q = conserved

$$\int_{x_1}^{x_2} q(x,t)\,dx = \text{mass}$$

mass can change by

$$\frac{d}{dt}\int_{x_1}^{x_2} q\,dx = F_1(t) - F_2(t)$$

$\uparrow$ flux

ex: $F = u \cdot q$

veloc. density

$\frac{m}{s}$ $\frac{kg}{m}$

$$F = f(q)$$

$$= f(q(x_1,t)) - f(q(x_2,t))$$

$$= -\int_{x_1}^{x_2} \frac{d}{dx} f(q(x,t))\,dx$$

$$\rightarrow \int_{x_1}^{x_2} \underbrace{\frac{\partial q}{\partial t} + \frac{\partial f(q(x,t))}{\partial x}}_{\text{cons. law}}\,dx = 0$$

$$q_t + \left(f(q)\right)_x = 0$$

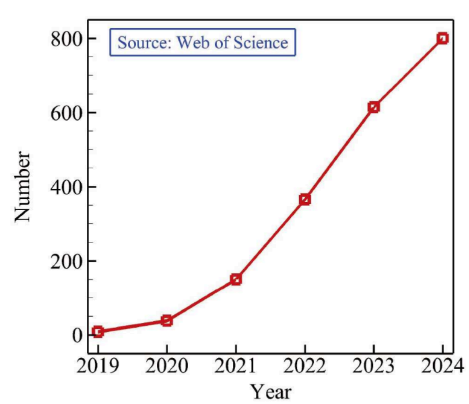


Figure 1 (Color online) Number of PINNs-related papers published from 2019 to 2024.

$u = u(x,t)$

Consider $u_t + (f(u))_x = 0$

$u_t + (\frac{u^2}{2})_x = 0$

$f = u^2/2$

$f' = u$

let $x(t)$ = curve in x-t.

satisfy $\frac{dx(t)}{dt} = f'(u)$

$$\frac{d\,u(x(t),t)}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \cdot \frac{dx}{dt}$$

$$= 0$$

① $\frac{du}{dt} = 0$ on this path. $\to$ u = constant

② $\frac{dx}{dt}$ = constant if $f(u)$ only depends on u

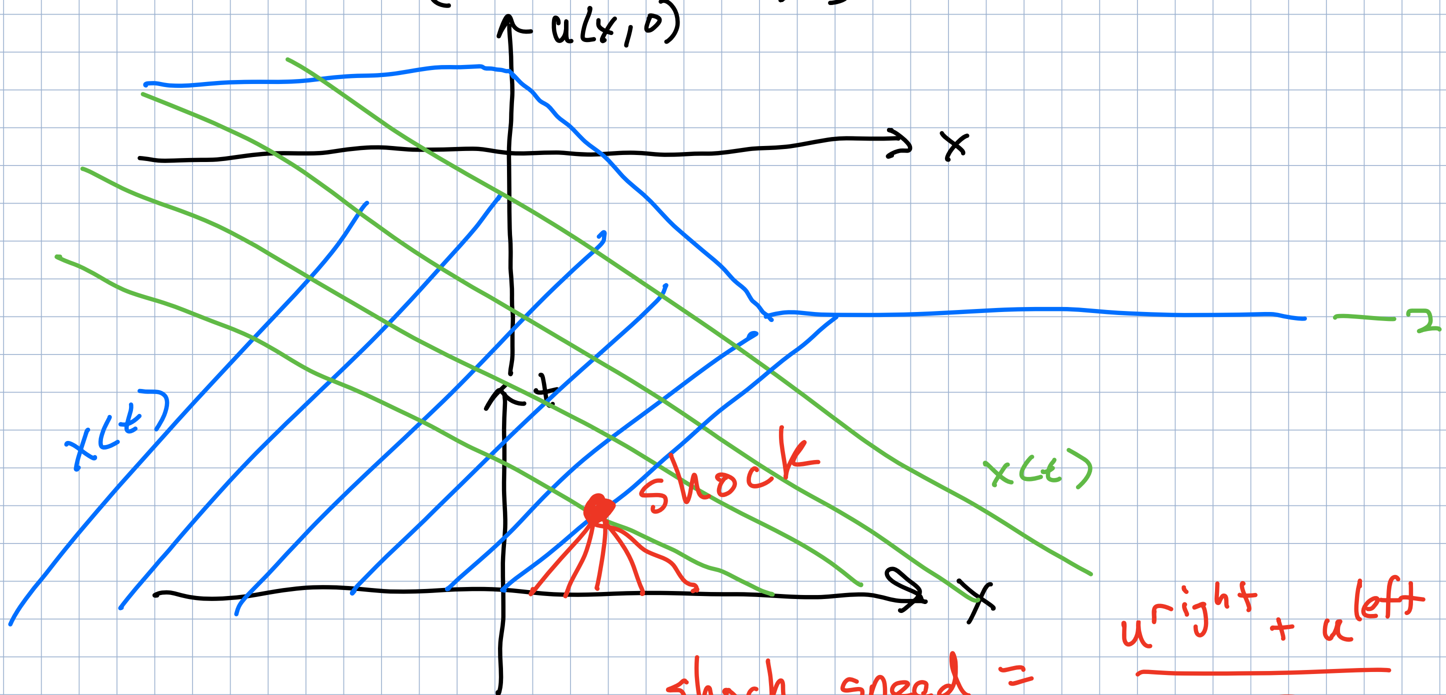
$\to$ straight lines.

Burgers: $u_t + (\frac{u^2}{2})_x = 0$ νu_{xx}

(or $u_t + u u_x = 0$)

curves: $x(t) = u(x_0, 0) \cdot t + x_0$

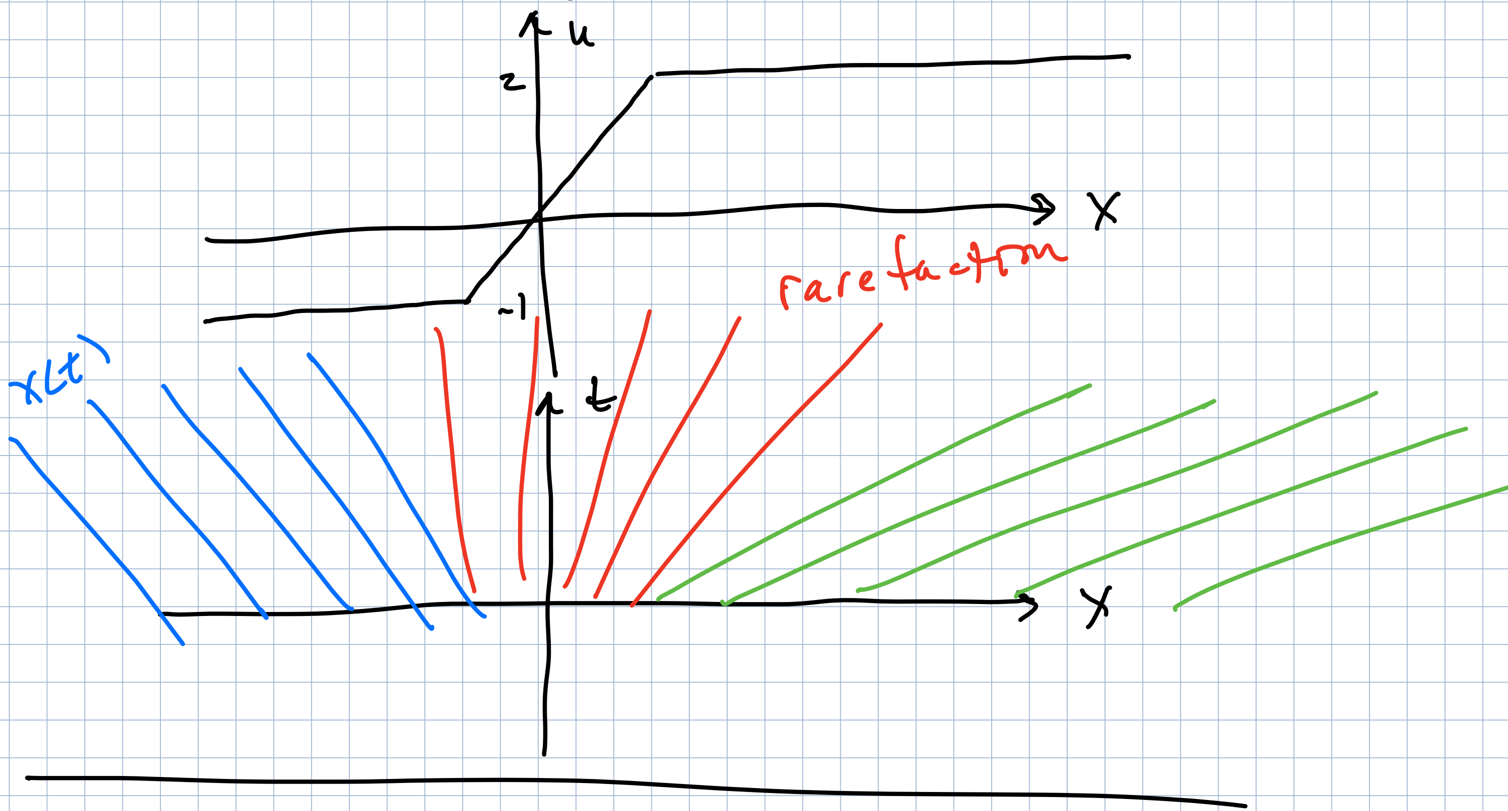
$$u(x,0) = \begin{cases} 1 & x \le 0 \\ 1-x & 0 \le x \le 3 \\ -2 & x \ge 3 \end{cases}$$

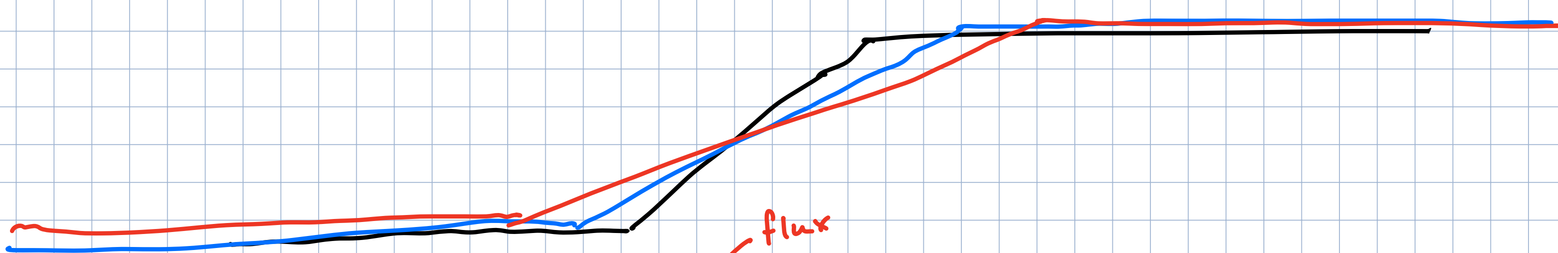


shock speed $= \dfrac{u^{right} + u^{left}}{2}$

Case 2

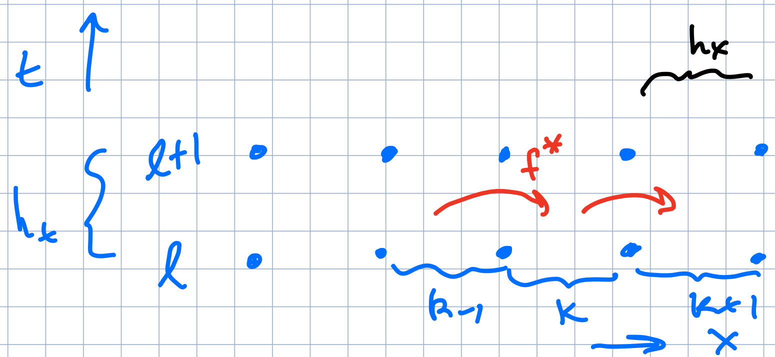
$$u = \begin{cases} -1 & x \le -1 \\ x & -1 \le x \le 2 \\ 2 & x \ge 2 \end{cases}$$





flux

$$u_t + \left(f(u)\right)_x = 0$$

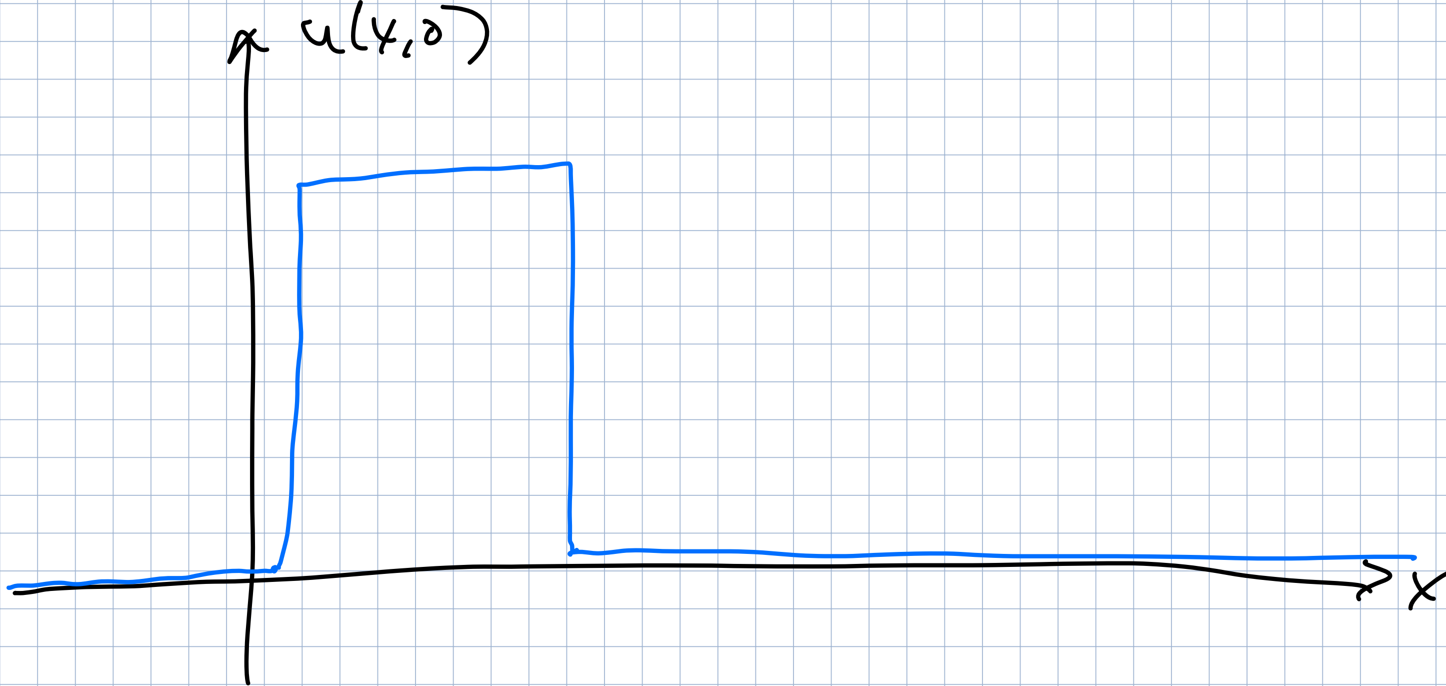


$$\frac{u_{k\,\ell+1} - u_{k\,\ell}}{h_t} + \frac{f^*(u_{k-1}, u_k) - f^*(u_k, u_{k+1})}{h_x} = 0$$

$$\cancel{f^*(u_k, u_{k+1}) = \alpha\, u_k}$$

$$f^*(u_{k,\ell}, u_{k+1,\ell}) = \frac{f(u_{k,\ell}) + f(u_{k+1,\ell})}{2} - \frac{\alpha_{k+\frac{1}{2}}}{2}(u_{k+1,\ell} - u_{k,\ell}),$$

$$\alpha_{k+\frac{1}{2}} = \max\left(|f'(u_{k,\ell})|, |f'(u_{k+1,\ell})|\right),$$



$f = u_t + u u_x = 0$ (or $\frac{.01}{\pi} u_{xx}$) $\quad x \in [-1,1]$, $t \in [0,1]$

$u(0,x) = -\sin(\pi x)$ I.C.

$u(t,-1) = u(t,1) = 0$ B.C.

Try it

$$u_t + u u_x = 0 \;\frac{.01}{\pi} u_{xx} \qquad x \in [-1,1]$$
$$t \in [0,1]$$

$$u(0,x) = -\sin(\pi x) \qquad \text{I.C.}$$
$$u(t,-1) = u(t,1) = 0 \qquad \text{B.C.}$$

PINNs.

2019

Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations

M. Raissi [a], P. Perdikaris [b,*], G.E. Karniadakis [a]

2021

Review Article | Published: 24 May 2021

Physics-informed machine learning

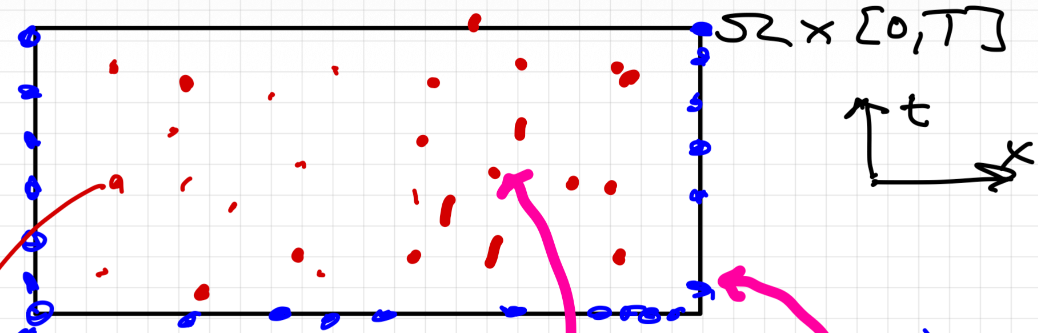
George Em Karniadakis, Ioannis G. Kevrekidis, Lu Lu, Paris Perdikaris, Sifan Wang & Liu Yang

Nature Reviews Physics 3, 422–440 (2021) | Cite this article

134k Accesses | 5884 Citations | 204 Altmetric | Metrics

New loss:

Domain

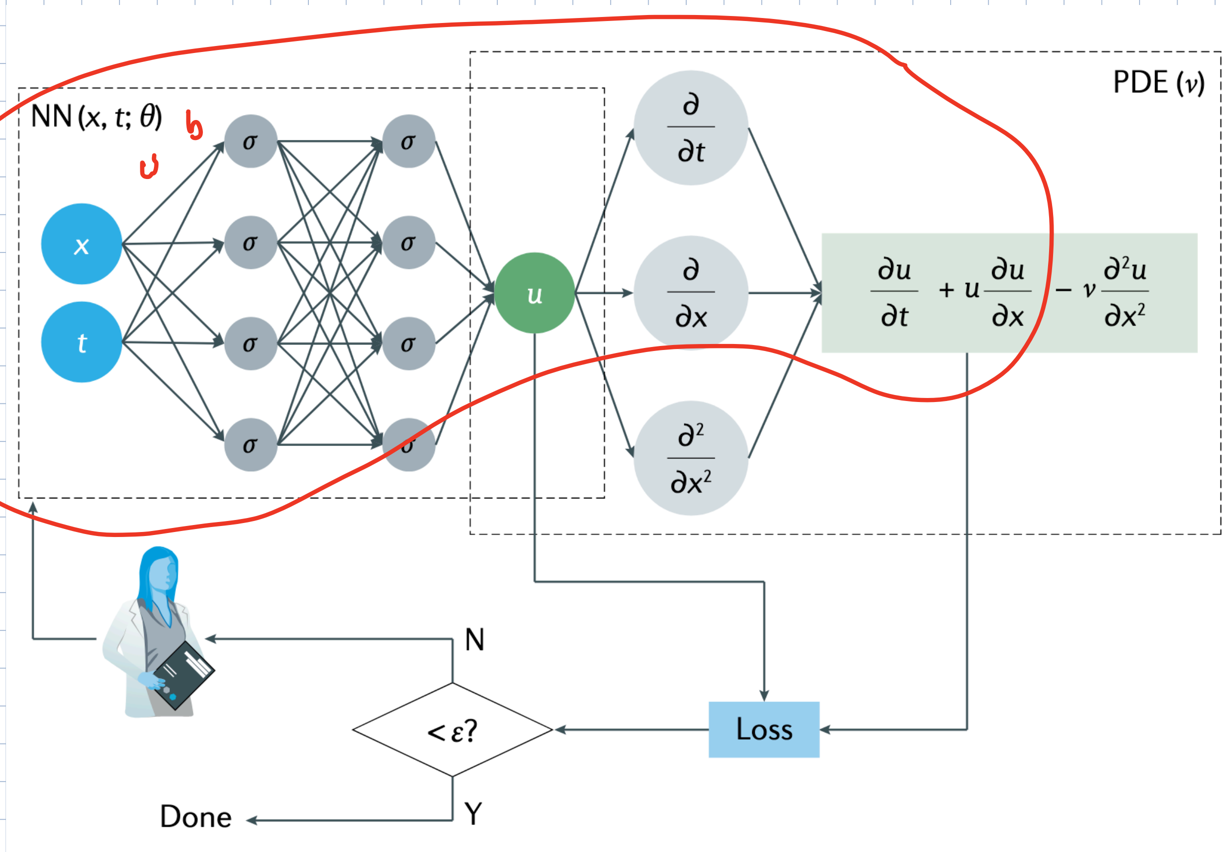


[N_u samples on the time-space bdy

N_f samples in $\Omega \times [0,T]$

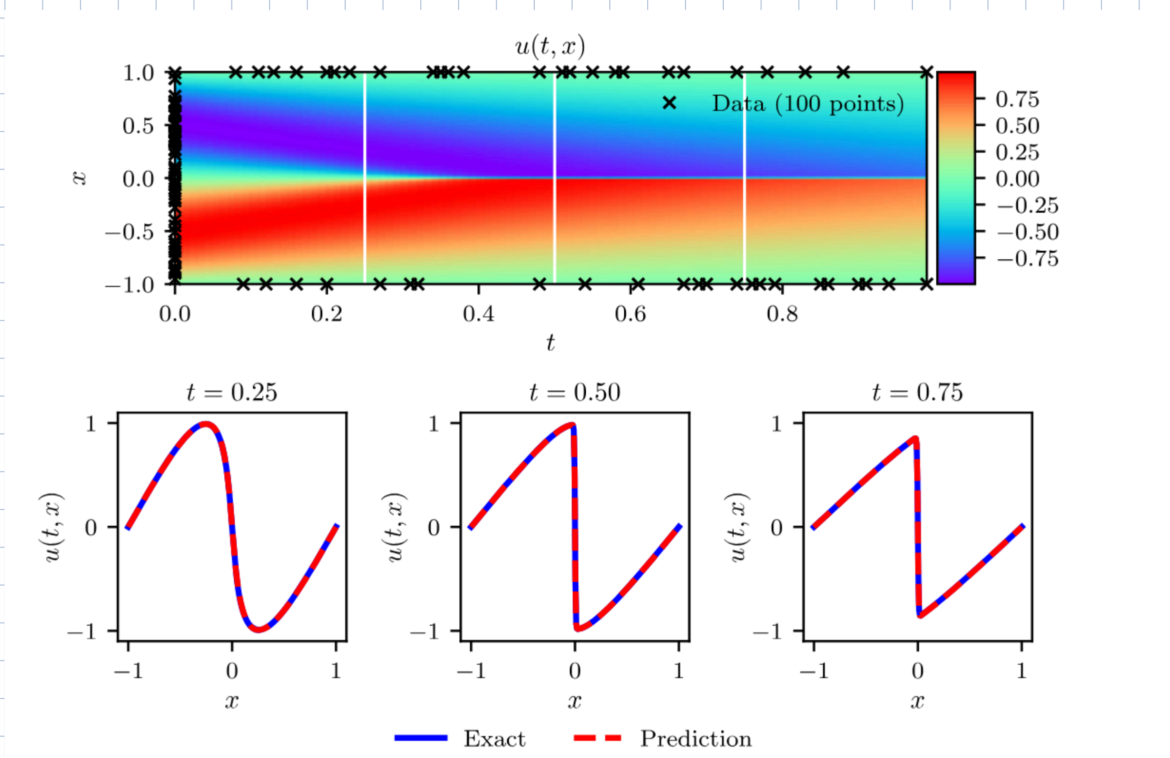
$$\text{loss} = \frac{1}{N_u} \sum_{i=1}^{N_u} \left| u^i_{model} - u(t^i_u, x^i_u) \right|^2$$
$$+ \frac{1}{N_f} \sum_{i=1}^{N_f} \left| f(t^i_f, u^i_f) \right|^2$$

$$f = u_t + u u_x = 0$$



PINNs: ① Given PDE parameters, find u

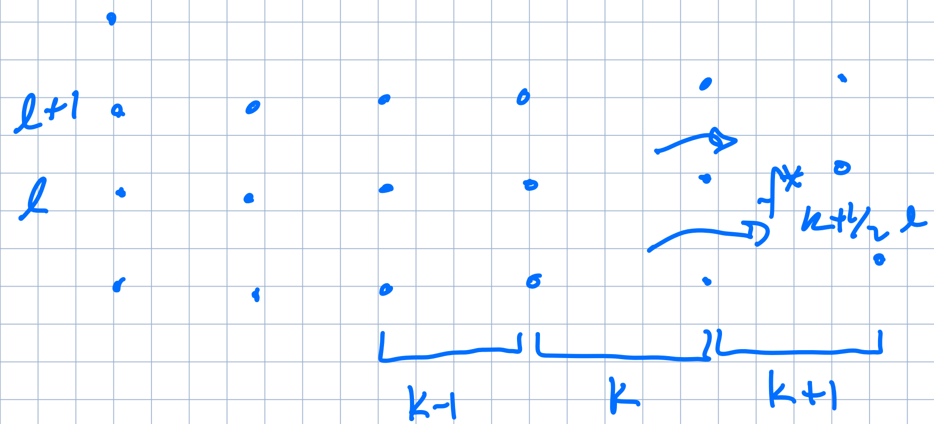
② Given some data, find PDE parameters



$$u_t + (f(u))_x = 0$$

$$\downarrow$$

$$\frac{u_{k\,\ell+1} - u_{k\ell}}{h_t} + \frac{f^*_{k+1/2,\ell} - f^*_{k-1/2\,\ell}}{h_x} = 0$$



$$f^*_{k+1/2,\ell} = f^*(u_{k\ell}, u_{k+1\,\ell})$$

$$f^*(u_{k,\ell}, u_{k+1,\ell}) = \frac{f(u_{k,\ell}) + f(u_{k+1,\ell})}{2} - \frac{\alpha_{k+\frac{1}{2}}}{2}(u_{k+1,\ell} - u_{k,\ell}),$$

...monly used choice for $\alpha_{k+\frac{1}{2}}$ is

$$\alpha_{k+\frac{1}{2}} = \max\left(|f'(u_{k,\ell})|, |f'(u_{k+1,\ell})|\right),$$