12-5 作业

13. 证明：$X_1, X_2, \cdots, X_n, i.i.d. \sim N(0,1)$, 则

$$X_1^2 \sim \chi^2(1), \sum_{i=2}^{n} \sim \chi^2(n-1),$$

且两者独立。由 F 分布的定义，

$$\frac{(n-1)X_1^2}{\sum_{i=2}^n X_i^2} = \frac{X_1^2}{\sum_{i=2}^n X_i^2/(n-1)} \sim F_{1,n-1}.$$

14. $X_1, X_2 i.i.d. \sim N(0,1)$. 则

$$X_1 - X_2 \sim N(0,2), \quad X_1 + X_2 \sim N(0,2)$$

样本均值及样本方差为

$$\overline{X} = \frac{X_1 + X_2}{2}$$
$$S^2 = \frac{1}{2-1}\sum_{i=1}^{2}(X_i - \overline{X}) = \left(\frac{X_1 - X_2}{2}\right)^2 + \left(\frac{X_2 - X_1}{2}\right)^2 = \frac{1}{2}(X_1 - X_2)^2$$

由 $\overline{X}$ 与 S^2 独立可知 $(X_1 - X_2)^2$ 与 $(X_1 + X_2)^2$ 独立。所以

$$Y = \frac{(X_1 - X_2)^2}{(X_1 + X_2)^2} = \frac{(\frac{X_1 - X_2}{2})^2}{(\frac{X_1 + X_2}{2})^2} \sim F_{1,1}.$$

15. $X_1, X_2, X_3, X_4 i.i.d. \sim N(0, 2^2)$, 则有

$$(X_1 - 2X_2) \sim N(0, 20), \quad (3X_3 - 4X_4) \sim N(0, 10^2)$$

要使 $T = a(X_1 - 2X_2)^2 + b(3X_3 - 4X_4)^2$ 服从 χ^2 分布，

$$\sqrt{a}(X_1 - 2X_2) \sim N(0,1), \quad \sqrt{b}(3X_3 - 4X_4) \sim N(0,1)$$
$$\Rightarrow \ a = \frac{1}{20}, \quad b = \frac{1}{100}$$

此时 $T \sim \chi^2(2)$. 或 $(a = 1/20, b = 0)$ 及 $(a = 0, b = 1/100)$ 也可，此时 $T \sim \chi^2(1)$.

16. $X_1, X_2, \cdots, X_9 i.i.d. \sim N(\mu, \sigma^2)$, 则有

$$Y_1 \sim N(\mu, \frac{1}{6}\sigma^2), \quad Y_2 \sim N(\mu, \frac{1}{3}\sigma^2), \quad \sqrt{2}(Y_1 - Y_2) \sim N(0, \sigma^2),$$

$\dfrac{2S^2}{\sigma^2} \sim \chi^2(2)$. ($Y_2$是样本$X_7, X_8, X_9$的样本均值，$S^2$ 为样本方差.)

$$Z = \frac{\frac{\sqrt{2}}{\sigma}(Y_1 - Y_2)}{S} = \frac{\sqrt{2}(Y_1 - Y_2)}{\sqrt{\frac{2S^2}{2\sigma^2}}} \sim t(2).$$

20. $X_1, \cdots, X_n, X_{n+1}$ $i.i.d. \sim N(a, \sigma^2)$, 则有

$$\begin{aligned}
&\bar{X} \sim N(a, \frac{\sigma^2}{n}), \quad X_{n+1} - \bar{X} \sim N(0, \frac{n+1}{n}\sigma^2), \\
&\frac{(n-1)S_n^2}{\sigma^2} \sim \chi^2(n-1), \\
&\frac{X_{n+1} - \bar{X}}{S_n}\sqrt{\frac{n}{n+1}} = \frac{\frac{X_{n+1}-\bar{X}}{\sqrt{(n+1)\sigma^2/n}}}{\sqrt{\frac{(n-1)S_n^2}{(n-1)\sigma^2}}} \sim t(n-1).
\end{aligned}$$

21. $X_1, \cdots, X_m i.i.d. \sim N(\mu_1, \sigma^2), Y_1, \cdots, Y_n$ $i.i.d. \sim N(\mu_2, \sigma^2)$, 且相互独立，则有

$$\begin{aligned}
&\alpha(\bar{X} - \mu_1) + \beta(\bar{Y} - \mu_2) \sim N\Big(0, \Big(\frac{\alpha^2}{m} + \frac{\beta^2}{n}\Big)\sigma^2\Big) \\
&\frac{(m-1)S_m^2}{\sigma^2} \sim \chi^2(m-1), \frac{(n-1)S_n^2}{\sigma^2} \sim \chi^2(n-1), \\
&\frac{(m-1)S_m^2 + (n-1)S_n^2}{\sigma^2} \sim \chi^2(m+n-2), \\
&T \sim t(m+n-2).
\end{aligned}$$