Supplementary File of Manuscript "Optimal Defense Resource Allocation Considering Nonlinear Attack Cost in Power Systems" to IEEE Transactions on Industrial Informatics

I. Proof of Lemma 1

Proof: The proof consists of two parts: (a) Firstly, if a certain $\boldsymbol{\phi}^*$ and a specific $\boldsymbol{b}_\alpha$ together form a feasible solution for (10), then there must exist a $\boldsymbol{b}_\beta$ such that $\boldsymbol{\phi}^*$ and $\boldsymbol{b}_\beta$ together form a feasible solution for (12). (b) Conversely, if a certain $\boldsymbol{\phi}^*$ and a certain $\boldsymbol{b}_\beta$ together form a feasible solution for (12), then there must exist a specific $\boldsymbol{b}_\alpha$ such that $\boldsymbol{\phi}^*$ and $\boldsymbol{b}_\alpha$ together form a feasible solution for (10).

(a) First, assuming that a certain $\boldsymbol{\phi}^*$ and a specific $\boldsymbol{b}_\alpha$ comprise a feasible solution for (10). Then clearly we have

$$\boldsymbol{h}_j^T[\phi_1^* f_1(b_{\alpha,1}), \ldots, \phi_m^* f_m(b_{\alpha,m})]^T \geq R,\ \forall j \in \mathcal{N} \tag{1}$$

which indicates that

$$\boldsymbol{h}_j^T \boldsymbol{\phi}^* > 0,\ \forall j \in \mathcal{N} \tag{2}$$

Then, recalling the fact that for each i, $\tilde{f}_i(\cdot)$ is a monotonically increasing function over the feasible region, and combining (14) and (23), there must exists a $\boldsymbol{b}_\beta \leq \boldsymbol{b}_{max}$ such that

$$\begin{aligned}&\boldsymbol{h}_j^T[\phi_1^* \tilde{f}_1(b_{max,1}), \ldots, \phi_m^* \tilde{f}_m(b_{max,m})]^T\\ &\geq \boldsymbol{h}_j^T[\phi_1^* \tilde{f}_1(b_{\beta,1}), \ldots, \phi_m^* \tilde{f}_m(b_{\beta,m})]^T\\ &\geq R,\ \forall j \in \mathcal{N}\end{aligned} \tag{3}$$

which indicates that $\boldsymbol{\phi}^*$ and $\boldsymbol{b}_\beta$ comprise a feasible solution for (12).

(b) Conversely, let's assume that a certain $\boldsymbol{\phi}^*$ and a specific $\boldsymbol{b}_\beta$ together form a feasible solution for (12). Therefore, we have

$$\boldsymbol{h}_j^T[\phi_1^* \tilde{f}_1(b_{\beta,1}), \ldots, \phi_m^* \tilde{f}_m(b_{\beta,m})]^T \geq R,\ \forall j \in \mathcal{N} \tag{4}$$

which indicates that

$$\boldsymbol{h}_j^T \boldsymbol{\phi}^* > 0,\ \forall j \in \mathcal{N} \tag{5}$$

Then, recalling that for each i, $f_i(\cdot)$ is a continuous, non-negative function that crosses the origin within the feasible region, and combining (13) and (26), there must exist a $\boldsymbol{b}_\alpha \leq \boldsymbol{b}_\gamma$ such that

$$\begin{aligned}&\boldsymbol{h}_j^T[\phi_1^* f_1(b_{\gamma,1}), \ldots, \phi_m^* f_m(b_{\gamma,m})]^T\\ &\geq \boldsymbol{h}_j^T[\phi_1^* f_1(b_{\alpha,1}), \ldots, \phi_m^* f_m(b_{\alpha,m})]^T\\ &\geq R,\ \forall j \in \mathcal{N}\end{aligned} \tag{6}$$

which indicates that $\boldsymbol{\phi}^*$ and $\boldsymbol{b}_\alpha$ comprise a feasible solution for (10). Hence, Lemma 1 is proven. ■