5.8

1. $|\theta,\varphi\rangle = \begin{bmatrix} e^{-i\frac{\varphi}{2}}\cos\frac{\theta}{2} \\ e^{i\frac{\varphi}{2}}\sin\frac{\theta}{2} \end{bmatrix}$

$$|\theta,\varphi+2\pi\rangle = \begin{bmatrix} e^{-i(\frac{\varphi}{2}+\pi)}\cos\frac{\theta}{2} \\ e^{i(\frac{\varphi}{2}+\pi)}\cos\frac{\theta}{2} \end{bmatrix}$$

$$= \begin{bmatrix} -e^{-i\frac{\pi}{2}}\cos\frac{\theta}{2} \\ -e^{i\frac{\pi}{2}}\cos\frac{\theta}{2} \end{bmatrix} = -|\theta,\varphi\rangle$$

$|\theta,\varphi+2\pi\rangle$ 与 $|\theta,\varphi\rangle$ 只差一个常系数 -1，

代表同一个量子状态

2. $|\pi+\theta,\varphi\rangle = \begin{bmatrix} e^{-i\frac{\varphi}{2}}\cos(\frac{\pi}{2}+\frac{\theta}{2}) \\ e^{i\frac{\varphi}{2}}\sin(\frac{\pi}{2}+\frac{\theta}{2}) \end{bmatrix}$

$$= \begin{bmatrix} -e^{-i\frac{\varphi}{2}} \sin\frac{\theta}{2} \\ e^{i\frac{\varphi}{2}} \cos\frac{\theta}{2} \end{bmatrix}$$

$$|\pi-\theta, \pi+\varphi\rangle = \begin{bmatrix} e^{-i(\frac{\varphi}{2}+\frac{\pi}{2})} \cos(\frac{\pi}{2}-\frac{\theta}{2}) \\ e^{i(\frac{\varphi}{2}+\frac{\pi}{2})} \sin(\frac{\pi}{2}-\frac{\theta}{2}) \end{bmatrix}$$

$$= \begin{bmatrix} -ie^{-i\frac{\varphi}{2}} \sin\frac{\theta}{2} \\ ie^{i\frac{\varphi}{2}} \cos\frac{\theta}{2} \end{bmatrix} = i|\pi+\theta, \varphi\rangle$$

只差一个常子数 i，故代表同个量子状态

3. $$\sigma_n = \sin\theta\cos\varphi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sin\theta\sin\varphi \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \cos\theta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta \end{bmatrix}$$

与本征值 $+1$ 对应的态矢假设为 $\begin{pmatrix} a \\ b \end{pmatrix}$，

有：

$$\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

有 $\cos\theta \cdot a + e^{-i\varphi} \sin\theta \cdot b = a$

$$\Rightarrow \frac{a}{b} = e^{-i\varphi} \frac{\sin\theta}{1-\cos\theta} = e^{-i\varphi} \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$$

$$= \frac{e^{-i\frac{\varphi}{2}} \cos\frac{\theta}{2}}{e^{i\frac{\varphi}{2}} \sin\frac{\theta}{2}}$$

所以可以取 $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\varphi}{2}} \cos\frac{\theta}{2} \\ e^{i\frac{\varphi}{2}} \sin\frac{\theta}{2} \end{pmatrix} = |\theta,\varphi\rangle$

与本征值 -1 对应的本征态满足方程

$$\cos\theta\, a' + e^{-i\varphi} \sin\theta\, b' = -a'$$

$$\Rightarrow \frac{a'}{b'} = -e^{-i\varphi} \frac{\sin\theta}{1+\cos\theta} = -e^{-i\varphi} \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$$

$$= -\frac{e^{-i\frac{\varphi}{2}} \sin\frac{\theta}{2}}{e^{i\frac{\varphi}{2}} \cos\frac{\theta}{2}}$$

所以可以取 $\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} -e^{-i\frac{\varphi}{2}} \sin\frac{\theta}{2} \\ e^{i\frac{\varphi}{2}} \cos\frac{\theta}{2} \end{pmatrix}$

而 $|\pi+\theta,\varphi\rangle = \begin{pmatrix} e^{-i\frac{\varphi}{2}} \cos(\frac{\pi}{2}+\frac{\theta}{2}) \\ e^{i\frac{\varphi}{2}} \sin(\frac{\pi}{2}+\frac{\theta}{2}) \end{pmatrix}$

$$= \begin{pmatrix} -e^{-i\frac{\varphi}{2}} \sin\frac{\theta}{2} \\ e^{i\frac{\varphi}{2}} \sin\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} a' \\ b' \end{pmatrix}$$

5.10.

1. 定态薛定谔方程为:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) + V_0\psi(x) = E\psi(x).$$

$$\Rightarrow -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) = (E-V_0)\psi(x)$$

与自由粒子的定态方程没有本质区别

⇒ 本征波函数为:

$$\psi_k(x) = e^{ikx} \quad (k\text{取实数})$$

对应的本征能量为:

$$E_k = \frac{\hbar^2 k^2}{2m} + V_0$$

2. 对 $\psi(x)$ 作平面波展开（fourier 变换）

$$\psi(x) = \int c(k) e^{ikx} dk$$

则：$$\psi(x,t) = \int c(k) e^{ikx} e^{-\frac{i}{\hbar}E_k t} dk$$

$$= \int c(k) e^{ikx} e^{-i\frac{\hbar k^2}{2m}t} e^{-\frac{i}{\hbar}V_0 t} dk$$