2015 - 2016.

一、$\lambda<0$ $\quad Y(x)=Ae^{\sqrt{-\lambda}x}+Be^{-\sqrt{-\lambda}x}$

代入边值条件 $\quad A=B=0$

$\lambda=0$ $\quad Y(x)=A+Bx \qquad Y'=B.$

故 $B=0$ 得 $Y(x)=1$ 此时 $\lambda=0$

$\lambda>0.$ $\quad Y(x)=A\cos\sqrt{\lambda}x+B\sin\sqrt{\lambda}x.$

$Y'(x)=-A\sqrt{\lambda}\sin\sqrt{\lambda}x+B\sqrt{\lambda}\cos\sqrt{\lambda}x$

$Y'(0)=0 \rightarrow B=0$

$Y(x)=\cos\sqrt{\lambda}x \qquad Y'(x)=-\sqrt{\lambda}\sin\sqrt{\lambda}x$

$Y'(16)=-\sqrt{\lambda}\sin\sqrt{\lambda}\cdot 16=0 \qquad 16\sqrt{\lambda}=n\pi$

$\lambda=\dfrac{n^2\pi^2}{256}. \qquad Y(x)=\cos\dfrac{n\pi x}{16}.$

二、$\begin{cases} u_t=4u_{xx} \\ u(t,0)=u(t,5)=0 \\ u(0,x)=\phi(x) \end{cases} \qquad u=TX$

$T'X=4TX'' \Rightarrow \dfrac{T'}{4T}=\dfrac{X''}{X}=\lambda$

$X=\sin\dfrac{n\pi x}{5}. \qquad \lambda=-\dfrac{n^2\pi^2}{25} \qquad T=e^{-\frac{4n^2\pi^2}{25}t}.$

故：$u=\sum\limits_{n=1}^{\infty}A_n e^{-\frac{4n^2\pi^2}{25}t}\sin\dfrac{n\pi x}{5}.$

$A_n=\dfrac{2}{5}\int_0^5\phi(x)\sin\dfrac{n\pi x}{5}.$

$\phi(x)=\delta(x-2) \qquad A_n=\dfrac{2}{5}\sin\dfrac{2n\pi}{5}.$

三、1. $a=2$

$u=\dfrac{1}{2}[(x+2t)^2+(x-2t)^2]+\dfrac{1}{4}\int_{x-2t}^{x+2t}\sin 2s\,ds$

$=x^2+4t^2+\dfrac{1}{8}\cos 2(x-2t)-\dfrac{1}{8}\cos 2(x+2t).$

$=x^2+4t^2+\dfrac{1}{4}\sin 2x\cos 4t.$

2. $u=v+w.$ $\quad v=$ 上一题 $u.$

$w=\int_0^t\dfrac{1}{4}\int_{x-2(t-\tau)}^{x+2(t-\tau)}f(\tau,s)\,ds\,d\tau$

$=\int_0^t x^2(t-\tau)\tau^2+\dfrac{8}{6}\tau^2(t-\tau)^3\,d\tau$

$=\dfrac{1}{12}x^2t^4+\dfrac{1}{45}t^6.$

$u=x^2+4t^2+\dfrac{1}{4}\sin 2x\cos 4t+\dfrac{1}{12}x^2t^4+\dfrac{1}{45}t^6.$

四、$F[u]=\bar{u}$ $\qquad \dfrac{d\bar{u}}{dt}=4(-i\lambda)^2\bar{u}+5\bar{u}=(5-4\lambda^2)\bar{u} \qquad \bar{u}|_{t=0}=\bar{\phi}(\lambda)$

$\bar{u}=\bar{\phi}(\lambda)e^{(5-4\lambda^2)t}$

$u=\phi(x)*F^{-1}[e^{5t}e^{-4\lambda^2t}]=\phi(x)*\dfrac{1}{4\sqrt{\pi t}}e^{-\frac{x^2}{16t}}e^{5t}$

$\phi(x)=e^{-x^2} \qquad u=\int_{-\infty}^{+\infty}\dfrac{e^{5t}}{4\sqrt{\pi t}}e^{-(x-s)^2}e^{-\frac{s^2}{16t}}ds$

$=\dfrac{e^{5t}e^{-\frac{t}{16+t}x^2}}{4\sqrt{\pi t}}\int_{-\infty}^{+\infty}e^{-\left(\sqrt{1+\frac{1}{16t}}s-\frac{1}{\sqrt{1+\frac{1}{16t}}}x\right)^2}ds$

$=\dfrac{e^{5t}e^{-\frac{t}{16+t}x^2}}{\sqrt{16t+1}}$

五. $u = TR$

$T'R = TR'' + \frac{1}{r}TR' \Rightarrow \frac{T'}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -\lambda$

$T' = -\lambda T \quad T = e^{-\lambda t}$.

$r^2R'' + rR' + \lambda r^2 R = 0 \quad \lambda > 0$

$\Rightarrow R = J_0(w_n r) \quad w_n$ 为: $J_0(w)$ 的第n个正根 $\lambda = w_n^2$

故 $u = \sum_{n=1}^{\infty} A_n e^{-w_n^2 t} J_0(w_n r)$

$$A_n = \frac{2}{J_1^2(w_n)} \int_0^1 r J_0(w_n r)\phi(r)\, dr.$$

$\phi(r) = J_0(ar) + 3J_0(br)$ 时

$$u = e^{-a^2 t}J_0(ar) + 3e^{-b^2 t}J_0(br).$$

六. 1.

$$G(M, M_0) = \frac{1}{4\pi r(M, M_0)} - \frac{1}{4\pi r(M, M_1)} \quad (x<0, \ x, y \in \mathbb{R})$$

$$= \frac{1}{4\pi\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}} - \frac{1}{4\pi\sqrt{(x+\xi)^2+(y-\eta)^2+(z-\zeta)^2}}$$

2.. 记 $2s = x \quad u_s = 2u_x \quad u_{ss} = 2u_{xx} \cdot 2 = 4u_{xx}$

故: $4u_{xx} + u_{yy} + u_{zz} = -\delta(x, y, z)$

$\Rightarrow u_{ss} + u_{yy} + u_{zz} = -\delta(2s, y, z) = -\frac{1}{2}\delta(s, y, z)$

$$G(M, M_0) = \frac{1}{8\pi\sqrt{(s-\xi)^2+(y-\eta)^2+(z-\zeta)^2}} - \frac{1}{8\pi\sqrt{(s-\xi)^2+(y-\eta)^2+(z+\zeta)^2}}.$$

$$\frac{\partial G}{\partial \zeta}\Big|_{\zeta=0} = \frac{1}{2}\frac{2(z-\zeta)}{8\pi(\sqrt{(s-\xi)^2+(y-\eta)^2+z^2})^3} + \frac{1}{2}\frac{2(z+\zeta)}{8\pi(\sqrt{(s-\xi)^2+(y-\eta)^2+(z+\zeta)^2})^3}$$

$$= \frac{2z}{8\pi\sqrt{(s-\xi)^2+(y-\eta)^2+z^2}} = \frac{z}{4\pi\sqrt{(s-\xi)^2+(y-\eta)^2+z^2}}$$

$x = 2s \quad \xi' = 2\xi$

$$\frac{\partial G}{\partial n} = \frac{\partial G}{\partial \zeta}\Big|_{\zeta=0} = \frac{z}{2\pi\sqrt{(x-\xi)^2+4(y-\eta)^2+4z^2}}$$

$$u(x, y) = -\iint_{\mathbb{R}^2} \frac{z \cdot \varphi(\xi, \eta)}{2\pi\sqrt{(x-\xi)^2+4(y-\eta)^2+4z^2}}\, d\xi\, d\eta$$

七. $\Delta_3 u \Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r}) = \frac{1}{r^2}(2r\frac{\partial u}{\partial r}) + \frac{1}{r^2}\cdot r^2 \cdot \frac{\partial^2 u}{\partial r^2} = \frac{2}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2}$

故 $u = TR$.

$\frac{T''R}{a^2} = \frac{2}{r}TR' + TR'' \Rightarrow \frac{T''}{a^2T} = \frac{2}{r}\frac{R'}{R} + \frac{R''}{R} = -\lambda$

$T = A_n \sin a\sqrt{\lambda_n}t + B_n \cos a\sqrt{\lambda_n}t \quad R = r^{-(n+1)} \quad \lambda_n = n(n+1)$

$u = TR = \sum_{n=0}^{\infty}(A_n e^{a\sqrt{\lambda_n}t} + B_n e^{-a\sqrt{\lambda_n}t})\, r^{-(n+1)}$

八. $a = 2n(2n+1) \quad y = P_{2n}(x). \quad 2x+1 = \sum_{n=0}^{\infty} A_{2n}P_{2n}(x).$

$A_{2n} = \frac{4n+1}{2}\int_0^1 (2x+1)P_{2n}(x)\, dx = \frac{1}{4}\int_0^1 (2x+1)\, d[P_{2n+1}(x) - P_{2n-1}(x)]$

$= \frac{1}{2}\int_0^1 P_{2n+1}(x) - P_{2n-1}(x)\, dx = \frac{1}{2(4n+3)}[P_{2n+2}(0) - P_{2n}(0)] - \frac{1}{2(4n-1)}[P_{2n}(0) - P_{2n-2}(0)]$

扫描全能王 创建

2016 - 2017 B.

一、 $u_x + y u_{xy} = 0$ 令 $u_x = v$

$v + y v_y = 0$ $\quad v = -y\frac{dv}{dy}$ $\quad \frac{dy}{y} = -\frac{dv}{v}$ $\quad \ln y = -\ln v + C$

$\ln v = -\ln y + C$ $\quad v = \frac{C}{y} = u_x$

$u = \frac{Cx}{y} + f(y).$

二、 对深问题去延拓：

$$\begin{cases} u_{tt} = 9u_{xx} & (t>0,\ -\infty<x<+\infty) \\ u|_{t=0} = x & u_t|_{t=0} = 2\sin x \end{cases}$$

$2\sin x \quad -2\cos x$

$$u = \frac{1}{2}[\varphi(x+at) + \varphi(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at}\psi(s)ds$$

$$= \frac{1}{2}(x+3t + x-3t) + \frac{1}{2a}[2\cos(x-at) - 2\cos(x+at)]$$

$$= x + \frac{2}{a}\sin x \sin at.$$

三、(1) $u_{tt} = u_{xx}$. 令 $u = XT$.

$X = A\sin wx + B\cos wx$ $\quad x=0 \quad X|_{x=0} = 0$

$B = 0$ $\quad X = \sin wx$ $\quad X' = w\cos wx$

$X'|_{x=\pi} = w\cos w\pi = 0$ $\quad w = n + \frac{1}{2}$

故 $X = \sin(n+\frac{1}{2})x.$ $\quad w = (n+\frac{1}{2})$

$T = A_n\cos(n+\frac{1}{2})t + B_n\sin(n+\frac{1}{2})t.$

故 $u_0(x) = \sum_{n=0}^{\infty}[A_n\cos(n+\frac{1}{2})t + B_n\sin(n+\frac{1}{2})t]\sin(n+\frac{1}{2})x.$

~~t=0.~~

$A_1 = 1$ $\quad \frac{1}{2}B_0 = 1$ $\quad B_0 = 2$

$u_1(x) = \cos\frac{3}{2}t\sin\frac{3}{2}x + 2\sin\frac{1}{2}t\sin\frac{1}{2}x.$

(2). 记 $v = A\sin\frac{x}{2}\sin wt.$ $\quad v_t = Aw\sin\frac{x}{2}\cos wt$

$v_{tt} - v_{xx} = (Aw^2 - \frac{A}{4})\sin\frac{x}{2}\sin wt. = A(\frac{1}{4} - w^2)\sin\frac{x}{2}\sin wt.$

$A = \frac{1}{\frac{1}{4} - w^2}$ $\quad u = v + w,$

边界条件：左端固定，右端自由
初始条件 $\sin\frac{3}{2}x$ 初位置
$\sin\frac{x}{2}$ 初速度
$f(t,x)$：外力

$$\begin{cases} w_{tt} = w_{xx} \\ w|_{x=0} = 0 \quad w_x|_{x=\pi} = 0 \\ w|_{t=0} = \sin\frac{3}{2}x \quad w_t|_{t=0} = \sin\frac{x}{2}(1 - \frac{w}{\frac{1}{4}-w^2}) \end{cases}$$

$w = \cos\frac{3}{2}t\sin\frac{3}{2}x + 2(1 - \frac{w}{\frac{1}{4}-w^2})\sin\frac{1}{2}t\sin\frac{1}{2}x.$

$$\Rightarrow u = \cos\frac{3}{2}t\sin\frac{3}{2}x + 2(1 - \frac{w}{\frac{1}{4}-w^2})\sin\frac{1}{2}t\sin\frac{1}{2}x + \frac{1}{\frac{1}{4}-w^2}\sin wt\sin\frac{x}{2}$$

四、

$\frac{\partial u}{\partial t} = \frac{1}{x}\frac{\partial}{\partial x}(x\frac{\partial u}{\partial x}) + u.$ $u = XT$

$T'X = \frac{\partial^2 u}{\partial x^2} + \frac{1}{x}\frac{\partial u}{\partial x} + u,$

$= TX'' + \frac{1}{x}TX' + TX$

$\frac{T'}{T} = \frac{X''}{X} + \frac{1}{x}\frac{X'}{X} + 1 = -\lambda$

$X'' + \frac{1}{x}X' + X = -\lambda X$

$x^2X'' + xX' + x^2(\lambda+1)X = 0.$

解为. $X = J_0(w_n x)$ $w_n^2 = \lambda_n + 1.$

w_n: $J_0'(wx)|_{x=1} = 0$ 的第n个正根.

$T = e^{-(w_n^2-1)t}$

故 $u = \sum_{n=1}^{\infty} C_n e^{-(w_n^2-1)t} J_0(w_n x)$

$C_n = \frac{1}{\frac{1}{2}J_0^2(w_n)} \int_0^1 x\varphi(x) J_0(w_n x)\, dx.$

五. $\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r})u + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta})u + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2} = r\cos\theta.$

由起意: $\frac{\partial u}{\partial\varphi} = 0.$ 故. $u = R$

$V = \frac{1}{6}z^3$ 故: $u = v + w$

$$\begin{cases} w_{xx} + w_{yy} + w_{zz} = 0 \\ w|_{x^2+y^2+z^2=1} = -\frac{1}{6}z^3 = -\frac{1}{6}\cos^3\theta \end{cases}$$

解: $w = \sum_{n=0}^{+\infty} A_n r^n P_n(\cos\theta).$

$A_n = \frac{2n+1}{2}\int_0^{\pi} -\frac{1}{6}\cos^3\theta\, P_n(\cos\theta)\sin\theta\, d\theta$

$= \frac{2n+1}{2}\int_{-1}^{1} -\frac{1}{6}x^3 P_n(x)\, dx.$

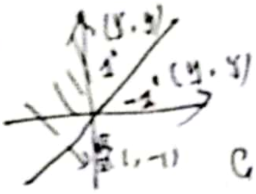
$P_0(x) = 1$ $P_1(x) = x$ $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$ $P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$

$A_3 = -\frac{1}{15}$ $A_2 = 0$ $A_1 = -\frac{1}{10}$ $A_0 = 0.$

故 $u = -\frac{1}{15}r^3 P_3(\cos\theta) - \frac{1}{10}r\cos\theta + \frac{1}{6}r^3\cos^3\theta,$

$(x = r\sin\theta\cos\varphi,\ y = r\sin\theta\sin\varphi\ \ z = r\cos\theta.$

六、(1)



故：由镜像法，

$$G(M,M_0)=\frac{1}{4\pi}\ln\frac{1}{(x-\xi)^2+(y-\eta)^2}-\frac{1}{4\pi}\ln\frac{1}{(x-\eta)^2+(y-\xi)^2}$$

$$=\frac{1}{4\pi}\ln\frac{(x-\eta)^2+(y-\xi)^2}{(x-\xi)^2+(y-\eta)^2}.$$

(2). $$u(x,y)=\iint f(M)G(M,M_0)dM_0-\int_{\partial D}\frac{\partial G(M,M_0)}{\partial \vec n_0}\varphi(M_0)\,dl.$$

$\vec n_0=\frac{\sqrt2}{2}(1,-1)$ $\quad dl=\sqrt2\,d\xi$,

$$\frac{\partial G}{\partial \xi}=\frac{1}{4\pi}\frac{2(\xi-y)}{(x-\eta)^2+(y-\xi)^2}-\frac{1}{4\pi}\frac{2(\xi-x)}{(x-\xi)^2+(y-\eta)^2}$$

$$\frac{\partial G}{\partial \eta}=\frac{1}{4\pi}\frac{2(\eta-x)}{(x-\eta)^2+(y-\xi)^2}-\frac{1}{4\pi}\frac{2(\eta-y)}{(x-\xi)^2+(y-\eta)^2}$$

$$\frac{\sqrt2}{2}\left(\frac{\partial G}{\partial \xi}-\frac{\partial G}{\partial \eta}\right)\Big|_{\xi=\eta}=\frac{1}{\sqrt2\pi}\frac{x-y}{(x-\xi)^2+(y-\xi)^2}$$

$$u(x,y)=-\int_{-\infty}^{+\infty}\frac{1}{\pi}\frac{x-y}{(x-\xi)^2+(y-\xi)^2}\varphi(\xi)\,d\xi$$

七. 1. $u_t=4u_{xx}+3u$ $\qquad \bar u=F[u]$.

$\frac{d\bar u}{dt}=(-4\lambda^2+3)\bar u$ $\qquad \bar u=e^{(-4\lambda^2+3)t}=e^{3t}e^{-4\lambda^2t}$.

$$u=e^{3t}\cdot e^{-\frac{x^2}{16t}}\cdot\frac{1}{4\sqrt{\pi t}}$$

$$u=\frac{1}{4\sqrt{\pi t}}e^{3t-\frac{x^2}{16t}}$$

2. $$u=U(t,x)*\varphi(x)=\frac{1}{4\sqrt{\pi t}}e^{3t}\int_{-\infty}^{+\infty}me^{-\frac{(x-m)^2}{16t}}dm$$

$$=\frac{4x\sqrt{\pi t}}{4\sqrt{\pi t}}e^{3t}=xe^{3t}.$$