1 Newmark Method

Within the following pages the Newmark time integration scheme is outlined. We will discuss the basic idea and derive the resulting formulas in terms of a displacement centric discretization. A simple example is given, in terms of a single mass oscillator, respectively. The oscillator is described by a differential equation, thats based on the spacial position of a mass, but also its first and second derivative in time. We will apply the Newmark integration to a numerical solution scheme for this differential equation.

1.1 The Newmark method for $\beta = \frac{1}{4}$, $\gamma = \frac{1}{2}$

For a numerical treatment, the idea of discretization is adopted to the temporal dimension. Here we divide time into a set of finite sized time steps Δt, while we want to calculate the values of certain quantities e.g. displacements at the discrete points in time that define the borders of each time step.

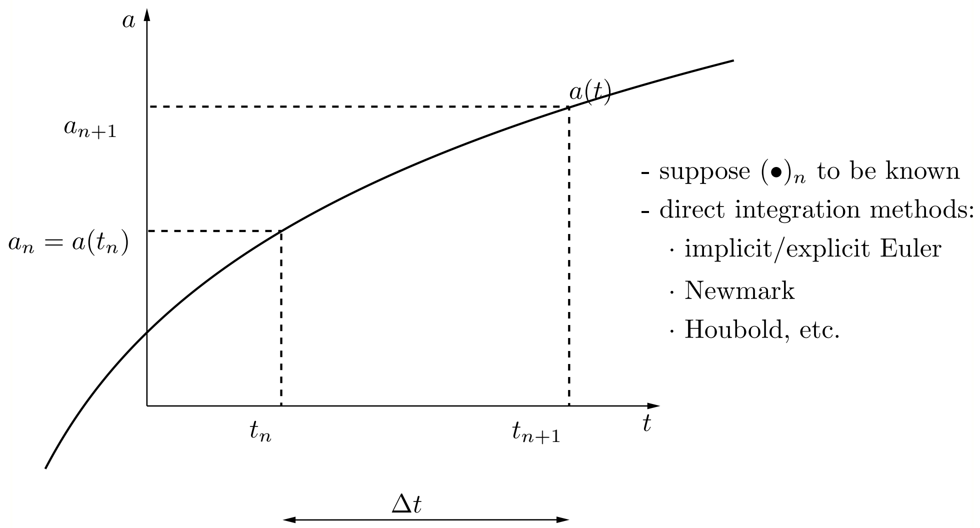


Figure 1: A function a(t) evaluated at discrete time points.

The main assumption of Newmarks method is a constant acceleration. In the following we will use the mean value of a time step Δt. If we apply integration of the acceleration to obtain velocity increments for this step, and further integration for displacement advancement in time, we gain

$$v_{n+1} = v_n + \int_{\Delta t} a(t)\ \mathrm{d}t\ = v_n + \frac{1}{2}(a_{n+1} + a_n)\,\Delta t \tag{1}$$

$$
\begin{aligned}
u_{n+1} &= u_n + v_n\,\Delta t + \int\limits_{\Delta t}\int\limits_{\Delta t} a(t)\ \mathrm{d}t \\
&= u_n + v_n\,\Delta t + \int\limits_{\Delta t} \frac{1}{2}\left(a_{n+1} + a_n\right)\Delta t\ \mathrm{d}t \\
&= u_n + v_n\Delta t + \frac{1}{4}(a_{n+1} + a_n)\,\Delta t^2
\end{aligned} \tag{2}
$$

In this form we were able to compute displacements and velocities from known values $(\bullet)_t$ if we would only know the acceleration for the new time step a_{n+1}. In the following we assume that we would obtain u_{n+1} instead, by means of an e.g. finite element formulation. Hence we need to make some reformulations.

From equation (2) : $u_{n+1} = \hat{u}(a_{n+1})\ \rightarrow a_{n+1} = \hat{a}(u_{n+1})$

$$
\begin{aligned}
u_{n+1} &= u_n + v_n\,\Delta t + \frac{1}{4}\left(a_{n+1} + a_n\right)\Delta t^2 \\
u_{n+1} - u_n - v_n\,\Delta t &= \frac{1}{4}\left(a_{n+1} + a_n\right)\Delta t^2 \quad .
\end{aligned} \tag{3}
$$

$$
\Rightarrow \frac{4}{\Delta t^2}\left(u_{n+1} - u_n - v_n\,\Delta t\right) - a_n = a_{n+1}
$$

Inserting eq. (3) into eq. (1) we can formulate $v_{n+1} = \hat{v}(a_{n+1})\ \rightarrow v_{n+1} = \hat{v}(u_{n+1})$,

$$
\begin{aligned}
v_{n+1} &= v_n + \frac{1}{2}\left(a_{n+1} + a_n\right)\Delta t \\
&= v_n + \frac{1}{2}\left(\frac{4}{\Delta t}\left(u_{n+1} - u_n - v_n\,\Delta t\right) - a_n + a_n\right)\Delta t \\
&= v_n + \frac{1}{2}\left(\frac{4}{\Delta t^2}\,\Delta t(u_{n+1} - u_n - v_n\,\Delta t)\right) \\
&= v_n + \frac{2}{\Delta t}\left(u_{n+1} - u_n - v_n\,\Delta t\right) \\
&= -v_n + \frac{2}{\Delta t}\left(u_{n+1} - u_n\right).
\end{aligned} \tag{4}
$$

With the Newmark time integration, we can thus draw the following two conclusions for a_{n+1} and v_{n+1} respectively,

$$
\boxed{
\begin{aligned}
a_{n+1} &= \frac{4}{\Delta t^2}(u_{n+1} - u_n - v_n\Delta t) - a_n\,, \quad (5) \\
v_{n+1} &= -v_n + \frac{2}{\Delta t}(u_{+1} - u_n)\,. \\
& \qquad\qquad\qquad\qquad\qquad (6)
\end{aligned}
}
$$

Furthermore, we can denote displacements, velocities, and accelerations in incremental form,

$$u_{n+1} = u_n + \Delta u \ , \ a_{n+1} = a_n + \Delta a \ \text{ and } \ v_{n+1} = v_n + \Delta v \, . \tag{7}$$

Now we solve equation (5) for Δa:

$$\begin{aligned} a_n + \Delta a &= \frac{4}{\Delta t^2} \left(u_{n+1} - u_n - v_n \, \Delta t \right) - a_n \\ &= \frac{4}{\Delta t^2} \left(u_{n+1} - u_n - v_n \, \Delta t \right) - 2a_n \\ \Rightarrow \Delta a &= \frac{4}{\Delta t^2} \left(\Delta u - v_n \, \Delta t \right) - 2a_n \end{aligned} \tag{8}$$

The same procedure is then applied to (6),

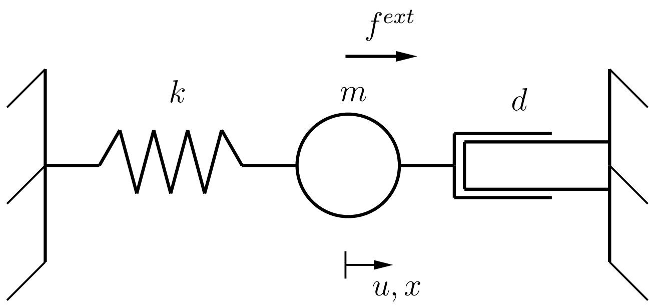
$$\begin{aligned} v_n + \Delta v &= -v_n + \frac{2}{\Delta t} \left(u_{n+1} - u_n \right) \\ \Delta v &= \frac{2}{\Delta t} \left(u_{n+1} - u_n \right) - 2v_n \\ \Rightarrow \Delta v &= \frac{2}{\Delta t} \, \Delta u - 2v_n \, . \end{aligned} \tag{9}$$

For the here discussed Newmark time integration we conclude the incremental form:

$$\boxed{\begin{aligned} \Delta \, a &= \frac{4}{\Delta \, t^2} \left(\Delta u - v_n \Delta t \right) - 2a_n && (10) \\ \Delta \, v &= \frac{2}{\Delta t} \, \Delta u - 2v_n && (11) \end{aligned}}$$

Remark: It is generally not necessary to formulate everything in terms of the displacements.

1.2 Single-mass-Oscillator: An Example of Newmark time integration



$k \rightarrow$ sprint stiffness
$m \rightarrow$ mass
$d \rightarrow$ damping parameter
$x \rightarrow$ position
$\dot{x} \rightarrow$ v, velocity
$\ddot{x} \rightarrow$ a, acceleration
$f^{ext} \rightarrow$ external force (given)
$u \rightarrow$ displacement
$t \rightarrow$ t

Figure 2: Single-mass-Oscillator

The system displayed in Figure (2) is described by the differential equation in equilibrium, for $t = t_{n+1}$.

$$m\, a_{n+1} + d\, v_{n+1} + k\, u_{n+1} = f^{ext}_{n+1}. \tag{12}$$

Our goal is to locally solve for each step of size Δt, for the displacement increment Δu. For this special problem, where we choose the mass to be located at the origin, u and x coincide. The differential equation in the incremental form is given with

$$m\,(a_n + \Delta a) + d(v_n + \Delta v) + k(u_n + \Delta u) = f^{ext}_{n+1}. \tag{13}$$

Bearing in mind that all quantities $(\bullet)_n$ are known, as well as the force f^{ext}_{n+1} at each time, we want so solve for Δu. In a first step we summarize all visible terms that can be known initially,

$$\begin{aligned} \Rightarrow \underbrace{m\, a_n + d\, v_n + k\, u_n}_{f^{int}_n} + m\Delta a + d\Delta v + k\Delta u &= f^{ext}_{n+1} \\ f^{int}_n \qquad + m\Delta a + d\Delta v + k\Delta u &= f^{ext}_{n+1}. \end{aligned} \tag{14}$$

$$\Rightarrow m\Delta a + d\Delta v + k\Delta u = f^{ext}_{n+1} - f^{int}_n. \tag{15}$$

Now the incremental Newmark equations (8) and (9) are inserted:

$$\begin{aligned} m\,[\frac{4}{\Delta t^2}\,(\Delta u - v_n\,\Delta t) - 2a_n] + d\,[\frac{2}{\Delta t}\,\Delta u - 2v_n] + k\Delta u &= f^{ext}_{n+1} - f^{int}_n \\ \frac{4m}{\Delta t^2}\,\Delta u + \frac{2d}{\Delta t}\,\Delta u + k\Delta u \underbrace{- \frac{4m v_n\,\Delta t}{\Delta t^2} - 2m\,a_n - 2d\,v_n}_{} &= f^{ext}_{n+1} - f^{int}_n \end{aligned} \tag{16}$$

$$\Rightarrow \frac{4m}{\Delta t^2}\,\Delta u + \frac{2d}{\Delta t}\,\Delta u + k\Delta u = f^{ext}_{n+1} - f^{int}_n + f^{NM}_n.$$

Finally we simplify the left hand side and solve for the unknown,

$$\begin{aligned} f^{\Delta}\Delta u &= f^{ext}_{n+1} - f^{int}_n + f^{NM}_n \\ \Delta u &= \frac{1}{f^{\Delta}}(f^{ext}_{n+1} - f^{int}_n + f^{NM}_n). \end{aligned} \tag{17}$$

The above terms summarized:

$$\begin{aligned}
f^{\Delta} &= \frac{4m}{\Delta t^2} + \frac{2d}{\Delta t} + k \\
f_n^{int} &= ma_n + dv_n + ku_n \\
f_n^{NM} &= m(\frac{4}{\Delta t} v_n + 2a_n) + 2du_n.
\end{aligned} \tag{18}$$

Once we have calculated Δu, we can update the system as follows:

$$\begin{aligned}
u_{n+1} &= u_n + \Delta u \\
v_{n+1} &= -v_n + \frac{2}{\Delta t} \Delta u \\
a_{n+1} &= \frac{4}{\Delta t} (\Delta u - v_n \Delta t) - a_n.
\end{aligned} \tag{19}$$

Within a numerical program we will advance in time, meaning all quantities at the end of this step $(\bullet)_{n+1}$ will serve as the known quantities $(\bullet)_n$ in the next time step.

1.3 The general Newmark method

The Newmark method for time integration introduces control parameters, that steer whether the integration is more explicit or more implicit.

$$
\begin{aligned}
v_{n+1} &= v_n[(1-\gamma)a_n + \gamma a_{n+1}]\,\Delta t \\
u_{n+1} &= u_n + v_n\Delta t + [(\frac{1}{2}-\beta)a_n + \beta a_{n+1}]\,\Delta t^2.
\end{aligned}
\tag{20}
$$

The following conclusions based on β and γ can be drawn:

$$
\begin{aligned}
\beta < \frac{1}{4} \text{ and } \gamma < \frac{1}{2} &\rightarrow \text{ more explicit} \\
\beta > \frac{1}{4} \text{ and } \gamma > \frac{1}{2} &\rightarrow \text{ more implicit} \\
\beta = 0 \text{ and } \gamma = 0 &\rightarrow \text{ coincides explicit Euler} \\
\beta = \frac{1}{2} \text{ and } \gamma = 1 &\rightarrow \text{ coincides implicit Euler.}
\end{aligned}
\tag{21}
$$

Commonly used values are $\beta = \frac{1}{4}$ and $\gamma = \frac{1}{2}$.