Mathematical Modeling and Simulation

Exercise Sheet 1

Fall 2023

Exercises on modeling with ODEs, evaluation of data (week 1 of ODEs):

Exercise 1: [Radiation]

The following counts of a radioactive source has been measured in time intervals of 15 seconds. The first time point is hence how many occurred during the first 15 seconds.
74, 64, 62, 54, 47, 39, 40, 35, 34, 29, 34, 30, 31, 22, 14, 14, 13, 25, 18, 11, 13, 16, 13, 12, 10, 12, 11, 13, 9, 8, 7, 5, 5, 4, 5, 2, 1, 3, 2, 2, 1, 1, 1, 0, 0, 1, 2, 0, 1, 1

a) Plot the counts as a function of time. (Let the first data point be at $t = 15$).

b) Radioactive sources usually decay at a rate proportional to their current strength. That is, for strength r,

$$\frac{dr}{dt} = -kr$$

for some proportionality constant $k > 0$. Solve this ODE when $r(0) = r_0 > 0$.

c) The counts are used as a measure of the strength of radioactive sources. By plotting various solutions for some choices of k and r_0, do you think this could be reasonable?

d) To find some qualified guesses for k and r_0, we can use estimated half-life; i.e. the time for the strength to halve.

i) Estimate the half-life from the data: Approximately how much later are the counts half of some earlier value? Consider several data points to get a good estimate of what the half-life could be.

ii) Using the solution of your ODE, how is the half-life connected to k?

iii) Using your results from i) and ii), fix k.

iv) By again using the half-life from i) and the data, what would the estimated value of counts at $t = 0$ be (hence, what is the expected value of the first data point, if the experiment had started 15 seconds earlier)? Use this for r_0.

v) Alternatively, one could have solved $\frac{dr}{dt} = -kr$ with $r(15) = 74$. How does this solution compare to the one found in iv)?

vi) Plot both solutions from iv) and v) (hence, having same k and different r_0) together with the data. How do they compare?

Exercise 2: [Population growth]

The world population in billions is given by

1800	1927	1960	1974	1987	1999	2011	2022
1	2	3	4	5	6	7	8

a) Plot the data as a function of year.

b) The exponential growth model

$$\frac{dy}{dt} = ry$$

assumes the population size y grows faster when there are more inhabitans, with r as growth rate.

i) Using this ODE and using $y(1800) = 1$ as initial condition, solve the ODE.

ii) Plot the solution together with the world population for various choices of r. Does this look like a reasonable model?

c) The logistic growth model

$$\frac{dy}{dt} = r(1 - \frac{y}{K})y$$

assumes there is a limit to the growth, indirectly given by K.

i) A critical point of an ODE is when $\frac{dy}{dt} = 0$. Find the critical point of the logistic growth model. How can you from this interpret K in terms of population size?

ii) Assuming $0 < y < K$, solve the ODE by separation of variables using $y(1800) = 1$ as initial condition.

iii) By trying out various values of r and K, does this seem like a reasonable model?

Exercises with classification of ODEs and uniqueness of solutions (week 2 of ODEs):

Exercise 3: [**Lotka-Volterra equations**]

The Lotka-Volterra equations are also known as predator-prey equations. For two populations, x and y, where x is the population density of the prey and y of the predator, the development of population densities with time are given by

$$\begin{aligned} \tfrac{dx}{dt} &= \alpha x - \beta xy, \\ \tfrac{dy}{dt} &= \delta xy - \gamma y \end{aligned} \tag{1}$$

where

- α is the net growth rate of prey
- β is the effect of predator on prey
- δ is the effect of prey on predator
- γ is the net death rate of predator

All $\alpha, \beta, \delta, \gamma$ are positive constants.

a) Study the system of ODEs (1). Convince yourself why the signs make sense. In particular, why is the effect of predator on prey negative, while effect of prey on predator is positive?

b) Classify the system. What is the order? Is it linear/nonlinear? Autonomous?

c) If an initial condition $x(0) = x_0, y(0) = y_0$ for some values x_0, y_0 are given, do you expect to find a unique solution? Why / Why not?

Exercise 4: [**Oscillating pendulum**]

A mass m is attached to the end of a rod of length L. The other end of the rod is attached to the ceiling and the rod can rotate freely. Hence, m is a pendulum. The position of the pendulum is described by the angle θ between the rod and downwards vertical direction, with counterclockwise position taken as positive. The pendulum experience a downwards gravity force mg and a damping force $c|\frac{d\theta}{dt}|$. The principle of angular momentum states that the time rate of change of angular momentum about any point is equal to the resultant force about that point. The angular momentum about the origin is here $mL^2\frac{d\theta}{dt}$.

a) Draw a figure of the rod and pendulum, mark the angle θ and place on the two forces as vectors.

b) Assuming that both θ and $\frac{d\theta}{dt}$ are positive, show that the resulting governing equation for the pendulum is

$$mL^2\frac{d^2\theta}{dt^2} = -cL\frac{d\theta}{dt} - mgL\sin\theta \tag{2}$$

c) For the other three sign combinations of θ and $\frac{d\theta}{dt}$, show that you still get the same resulting equation.

d) Classify (2). What is the order? Is it linear/nonlinear? Autonomous?

e) Rewrite (2) into a system of two coupled first order ODEs.

Exercise 5: [**Lorenz equations**]

When studying atmospheric (in)stabilities, meteorologist Edward Lorenz ended up with the following system of ODEs, now called the Lorenz equations:

$$\begin{aligned} \tfrac{dx}{dt} &= \sigma(-x+y) \\ \tfrac{dy}{dt} &= rx - y - xz \\ \tfrac{dz}{dt} &= -bz + xy \end{aligned} \tag{3}$$

for positive constants σ, r, b. In this system of equations, x represents intensity of fluid motion, while y and z come from temperature variations in the horizontal and vertical directions. The constants σ and b are related to material properties of the fluid in the atmosphere, while r is proportional to a temperature difference. Many therefore study how these equations behave for different values of r, while σ and b would usually be kept fixed.

a) Classify (3).

b) Stationary points of an ODE $\frac{du}{dt} = f$ is where $\frac{du}{dt} = 0$, which has to be fulfilled for all equations at the same time in the case of a system of ODEs. Depending on r, (3) has up to three real critical points. Find these critical points, expressed as functions of r.

Exercise 6: [**PID controller**]

A PID controller (PID = Proportional–Integral–Derivative) is a control loop mechanism that is widely used in industrial control systems. Say a wished value $r(t)$ is desired (setpoint value), for example the set velocity of the cruise control on the car. The measured value $y(t)$ is instead the current value (process variable), so the actual velocity of the car. We then have an error $e(t) = r(t) - y(t)$. A PID controller tries to minimize this error by adjustment of a control variable $u(t)$, for example adjusting the engine of the car. This is done in three ways (PID). The overall control function is

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau) d\tau + K_d \frac{de}{dt} \tag{4}$$

where K_p, K_i, K_d are non-negative coefficients for the proportional, integral and derivative terms.

a) Differentiate (4) to get an ODE for u.

b) Classify the ODE. *Hint: $e(t)$ is assumed known*

Exercise 7: [**Multiple solutions to IVP**]

Consider the initial value problem

$$y'(t) = y^{1/3}, \quad y(0) = 0$$

a) There are three possible solutions to this IVP. Find all of them.

b) Why was there no unique solution here?

Exercises with numeric discretization of ODEs (week 3 of ODEs):

Exercise 8: [**Discretizing population growth models**]
Recall the exponential growth model

$$\frac{dy}{dt} = ry, \quad y(1800) = 1$$

and logistic growth model

$$\frac{dy}{dt} = r(1 - \frac{y}{K})y, \quad y(1800) = 1$$

a) Discretize each of them using Euler forward and Euler backward method, using a fixed time-step size h which can be varied. Find solutions up to year 2100.

b) Compare the numerical solutions with the exact solutions. What is the global error?

c) Change h by factors of 2 smaller and larger in a reasonable range. How does the global error change? Make a table similar as on slide 51. Do the errors behave as expected?

Exercise 9: [**Discretizing Lotka-Volterra equations**]
Recall the Lotka-Volterra (predator-prey) equations

$$\begin{aligned} \tfrac{dx}{dt} &= \alpha x - \beta xy, \\ \tfrac{dy}{dt} &= \delta xy - \gamma y \end{aligned}$$

a) A stationary point, which is when $\frac{dx}{dt} = \frac{dy}{dt} = 0$, corresponds to solutions that are constant in time. Find the stationary points, expressed as x, y as functions of $\alpha, \beta, \delta, \gamma$.

Now let $\alpha = 2/3, \beta = 4/3, \gamma = 1, \delta = 1$ for the rest of the exercise.

b) What are the stationary points in this case?

c) Let $x(0) = 0.9, y(0) = 0.9$ and discretize the system of equations using Euler forward and Euler backward, using a fixed time-step size. Solve them up to time $t = 10$. Plot the two solutions as a function of time.

d) Interpret the solution: How are predator and prey populations developing?

Exercise 10: [**Discretizing pendulum**]

Recall the pendulum

$$mL^2\frac{d^2\theta}{dt^2} = -cL\frac{d\theta}{dt} - mgL\sin\theta$$

We not let length of rod and mass of pendulum be $L = m = 1$. Gravity is $g = 9.81$. In the following, use the system rewritten into two coupled first order ODEs, with θ and $v = \frac{d\theta}{dt}$ as the two coupled unknowns.

a) Set up a solving scheme for the two coupled first order ODEs using `odeint` or `solve_ivp` according to your choice. Make sure it is easy to change values of c and the initial conditions for θ and v.

b) Let $c = 0$. With $v(0) = 0$ and $\theta(0) = \pi/2$. Without solving, what behavior would you now expect from the solution? Then solve the equations using your scheme from **a)**. Is the solution behaving as you expected?

c) Let $c = 0.1$ and use same initial condition as in **b)**. Without solving, what behavior would you now expect from the solution? Then solve the equations using your scheme from **a)**. Is the solution behaving as you expected?

d) Let $c = 0.1$ and use initial condition $v(0) = 0$ together with $\theta(0) = 0$ and afterwards $\theta(0) = \pi$. Without solving, what behavior would you now expect in the two cases? Then solve the equations using your scheme from **a)**. Are the solutions behaving as you expected?

Exercise 11: [Discretizing Lorenz equations]

Recall the Lorenz equations

$$\begin{aligned}\tfrac{dx}{dt} &= \sigma(-x+y)\\ \tfrac{dy}{dt} &= rx - y - xz\\ \tfrac{dz}{dt} &= -bz + xy\end{aligned}$$

We now take the same numbers as Lorenz used: $\sigma = 10$, $b = 8/3$ and $r = 28$. Let the initial condition be $x(0) = y(0) = z(0) = 1$.

a) Solve the Lorenz equations using `odeint` and `solve_ivp`. Find solutions up to time $t = 100$.

b) Plot the solutions x, y, z found by `odeint` and `solve_ivp` as function of time. Consider especially later times: Do the two solutions compare?

Exercises with phase plane plots and fitting of numbers in ODEs (week 4 of ODEs):

Exercise 12: [**Phase plane of Lotka-Volterra equations**]
We consider the predator-prey system

$$\frac{dx}{dt} = \alpha x - \beta xy,$$
$$\frac{dy}{dt} = \delta xy - \gamma y$$

with $\alpha = 2/3, \beta = 4/3, \gamma = 1, \delta = 1$ as earlier.

a) Make a phase plane plot for the equations. Decide for a reasonable range of the solution variables. *Hint: Including the stationary point(s) in that range is a good idea*

b) Interpret the phase plane. What does this tell us about the behavior of the system?

c) Include also some solution curves in the phase plane. Choose initial conditions such that different type of curves appear. Can you interpret these behaviors? *Hint: It can be helpful to also plot these as function of time; to have time on the horizontal axis and the two solution variables on a common vertical axis.*

Exercise 13: [**Phase plane of the Lorenz equations**]
We consider the Lorenz equations

$$\frac{dx}{dt} = \sigma(-x + y)$$
$$\frac{dy}{dt} = rx - y - xz$$
$$\frac{dz}{dt} = -bz + xy$$

with $\sigma = 10$, $b = 8/3$ and $r = 28$ as before.

a) Make a phase plane plot for the equations. Decide for a reasonable range of the solution variables. Choose number of points carefully. `quiver` can also be used for 3D, visit `https://matplotlib.org/stable/gallery/mplot3d/quiver3d.html`

b) Interpret the phase plane. What does this tell us about the behavior of the system?

c) Include also some solution curves in the phase plane. Choose initial conditions such that different type of curves appear.

d) Instead of plotting arrows, only plot solution curves in the phase plane. Choose initial conditions that are almost identical to each other. What happens?

Exercise 14: [Fermentation model]

Load and rerun the fermentation model from the lectures. The code (`fermentation.py`) and the data (`fermentation.csv`) is available on the canvas-page.

- Estimate the value for k in the model.
- Why will it be important to have a fast ODE-solver in this example?
- Discuss methods to make the ODE-solver faster
- CHALLENGE: extend the implementation to make it capable of handling time varying temperatures. Use the temperature observations to make an improved model.

Fifth (and final) week of ODEs: No exercise sheet, students work on project plan which is due this week.