Mathematical Modeling and Simulation

Exercise Sheet 1

Fall 2023

Exercises will be discussed on 22.08

Exercise 1: [Blood flow in capillary network]

A capillary is a small blood vessel. Inside these vessels, the flow is laminar and depends on the pressure drop between entry point and exit point. Hence, for one capillary, the volumetric flow rate Q (m^3/s) through a capillary of radius r (m) and length L (m) having pressures p_{entry} and p_{exit} (kg/ms^2) at entry and exit, is

$$Q = \frac{\pi r^4}{8\mu} \frac{p_{\text{entry}} - p_{\text{exit}}}{L},$$

where μ (kg/ms) is the blood viscosity. We now consider a small capillary network, as seen in Figure 1. All capillaries have the same radius and viscosity is always the same, hence the flow between two connected points i and j can be written as

$$q_{ij} = \frac{p_i - p_j}{L_{ij}},$$

where L_{ij} is the distance between the two points. This new variable q_{ij} is not the volumetric flow rate itself, but is proportional to it through the (constant and assumed known) factor $\frac{\pi r^4}{8\mu}$, and we use it to represent the volumetric flow rate in the following. Note that $q_{ij} < 0$ means that flow goes from point j to point i. In the following we will set up equations to model the flow through this capillary network.

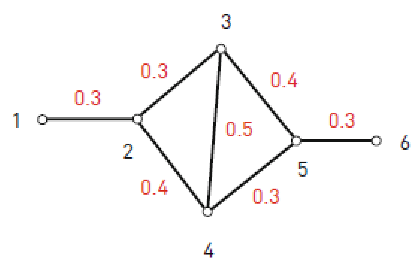

Figure 1: Capillary network: Black lines are the capillaries, while numbered circles (1-6) mark their connections, which are entry/exit points for the various capillaries. Numbers in red are the lengths of each capillary. Figure from Quarteroni and Gervasio.

a) To formulate model equations we use a balance law: What flows into a connection point must flow out of it. Consider point number 3: Which q_{ij} are here relevant? Which balance law must these q_{ij}'s fulfill?

b) Formulate the needed balance laws for all connection points 2-5. Insert what each q_{ij} is using the given lengths. You can additionally formulate equations saying that p_1 and p_6 are equal to some given pressures. This gives you a system of 6 linear equations for the pressures $p_1, \ldots, p_6$.

c) Formulate the linear system of equation from b) as matrix-vector system $\mathbf{Ap} = \mathbf{b}$, with $\mathbf{p} = \left[p_1, p_2, p_3, p_4, p_5, p_6\right]^T$ as the solution vector. If $p_1 = 1$ and $p_6 = 0$, how is the flow through the capillary network? That is, what are the q_{ij}'s?
Hint: Solve matrix equation by row reduction or using python to find p_i's, then find the q_{ij}'s.

d) What are S, Q and M here? What are the state variables and system parameters for the capillary network?

Exercise 2: [**Error when estimating integrals**]

We here explore errors for numerical integration. For a continuous and smooth function $f(x)$, we can approximate the integral $\int_a^b f(x)dx$ by

- Rectangle rule (midpoint rule): $\int_a^b f(x)dx \approx \sum_{k=1}^N f(\frac{x_{k-1}+x_k}{2})\Delta x_k$
- Trapezoidal rule: $\int_a^b f(x)dx \approx \sum_{k=1}^N \frac{f(x_{k-1})+f(x_k)}{2}\Delta x_k$

where $a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b$, with $\Delta x_k = x_k - x_{k-1}$. We here use equidistant points, such that $x_k = a + \frac{b-a}{N}k$ and $\Delta x_k = \frac{b-a}{N} = h$.

a) Make sketches with four-five intervals of how the midpoint rule and trapezoidal rule approximates the integral of some function. Which of the two rules do you expect to approximate the integral better?

b) It is possible to show that the error is
- Midpoint rule: $|\int_a^b f(x)dx - \sum_{k=1}^N f(\frac{x_{k-1}+x_k}{2})\Delta x_k| \leq h^2 \frac{(b-a)}{24} \max_{\xi\in[a,b]} |f''(\xi)|$
- Trapezoidal rule: $|\int_a^b f(x)dx - \sum_{k=1}^N \frac{f(x_{k-1})+f(x_k)}{2}\Delta x_k| \leq h^2 \frac{(b-a)}{12} \max_{\xi\in[a,b]} |f''(\xi)|$.

Which type of error is this? Are the methods convergent? If so, which order?

c) Implement the midpoint rule and trapezoidal rule in python, giving N as input. Make the implementation general so that it is easy to change which function $f(x)$ to integrate and the interval to integrate over.

d) Use $f(x) = e^x$ on $x \in [0, 1]$ and the scripts you made in c). By comparing to the known exact solution of this integral, how does the error for midpoint and trapezoidal rule behave as h becomes smaller? Does the observed error fit with the formulas in b)?

e) Use $f(x) = x$ on $x \in [0, 1]$ and the scripts you made in c). How does the error behave as h becomes smaller? Why is this case so different?

f) What are S, Q and M when approximating integrals?