymbol{w}^{k+1})$
  $\boldsymbol{\delta}^{k+1} = \boldsymbol{\delta}^k - \zeta(\boldsymbol{\Psi}(\boldsymbol{X}^{k+1}) - \boldsymbol{Y})$

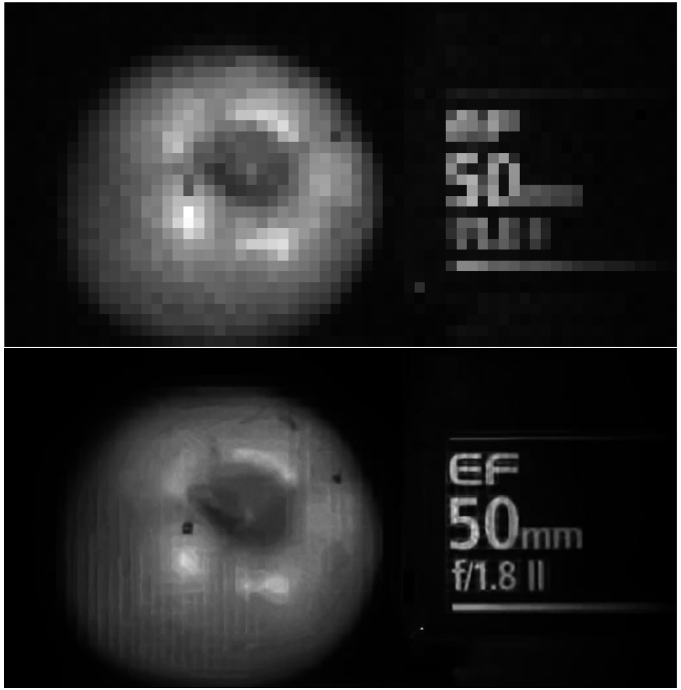
Prototype

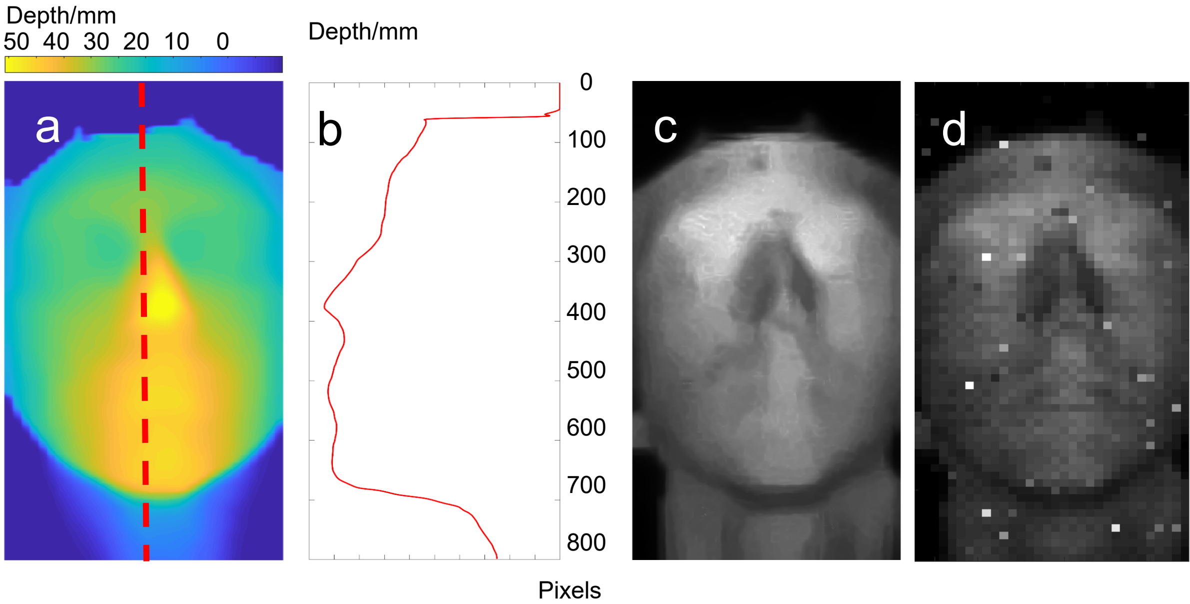

a
Imaging lens
50MHz Synchronizing Signal
Reimaging Lens
Edmund
SPAD
DMLA
DLP4500




a
E/F Mounts
655nm/15nm bp Filter
TIR Prism
DMD
b
α_1
α_2
$\alpha_2 - \alpha_1$
γ
θ_{i1}
θ_{o1}
θ_{i2}
θ_{o2}
c
d
Qilin Sun
DLP FLEX

Results

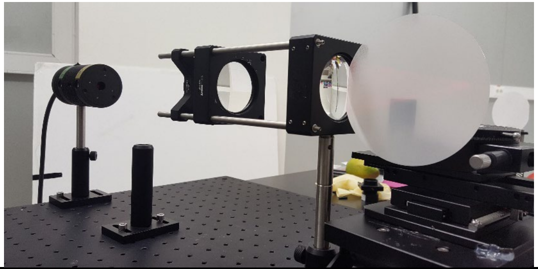




Compressive in Intensity

Compressive in Depth

Results



Compressive in Transient

References and Further Reading

Must read: EE367 course notes on Solving Regularized Inverse Problems with ADMM!

Optional read: EE367 course notes on Noise, Denoising, and Image Reconstruction with Noise

ADMM

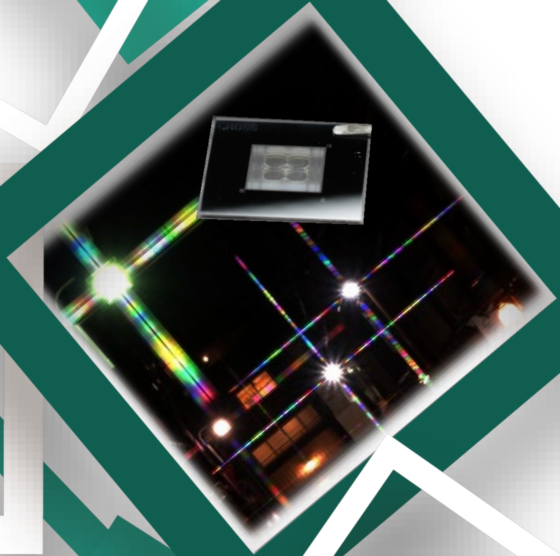
S. Boyd, N. Parikh, E. Chu, B. Peleato, J. Eckstein "Distributed optimization and statistical learning via the alternating direction method of multipliers", Foundation and Trends in Machine Learning, 2001

Single-pixel Imaging

M. Duarte, M. Davenport, D. Takhar, J. Laska, T. Sun, K. Kelly, R. Baraniuk "Single-pixel imaging via compressive sampling", IEEE Signal Processing Magazine 2008

Overview

- Inverse Problem
- Single-pixel Camera and ADMM
- Bayesian Perspective of Inverse Problems
- High-dimensional Inverse Problems



Thank You!

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