Mathematical Modeling and Simulation

Exercise Sheet PDE

Fall 2023

Exercises on modeling with PDEs (week 1 of PDEs):

Exercise 1: [Conservation of mass]

Consider a fluid with varying density $\rho(t, \mathbf{x})$, measured in kg m^{-3}, which is e.g. water. The fluid flows with a given velocity $\mathbf{v}(t, \mathbf{x})$.

a) Consider a volume V. What is the total mass of the fluid inside that volume? What is then the rate of change of mass inside that volume?

b) Mass can only leave the volume by flowing (with $\mathbf{v}$) across the boundary of V. Argue why this mass flux is $\int_{\partial V} \rho \mathbf{v} \cdot \mathbf{n} dS$, where $\mathbf{n}$ is the unit normal pointing out of the boundary ∂V.

c) Using Gauss' theorem on the boundary term, what is the resulting mass conservation equation?

d) Classify the PDE. What is its order? Is it linear or nonlinear?

Exercise 2: [Incompressible Navier-Stokes]

For an incompressible fluid (i.e., ρ is a known constant), the equations describing its velocity $\mathbf{v}(t, \mathbf{x})$ and pressure $p(t, \mathbf{x})$ are the incompressible Navier-Stokes equations

$$\nabla \cdot \mathbf{v} = 0$$
$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} = \mu \nabla^2 \mathbf{v} - \nabla p$$

where μ is the given viscosity of the fluid. The equations come from conservation of mass and conservation of linear momentum.

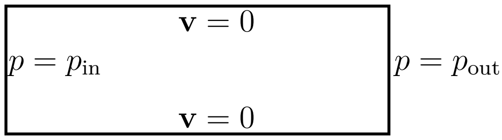
a) In a three-dimensional world, how many equations are there? How many unknowns are there?

b) Classify this system of PDEs. What is the order? Is it linear or nonlinear?

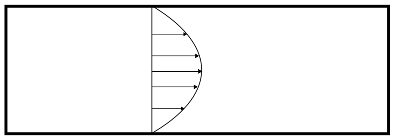
c) For solving these equations in a three-dimensional domain, how many initial and boundary conditions do you presumably need to get a well-posed problem?
Remark: Proving well-posedness of the Navier-Stokes equations is one of the (still unsolved) Millennium problems in mathematics.

Exercise 3: [**Incompressible Navier-Stokes in a channel**]

We consider here the incompressible Navier-Stokes equations as in Exercise 2, and flow through a channel. Boundary conditions are as following:



When starting from $\mathbf{v} = 0$ initially and for low pressure difference $\Delta p = p_{\text{in}} - p_{\text{out}}$, the solution will approach a parabolic velocity profile through the channel:



a) What symmetry do you find in the given solution? How would you advice to discretize the system to reduce computational costs?

b) Do you see any risks of using symmetry when discretizing flow through channels, if the pressure difference Δp is larger?
Hint: If Δp is larger, turbulence can occur instead of a parabolic profile.

Exercise 4: [**Conserving energy**]

Consider the heat equation inside some domain Ω:

$$c\rho \frac{\partial T}{\partial t} = k \Delta T \text{ in } \Omega$$

a) On the boundary of Ω, assume you have homogeneous Neumann boundary conditions:

$$\nabla T \cdot \mathbf{n} = 0 \text{ on } \partial\Omega$$

Show that in this case, the amount of energy $(c\rho T)$ inside Ω remains constant with time.
Hint: integrate over Ω and use Gauss' theorem

b) Instead, consider now homogeneous Dirichlet conditions:

$$T = 0 \text{ on } \partial\Omega$$

What can you say about the amount of energy inside Ω in this case?

c) From the two calculations above, when would you say a system is *thermally insulated*?

Exercises on FD (week 2 of PDEs):

Exercise 5: [**Neumann boundary conditions in FD**]
Look at slide 31, 32 and 35 from the PDE slides

a) Formulate the corresponding discretization matrix $\mathbf{A}$ for FD of 1-dimensional Poisson equation such that the Neumann boundary conditions are incorporated on the two edges. Try both versions where T_0, T_N are either part of the solution vector or not. (So that $\mathbf{A}$ is either $(N-1) \times (N-1)$ or $(N+1) \times (N+1)$).

b) Assemble these matrices in python (or some other language). What is the condition number of the matrix? Why?

Exercise 6: [**Solving steady-state heat equation in 1D**]
Consider the model problem

$$\frac{d^2T}{dx^2} = x \quad 0 < x < 1$$
$$T(0) = 1$$
$$\frac{dT}{dx}(1) = 0$$

a) Write a code in python which can discretize and solve the problem using FD using $N+1$ equally spaced grid points.

b) Find the exact solution and compare the numeric solution with it.

c) Vary the number of grid points and hence the grid size h. How does the maximum absolute difference between exact and numeric solution behave?

Exercise 7: [**Solving steady-state heat equation in 2D**]

Consider the model problem

$$\begin{aligned}
\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} &= 1 && 0 < x < 1, 0 < y < 1 \\
T(0, y) &= 0 && 0 \leq y \leq 1 \\
T(1, y) &= 1 && 0 \leq y \leq 1 \\
\frac{\partial T}{\partial y}(x, 0) &= 0 && 0 < x < 1 \\
\frac{\partial T}{\partial y}(x, 1) &= 0 && 0 < x < 1
\end{aligned}$$

a) Write a code in python which can discretize and solve the problem using FD in 2D using $(N + 1) \cdot (N + 1)$ grid points.

b) What is the dimensionality of the problem? Could the problem be solved as one-dimensional instead? If so, make a corresponding code for the 1D problem and find the exact solution. Compare with the solution you found in a).

Exercise 8: [**Heat equation using FD and backward Euler**]

Consider the model problem from Example 4.28.

a) Formulate the corresponding linear system of equations using backward Euler.

b) Write a code in python that discretizes and solves the problem using FD with $N + 1$ grid points and backward Euler with M time steps.

c) Run the code and vary the grid size and time step size. Do you by trial and error find when the code becomes unstable? Is the stability criterion similar as in Example 4.28 where forward Euler was used?

Exercise 9: [**Heat equation using FD and `solve_ivp`**]

Consider the model problem from Example 4.28. Define the right-hand side for the spatially discretized PDE (using FD with $N + 1$ equally spaced grid points) and give this as input to `solve_ivp`.

Hint: `solve_ivp` *simply needs the right-hand side. There is hence no need to assemble a matrix, but each line of the vector on the right-hand side has to be specified.*

Exercises on weak formulation and FEM (week 3 of PDEs):

Exercise 10: [**Weak form of Poisson equation**]

Consider the Poisson equation with homogeneous Dirichlet boundary conditions:

$$\begin{aligned} &\Delta u + f(\mathbf{x}) = 0 && \mathbf{x} \in \Omega \\ &u = 0 && \mathbf{x} \in \partial\Omega \end{aligned}$$

where Ω is some bounded domain in 2D or 3D with a smooth boundary $\partial\Omega$. By using a function $v \in H_0^1(\Omega)$, find the weak formulation of this problem.

Exercise 11: [**Weak form of heat equation**]

Consider the heat equation with given initial condition and homogeneous Dirichlet boundary conditions:

$$\begin{aligned} &\tfrac{\partial T}{\partial t} = \alpha \Delta T && \mathbf{x} \in \Omega, t > 0 \\ &T = 0 && \mathbf{x} \in \partial\Omega, t > 0 \\ &T = T_0(\mathbf{x}) && \mathbf{x} \in \Omega, t = 0 \end{aligned}$$

for some given, smooth function T_0, and where Ω is some bounded domain in 2D or 3D with a smooth boundary $\partial\Omega$. By using a function $v \in H_0^1(\Omega)$, find the weak formulation of this problem.

Exercise 12: [**Hat functions in 1D**]

Consider the interval $[0, 1]$ and divide it into N equally sized subintervals with endpoints $x_i = i/N, i = 0, \ldots, N$. Using the hat functions ($i = 1, \ldots, N-1$)

$$v_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & x \in [x_{i-1}, x_i] \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & x \in [x_i, x_{i+1}] \\ 0 & \text{otherwise} \end{cases}$$

and the weak form of the PDE $-u''(x) = f(x)$, find the resulting matrix $\mathbf{A}$. Compare with the resulting matrix when using FD to discretize the same problem.

Fourth and fifth week of PDEs: No exercise sheet, students work on project presentation/report.