11-7 作业

32. (1) 由题意知

$$\begin{aligned}
&\operatorname{Cov}(\alpha X+\beta Y,\alpha X-\beta Y)\\
=&\operatorname{Cov}(\alpha X,\alpha X-\beta Y)+\operatorname{Cov}(\beta Y,\alpha X-\beta Y)\\
=&\operatorname{Cov}(\alpha X,\alpha X)-\operatorname{Cov}(\alpha X,\beta Y)+\operatorname{Cov}(\beta Y,\alpha x)-\operatorname{Cov}(\beta Y,\beta Y)\\
=&\alpha^2\operatorname{Cov}(X,X)-0+0-\beta^2\operatorname{Cov}(Y,Y)\\
=&\left(\alpha^2-\beta^2\right)\sigma^2
\end{aligned}$$

(2) $X,Y\sim N(\mu,\sigma^2)$, 且相互独立. 则 $\alpha X+\beta Y$ 与 $\alpha X-\beta Y$ 都服从正态分布. 由正态分布的不相关与独立等价，有

$$\operatorname{Cov}(\alpha X+\beta Y,\alpha X-\beta Y)=\left(\alpha^2-\beta^2\right)\sigma^2=0$$
$$\Rightarrow\alpha=\pm\beta.$$

34. 法一：
(1)(X,Y) 的联合密度为

$$f_{X,Y}(x,y)=\begin{cases}\frac{1}{2}, & |x|+|y|\le 1,\\ 0 & \text{其他}.\end{cases}$$

由对称性可知，$EX=EY=EXY=0$，所以

$$\operatorname{Cov}(X,Y)=EXY-EXEY=0.$$

(2) 不独立。
证：画图通过计算面积之比易得：$P(X\le-\frac{1}{2})=P(Y\le-\frac{1}{2})=\frac{1}{8}$,
$P(X\le-\frac{1}{2},Y\le-\frac{1}{2})=0$，所以

$$P(X\le-\frac{1}{2},Y\le-\frac{1}{2})\ne P(X\le-\frac{1}{2})P(Y\le-\frac{1}{2}).$$

法二：
X,Y 的边际密度为

$$f_X(x)=\begin{cases}\int_{-x-1}^{x+1}\frac{1}{2}dy=x+1, & -1<x<0,\\ \int_{x-1}^{-x+1}\frac{1}{2}dy=-x+1, & 0<x<1.\end{cases}\quad \text{类似地，}f_Y(y)=\begin{cases}y+1, & -1<y<0,\\ -y+1, & 0<y<1.\end{cases}$$

(1) $\mathrm{Cov}(X,Y)=E(XY)-E(X)E(Y)$. 其中

$$E(X)=\int_{-1}^{0}x\cdot(x+1)dx+\int_{0}^{1}x\cdot(-x+1)dx=0,\quad \text{同理},E(Y)=0$$

$$E(XY)=\int_{-1}^{0}\int_{-x-1}^{x+1}\frac{xy}{2}dydx+\int_{0}^{1}\int_{x-1}^{-x+1}\frac{xy}{2}dydx=0$$

$\mathrm{Cov}(X,Y)=E(XY)-E(X)E(Y)=0.$

(2) $f(x,y)\neq f_X(x)f_Y(y)$, X,Y 不独立.

36. (1) $Z=\pi X+(1-\pi)Y$, 则方差为

$$\begin{aligned}\mathrm{Var}(Z)&=\mathrm{Var}(\pi X+(1-\pi)Y)\\&=\pi^2\,\mathrm{Var}(X)+(1-\pi)^2\,\mathrm{Var}(Y)+2\pi(1-\pi)\,\mathrm{Cov}(X,Y)\\&=\pi^2\sigma^2+(1-\pi)^2\sigma^2+2\pi(1-\pi)\cdot\left(-\frac{1}{2}\right)\sigma^2\\&=\left(3\pi^2-3\pi+1\right)\sigma^2\cdot\pi\in(0,1)..\end{aligned}$$

在 $\pi\in(0,1)$ 上, $3\pi^2-3\pi+1=3\pi(\pi-1)+1<1$ 恒成立, $\mathrm{Var}(Z)<\sigma^2$, 即证投资组合 Z 的风险小于将所有资本投资于其中一个的风险.

(2) $f(\pi)=3\pi^2-3\pi+1,\pi\in(0,1)$ 的极小值点为 $\pi=1/2$. 所以使得投资组合风险最小的分配比例为 $\pi=1/2$.

40. 由 $N(t)$ 的分布律可知，$N(T)|T=t\sim Possion(\lambda t)$. 记 T 的概率密度为 $f_T(t)$, 由重期望公式有

$$\begin{aligned}E[N(T)]&=E\Big[E\big(N(T)|T\big)\Big]=\int_0^\infty E\big(N(T)|T=t\big)f_T(t)dt\\&=\int_0^\infty\lambda t\cdot f_T(t)dt=\lambda E(T)=\lambda a\end{aligned}$$

同理，$$\begin{aligned}E[N(T)^2]&=E\Big[E\big(N(T)^2|T\big)\Big]=\int_0^\infty E\big(N(T)^2|T=t\big)f_T(t)dt\\&=\int_0^\infty(\lambda^2t^2+\lambda t)\cdot f_T(t)dt=\lambda^2E(T^2)+\lambda E(T)=\lambda^2(a^2+b)+\lambda a\end{aligned}$$

$$\begin{aligned}E[TN(T)]&=E\Big[E\big(TN(T)|T\big)\Big]=\int_0^\infty E\big(TN(T)|T=t\big)f_T(t)dt\\&=\int_0^\infty tE\big(N(T)|T=t\big)\cdot f_T(t)dt=\int_0^\infty\lambda t^2\cdot f_T(t)dt\\&=\lambda E(T^2)=\lambda(a^2+b)\end{aligned}$$

则可求得：

(1)

$$\mathrm{Cov}\Big(T, N(T)\Big) = E(TN(T)) - E(T)E(N(T)) = \lambda(a^2+b) - a\cdot\lambda a = \lambda b.$$

(2)

$$\mathrm{Var}(N(T)) = E[N(T)^2] - [E(N(T))]^2 = \lambda^2(a^2+b) + \lambda a - \lambda^2 a^2 = \lambda^2 b + \lambda a.$$

41. 由教材例 3.14 可知：$(X,Y) \sim N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$, 则

$$X|Y=y \sim N\Big(\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(y-\mu_2), \sigma_1^2(1-\rho^2)\Big).$$

本题中 $(X,Y) \sim N(1,2,4,9,0.3)$, 则

$$\begin{aligned} E(X|Y=2) &= \mu_1 + \rho\frac{\sigma_1}{\sigma_2}(y-\mu_2) = 1 \\ E(XY^2+Y|Y=1) &= E(XY^2|Y=1) + E(Y|Y=1) \\ &= E(X|Y=1) + E(Y|Y=1) \\ &= 1 + 0.3(2/3)(1-2) + 1 \\ &= \frac{9}{5} \end{aligned}$$