第三章特殊函数

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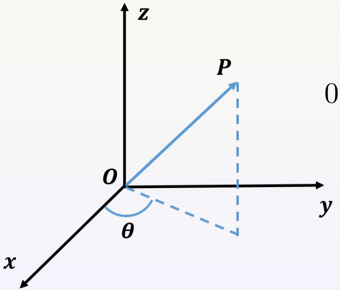
利用分离变量法解定解问题时，要对方程和边界条件进行变量分离，如果边界为球面或柱面的一部分，则在直角坐标系下无法分离变量，要根据边界情况引入新的坐标系.

选择坐标系，应使所研究问题的边界面和一个或几个坐标面平行。

边界	矩形	圆环，扇形	圆柱	球	圆锥
坐标系	直角坐标	极坐标	柱坐标	球坐标	球坐标

1. 极坐标系： $x = r\cos\theta,\ y = r\sin\theta$
2. 柱坐标系: $x = r\cos\theta,\ y = r\sin\theta,\ z = z$
3. 球坐标系: $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$

柱坐标系空间中一点$P(x,y,z) \leftrightarrow P(r,\theta,z)$

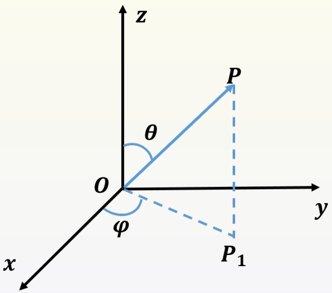


$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$$0 \le r \le +\infty, 0 \le \theta \le 2\pi$$

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \arctan\frac{y}{x}(\text{或} + \pi, +2\pi) \\ z = z \end{cases}$$

球坐标系



$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

$$\theta \in [0,\pi],\ \ \varphi \in [0,2\pi]$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \arctan\frac{\sqrt{x^2+y^2}}{z} \\ \varphi = \arctan\frac{y}{x} \end{cases}$$

1. 平面极坐标系中$\Delta_2 u = u_{xx} + u_{yy}$

经过极坐标变换可得 $\Delta_2 u = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}$

或可写成$\Delta_2 u = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}$

2. 柱坐标系中$\Delta_3 u = u_{xx} + u_{yy} + u_{zz}$

$$\Delta_3 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

或可写成$\Delta_3 u = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} + \dfrac{\partial^2 u}{\partial z^2}$

3. 球坐标系中$\Delta_3 u = u_{xx} + u_{yy} + u_{zz}$

$$\Delta_3 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}$$

一、 **Bessel函数**

二、 Bessel函数的性质

三、 Bessel方程的固有值问题

四、 Legendre多项式

五、 函数的Fourier-Legendre展开

Bessel方程的导出

考虑如下定解问题

$$\begin{cases} u_{xx}+u_{yy}+u_{zz}=0, \qquad x^2+y^2<R, 0<z<H \\ u\big|_{x^2+y^2=R^2}=0 \\ u\big|_{z=0}=g_1(r,\theta), u\big|_{z=H}=g_2(r,\theta) \end{cases}$$

在圆柱内求解Laplace方程，需要在柱坐标系中分离变量，方程化为

$$\Delta_3 u=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial z^2}=0$$

设$u=R(r)\Theta(\theta)Z(z)$代入方程

$$R''\Theta Z+\frac{1}{r}R'\Theta Z+\frac{1}{r^2}R\Theta''Z+R\Theta Z''=0$$

两边除以$R(r)\Theta(\theta)Z(z)$, 方程化为

$$\frac{R''}{R}+\frac{1}{r}\frac{R'}{R}+\frac{1}{r^2}\frac{\Theta''}{\Theta}=-\frac{Z''}{Z}=-\lambda$$

$$Z''-\lambda Z=0 \tag{1}$$

$$r^2\frac{R''}{R}+r\frac{R'}{R}+\frac{\Theta''}{\Theta}+\lambda r^2=0 \tag{2}$$

上式进一步分离变量

$$r^2\frac{R''}{R}+r\frac{R'}{R}+\lambda r^2=-\frac{\Theta''}{\Theta}=\mu$$

$$r^2R'' + rR' + (\lambda r^2 - \mu)R = 0 \tag{3}$$
$$\Theta'' + \mu\Theta = 0 \tag{4}$$

令$x = \sqrt{\lambda}r, R(r) \triangleq y(x), \mu = \nu^2(\nu \geq 0)$, 最后一个方程可化为

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 - - - - -\nu\text{阶Bessel方程}$$

接下来用幂级数方法求解Bessel方程。

二阶线性常微分方程的解析理论

$$w''(z)+p(z)w'(z)+q(z)w(z)=0 \tag{5}$$

(1)若z_0是$p(z)$，$q(z)$的解析点，称z_0是方程的常点。

Cauchy定理

在常点z_0 的邻域$\{z||z-z_0|<R\}$ 内,定解问题

$$\begin{cases} w''(z)+p(z)w'(z)+q(z)w(z)=0 \\ w(z_0)=a_0, w'(z_0)=a_1 \end{cases}$$

有唯一的解可写为幂级数形式$w(z)=\sum\limits_{n=0}^{+\infty} a_n(z-z_0)^n$

$$\boxed{w''(z) + p(z)w'(z) + q(z)w(z) = 0}$$

(2)若z_0是$p(z)$的至多一级极点，是$q(z)$的至多2级极点，则称z_0是方程（10）的正则奇点.

Fuchs定理

在正则奇点z_0的邻域内，方程（10）有两个线性无关的广义幂级数解

$$w_1(z) = (z - z_0)^{\rho_1} \sum_{n=0}^{+\infty} a_n (z - z_0)^n$$

$$w_2(z) = \alpha w_1(z) \ln(z - z_0) + (z - z_0)^{\rho_2} \sum_{n=0}^{+\infty} b_n (z - z_0)^n$$

其中$a_0 b_0 \neq 0$, $\rho_2 - \rho_1$不为整数时α可为0, ρ_1, ρ_2称为方程的指标.

二、Bessel方程求解

ν阶贝塞尔方程

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0(\nu \geqslant 0)$$

$$y'' + \frac{1}{x}y' + (1 - \frac{\nu^2}{x^2})y = 0$$

(1)$x = 0$是Bessel方程的正则奇点，它的解具有广义幂级数的形式。设$y = x^\rho \sum\limits_{n=0}^{+\infty} a_n x^n$。

$$xy' = \sum_{n=0}^{+\infty}(n+\rho)a_n x^{n+\rho}, \qquad x^2y'' = \sum_{n=0}^{+\infty}(n+\rho)(n+\rho-1)a_n x^{n+\rho}$$

代入Bessel方程

$$\sum_{n=0}^{+\infty}[(n+\rho)^2-\nu^2]a_nx^{n+\rho}+\sum_{n=2}^{+\infty}a_{n-2}x^{n+\rho}=0$$

(2)比较$x^{n+\rho}$的系数，应有

$$(\rho^2-\nu^2)a_0=0 \tag{6}$$

$$[(1+\rho)^2-\nu^2]a_1=0 \tag{7}$$

$$[(n+\rho)^2-\nu^2]a_n+a_{n-2}=0 \quad (n=2,3,\cdots) \tag{8}$$

由于a_0是广义幂级数的第一项系数，所以$a_0\neq 0$，故由第一个式子，

$\boxed{\rho^2=\nu^2}$——指标方程

后面两式化为

$$a_1(1+2\rho)=0 \tag{9}$$

$$a_nn(n+2\rho)+a_{n-2}=0 \quad (n=2,3,\cdots) \tag{10}$$

(3)下面分$\rho=\nu,\rho=-\nu$两种情况讨论。

①$\rho=\nu(\nu\geqslant 0)$. 由（10）式,得到递推关系

$$a_n=\frac{-a_{n-2}}{n(n+2\nu)}. \tag{11}$$

只需要选取两个线性无关的解即可组成基本解组，所以不妨取，$a_1=0.$
由（11）式，$a_{2n-1}=0,n=1,2,\cdots.$

$$\begin{aligned}a_{2n}&=\frac{-a_{2(n-1)}}{2n(2n+2\nu)}=\frac{-a_{2(n-1)}}{2^2n(n+\nu)}=\frac{a_{2(n-2)}}{2^4n(n-1)(n+\nu)(n+\nu-1)}\\&=\cdots\\&=\frac{(-1)^na_0}{2^{2n}n!(1+\nu)(2+\nu)\cdots(n+\nu)}=\frac{(-1)^na_0\Gamma(\nu+1)}{2^{2n}n!\Gamma(n+\nu+1)}\end{aligned}$$

a_0可以取任意常数，特别取$a_0 = \dfrac{1}{2^\nu \Gamma(\nu+1)}$.

$$a_{2n} = \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{1}{2}\right)^{2n+\nu}$$

$$\boxed{y_1(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n}}$$

解的敛散性：

$$\lim_{n\to+\infty} \left|\frac{a_{2n}}{a_{2n-2}}\right| = \lim_{n\to+\infty} \frac{1}{4n(n+\nu)} = 0,$$ 所以幂级数$y_1(x)$的收敛半径是$+\infty$,可以逐项求导，是Bessel方程的解。

②$\rho=-\nu(\nu\geqslant 0)$,设广义幂级数$x^{-\nu}\sum\limits_{n=0}^{\infty}a_nx^n$满足Bessel 方程，

同理有递推公式

$$a_n=\frac{-a_{n-2}}{n(n-2\nu)}$$

取$a_{2n-1}=0, a_0=\dfrac{1}{2^{-\nu}\Gamma(-\nu+1)}$，可得

$$a_{2n}=\frac{(-1)^n}{n!\Gamma(n-\nu+1)}\left(\frac{1}{2}\right)^{2n-\nu}$$

$$\boxed{y_2(x)=\left(\frac{x}{2}\right)^{-\nu}\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!\Gamma(n-\nu+1)}\left(\frac{x}{2}\right)^{2n}}$$

Gamma函数

1.定义 $\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} \mathrm{d}t (x > 0)$

2.递推公式 $\Gamma(x+1) = x\Gamma(x)$

$$\Gamma(1) = \int_0^{+\infty} t^0 e^{-t} \mathrm{d}t = -e^{-t}\Big|_0^{+\infty} = 1$$

$$\Gamma(n+1) = n\Gamma(n) = \cdots = n!$$

$$\Gamma(\frac{1}{2}) = \int_0^{+\infty} t^{-\frac{1}{2}} e^{-t} \mathrm{d}t = 2\int_0^{+\infty} e^{-t} \mathrm{d}\sqrt{t} = \sqrt{\pi}$$

$$\Gamma(n+\frac{1}{2}) = \cdots = \frac{(2n-1)(2n-3)\cdots 1}{2^n}\Gamma(\frac{1}{2}) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$$

3.利用$\Gamma(x)=\dfrac{\Gamma(x+1)}{x}$，可以将$\Gamma(x)$的定义推广到$\mathbb{R}$除去负整数的部分。

$$\Gamma(-\frac{1}{2})=-2\Gamma(\frac{1}{2})=-2\sqrt{\pi},\Gamma(-\frac{3}{2})=-\frac{2}{3}\Gamma(-\frac{1}{2})=\frac{4}{3}\sqrt{\pi}$$

$m=0,-1,-2,-3,\cdots$时，

$$\lim_{x\to m}\Gamma(x)=\lim_{x\to m}\frac{\Gamma(x+1)}{x}=\lim_{x\to m}\frac{\Gamma(x-m+1)}{x(x+1)\cdots(x-m)}=\infty$$

三、贝塞尔函数

定义：第一类Bessel 函数

$$J_\nu(x)=\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!\Gamma(n+\nu+1)}\left(\frac{x}{2}\right)^{2n+\nu} \tag{12}$$

$$J_{-\nu}(x)=\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!\Gamma(n-\nu+1)}\left(\frac{x}{2}\right)^{2n-\nu} \tag{13}$$

当ν不是整数时，只看级数中第一项，就可知道

$$\lim_{x\to 0}J_\nu(x)=\begin{cases}0, & \nu>0\\ 1 & \nu=0\end{cases}\qquad \lim_{x\to 0}J_{-\nu}(x)=\infty$$

两个级数不成比例，是线性无关的函数。Bessel 方程的解是

$$y(x)=c_1J_\nu(x)+c_2J_{-\nu}(x)$$

当ν是整数时，$n=0,1,2,\cdots,\nu-1$时

$$\frac{1}{\Gamma(n-\nu+1)}=\frac{1}{\infty}=0$$

$$\begin{aligned}
J_{-n}(x) &= \sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma(k-n+1)}\left(\frac{x}{2}\right)^{2k-n}\\
&= \sum_{k=n}^{+\infty}\frac{(-1)^k}{k!\Gamma(k-n+1)}\left(\frac{x}{2}\right)^{2k-n}\\
&= \sum_{m=0}^{+\infty}\frac{(-1)^{m+n}}{(n+m)!\Gamma(m+1)}\left(\frac{x}{2}\right)^{2m+n}\\
&= (-1)^n\sum_{m=0}^{+\infty}\frac{(-1)^m}{m!\Gamma(m+n+1)}\left(\frac{x}{2}\right)^{2m+n}=(-1)^nJ_n(x)
\end{aligned}$$

$J_n(x)$和$J_{-n}(x)$线性相关，不能得到方程的通解。

第二类ν阶Bessel函数或Neumann函数

$$N_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi} \qquad (\nu\text{不是整数})$$

$$N_n(x) = \lim_{\nu\to n} N_\nu(x) = \lim_{\nu\to n}\frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}$$

当ν不是整数时，$N_\nu(x)$是$J_\nu(x)$，$J_{-\nu}(x)$的线性组合，所以是Bessel方程的解，且$N_\nu(x)$与$J_\nu(x)$线性无关。

当ν是整数时，由洛必达法则,分子分母同时对ν求导

$$N_n(x) = \frac{\left.\left[\frac{\partial J_\nu}{\partial\nu}\cos n\pi - \frac{\partial J_{-\nu}}{\partial\nu}\right]\right|_{\nu=n}}{\pi\cos n\pi} = \frac{1}{\pi}\left.\left[\frac{\partial J_\nu}{\partial\nu} - (-1)^n\frac{\partial J_{-\nu}}{\partial\nu}\right]\right|_{\nu=n}$$

$J_n(x)$与$N_n(x)$线性无关。

从$N_n(x)$的表达式可以看出，$\lim_{x\to 0} N_n(x) = \infty$.

而$\lim_{x\to 0} J_n(x) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$ 所以$N_n(x)$与$J_n(x)$线性无关。

结论

Bessel方程$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \quad (\nu \geqslant 0)$的通解是

$$y(x) = C_1 J_\nu(x) + C_2 N_\nu(x).$$

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五、 函数的Fourier-Legendre展开

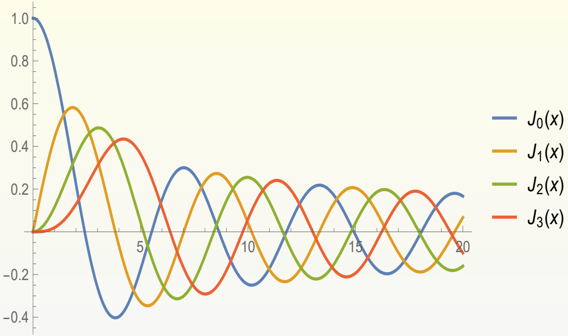
一、Bessel函数的图形，特殊函数值

$$
\begin{aligned}
J_{\frac{1}{2}}(x) &= \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!\Gamma(k+\frac{1}{2}+1)} \left(\frac{x}{2}\right)^{2k+\frac{1}{2}} \\
&= \left(\frac{x}{2}\right)^{-\frac{1}{2}} \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!(k+\frac{1}{2})(k-\frac{1}{2})\cdots\frac{1}{2}\sqrt{\pi}} \left(\frac{x}{2}\right)^{2k+1} \\
&= \sqrt{\frac{2}{\pi x}} \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = \sqrt{\frac{2}{\pi x}} \sin x
\end{aligned}
$$

同样可以计算

$$
J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x
$$

半整数阶的Bessel函数是初等函数，其它都是非初等函数。

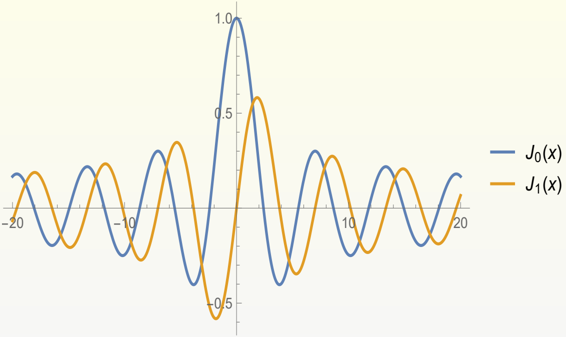


Bessel 函数的特点

1.震荡衰减.

当x值很大时,

$$J_\nu(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{1}{4}\pi - \frac{\nu\pi}{2}\right)$$



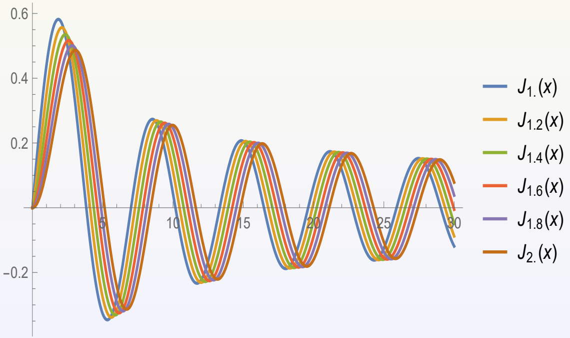
Bessel 函数的特点

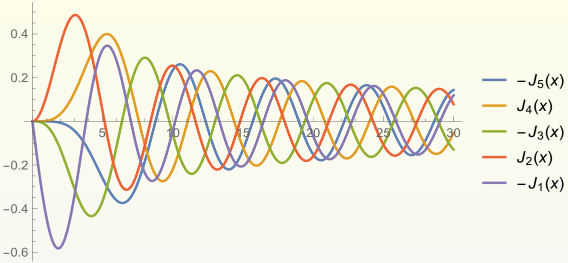
2.零点特性

1. $J_n(x)$有无穷多个实零点，关于原点对称分布。
2. $J_n(x)$与$J_{n+1}(x)$的零点交错分布。
3. 由Rolle定理，$J_n(x)$的两个零点之间存在$J_n'(x)$的零点。

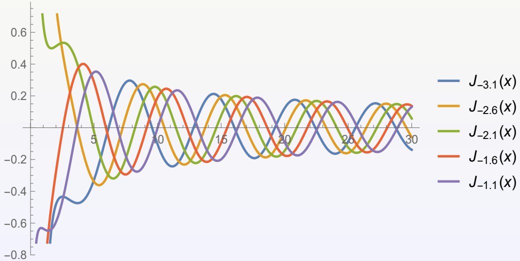
$J_n(x)$的零点分布

$J_0(x)$	2.40483,	5.52008,	8.65373,	11.7915,	14.9309,	18.0711
$J_1(x)$	3.83171,	7.01559,	10.1735,	13.3237,	16.4706,	19.6159
$J_2(x)$	5.13562,	8.41724,	11.6198,	14.796,	17.9598,	21.117




0.4
0.2
−0.2
−0.4
−0.6
5
10
15
20
25
30
$-J_5(x)$
$J_4(x)$
$-J_3(x)$
$J_2(x)$
$-J_1(x)$

Out[18]=


0.6
0.4
0.2
−0.2
−0.4
−0.6
−0.8
5
10
15
20
25
30
$J_{-3.1}(x)$
$J_{-2.6}(x)$
$J_{-2.1}(x)$
$J_{-1.6}(x)$
$J_{-1.1}(x)$

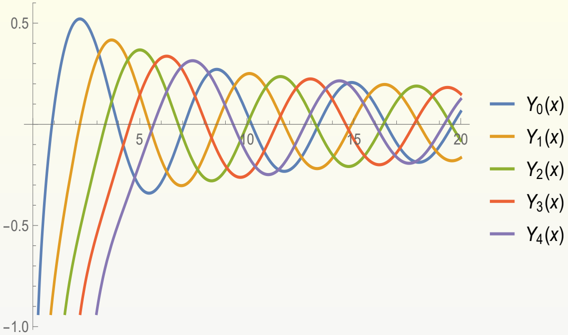


Figure: Neumann 函数

1. 当$x \to \infty$时，$N_\nu(x) \approx \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{1}{4}\pi - \frac{\nu\pi}{2}\right)$
2. 当$x \to 0$时$N_\nu(x) \to -\infty$.

二、整数阶Bessel函数母函数

函数$f(z)=exp\left\{\frac{x}{2}\left(z-\frac{1}{z}\right)\right\}$的洛朗级数$f(z)=\sum\limits_{n=-\infty}^{+\infty}a_nz^n$

$a_n=\frac{1}{2\pi i}\int\limits_C\frac{exp\left\{\frac{x}{2}\left(z-\frac{1}{z}\right)\right\}}{z^{n+1}}\mathrm{d}z$.通过间接法可以证明

$$f(z)=exp\left\{\frac{x}{2}\left(z-\frac{1}{z}\right)\right\}=\sum_{n=-\infty}^{+\infty}J_n(x)z^n$$

所以，$f(z)$称为$J_n(x)$的母函数。

$$J_n(x)=\frac{1}{2\pi i}\int\limits_C\frac{exp\left\{\frac{x}{2}\left(z-\frac{1}{z}\right)\right\}}{z^{n+1}}\mathrm{d}z=\frac{1}{2\pi}\int_0^{2\pi}\cos(x\sin\theta-n\theta)\mathrm{d}\theta$$

Example 1 (证明)

$$J_n(x+y) = \sum_{k=-\infty}^{+\infty} J_k(x) J_{n-k}(y)$$

证明:

$$\sum_{n=-\infty}^{+\infty} J_n(x+y) z^n = \exp\left(\frac{x+y}{2}(z-\frac{1}{z})\right)$$

$$= \exp\left(\frac{x}{2}(z-\frac{1}{z})\right) \exp\left(\frac{y}{2}(z-\frac{1}{z})\right) = \sum_{k=-\infty}^{+\infty} J_k(x) z^k \cdot \sum_{m=-\infty}^{+\infty} J_m(y) z^m$$

$$= \sum_{n=-\infty}^{+\infty}\left[\sum_{k=-\infty}^{+\infty} J_k(x) J_{n-k}(y)\right] z^n$$

Example 2（证明）

$$\cos x = J_0(x)+2\sum_{n=1}^{+\infty}(-1)^n J_{2n}(x) \qquad \sin x = 2\sum_{n=0}^{+\infty}(-1)^n J_{2n+1}(x)$$

证明:

$$e^{ix} = e^{\frac{x}{2}(i-\frac{1}{i})} = J_0(x) + \sum_{m=1}^{+\infty} J_m(x)i^m + \sum_{m=-\infty}^{-1} J_m(x)i^m$$

$$= J_0(x) + \sum_{m=1}^{+\infty}(J_m(x)i^m + J_{-m}(x)i^{-m}) = J_0(x) + 2\sum_{m=1}^{+\infty} i^m J_m(x)$$

$$= J_0(x) + 2\sum_{n=1}^{+\infty}(-1)^n J_{2n}(x) + 2i\sum_{n=0}^{+\infty}(-1)^n J_{2n+1}(x)$$

Example 3 (证明恒等式)

$$J_0^2(x)+2\sum_{n=1}^{\infty}J_n^2(x)=1$$

证明: $$e^{\frac{x}{2}(z-\frac{1}{z})}=\sum_{n=-\infty}^{+\infty}J_n(x)z^n,\qquad 0<|z|<+\infty$$

将$\frac{1}{z}$代入上式可得 $$e^{-\frac{x}{2}(z-\frac{1}{z})}=\sum_{m=-\infty}^{+\infty}J_m(x)z^{-m},$$

$$1=\left(\sum_{m=-\infty}^{+\infty}J_m(x)z^{-m}\right)\left(\sum_{n=-\infty}^{+\infty}J_n(x)z^n\right)$$
$$=\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}J_n(x)J_m(x)z^{n-m}=\sum_{k=-\infty}^{+\infty}\left(\sum_{m=-\infty}^{+\infty}J_n(x)J_{n-k}(x)\right)z^k$$

比较z^0系数可知结论成立.

三、微分递推关系

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^{\nu}J_{\nu}(x)] = x^{\nu}J_{\nu-1}(x) \tag{14}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^{-\nu}J_{\nu}(x)] = -x^{-\nu}J_{\nu+1}(x) \qquad (\forall\nu) \tag{15}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^{\nu}J_{\nu}(x)] = \sum_{n=0}^{+\infty}\frac{(-1)^n(2n+2\nu)}{n!\Gamma(n+\nu+1)2^{2n+\nu}}x^{2n+2\nu-1}$$

$$=\sum_{n=0}^{+\infty}\frac{(-1)^n(n+\nu)x^{\nu}}{n!\Gamma(n+\nu+1)2^{2n+\nu-1}}x^{2n+\nu-1}$$

$$=x^{\nu}\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!\Gamma(n+(\nu-1)+1)}\left(\frac{x}{2}\right)^{2n+\nu-1} = x^{\nu}J_{\nu-1}(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^{\nu}J_{\nu}(x)] = x^{\nu}J'_{\nu}(x) + \nu x^{\nu-1}J_{\nu}(x) = x^{\nu}J_{\nu-1}(x) \tag{16}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^{-\nu}J_{\nu}(x)] = x^{-\nu}J'_{\nu}(x) - \nu x^{-\nu-1}J_{\nu}(x) = -x^{-\nu}J_{\nu+1}(x) \tag{17}$$

$$J'_{\nu}(x) = J_{\nu-1}(x) - \frac{\nu}{x}J_{\nu}(x) \tag{18}$$

$$J'_{\nu}(x) = \frac{\nu}{x}J_{\nu}(x) - J_{\nu+1}(x) \tag{19}$$

$$J_{\nu+1}(x) = \frac{2\nu}{x}J_{\nu}(x) - J_{\nu-1}(x) \tag{20}$$

$$2J'_{\nu}(x) = J_{\nu-1}(x) - J_{\nu+1}(x) \tag{21}$$

$J_n(x)$可以表示成$J_0(x)$与$J_1(x)$的组合形式，例如

$$J_3(x) = \frac{4}{x}J_2(x) - J_1(x) = \left(\frac{8}{x^2} - 1\right)J_1(x) - \frac{4}{x}J_0(x)$$

Neaumann函数的递推公式

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}x}[x^\nu N_\nu(x)] = x^\nu N_{\nu-1}(x) \\ \dfrac{\mathrm{d}}{\mathrm{d}x}[x^{-\nu} N_\nu(x)] = -x^{-\nu} N_{\nu+1}(x) \\ N_{\nu+1}(x) + N_{\nu-1}(x) = \dfrac{2\nu}{x} N_\nu(x) \\ N_{\nu-1}(x) - N_{\nu+1}(x) = 2N'_\nu(x) \end{cases} \tag{22}$$

证明:(1)

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^\nu N_\nu(x)] = \frac{\cos\pi\nu}{\sin\pi\nu}\frac{\mathrm{d}}{\mathrm{d}x}[x^\nu J_\nu(x)] - \frac{1}{\sin\pi\nu}\frac{\mathrm{d}}{\mathrm{d}x}[x^\nu J_{-\nu}(x)]$$

$$= \frac{\cos\pi\nu}{\sin\pi\nu}(x^\nu J_{\nu-1}(x)) - \frac{1}{\sin\pi\nu}(-x^\nu J_{-\nu+1}(x))$$

$$= \frac{\cos\pi(\nu-1)}{\sin\pi(\nu-1)}(x^\nu J_{\nu-1}(x)) - \frac{1}{\sin\pi(\nu-1)}(x^\nu J_{-(\nu-1)}(x))$$

$$= x^\nu N_{\nu-1}(x)$$

Example 4 (计算积分)

$$\int x^4 J_1(x)\mathrm{d}x.$$

解:(法1)

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^\nu J_\nu(x)) = x^\nu J_{\nu-1}(x)$$

$$\begin{aligned}\int x^4 J_1(x)\mathrm{d}x &= \int x^2(x^2 J_2(x))'\mathrm{d}x\\ =&x^4 J_2(x) - \int 2x^3 J_2(x)\mathrm{d}x\\ =&x^4 J_2(x) - 2x^3 J_3(x) + C\end{aligned}$$

计算积分

$$\int x^4 J_1(x)\mathrm{d}x.$$

解:(法2) $\frac{\mathrm{d}}{\mathrm{d}x}(x^{-\nu}J_\nu(x)) = -x^{-\nu}J_{\nu+1}(x)$

$$\int x^4 J_1(x)\mathrm{d}x = -\int x^4 \mathrm{d}J_0(x) = -x^4 J_0(x) + \int 4x^3 J_0(x)\mathrm{d}x$$

$$= -x^4 J_0(x) - \int 4x^2 (xJ_{-1}(x))'\mathrm{d}x$$

$$= -x^4 J_0(x) - 4x^3 J_{-1}(x) + 8\int x^2 J_{-1}(x)\mathrm{d}x$$

$$= -x^4 J_0(x) - 4x^3 J_{-1}(x) - 8x^2 J_{-2}(x) + C$$

Example 5 (计算积分)

$$\int x^2 J_{-2}(x)\mathrm{d}x$$

$$\int x^2 J_{-2}(x)\mathrm{d}x = \int x^2 J_2(x)\mathrm{d}x = -\int x^3 (x^{-1}J_1(x))'\mathrm{d}x$$

$$= -\int x^3 \mathrm{d}(x^{-1}J_1(x)) = -x^2 J_1(x) + \int 3x J_1(x)\mathrm{d}x$$

$$= -x^2 J_1(x) - \int 3x (J_0(x))'\mathrm{d}x$$

$$= -x^2 J_1(x) - 3x J_0(x) + 3\int J_0(x)\mathrm{d}x$$

形如$\int x^p J_q(x)\mathrm{d}x$的积分，如果$p, q$是整数，且$p+q \geqslant 0$，若$p+q$是奇数，积分可以用$J_0(x), J_1(x)$表示。
若$p+q$是偶数，积分结果要用$\int J_0(x)\mathrm{d}x$表示。

Example 6

(1) 计算积分$I=\int_0^{+\infty}e^{-ax}J_0(bx)\mathrm{d}x$

(2) 计算Laplace变换$\mathcal{L}[J_0(t)]$，$\mathcal{L}[J_1(t)]$.

$$\int_0^{+\infty}e^{-ax}J_0(bx)\mathrm{d}x=\int_0^{+\infty}\frac{e^{-ax}}{\pi}\int_0^{\pi}\cos(bx\sin\theta)\mathrm{d}\theta\mathrm{d}x$$

$$=\int_0^{+\infty}e^{-ax}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{ibx\sin\theta}d\theta\right)dx=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\theta\int_0^{\infty}e^{-(a-ib\sin\theta)x}dx$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta}{a-ib\sin\theta}=\frac{1}{2\pi i}\oint_{|z|=1}\frac{2}{-bz^2+2az+b}dz=\frac{1}{\sqrt{a^2+b^2}}$$

$\operatorname{Re} p > 0$时：

$$\mathcal{L}[J_0(t)] = \int_0^{+\infty} J_0(t)e^{-pt}\mathrm{d}t = \frac{1}{\sqrt{1+p^2}}$$

$$\begin{aligned}\mathcal{L}[J_1(t)] &= \int_0^{+\infty} J_1(t)e^{-pt}\mathrm{d}t = \int_0^{+\infty} -J_0'(t)e^{-pt}\mathrm{d}t \\ &= -J_0(t)e^{-pt}\Big|_0^{+\infty} - \int_0^{+\infty} pe^{-pt}J_0(t)\mathrm{d}t \\ &= 1 - \frac{p}{\sqrt{1+p^2}}\end{aligned}$$

$$\int_0^{+\infty} e^{-ax} J_0(bx)\mathrm{d}x = \int_0^{+\infty} e^{-ax} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{bx}{2}\right)^{2k} \mathrm{d}x$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{b}{2}\right)^{2k} \int_0^{+\infty} e^{-ax} x^{2k} \mathrm{d}x$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2 a^{2k+1}} \left(\frac{b}{2}\right)^{2k} \int_0^{+\infty} e^{-t} t^{2k} \mathrm{d}t$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2 a^{2k+1}} \left(\frac{b}{2}\right)^{2k} (2k)!$$

$$= \frac{1}{a} \sum_{k=0}^{\infty} \frac{1}{k!} (-\frac{1}{2})(-\frac{3}{2}) \cdots (-\frac{2k-1}{2}) \left(\frac{b}{a}\right)^{2k}$$

$$= \frac{1}{a} \left[1 + \left(\frac{b}{a}\right)^2\right]^{-\frac{1}{2}} = \frac{1}{\sqrt{a^2+b^2}}$$

一、 Bessel函数

二、 Bessel函数的性质

三、 **Bessel方程的固有值问题**

四、 Legendre多项式

五、 函数的Fourier-Legendre展开

§3.3 贝塞尔方程的固有值问题

贝塞尔方程的固有值问题

$$\begin{cases} x^2y'' + xy' + (\lambda x^2 - \nu^2)y = 0(0 < x < a, \nu \geqslant 0) \\ \alpha y(a) + \beta y'(a) = 0 \\ y(0)\text{有界} \end{cases}$$

α, β是非负常数，不全为零。

转化成S-L形方程$(xy')' + (\lambda x - \frac{\nu^2}{x})y = 0.$

$$k(x) = x,\ q(x) = \frac{\nu^2}{x},\ \rho(x) = x.$$

所以$\lambda \geqslant 0$，记$\lambda = \omega^2$.

作变量代换$t=\omega x$ 就可以转化成标准的Bessel方程

$$t^2y''+ty'+(t^2-\nu^2)y=0$$

所以方程的解是 $y(x)=AJ_\nu(\omega x)+BN_\nu(\omega x).$

由$y(0)$有界，所以$B=0$,

由$\alpha y(a)+\beta y'(a)=0$,得

$$\alpha J_\nu(\omega a)+\beta\omega J_\nu'(\omega a)=0$$

满足等式的无穷多个非负零点记为$\omega_n, n=1,2,\cdots$.

固有值$\lambda_n=\omega_n^2$, 固有函数是$y_n(x)=J_\nu(\omega_n x)$.

Theorem 7

$f(x)$是定义在$(0,a)$上的逐段光滑函数，$\int_0^a \sqrt{x}|f(x)|\mathrm{d}x$是有限数，且$f(x)$满足相应固有值问题的边界条件，则可以将$f(x)$写成固有函数的广义傅里叶级数（傅里叶-贝塞尔级数）。

$$f(x) = \sum_{n=1}^{+\infty} a_n J_\nu(\omega_n x),$$

其中，$a_n = \dfrac{1}{\|J_\nu(\omega_n x)\|^2} \displaystyle\int_0^a x f(x) J_\nu(\omega_n x)\mathrm{d}x$

级数收敛于$\dfrac{1}{2}[f(x+0)+f(x-0)]$.

将函数展开成Bessel函数的级数时，需要计算$\|J_\nu(\omega x)\|^2$.

$$\|J_\nu(\omega x)\|^2 = \int_0^a xJ_\nu^2(\omega x)\mathrm{d}x$$

记$y(x) = J_\nu(\omega x)$满足方程

$$x^2y'' + xy' + (\omega^2x^2 - \nu^2)y = 0$$

S-L标准型是$(xy')' - \dfrac{\nu^2}{x}y + \omega^2xy = 0$

两边乘以$2xy'$得: $2xy'(xy')' - 2\nu^2yy' + 2\omega^2x^2yy' = 0$

在$[0,a]$积分得: $\int_0^a 2xy'\mathrm{d}(xy') + \int_0^a(\omega^2x^2 - \nu^2)2y\mathrm{d}y = 0$

分部积分得: $(xy')^2\big|_0^a + (\omega^2x^2 - \nu^2)y^2\Big|_0^a - \int_0^a 2\omega^2xy^2\mathrm{d}x = 0$

所以 $2\omega^2\|J_\nu(\omega x)\|^2 = (xy')^2\big|_0^a + (\omega^2x^2 - \nu^2)y^2\Big|_0^a$.

按照边界条件的类型确定$\|J_\nu(\omega x)\|^2$

$$\|J_\nu(\omega x)\|^2 = \frac{1}{2\omega^2}\left[(xy')^2\Big|_0^a + (\omega^2x^2-\nu^2)y^2\Big|_0^a\right]$$
$$= \frac{1}{2\omega^2}\left[\omega^2a^2(J_\nu'(\omega a))^2 + (\omega^2a^2-\nu^2)J_\nu^2(\omega a)\right]$$
$$J_\nu'(x) = \frac{\nu}{x}J_\nu(x) - J_{\nu+1}(x)$$

$J_\nu(\omega a) = 0$

$$J_\nu'(\omega a) = \frac{\nu}{\omega a}J_\nu(\omega a) - J_{\nu+1}(\omega a) = -J_{\nu+1}(\omega a)$$

$$N_{\nu 1}^2 = \|J_\nu(\omega x)\|_1^2 = \frac{1}{2\omega^2}\left[\omega^2a^2(J_\nu'(\omega a))^2\right] = \frac{a^2}{2}J_{\nu+1}^2(\omega a)$$

$J'_\nu(\omega a) = 0$

$$N_{\nu 2}^2 = \|J_\nu(\omega x)\|_2^2 = \frac{1}{2}\left[a^2 - \frac{\nu^2}{\omega^2}\right] J_\nu^2(\omega a).$$

$\alpha J_\nu(\omega a) + \beta\omega J'_\nu(\omega a) = 0(\alpha\beta \neq 0)$

$$J'_\nu(\omega a) = -\frac{\alpha}{\beta\omega} J_\nu(\omega a)$$

$$N_{\nu 3}^2 = \|J_\nu(\omega x)\|_3^2 = \frac{1}{2}\left(a^2 - \frac{\nu^2}{\omega^2} - \left(\frac{a\alpha}{\omega\beta}\right)^2\right) J_\nu^2(\omega a).$$

Example 8

设ω_n是$J_0(x)$的全体正根，将$f(x)=1-x^2$在$0<x<1$按$\{J_0(\omega_n x)\}$作广义Fourier展开.

解:设$1-x^2=\sum\limits_{n=1}^{\infty} C_n J_0(\omega_n x)$ 作变量代换$\omega_n x=t$,则

$$
\begin{aligned}
C_n &= \frac{1}{N_{01n}^2}\int_0^1 x(1-x^2)J_0(\omega_n x)\mathrm{d}x \\
&= \frac{1}{\|J_0(\omega_n x)\|^2}\int_0^{\omega_n} \frac{t}{\omega_n^2}\left(1-\frac{t^2}{\omega_n^2}\right)J_0(t)\mathrm{d}t \\
&= \frac{2}{\omega_n^2 J_1^2(\omega_n)}\int_0^{\omega_n}\left(1-\frac{t^2}{\omega_n^2}\right)\mathrm{d}(tJ_1(t)) \\
&= \frac{2}{\omega_n^2 J_1^2(\omega_n)}\left[\left(1-\frac{t^2}{\omega_n^2}\right)tJ_1(t)\Big|_{t=0}^{\omega_n}+\frac{2}{\omega_n^2}\int_0^{\omega_n} t^2 J_1(t)\mathrm{d}t\right]
\end{aligned}
$$

$$= \frac{2}{\omega_n^2 J_1^2(\omega_n)} \left[\frac{2}{\omega_n^2} t^2 J_2(t) \Big|_0^{\omega_n} \right] = \frac{4J_2(\omega_n)}{\omega_n^2 J_1^2(\omega_n)}$$

由递推关系

$$J_2(x) = \frac{2}{x} J_1(x) - J_0(x), \quad J_0(\omega_n) = 0$$

得$J_2(\omega_n) = \dfrac{2}{\omega_n} J_1(\omega_n)$, 所以$C_n = \dfrac{8}{\omega_n^3 J_1(\omega_n)}$

$$1 - x^2 = \sum_{n=1}^{\infty} \frac{8}{\omega_n^3 J_1(\omega_n)} J_0(\omega_n x).$$

Example 9

若$f(x) = \sum_{n=1}^{\infty} A_n J_0(\omega_n x)$，其中$\omega_n$满足$J_0(\omega_n) = 0, n = 1, 2, \cdots$，证明

$$\int_0^1 x f^2(x) \mathrm{d}x = \frac{1}{2} \sum_{n=1}^{\infty} A_n^2 J_1^2(\omega_n).$$

证明:$J_0(\omega_n x)$是固有值问题

$$\begin{cases} x^2 y'' + xy' + \lambda x^2 y = 0, & 0 < x < 1 \\ |y(0)| < \infty, y(1) = 0 \end{cases}$$

的解，根据正交性$\int_0^1 J_0(\omega_n x) J_0(\omega_m x) x \mathrm{d}x = 0 (m \neq n)$.

$$
\begin{aligned}
\int_0^1 x f^2(x)\mathrm{d}x &= \int_0^1 x \left(\sum_{n=1}^{\infty} A_n J_0(\omega_n x)\right)\left(\sum_{m=1}^{\infty} A_m J_0(\omega_m x)\right)\mathrm{d}x \\
&= \sum_{n=1}^{\infty} A_n^2 \int_0^1 x J_n^2(\omega_n x)\mathrm{d}x \\
&= \sum_{n=1}^{\infty} A_n^2 \|J_0(\omega_n x)\|^2 = \frac{1}{2}\sum_{n=1}^{\infty} A_n^2 J_1^2(\omega_n)
\end{aligned}
$$

Example 10

设半径为1的均匀薄圆盘，边界温度为0度，初始时刻圆内温度$u(r,\theta,0)=1-r^2$, 求圆内温度分布规律.

解:初始温度与θ无关，所以设$u=u(r,t)$,u满足的定解问题是

$$\begin{cases} u_t = a^2(u_{rr}+\frac{1}{r}u_r), & 0<r<1, t>0 \\ u|_{r=1}=0 \\ u|_{t=0}=1-r^2 \end{cases}$$

1° 分离变量：设$u=R(r)T(t)$, 代入方程和边界条件得

$$RT' = a^2(R''+\frac{1}{r}R')T, \qquad R(1)T(t)=0$$

所以$\dfrac{T'}{a^2T}=\dfrac{R''+\frac{1}{r}R'}{R}=-\lambda, \qquad R(1)=0.$

2° 解固有值问题$\begin{cases} r^2R''+rR'+\lambda r^2R=0 \\ R(1)=0, R(0)\text{有界} \end{cases}$

化成S-L标准型为$(rR')'+\lambda rR=0$，$\rho(r)=r$.

记$\lambda=\omega^2>0$, 此方程通解是$R(r)=CJ_0(\omega r)+DN_0(\omega r)$.

由$R(0)$有界，取$D=0$, $R(1)=CJ_0(\omega)=0$

设ω_n是$J_0(\omega)$的所有实零点，固有值为$\lambda_n=\omega_n^2$,

固有函数$R_n(r)=J_0(\omega_n r)$

3° 解方程$T_n'+a^2\lambda_nT_n=0$

$T_n(T)=C_ne^{-a^2\lambda_nt}$

4° $u(r,t)=\sum_{n=1}^{\infty}C_n e^{-a^2\lambda_n t}J_0(\omega_n r)$

$u(r,0)=\sum_{n=1}^{\infty}C_n J_0(\omega_n r)=1-r^2$

$C_n=\frac{1}{N_{01}^2}\int_0^1 r(1-r^2)J_0(\omega_n r)\mathrm{d}r,\quad N_{01}^2=\frac{J_1^2(\omega_n)}{2}$

$\int_0^1 r(1-r^2)J_0(\omega_n r)\mathrm{d}r=\frac{4}{\omega_n^3}J_1(\omega_n)$，所以$C_n=\frac{8}{\omega_n^3 J_1(\omega_n)}$

定解问题的解是$u(r,t)=\sum_{n=1}^{\infty}\frac{8}{\omega_n^3 J_1(\omega_n)}e^{-a^2\lambda_n t}J_0(\omega_n r)$.

Example 11

求解定解问题

$$\begin{cases} \Delta_3 u = 0, \qquad 0 < r < a, 0 < z < h, 0 < \theta < 2\pi \\ \left.\dfrac{\partial u}{\partial r}\right|_{r=a} = 0 \\ u|_{z=0} = 0, u|_{z=h} = f(r) \end{cases}$$

解: $\Delta_3 u = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} + \dfrac{\partial^2 u}{\partial z^2}$

定解条件与θ无关，$\dfrac{\partial^2 u}{\partial \theta^2} = 0$，设$u = u(r, z)$是方程的解。

1° 令$u(r, z) = R(r)Z(z)$,代入方程分离变量得:

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -\frac{Z''}{Z} = -\lambda$$

$$R'' + \frac{1}{r}R' + \lambda R = 0, \qquad Z'' - \lambda Z = 0$$

由齐次边界条件$\left.\dfrac{\partial u}{\partial r}\right|_{r=a} = 0$得：$R'(a) = 0$ 得到固有值问题

$$\begin{cases} R'' + \dfrac{1}{r}R' + \lambda R = 0 \\ R'(a) = 0,\ R(0)\text{有界} \end{cases}$$

2° 由S-L定理，可知$\lambda \geqslant 0$, 记$\lambda = \omega^2$.

方程的解是$R(r) = CJ_0(\omega r)$.

由边界条件得$R'(a) = -C\omega J_1(\omega a) = 0$.

设ω_n是$J_0'(\omega a)$的第n个正零点，$n = 0, 1, 2, \cdots$

固有值$\lambda_0 = 0, \lambda_n = \omega_n^2$,

对应的固有函数分别是$R_0(r) = 1, R_n(r) = J_0(\omega_n r)$.

3° 将固有值代入$Z(z)$的方程得

$$Z_0(z) = C_0 + D_0 z$$
$$Z_n(z) = C_n \cosh \omega_n z + D_n \sinh \omega_n z \quad (n = 1, 2, \cdots)$$

$$u(r, z) = C_0 + D_0 z + \sum_{n=1}^{+\infty} (C_n \cosh \omega_n z + D_n \sinh \omega_n z) J_0(\omega_n r)$$

4° 由非齐次的边界条件定系数。

$$0 = C_0 + \sum_{n=1}^{+\infty} C_n J_0(\omega_n r) \Rightarrow C_n = 0 \tag{23}$$

$$f(r) = D_0 h + \sum_{n=1}^{+\infty} D_n \sinh(\omega_n h) J_0(\omega_n r) \tag{24}$$

根据固有函数的正交性

$$D_0 h = \frac{1}{N_{020}^2} \int_0^a f(r) r \mathrm{d}r$$

$$D_n \sinh(\omega_n h) = \frac{1}{N_{02n}^2} \int_0^a f(r) J_0(\omega_n r) r \mathrm{d}r$$

在第二类齐次边界条件下

$$N_{\nu 2n}^2 = \|J_\nu(\omega_n r)\|^2 = \frac{1}{2}\left(a^2 - \frac{\nu^2}{\omega_n^2}\right) J_\nu^2(\omega_n a)$$

$$N_{020}^2 = \frac{a^2}{2} J_0^2(0) = \frac{a^2}{2}, \quad N_{02n}^2 = \frac{1}{2} a^2 J_0^2(\omega_n a)$$

$$D_0 = \frac{2}{a^2 h} \int_0^a f(r) r \mathrm{d}r \qquad D_n = \frac{2 \int_0^a f(r) J_0(\omega_n r) r \mathrm{d}r}{a^2 J_0^2(\omega_n a) \sinh(\omega_n h)}$$

Example 12

$$\begin{cases} u_{tt} = a^2(u_{xx} + u_{yy}), & x^2 + y^2 < l^2, t > 0 \\ u|_{r=l} = 0, u|_{r=0}\text{有界} \\ u|_{t=0} = f(r,\theta), u_t|_{t=0} = 0 \end{cases}$$

解:在极坐标下，定解问题改写为

$$\begin{cases} u_{tt} = a^2(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}), & 0 < r < l, t > 0 \\ u|_{r=l} = 0, u|_{r=0}\text{有界} \\ u|_{\theta=0} = u|_{\theta=2\pi}, u_\theta|_{\theta=0} = u_\theta|_{\theta=2\pi} \\ u|_{t=0} = f(r,\theta), u_t|_{t=0} = 0 \end{cases}$$

1°设$u = R(r)\Theta(\theta)T(t)$, 代入方程

$$T''R\Theta = a^2\left(R''T\Theta + \frac{1}{r}R'T\Theta + \frac{1}{r^2}RT\Theta''\right)$$

逐步分离变量 $\dfrac{T''}{a^2T} = \dfrac{R''}{R} + \dfrac{1}{r}\dfrac{R'}{R} + \dfrac{1}{r^2}\dfrac{\Theta''}{\Theta} = -\lambda$

所以有

$$T'' + \lambda a^2 T = 0$$

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \lambda r^2 = -\frac{\Theta''}{\Theta} = m^2$$

又可以得到$r^2R''+rR'+(\lambda r^2-m^2)R = 0, \qquad \Theta''+m^2\Theta = 0.$

2°解固有值问题

$$\begin{cases} \Theta'' + m^2\Theta = 0 \\ u|_{\theta=0} = u|_{\theta=2\pi}, u_\theta|_{\theta=0} = u_\theta|_{\theta=2\pi} \end{cases}$$

固有值m取非负整数，固有函数$\Theta_m = C_m \cos m\theta + D_m \sin m\theta,$

$m = 0, 1, 2, \cdots$

$$\begin{cases} r^2R'' + rR' + (\lambda r^2 - m^2)R = 0 \\ R(0)\text{有界，} R(l) = 0 \end{cases}$$

有第一类边界条件，所以$\lambda > 0$, 记$\lambda = k^2$, 方程的通解是

$$R(r) = CJ_m(kr) + DN_m(kr),\ k > 0$$

$R(0)$有界，所以$D = 0$,

$R(l) = 0$可得$J_m(kl) = 0$.

记$\omega_i^{(m)}$是$J_m(x)$的第i个零点，$k_{im} = \frac{\omega_i^{(m)}}{l}$, $i = 1, 2, \cdots, m = 0, 1, 2\cdots$

固有值$\lambda_{im} = k_{im}^2$, $R_{im} = J_m(k_{im}r)$

3° $T'' + k_{im}^2a^2T = 0$

通解是$T_{im} = A_{im}\cos\frac{a\omega_i^{(m)}t}{l} + B_{im}\sin\frac{a\omega_i^{(m)}t}{l}$.

4° 叠加定系数

$$u(r,\theta,t)=\sum_{m=0}^{\infty}\left[\sum_{i=1}^{\infty}\left(A_{im}\cos\frac{a\omega_i^{(m)}t}{l}+B_{im}\sin\frac{a\omega_i^{(m)}t}{l}\right)\right.$$
$$\left.J_m\left(\frac{a\omega_i^{(m)}r}{l}\right)\right]\cdot(C_m\cos m\theta+D_m\sin m\theta)$$

由初始条件

$$\begin{cases}\displaystyle\sum_{m=0}^{\infty}\left[\sum_{i=1}^{\infty}A_{im}J_m\left(\frac{a\omega_i^{(m)}r}{l}\right)\right]\cdot(C_m\cos m\theta+D_m\sin m\theta)=f(r,\theta)\\ \displaystyle\sum_{m=0}^{\infty}\left[\sum_{i=1}^{\infty}B_{im}\frac{a\omega_i^{(m)}}{l}J_m\left(\frac{a\omega_i^{(m)}r}{l}\right)\right]\cdot(C_m\cos m\theta+D_m\sin m\theta)=0\end{cases}$$

$$A_{im}C_m = \frac{\alpha_m}{\pi l^2 J_{m+1}^2(k_{im}l)} \int_0^l \int_0^{2\pi} rf(r,\theta)J_m\left(k_{mi}r\right)\cos m\theta \mathrm{d}r\mathrm{d}\theta$$

$$\alpha_m = \begin{cases} 1, & m = 0 \\ 2, & m \neq 0 \end{cases}$$

$$A_{im}D_m = \frac{2}{\pi l^2 J_{m+1}^2(k_{im}l)} \int_0^l \int_0^{2\pi} rf(r,\theta)J_m\left(k_{mi}r\right)\sin m\theta \mathrm{d}r\mathrm{d}\theta$$

$$B_{im} = 0$$

一、 Bessel函数

二、 Bessel函数的性质

三、 Bessel方程的固有值问题

四、 **Legendre多项式**

五、 函数的Fourier-Legendre展开

一、Legendre方程导出

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$

在球坐标系

$$x=r\sin\theta\cos\varphi,\ \ y=r\sin\theta\sin\varphi,\ \ z=r\cos\theta$$

$$\Delta_3 u=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}=0$$

对此方程运用分离变量法：令$u(r,\theta,\varphi)=R(r)\Theta(\theta)\Phi(\varphi)$，代入方程

$$\frac{\Theta\Phi}{r^2}(r^2R')'+\frac{R\Phi}{r^2\sin\theta}(\sin\theta\Theta')'+\frac{R\Theta}{r^2\sin^2\theta}\Phi''=0$$

两边乘以$\frac{r^2}{R\Theta\Phi}$

$$\frac{1}{\sin\theta}\frac{(\sin\theta\Theta')'}{\Theta}+\frac{1}{\sin^2\theta}\frac{\Phi''}{\Phi}=-\frac{(r^2R')'}{R}=-\lambda$$

得到$(r^2R')'-\lambda R=0$————Euler方程

$$\sin\theta\frac{(\sin\theta\Theta')'}{\Theta}+\mu\sin^2\theta=-\frac{\Phi''}{\Phi}=\sigma$$

$$\sin\theta(\sin\theta\Theta')'+(\lambda\sin^2\theta-\sigma)\Theta=0 \tag{25}$$

$$\Phi''+\sigma\Phi=0 \tag{26}$$

(31)式搭配周期条件可得固有值问题：

$$\begin{cases}\Phi''+\sigma\Phi=0\\ \Phi(0)=\Phi(2\pi),\ \Phi'(0)=\Phi'(2\pi)\end{cases}$$

当$\sigma = m^2, m = 0, 1, 2, \cdots$时有解，
$\Phi(\varphi) = B_1 \cos m\varphi + B_2 \sin m\varphi$
$\sigma = m^2$代入，变形可得

$$\Theta_m'' + \frac{\cos\theta}{\sin\theta}\Theta_m' + [\mu - \frac{m^2}{\sin^2\theta}]\Theta_m = 0$$

令$x = \cos\theta$，记$y(x) = \Theta(\arccos x)$,则

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\theta} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}\theta} = -\frac{\mathrm{d}y}{\mathrm{d}x}\sin\theta, \quad \frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2} = \frac{\mathrm{d}^2y}{\mathrm{d}x^2}\sin^2\theta - \frac{\mathrm{d}y}{\mathrm{d}x}\cos\theta$$

方程化为

$$\sin^2\theta y'' - 2\cos\theta y' + [\lambda - \frac{m^2}{1-\cos^2\theta}]y = 0$$

$$(1-x^2)y'' - 2xy' + [\lambda - \frac{m^2}{1-x^2}]y = 0$$

此方程称为**连带Legendre方程**

若$u(r,\theta,\varphi)$与φ无关，即问题及边界条件有轴对称特性，则$m=0$ 方程应是

$$(1-x^2)y'' - 2xy' + \lambda y = 0$$

此方程称为**Legendre方程**

Legendre方程求解

$$(1-x^2)y''-2xy'+\lambda y=0\longrightarrow y''-\frac{2x}{1-x^2}y'+\frac{\lambda}{1-x^2}y=0$$

◇ Legendre方程有三个奇点$x=\pm1,\infty$,且都是正则奇点，因此除这三个点以外，Legendre方程的解在全平面解析.

◇ $x=0$是方程的常点，$x=\pm1$是正则奇点，方程的解在$x=0$的邻域$|x|<1$内可以展开成幂级数。

Legendre方程固有值问题

$$\begin{cases} [(1-x^2)y']' + \lambda y = 0 \qquad -1 < x < 1 \\ |y(\pm 1)| < \infty \end{cases}$$

设$y(x) = \sum\limits_{n=0}^{+\infty} c_n x^n$是方程的解，代入方程得

$$(1-x^2)\sum_{n=2}^{+\infty} c_n n(n-1)x^{n-2} - 2x\sum_{n=1}^{+\infty} c_n n x^{n-1} + \lambda \sum_{n=0}^{+\infty} c_n x^n = 0$$

$$\sum_{k=0}^{+\infty} c_{k+2}(k+2)(k+1)x^k - \sum_{n=2}^{+\infty} c_n n(n-1)x^n - \sum_{n=1}^{+\infty} c_n 2n x^n$$

$$+ \lambda \sum_{n=0}^{+\infty} c_n x^n = 0$$

合并同类项:

$$
\begin{aligned}
&[2c_2+\lambda c_0]x^0+[c_3 3\cdot 2-2c_1+\lambda c_1]x \\
&+\sum_{n=2}^{\infty} c_{n+2}(n+2)(n+1)-c_n[n(n+1)-\lambda]x^n=0
\end{aligned}
$$

$$
\begin{aligned}
x^0: &\quad 2c_2+\lambda c_0 = c_2 2\cdot 1 - c_0[0\cdot 1-\lambda] \\
x^1: &\quad c_3 3\cdot 2 - 2c_1+\lambda c_1 = c_3 3\cdot 2 - c_1[1\cdot 2-\lambda]
\end{aligned}
$$

前两项与$x^n(n\geq 2)$的系数可以写成统一形式，所以

$$
\sum_{n=0}^{\infty}\left(c_{n+2}(n+2)(n+1)-c_n[n(n+1)-\lambda]\right)x^n=0
$$

令$\lambda = l(l+1)$(任意常数λ都可以写成这种形式)

$$c_{n+2} = \frac{n(n+1) - l(l+1)}{(n+2)(n+1)} c_n = \frac{(n-l)(n+l+1)}{(n+2)(n+1)} c_n, \quad n = 0, 1, 2, \cdots$$

$$\begin{aligned} c_{2n} &= \frac{(2n-l-2)(2n+l-1)}{2n(2n-1)} c_{2n-2} \\ &= \frac{(2n-l-2)(2n-l-4)(2n+l-1)(2n+l-3)}{2n(2n-1)(2n-2)(2n-3)} c_{2n-4} = \cdots \\ &= \frac{c_0}{(2n)!}(2n-l-2)\cdots(-l) \times (2n+l-1)\cdots(l+1) \\ &= \frac{2^{2n}}{(2n)!} \frac{\Gamma(n-\frac{l}{2})\Gamma(n+\frac{l+1}{2})}{\Gamma(-\frac{l}{2})\Gamma(\frac{l+1}{2})} c_0 \end{aligned}$$

同理可以计算得到

$$
\begin{aligned}
c_{2n+1} &= \frac{c_1}{(2n+1)!}(2n-l-1)\cdots(-l+1)\times(2n+l)\cdots(l+2)\\
&= \frac{2^{2n}}{(2n+1)!}\frac{\Gamma(n-\frac{l-1}{2})\Gamma(n+1+\frac{l}{2})}{\Gamma(-\frac{l-1}{2})\Gamma(1+\frac{l}{2})}c_1
\end{aligned}
$$

Legendre方程的通解是$y(x)=c_0y_1(x)+c_1y_2(x)$.其中

$$
y_1(x)=\sum_{n=0}^{+\infty}\frac{2^{2n}}{(2n)!}\cdot\frac{\Gamma(n-\dfrac{l}{2})\Gamma(n+\dfrac{l+1}{2})}{\Gamma(-\dfrac{l}{2})\Gamma(\dfrac{l+1}{2})}x^{2n}
$$

$$
y_2(x)=\sum_{n=0}^{+\infty}\frac{2^{2n}}{(2n+1)!}\cdot\frac{\Gamma(n-\dfrac{l-1}{2})\Gamma(n+1+\dfrac{l}{2})}{\Gamma(-\dfrac{l-1}{2})\Gamma(1+\dfrac{l}{2})}x^{2n+1}
$$

(1)由于 $\lim\limits_{n\to\infty}\dfrac{c_{n+2}}{c_n}=\lim\limits_{n\to\infty}\dfrac{(n-l)(n+l+1)}{(n+2)(n+1)}=1$,
所以$y_1(x),y_2(x)$的收敛半径均为1.可以（用Gauss判别法）证明在$x=\pm1$处,级数都发散

(2)在Legendre方程固有值问题中，$x=1$，-1对应$\theta=0,\pi$,相应解须在$\theta=0,\pi$有意义且满足可导条件，即$|y(\pm1)|<\infty$，但无穷级数在这两点发散，所以$y_1(x)$或$y_2(x)$不能是无穷级数。

(3)当l为整数时,$y_1(x)$或$y_2(x)$就成为多项式,
$l=2k$时，$y_1(x)$是l次多项式,$y_2(x)$是无穷级数;
$l=2k+1$时，$y_2(x)$是l次多项式,$y_1(x)$是无穷级数.

固有值问题$\begin{cases}(1-x^2)y''-2xy'+\lambda y=0 & -1<x<1\\ |y(\pm1)|<\infty\end{cases}$

方程的通解是$y(x)=c_0y_1(x)+c_1y_2(x)$,
当$\lambda=l(l+1)$且l为非负整数时，方程有满足有界性条件的解，

固有值$l(l+1), l=0,1,2,\cdots,$固有函数

$$\begin{cases}l=2n: & y(x)=c_0+c_2x^2+\cdots+c_lx^l\\ l=2n+1 & y(x)=c_1x+c_3x^3+\cdots+c_lx^l\end{cases}$$

可以直接证明方程的解具有下述微分形式.

Theorem 13

*Legendre*方程固有值问题

$$\begin{cases} (1-x^2)y'' - 2xy' + \lambda y = 0 & \quad -1 < x < 1 \\ |y(\pm 1)| < \infty \end{cases}$$

固有值是$\lambda_n = n(n+1), n = 0, 1, 2, \cdots,$
固有函数是$y_n(x) = C_n \dfrac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n.$

证明: $y_n(x)$是n次多项式，满足有界性条件，所以只需证明$y_n(x)$满足方程

$$(1-x^2)y'' - 2xy' + \lambda y = 0.$$

令$u=(x^2-1)^n$, 则$u'=2nx(x^2-1)^{n-1}$,

所以$u'(x^2-1)=2nx(x^2-1)^n=2nxu$

两边求$n+1$阶导数,

$$(x^2-1)u^{(n+2)}+(n+1)2xu^{(n+1)}+n(n+1)u^{(n)}=2nxu^{(n+1)}+2n(n+1)u^{(n)}$$

整理可得 $(1-x^2)u^{(n+2)}-2xu^{(n+1)}+n(n+1)u^{(n)}=0.$

所以$y_n(x)=u^{(n)}(x)$是方程的解.

固有函数只需找一个特解即可，一般取$C_n=\dfrac{1}{2^n n!}$,

$P_n(x)=\dfrac{1}{2^n n!}\dfrac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n$称为Legendre多项式.

Legendre多项式的降幂形式

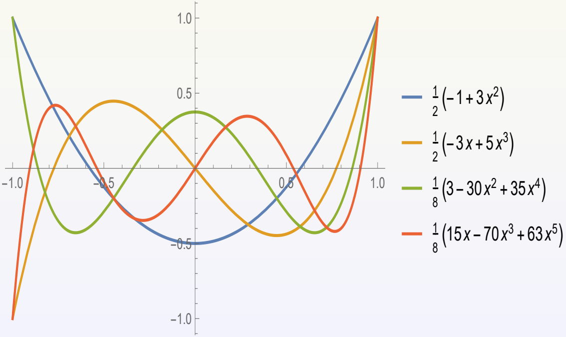
$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n = \sum_{k=0}^{[\frac{n}{2}]}\frac{(-1)^k(2n-2k)!}{2^n k!(n-k)!(n-2k)!}x^{n-2k}$$

证明:

$$(x^2-1)^n = \sum_{k=0}^{n}\frac{n!}{k!(n-k)!}(-1)^k x^{2n-2k}$$

$$\begin{aligned}\frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n &= \frac{1}{2^n}\sum_{k=0}^{n}\frac{(-1)^k}{k!(n-k)!}(x^{2n-2k})^{(n)}\\ &= \sum_{k=0}^{[\frac{n}{2}]}\frac{(-1)^k(2n-2k)!}{2^n k!(n-k)!(n-2k)!}x^{n-2k}\end{aligned}$$

$$P_0(x) = 1, \qquad P_1(x) = x,$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1), \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x)$$
$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \qquad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$



Legendre多项式的性质

1. 正交性　根据$S-L$固有值理论

$$\int_{-1}^{1} P_n(x)P_m(x)\mathrm{d}x = 0, \quad m \neq n$$

2. 特殊值

$$P_n(0) = \begin{cases} 0, & n = 2m+1 \\ \dfrac{(-1)^m(2m)!}{2^{2m}m!m!} = \dfrac{(-1)^m(2m-1)!!}{(2m)!!}, & n = 2m \end{cases}$$

3.母函数 $w(x,t)=\dfrac{1}{\sqrt{1-2xt+t^2}}=\sum\limits_{n=0}^{\infty}P_n(x)t^n \quad (|t|<1).$

证明:将$w(x,t)$看做t的函数，使分母为零的的$t_{12}=x\pm\sqrt{1-x^2}i$, $|t_{12}|=1$, 所以$|t|<1$ 时，$w(x,t)$解析。可以展开成t的幂级数，设

$$w(x,t)=\sum_{n=0}^{+\infty}C_n(x)t^n$$

$$C_n(x)=\frac{1}{2\pi i}\int\limits_C\frac{(1-2xt+t^2)^{-\frac{1}{2}}}{t^{n+1}}\mathrm{d}t$$

C是$|t|<1$内包含原点的任意闭路。

作欧拉代换$\sqrt{1-2xt+t^2}=1-tu$,

两边平方并整理得$t = \dfrac{2(u-x)}{u^2-1}$

$$\mathrm{d}t = \frac{2(1-tu)}{u^2-1}\mathrm{d}u, \quad \frac{1}{t^{n+1}} = \frac{(u^2-1)^{n+1}}{2^{n+1}(u-x)^{n+1}}$$

$$\frac{1}{2\pi i}\int_C \frac{(1-2xt+t^2)^{-\frac{1}{2}}}{t^{n+1}}\mathrm{d}t = \frac{1}{2\pi i}\int_{C'} \frac{(u^2-1)^n}{2^n(u-x)^{n+1}}\mathrm{d}u.$$

根据Cauchy积分公式

$$C_n(x) = \frac{1}{2^n n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}u^n}(u^2-1)\right]\Big|_{u=x} = P_n(x)$$

4. 奇偶性 $P_n(-x) = (-1)^n P_n(x)$

证明:

$$\frac{1}{\sqrt{1-2xt+t^2}} = \frac{1}{\sqrt{1-2(-x)(-t)+(-t)^2}}$$

$$\sum_{n=0}^{+\infty} P_n(x)t^n = \sum_{n=0}^{+\infty} P_n(-x)(-t)^n = \sum_{n=0}^{+\infty} (-1)^n P_n(-x)t^n$$

$$P_n(-x) = (-1)^n P_n(x)$$

n是奇数时，$P_n(x)$是奇函数，n是偶数时，$P_n(x)$是偶函数。

类似可以得到$\dfrac{1}{\sqrt{1-2t+t^2}} = \dfrac{1}{1-t} = \sum\limits_{n=0}^{\infty} t^n = \sum\limits_{n=0}^{+\infty} P_n(1)t^n.$

比较系数可得 $P_n(1) = 1$, $P_n(-1) = (-1)^n$, $n = 0, 1, 2, \cdots.$

5. 递推公式

$$\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{+\infty}P_n(x)t^n$$

两边对t求导

$$\frac{x-t}{(1-2xt+t^2)^{\frac{3}{2}}}=\sum_{n=1}^{+\infty}nP_n(x)t^{n-1}$$

$$\frac{x-t}{\sqrt{1-2xt+t^2}}=(1-2xt+t^2)\sum_{n=1}^{+\infty}nP_n(x)t^{n-1}=(x-t)\sum_{n=0}^{+\infty}P_n(x)t^n$$

比较两边t的同次幂系数

$$xP_n(x)-P_{n-1}(x)=(n+1)P_{n+1}(x)-2xnP_n(x)+(n-1)P_{n-1}(x)$$

$$\boxed{(n+1)P_{n+1}(x)-x(2n+1)P_n(x)+nP_{n-1}(x)=0 \qquad \text{(I)}}$$

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{+\infty} P_n(x)t^n$$

两边对x求导

$$\frac{t}{(1-2xt+t^2)^{\frac{3}{2}}} = \sum_{n=0}^{+\infty} P_n'(x)t^n$$

$$\frac{t}{\sqrt{1-2xt+t^2}} = (1-2xt+t^2)\sum_{n=0}^{+\infty} P_n'(x)t^n = \sum_{n=0}^{+\infty} P_n(x)t^{n+1}$$

比较t的同次幂系数

$$\boxed{P_n(x) = P_{n+1}'(x) - 2xP_n'(x) + P_{n-1}'(x) \qquad \text{(II)}}$$

(I)式两边对x求导得

$$(2n+1)P_n(x)+x(2n+1)P_n'(x)=(n+1)P_{n+1}'(x)+nP_{n-1}'(x)$$

与（II）式联立，消去$P_{n+1}'(x)$得

$$\boxed{nP_n(x)-xP_n'(x)+P_{n-1}'(x)=0 \qquad \text{(III)}}$$

消去$P_{n-1}'(x)$得

$$\boxed{(n+1)P_n(x)-P_{n+1}'(x)+xP_n'(x)=0 \qquad \text{(IV)}}$$

(III)+(IV)得

$$\boxed{P_{n+1}'(x)-P_{n-1}'(x)=(2n+1)P_n(x) \qquad \text{(V)}}$$

递推关系的用途是计算某些类型的积分。

Example 14

计算积分 $\int_0^1 xP_{2n}(x)\mathrm{d}x$

分析 $\boxed{P_n(x) = \frac{1}{2n+1}[P'_{n+1}(x) - P'_{n-1}(x)]}$

解：

$$\begin{aligned}\int_0^1 xP_{2n}(x)dx &= \frac{1}{4n+1}\int_0^1 x[P'{}_{2n+1}(x) - P'{}_{2n-1}(x)]\mathrm{d}x \\ &= \frac{1}{4n+1}\left[x(P_{2n+1}(x) - P_{2n-1}(x))\Big|_0^1\right. \\ &\qquad \left. - \int_0^1 P_{2n+1}(x) - P_{2n-1}(x)\mathrm{d}x\right] \\ &= \frac{1}{4n+1}\int_0^1 P_{2n-1}(x) - P_{2n+1}(x)\mathrm{d}x\end{aligned}$$

$$\int_0^1 P_{2n+1}(x)dx = \frac{1}{2(2n+1)+1}\int_0^1 [P'_{2n+1+1}(x) - P'_{2n+1-1}(x)]dx$$
$$= \frac{1}{4n+3}\left[P_{2n+2}(x) - P_{2n}(x)\right]\Big|_0^1 = \frac{1}{4n+3}[P_{2n}(0) - P_{2n+2}(0)]$$
$$= \frac{1}{4n+3}\left[\frac{(-1)^n(2n-1)!!}{(2n)!!} - \frac{(-1)^n(2n+1)!!}{(2n+2)!!}\right]$$
$$= \frac{(-1)^n(2n-1)!!}{(2n+2)!!}$$

所以 $\displaystyle\int_0^1 P_{2n-1}(x)dx = \frac{(-1)^{n-1}(2n-3)!!}{(2n)!!}$

原积分 $\displaystyle\int_0^1 xP_{2n}(x)dx = \frac{(-1)^{n-1}(2n-3)!!}{(2n+2)!!}$

Example 15

设$m \geq 1$，$n \geq 1$，证明：

$$\int_0^1 x^m P_n(x)\mathrm{d}x = \frac{m}{m+n+1}\int_0^1 x^{m-1}P_{n-1}(x)\mathrm{d}x$$

证明:利用递推公式$nP_n(x) = xP_n'(x) - P_{n-1}'(x)$

$$n\int_0^1 x^m P_n(x)\mathrm{d}x = \int_0^1 x^m(xP_n'(x) - P_{n-1}'(x))\mathrm{d}x$$

$$=[x^{m+1}P_n(x) - x^m P_{n-1}(x)]\Big|_0^1 - \int_0^1 (m+1)x^m P_n(x) - mx^{m-1}P_{n-1}(x)$$

$$= -\int_0^1 (m+1)x^m P_n(x)\mathrm{d}x + m\int_0^1 x^{m-1}P_{n-1}(x)\mathrm{d}x$$

移项整理得证.

一、 Bessel函数

二、 Bessel函数的性质

三、 Bessel方程的固有值问题

四、 Legendre多项式

五、 **函数的Fourier-Legendre展开**

Theorem 16

$f(x)$是$(-1,1)$内任意实值函数，满足：

(1) $f(x)$在$(-1,1)$分段光滑；*(2)* $\int_{-1}^{1} f^2(x)\mathrm{d}x$有限。

则$f(x)$可以按*Legendre*多项式展开成无穷级数。

$$f(x) = \sum_{n=0}^{+\infty} C_n P_n(x)$$

其中 $C_n = \dfrac{2n+1}{2} \displaystyle\int_{-1}^{1} f(x)P_n(x)\mathrm{d}x.$

右端级数收敛于$\dfrac{1}{2}(f(x+0)+f(x-0))$, 如果$f(x)$在点$x$连续，则级数收敛于$f(x)$。

证明:只需证明Legendre多项式的模平方为$\dfrac{2}{2n+1}$,再根据Legendre多项式的正交性，结论可证.

$$\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{\infty}P_n(x)t^n, \qquad |t|<1$$

两边平方

$$\frac{1}{1-2xt+t^2}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}P_n(x)P_m(x)t^{n+m}$$

两边对x从-1到1积分

$$\text{左边}=\int_{-1}^{1}\frac{1}{1-2xt+t^2}\mathrm{d}x=\frac{1}{t}(\ln(1+t)-\ln(1-t))=\sum_{n=0}^{\infty}\frac{2}{2n+1}t^{2n}$$

$$\text{右边}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\int_{-1}^{1}P_n(x)P_m(x)\mathrm{d}x\right)t^{2n}=\sum_{n=0}^{\infty}\|P_n(x)\|^2t^{2n}$$

Example 17

将$f(x)=x^2$按Legendre多项式展开。

解法1. 设$x^2=\sum_{n=0}^{+\infty} C_n P_n(x)$.

$$C_n=\frac{2n+1}{2}\int_{-1}^{1} x^2 P_n(x)\mathrm{d}x$$

已知当$n>2$时，$C_n=0$, $n=1$时，$2+1$是奇数，$C_1=0$.

$$C_0=\frac{1}{2}\int_{-1}^{1} x^2\mathrm{d}x=\frac{1}{3}.$$

$$C_2=\frac{5}{2}\int_{-1}^{1} x^2 P_2(x)\mathrm{d}x=\frac{5}{2}2^3\frac{2!2!}{5!}=\frac{2}{3}$$

$$x^2=\frac{1}{3}P_0(x)+\frac{2}{3}P_2(x).$$

解法2：（待定系数法）利用Legendre多项式的奇偶性，设$x^2 = C_0P_0(x) + C_2P_2(x)$.

$$x^2 = C_0 \cdot 1 + C_2 \cdot \frac{1}{2}(3x^2 - 1)$$

比较两边x同次幂的系数可得$C_0 = \dfrac{1}{3}, \qquad C_2 = \dfrac{2}{3}$.

解法3：在用积分定出C_0后，还可以利用$P_n(1) = 1$定出C_2.

$$1^2 = C_0 + C_2P_2(1) = C_0 + C_2$$

Example 18

将函数$f(x)=\begin{cases}0, & -1\leqslant x<\alpha\\ \frac{1}{2}, & x=\alpha\\ 1 & \alpha<x\leqslant 1\end{cases}$ 用Legendre多项式展开.

解:设$f(x)=\sum\limits_{n=0}^{\infty}C_nP_n(x)$,$C_0=\frac{1}{2}\int_{\alpha}^{1}P_0(x)\mathrm{d}x=\frac{1-\alpha}{2}$

$$C_n=\frac{2n+1}{2}\int_{-1}^{1}f(x)P_n(x)\mathrm{d}x=\frac{2n+1}{2}\int_{\alpha}^{1}P_n(x)\mathrm{d}x$$

$$=\frac{1}{2}\int_{\alpha}^{1}P'_{n+1}(x)-P'_{n-1}(x)\mathrm{d}x=\frac{1}{2}[P_{n-1}(\alpha)-P_{n+1}(\alpha)]$$

$$f(x)=\frac{1-\alpha}{2}+\sum_{n=1}^{\infty}\frac{1}{2}[P_{n-1}(\alpha)-P_{n+1}(\alpha)]P_n(x).$$

Example 19

计算积分 $\int_{-1}^{1} x^l P_n(x) \mathrm{d}x$

解: 1.$l+n$是奇数时，$x^l P_n(x)$是奇函数，积分 $\int_{-1}^{1} x^l P_n(x) \mathrm{d}x = 0.$

2.$l+n$是偶数时，

(1)$l < n$时，由Legendre多项式的正交性和完备性，存在一组常数$c_0, c_1, c_2, \cdots, c_l$ 使得$x^l = c_0 P_0(x) + c_1 P_1(x) + \cdots + c_l P_l(x)$, 所以

$$\int_{-1}^{1} x^l P_n(x) \mathrm{d}x = \int_{-1}^{1} \sum_{k=0}^{l} c_k P_k(x) P_n(x) \mathrm{d}x = 0.$$

(2)当$l \geqslant n$时，

$$\int_{-1}^{1} x^l P_n(x) = \frac{1}{2^n n!}\int_{-1}^{1} x^l \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n \mathrm{d}x$$

$$= \frac{1}{2^n n!}\int_{-1}^{1} x^l \mathrm{d}\left([(x^2-1)^n]^{(n-1)}\right)$$

$$= \frac{1}{2^n n!}\left[x^l[(x^2-1)^n]^{(n-1)}\right]\Big|_{-1}^{1} - \frac{l}{2^n n!}\int_{-1}^{1} x^{l-1}[(x^2-1)^n]^{(n-1)}\mathrm{d}x$$

$$= -\frac{l}{2^n n!}\int_{-1}^{1} x^{l-1}[(x^2-1)^n]^{(n-1)}\mathrm{d}x = \cdots\cdots (n\text{次分部积分})$$

$$= \frac{(-1)^n l!}{2^n n!(l-n)!}\int_{-1}^{1}(x^2-1)^n x^{l-n}\mathrm{d}x$$

$$
\begin{aligned}
&= \frac{l!}{2^n n!(l-n)!} \int_{-1}^{1} (1-x^2)^n x^{l-n} \mathrm{d}x \\
&= \frac{l!}{2^n n!(l-n)!} \int_{0}^{1} (1-t)^n t^{\frac{l-n-1}{2}} \mathrm{d}t \\
&= \frac{l!}{2^n n!(l-n)!} B(n+1, \frac{l-n+1}{2}) \\
&= \frac{l!}{2^n n!(l-n)!} \cdot \frac{n!}{\frac{l+n+1}{2} \frac{l+n-1}{2} \cdots \frac{l-n+1}{2}} \\
&= \frac{l!}{(l-n)!(l+n+1)!!}
\end{aligned}
$$

Example 20

计算$\displaystyle\int_{-1}^{1} xP_n(x)P_m(x)\mathrm{d}x$.

解: 1° $m=n$时，$xP_n^2(x)$是奇函数，积分为零.

2° 当$|m-n|>1$, 不妨设$n>m+1$, 则设$xP_m(x)=\displaystyle\sum_{k=0}^{m+1} c_kP_k(x)$,

根据Legendre多项式的正交性

$$\int_{-1}^{1} xP_n(x)P_m(x)\mathrm{d}x=\sum_{k=0}^{m+1}\int_{-1}^{1} c_kP_k(x)P_n(x)\mathrm{d}x=0$$

3° $|m-n|=1$, 设$m=n+1$ 利用递推公式

$$(2n+1)xP_n(x)=(n+1)P_{n+1}(x)+nP_{n-1}(x)$$

$$\begin{aligned}
&\int_{-1}^{1} xP_n(x)P_{n+1}(x)\mathrm{d}x \\
=&\int_{-1}^{1} \frac{n+1}{2n+1}P_{n+1}^2(x)+\frac{n}{2n+1}P_{n-1}(x)P_{n+1}(x)\mathrm{d}x \\
=&\frac{n+1}{2n+1}\frac{2}{2(n+1)+1}=\frac{2(n+1)}{(2n+1)(2n+3)}
\end{aligned}$$

$$\int_{-1}^{1} xP_n(x)P_m(x)\mathrm{d}x=\begin{cases} 0, & |m-n|>1, \text{或} m=n \\ \dfrac{2(n+1)}{(2n+1)(2n+3)}, & m=n+1 \\ \dfrac{2n}{4n^2-1}, & m=n-1 \end{cases}$$

Example 21

在半径为a的接地金属球面内，放一个点电荷$4\pi\varepsilon_0 q$(ε_0为真空介电常数)，与球心距离为$b(b < a)$,求球面内的电势分布。

分析

1. 由于静电感应，在导体球面内部会形成一定的感应电荷；
2. 球内一点的电势为点电荷电势与感应电荷产生电势的叠加；点电荷的电势$u_1 = \dfrac{q}{\rho}$,ρ是球内一点到点电荷的距离。
   由于球面内部没有感应电荷，所以感应电荷产生电势在球面内部满足Laplace方程。
3. 球面接地，所以球面上电势为零。

解：选取球心为坐标原点，使点电荷位于z轴$(0, 0, b)$,
球面上点的坐标是$(a\sin\theta\cos\varphi, a\sin\theta\cos\varphi, a\cos\theta)$
点电荷与球面上点的距离$\rho_a = \sqrt{a^2 - 2ab\cos\theta + b^2}$

设球内一点的电势$v = u_1 + u$，

其中$u_1 = \dfrac{q}{\sqrt{r^2 - 2rb\cos\theta + b^2}}$

u满足 $\Delta u = 0$

$r = a$时，$v = u_1 + u = 0$, 所以

$$u\Big|_{r=a} = -\frac{q}{\sqrt{a^2 - 2ab\cos\theta + b^2}}$$

考虑球的对称性, 及点电荷的位置，u_1, u都与φ无关，设$u = u(r,\theta)$, 则u满足的定解问题是

$$\begin{cases} \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial u}{\partial r}\right) + \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial u}{\partial\theta}\right) = 0, & 0 < r < a, 0 < \theta < \pi \\ u\Big|_{\theta=0}\text{有界},\ u\Big|_{\theta=\pi}\text{有界}, & 0 \leqslant r \leqslant a \\ u\Big|_{r=0}\text{有界},\ u\Big|_{r=a} = -\dfrac{q}{\sqrt{a^2 - 2ab\cos\theta + b^2}}, & 0 \leqslant \theta \leqslant \pi \end{cases}$$

1° 将方程和边界条件分离变量

设$u = R(r)\Theta(\theta)$, 代入方程分离变量得

$$(I) \qquad \frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) + \lambda\Theta = 0$$

$$(II) \qquad \frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) - \lambda R = 0$$

2° 方程(I)是Legendre方程，当$\lambda = n(n+1)$（n是非负整数）时，方程有解固有值$\lambda_n = n(n+1)$对应的固有函数是$P_n(\cos\theta)$.

3° 将固有值代入方程（II）可得

$$r^2 R_n'' + 2rR_n' - n(n+1)R = 0$$

这是Euler方程，可作变量代换$t=\ln r$.方程化为

$$\frac{\mathrm{d}^2 R_n}{\mathrm{d}t^2}+\frac{\mathrm{d}R_n}{\mathrm{d}t}-n(n+1)R_n=0$$

方程的解是

$$R_n(r)=\left[A_n e^{nt}+B_n e^{-(n+1)t}\right]_{t=\ln r}=A_n r^n+B_n r^{-(n+1)}$$

一般解

$$u(r,\theta)=\sum_{n=0}^{+\infty}A_n r^n P_n(\cos\theta)$$

4°定系数。

由$u\big|_{r=0}$有界，可得$B_n=0$.
由$u\big|_{r=a}=f(\theta)$ 可得

$$\sum_{n=0}^{+\infty}A_n a^n P_n(\cos\theta)=-\frac{q}{\sqrt{a^2-2ab\cos\theta+b^2}}$$

根据Legendre多项式的正交性，

$$
\begin{aligned}
A_n a^n &= \frac{1}{\|P_n(\cos\theta)\|^2}\int_0^{\pi} -\frac{q}{\sqrt{a^2-2ab\cos\theta+b^2}}P_n(\cos\theta)\sin\theta\mathrm{d}\theta \\
&= -\frac{(2n+1)q}{2a}\int_{-1}^{1}\frac{1}{\sqrt{1-2\left(\frac{b}{a}\right)x+\left(\frac{b}{a}\right)^2}}P_n(x)\mathrm{d}x \\
&= -\frac{(2n+1)q}{2a}\int_{-1}^{1}\sum_{k=0}^{+\infty}P_k(x)\left(\frac{b}{a}\right)^k P_n(x)\mathrm{d}x \\
&= -\frac{(2n+1)q}{2a}\left(\frac{b}{a}\right)^n\int_{-1}^{1}P_n(x)P_n(x)\mathrm{d}x \\
&= -\frac{(2n+1)q}{2a}\left(\frac{b}{a}\right)^n\frac{2}{2n+1} = -\frac{q}{a}\left(\frac{b}{a}\right)^n
\end{aligned}
$$

$$
\begin{aligned}
u(r,\theta) &= -\frac{q}{a}\sum_{n=0}^{+\infty}\left(\frac{br}{a^2}\right)^n P_n(\cos\theta) \\
&= -\frac{q}{a}\frac{1}{\sqrt{1-\frac{2br}{a^2}\cos\theta+\left(\frac{br}{a^2}\right)^2}} \\
&= -\frac{aq}{b}\frac{1}{\sqrt{\left(\frac{a^2}{b}\right)^2-2r\left(\frac{a^2}{b}\right)\cos\theta+r^2}} \\
&= \frac{q'}{\rho'}
\end{aligned}
$$

球内一点(r,θ,φ)的电势为

$$
v = \frac{q}{-} + \frac{q'}{\prime}
$$

Example 22 (解球域外部的Dirichlet问题)

$$\begin{cases} u_{xx}+u_{yy}+u_{zz}=0, & x^2+y^2+z^2>a^2 \\ u|_{r=a}=\cos^2\theta, & 0\leqslant\theta\leqslant\pi, 0\leqslant\varphi\leqslant 2\pi \\ u|_{r\to\infty}\text{有界} \end{cases}$$

解：边界条件与φ无关，考虑球的对称性，在球坐标系求解，设$u=u(r,\theta)$.
在球坐标系下，Laplace方程可转化为

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)=0$$

可设方程的解为

$$u(r,\theta)=\sum_{n=0}^{+\infty}(A_nr^n+B_nr^{-n-1})P_n(\cos\theta)$$

由于$r \to \infty$时，$u(r,\theta)$有界，所以$A_n = 0, n \geqslant 1$.

由$u(a,\theta) = (A_0 + B_0 a^{-1}) + \sum_{n=1}^{+\infty} B_n a^{-n-1} P_n(\cos\theta) = \cos^2\theta$

$$A_0 + B_0 a^{-1} = \frac{1}{2}\int_0^{\pi} \cos^2\theta P_0(\cos\theta)\sin\theta \mathrm{d}\theta = \frac{1}{3}$$

$$B_n a^{-n-1} = \frac{2n+1}{2}\int_0^{\pi} \cos^2\theta P_n(\cos\theta)\sin\theta \mathrm{d}\theta$$

$$= \frac{2n+1}{2}\int_{-1}^{1} x^2 P_n(x)\mathrm{d}x = \begin{cases} 0, & n \neq 2 \\ \frac{2}{3}, & n = 2 \end{cases}$$

所求解为

$$u(r,\theta) = A_0(1 - \frac{a}{r}) + \frac{a}{3}r^{-1} + \frac{2a^3}{3}r^{-3}P_2(\cos\theta)$$

$$= A_0(1 - \frac{a}{r}) + \frac{a}{3r} + \frac{a^3}{3r^3}(3\cos^2\theta - 1)$$

Example 23

均匀半球的球面保持恒温u_0，底面温度为0,内部无热源，求半球内部温度的稳态分布.

解：取球心为原点，底面为xOy面，建立坐标系.

则温度满足定解问题 $\begin{cases} \Delta u = 0, & 0 < r < a, 0 < \theta < \dfrac{\pi}{2} \\ u\big|_{\theta=\frac{\pi}{2}} = 0, \ u\big|_{r=a} = u_0 \end{cases}$

1° 温度与ϕ无关，设$u = R(r)\Theta(\theta)$,分离变量可得

$$(I) \qquad r^2 R'' + 2rR' - \lambda R = 0$$

$$(II) \qquad \frac{1}{\sin\theta}(\sin\theta\Theta')' + \lambda\Theta = 0$$

2° 解固有值问题 $\begin{cases} \dfrac{1}{\sin\theta}(\sin\theta\Theta')' + \lambda\Theta = 0 \\ \Theta(0)\text{有界}, \Theta(\dfrac{\pi}{2}) = 0 \end{cases}$

$$\lambda_n = (2n{+}1)(2n{+}2), \qquad \Theta_n = P_{2n+1}(\cos\theta),\ n = 0, 1, 2, \cdots$$

3° 求R_n: $r^2R_n'' + 2rR_n' - (2n+1)(2n+2)R_n = 0$

$$R_n = A_n r^{2n+1},\ n = 0, 1, 2, \cdots$$

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n r^{2n+1} P_{2n+1}(\cos\theta)$$

4°定系数：

$$N_n^2=\int_0^{\frac{\pi}{2}}P_{2n+1}^2(\cos\theta)\sin\theta\mathrm{d}\theta$$
$$=\int_0^1 P_{2n+1}^2(x)\mathrm{d}x=\frac{1}{2}\int_{-1}^1 P_{2n+1}^2(x)\mathrm{d}x$$
$$=\frac{1}{2}\cdot\frac{2}{2(2n+1)+1}=\frac{1}{4n+3}$$

$$u(a,\theta)=\sum_{n=0}^{\infty}A_n a^{2n+1}P_{2n+1}(\cos\theta)$$

$$A_n=\frac{4n+3}{a^{2n+1}}\int_0^1 u_0P_{2n+1}(x)\mathrm{d}x=\frac{4n+3}{a^{2n+1}}\frac{(-1)^n(2n+2)!u_0}{2^{2n+2}[(n+1)!]^2(2n+1)}$$