# 第五章基本解和解的积分表达式

中国科学技术大学 数学科学学院

April 18, 2022

# 问题引入

## 已学过的解方程方法:

行波法——求解无界波动初值问题.
分离变量法——各种有界问题，解为无穷级数表示，不易研究.
积分变换法——各种无界问题，解为无穷积分.

引入新的**Green 函数法**

## 基本思想

叠加原理——将连续元分成点源的叠加
将每个点源产生的影响求出来，整个问题的解是点源影响的叠加。

## 优点

1.对于线性问题，只要求出点源的解，就可以算出任意源的解;
2.解的形式用积分表示，便于理论研究。

# §5 基本解和解的积分表达式

一、 $\delta$函数

二、 场势方程的边值问题

三、 $u_t = Lu$型方程Cauchy问题的基本解

四、 $u_{tt} = Lu$型方程Cauchy问题的基本解

# 一、$\delta$函数引入

**例1.点电荷的线密度函数**

总电量为1的电荷均匀分布在$[-\varepsilon, +\varepsilon]$,则电荷密度函数

$$\rho_\varepsilon(x) = \begin{cases} \dfrac{1}{2\varepsilon}, & |x| < \varepsilon \\ 0, & |x| \geqslant \varepsilon \end{cases}$$

抽象为点电荷，令$\varepsilon \to 0$, 则有

$$\delta(x) = \lim_{\varepsilon \to 0} \rho_\varepsilon(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

且满足

$$\int_{-\infty}^{+\infty} \delta(x)\mathrm{d}x = \lim_{\varepsilon \to 0} \int_{-\infty}^{+\infty} \rho_\varepsilon(x)\mathrm{d}x = 1.$$

称$\delta(x)$为**Dirac 函数.**

**例2**

一根线密度为$\rho$的导热杆，初始温度为0, 比热为$C$. 用火焰在$x=0$点烧一下，传给杆的热量为$Q$, 立刻移开热源，考虑开始一瞬间杆身温度分布情况$T(x)$.

$$\lim_{\Delta x \to 0} \rho \Delta x C T(x) = Q \Rightarrow T(x) = \lim_{\Delta x \to 0} \frac{Q}{C\rho\Delta x}$$

$$T(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases}, \qquad \int_{-\infty}^{+\infty} T(x)\mathrm{d}x = \frac{Q}{C\rho}.$$

$$T(x) = \frac{Q}{C\rho}\delta(x)$$

**例3:**

一个小球放在光滑平面上，$t=0$时，小球受到瞬时外力作用，得到冲量，转化为初速度$V$，描述小球所受外力情况$F(t)$.

$$F(0)\Delta t = mV \Longrightarrow F(0) = \lim_{\Delta t \to 0} \frac{mV}{\Delta t} = \infty.$$

$$F(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

$$\int_{-\infty}^{+\infty} F(t)\mathrm{d}t = mV.$$

$$F(t) = mV\delta(x).$$

# $\delta$函数的解释

1. 由物理学家Dirac首先引进，在近代物理学中有广泛的应用.
2. 用于描述物理学中的点量，如点质量，点电量，脉冲等.
3. 在数学上$\delta$函数是一种广义函数。

**例：广义函数**

设$f(x)$在任意有界区间可积，
$\mathbb{K} = \{\varphi(x) | \varphi(x) \in C^{\infty}(\mathbf{R}), \exists M > 0, s.t. \varphi(x) = 0, |x| > M\}$
具有有界支集的任意阶可导函数全体。

定义：$F : \mathbb{K} \to \mathbf{R}$,即对任意$\varphi(x) \in \mathbb{K}$

$$F[\varphi(x)] = \int_{-\infty}^{+\infty} f(x)\varphi(x)\mathrm{d}x.$$

$F[\varphi(x)]$是一个广义函数.

## $\delta$函数的筛选性

对任意连续函数$f(x)$

$$\int_{-\infty}^{+\infty} f(x)\delta(x)\mathrm{d}x = f(0), \qquad \int_{-\infty}^{+\infty} f(x)\delta(x-\xi)\mathrm{d}x = f(\xi)$$

**证明:**由于$x \neq 0$时，$\delta(x) = 0$，所以

$$\int_{-\infty}^{+\infty} f(x)\delta(x)\mathrm{d}x = \int_{-\infty}^{+\infty} f(0)\delta(x)\mathrm{d}x = f(0)\int_{-\infty}^{+\infty} \delta(x)\mathrm{d}x = f(0)$$

根据$\delta$函数的定义

$$\delta(x-\xi) = \begin{cases} +\infty, & x = \xi \\ 0, & x \neq \xi \end{cases}$$

可得第二式.

## 说明

- $\delta(x)$并没有给出函数值与自变量之间的对应关系，不是通常意义下的函数.
- $\delta(x)=\begin{cases}\infty, & x=0\\ 0, & x\neq 0\end{cases}$只在积分运算中才有意义
- 积分限不一定是$\infty$, 只要$0\in[a,b]$,则

$$\int_a^b f(x)\delta(x)\mathrm{d}x=\int_a^b f(0)\delta(x)\mathrm{d}x=f(0).$$

若$\xi\in[a,b]$

$$\int_a^b f(x)\delta(x-\xi)\mathrm{d}x=\int_a^b f(\xi)\delta(x-\xi)\mathrm{d}x=f(\xi).$$

# $\delta$函数的性质与运算

**1.$c\delta(x), c \in \mathbf{R}$**

$$\int_{-\infty}^{+\infty} f(x)c\delta(x)\mathrm{d}x = \int_{-\infty}^{+\infty} (cf(x))\delta(x)\mathrm{d}x = cf(0)$$

**2.对称性$\delta(x-\xi) = \delta(\xi - x)$**

$$\int_{-\infty}^{+\infty} \varphi(x)\delta(x-\xi)\mathrm{d}x = \int_{-\infty}^{+\infty} \varphi(\xi+t)\delta(t)\mathrm{d}t = \varphi(\xi)$$

$$\int_{-\infty}^{+\infty} \varphi(x)\delta(\xi-x)\mathrm{d}x = \int_{-\infty}^{+\infty} \varphi(\xi-s)\delta(s)\mathrm{d}s = \varphi(\xi)$$

## 3.放缩$\delta(cx), c \neq 0$

$$\int_{-\infty}^{+\infty} f(x)\delta(cx)\mathrm{d}x = \int_{-\infty}^{+\infty} \frac{1}{|c|} f(\frac{t}{c})\delta(t)\mathrm{d}t = \frac{1}{|c|} f(0)$$

$$\delta(cx) = \frac{1}{|c|}\delta(x), \qquad \delta(-x) = \delta(x).$$

## 4.导数$\delta'(x)$

$$\int_{-\infty}^{+\infty} f(x)\delta'(x)\mathrm{d}x = f(x)\delta(x)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} f'(x)\delta(x)\mathrm{d}x = -f'(0)$$

更一般地

$$\int_{-\infty}^{+\infty} f(x)\delta^{(n)}(x)\mathrm{d}x = (-1)^n f^{(n)}(0).$$

$\delta^{(n)}(x)$也是广义函数，定义了函数集合$\mathbb{K}$到**R** 的运算.

## 5. $\delta(x)$的原函数

$$\int_{-\infty}^{x} \delta(t)\mathrm{d}t = \begin{cases} 0, & x < 0 \\ \frac{1}{2}, & x = 0 \\ 1, & x > 0 \end{cases} \widehat{=} h(x)$$

$$\int_{-\infty}^{+\infty} h'(x)\varphi(x)\mathrm{d}x = h(x)\varphi(x)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \varphi'(x)h(x)\mathrm{d}x = \varphi(0)$$

## 6. $\delta(x)$与一般函数的卷积

$$f(x) * \delta(x) = \int_{-\infty}^{+\infty} f(\xi)\delta(x-\xi)\mathrm{d}\xi = f(x)$$

## 7. $\delta(x)$的Fourier级数展开

$x, \xi \in (-l, l)$时

$$\delta(x-\xi) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

其中

$$a_n = \int_{-l}^{l} \delta(x-\xi)\cos\frac{n\pi x}{l}\mathrm{d}x, \quad n = 0, 1, 2, \cdots$$

$$b_n = \int_{-l}^{l} \delta(x-\xi)\sin\frac{n\pi x}{l}\mathrm{d}x, \quad n = 1, 2, \cdots$$

## 8. $\delta(x)$的Fourier变换

$$\mathcal{F}[\delta(x)] = \int_{-\infty}^{+\infty} \delta(x) e^{i\lambda x} \mathrm{d}x = e^0 = 1$$

Fourier逆变换$\mathcal{F}^{-1}[1] = \delta(x)$,即$\frac{1}{2\pi}\int_{-\infty}^{+\infty} 1 \cdot e^{-i\lambda x} \mathrm{d}\lambda = \delta(x)$.

Fourier变换的解释：在广义函数意义下

对任意$f(x) \in \mathbb{K}$,$F(\lambda) = \int_{-\infty}^{+\infty} f(\xi) e^{i\lambda\xi} \mathrm{d}\xi$

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\lambda) e^{-i\lambda x} \mathrm{d}\lambda \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} f(\xi) e^{i\lambda(\xi - x)} \mathrm{d}\xi\right] \mathrm{d}\lambda \end{aligned}$$

$$
\begin{aligned}
f(0) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} f(\xi)e^{i\lambda\xi}\mathrm{d}\xi\right]\mathrm{d}\lambda \\
&= \int_{-\infty}^{+\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\lambda\xi}\mathrm{d}\lambda\right] f(\xi)\mathrm{d}\xi
\end{aligned}
$$

所以 $$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 1\cdot e^{i\lambda x}\mathrm{d}\lambda = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\cos\lambda x\mathrm{d}\lambda.$$

## Example 1 (利用$\delta$函数的性质计算)

$$(1) \int_{-2l}^{2l} \delta(x-l)\cos x \mathrm{d}x \quad (2) \int_{-\infty}^{+\infty} \delta(x-\frac{\pi}{2})\sin x \mathrm{d}x$$

$$(3) \delta(2x-1) * x^2 \quad (4) \int_{-2}^{2} \delta'(x+1)e^{-2x}\mathrm{d}x$$

$$(1) \int_{-2l}^{2l} \delta(x-l)\cos x \mathrm{d}x = \int_{-3l}^{l} \delta(t)\cos(l+t)\mathrm{d}t = \cos(l+t)\Big|_{t=0} = \cos l$$

$$(2) \int_{-\infty}^{+\infty} \delta(x)\sin x \mathrm{d}x = \sin x\Big|_{x=\frac{\pi}{2}} = \sin\frac{\pi}{2} = 1$$

$$(3) \delta(2x-1) * x^2 = \int_{-\infty}^{+\infty} \delta(2\xi-1)(x-\xi)^2\mathrm{d}\xi$$

$$= \int_{-\infty}^{+\infty} \delta(t)(x-\frac{t+1}{2})^2\mathrm{d}\frac{t+1}{2} = \frac{1}{2}(x-\frac{1}{2})^2.$$

$$(4)\ \int_{-2}^{2}\delta'(x+1)e^{-2x}\mathrm{d}x = \int_{-2}^{2}e^{-2x}\mathrm{d}\delta(x+1)$$
$$= -(-2)\int_{-2}^{2}\delta(x+1)e^{-2x}\mathrm{d}x = 2e^{2}$$

**Example 2**

证明: $x\delta'(x) = -\delta(x)$.

**证明:**对任意函数$f(x) \in \mathbb{K}$

$$\int_{-\infty}^{+\infty} x\delta'(x)f(x)\mathrm{d}x = xf(x)\delta(x)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty}\delta(x)(xf(x))'\mathrm{d}x$$
$$= -\left.(xf(x))'\right|_{x=0} = -f(0) - 0f'(0) = -f(0)$$
$$\int_{-\infty}^{+\infty} -\delta(x)f(x)\mathrm{d}x = -f(0)$$

所以$x\delta'(x) = -\delta(x)$.

## Example 3 (例题：$\delta$函数的应用)

1.计算$\mathcal{F}[\cos x]$.

2.计算$\displaystyle\int_{-\infty}^{+\infty}\frac{\sin x}{x}\mathrm{d}x$.

**解**:1. $\cos x = \frac{e^{ix}+e^{-ix}}{2}$

$$\mathcal{F}[\cos x] = \int_{-\infty}^{+\infty}\frac{e^{ix}+e^{-ix}}{2}e^{i\lambda x}\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{+\infty} e^{ix(\lambda+1)} + e^{ix(\lambda-1)}\mathrm{d}x$$
$$= \pi(\delta(\lambda+1)+\delta(\lambda-1))$$

2. 令$F(\lambda) = \int_{-\infty}^{+\infty}\dfrac{\sin\lambda x}{x}\mathrm{d}x$, 则$F'(\lambda) = \int_{-\infty}^{+\infty}\cos\lambda x\mathrm{d}x = 2\pi\delta(\lambda)$

积分可得$F(\lambda) = 2\pi h(\lambda) + C$,由$F(0) = 0$，取$C = -\pi$.

$$\int_{-\infty}^{+\infty}\frac{\sin x}{x}\mathrm{d}x = F(1) = \pi.$$

## Example 4

用Fourier变换法求解初值问题

$$\begin{cases} u_t = u_{xx}, & (-\infty < x < +\infty, t > 0) \\ u\big|_{t=0} = \cos x \end{cases}$$

**解**:记$\bar{u} = \mathcal{F}[u(x,t)] = \int_{-\infty}^{+\infty} u(x,t)e^{i\lambda x}\mathrm{d}x$

$$\mathcal{F}[\cos x] = \pi[\delta(\lambda+1) + \delta(\lambda-1)]$$

对定解问题作Fourier变换

$$\begin{cases} \bar{u}_t = (-i\lambda)^2\bar{u} \\ \bar{u}\big|_{t=0} = \pi[\delta(\lambda+1) + \delta(\lambda-1)] \end{cases}$$

解得

$$\bar{u} = \pi[\delta(\lambda+1) + \delta(\lambda-1)]e^{-\lambda^2 t}$$

作逆变换得

$$
\begin{aligned}
u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \pi[\delta(\lambda+1)+\delta(\lambda-1)]e^{-\lambda^2 t}e^{-i\lambda x}\mathrm{d}\lambda \\
&= \frac{1}{2}\left[e^{-\lambda^2 t - i\lambda x}\Big|_{\lambda=-1} + e^{-\lambda^2 t - i\lambda x}\Big|_{\lambda=1}\right] \\
&= \frac{1}{2}e^{-t}[e^{ix}+e^{-ix}] = e^{-t}\cos x.
\end{aligned}
$$

## Example 5

求解定解问题
$$\begin{cases} u_t = a^2 u_{xx} & t > 0, 0 < x < 2l \\ u_x\big|_{x=0} = u_x\big|_{x=2l} = 0 \\ u\big|_{t=0} = \delta(x-l) \end{cases}$$

**解:**(1) 设方程的解$u(x,t) = X(x)T(t)$, 代入方程和边界条件得

固有值问题
$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = X'(2l) = 0 \end{cases}$$

常微分方程$T' + a^2\lambda T = 0.$

(2)求解固有值问题得$\lambda_n = \left(\frac{n\pi}{2l}\right)^2, \quad n = 0, 1, 2, \cdots$

$$X_0 = 1, \qquad X_n = \cos\frac{n\pi}{2l}x$$

(3) $T$满足的常微分方程解为$T_n = C_n e^{-\left(\frac{n\pi a}{2l}\right)^2 t}$,

方程的一般解是$u(x,t) = C_0 + \sum\limits_{n=1}^{\infty} C_n e^{-\left(\frac{n\pi a}{2l}\right)^2 t} \cos\frac{n\pi}{2l}x$

(4)定系数$t=0$时，$u(x,0) = C_0 + \sum\limits_{n=1}^{\infty} C_n \cos\frac{n\pi}{2l}x = \delta(x-l)$

$$C_0 = \frac{1}{2l}\int_0^{2l} \delta(x-l)\mathrm{d}x = \frac{1}{2l}.$$

$$C_n = \frac{2}{2l}\int_0^{2l} \delta(x-l)\cos\frac{n\pi}{2l}x\mathrm{d}x = \frac{1}{l}\cos\frac{n\pi}{2}$$

$$= \begin{cases} 0, & n = 2k-1 \\ (-1)^k \frac{1}{l}, & n = 2k \end{cases}, (k = 1, 2, \cdots)$$

所以$u(x,t) = \frac{1}{2l} + \sum\limits_{k=1}^{\infty} \frac{(-1)^k}{l} e^{-\left(\frac{k\pi a}{l}\right)^2 t} \cos\frac{k\pi}{l}x$

# 多元$\delta$函数

**定义：**

1. $\delta(x,y,z) = \delta(x)\delta(y)\delta(z)$
2. $\iiint\limits_{\mathbf{R}^3} \delta(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = 1$

**性质：**

1. $\iiint\limits_{\mathbf{R}^3} \delta(x,y,z)f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = f(0,0,0).$
2. $\iiint\limits_{\mathbf{R}^3} \delta(x-x_0,y-y_0,z-z_0)f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = f(x_0,y_0,z_0)$
3. $\delta(x,y,z) * f(x,y,z) = f(x,y,z).$
4. $\mathcal{F}[\delta(x,y,z)] = 1.$

# §5 基本解和解的积分表达式

一、 $\delta$函数

二、 场势方程的边值问题

三、 $u_t = Lu$型方程Cauchy问题的基本解

四、 $u_{tt} = Lu$型方程Cauchy问题的基本解

## 数学工具I

Green第一公式:
$\Omega$是由封闭光滑曲面$S$围成的空间立体区域。

$$\iint\limits_S u\nabla v\cdot\vec{n}\mathrm{d}S=\iiint\limits_\Omega u\Delta v\mathrm{d}V+\iiint\limits_\Omega \nabla u\cdot\nabla v\mathrm{d}V$$

其中$\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$, $\vec{n}\mathrm{d}S = (\mathrm{d}y\mathrm{d}z, \mathrm{d}z\mathrm{d}x, \mathrm{d}x\mathrm{d}y)$, $\vec{n}$ 是封闭曲面$\partial\Omega$的单位外法向量.

**证明:** 由Gauss公式

$$\iint\limits_S P\mathrm{d}y\mathrm{d}z+Q\mathrm{d}z\mathrm{d}x+R\mathrm{d}x\mathrm{d}y=\iiint\limits_\Omega \frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

设$\vec{F}=(P,Q,R),\nabla=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)$,Gauss 公式可改写为

$$\iint\limits_S \vec{F}\cdot\vec{n}\mathrm{d}S=\iiint\limits_\Omega \nabla\cdot\vec{F}\mathrm{d}V$$

设$u(x,y,z), v(x,y,z) \in C^2(\Omega)$, 上式中取$\vec{F} = u\nabla v = u\left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z}\right)$

则有$\iint\limits_S (u\nabla v)\cdot \vec{n}\mathrm{d}S = \iiint\limits_\Omega \nabla\cdot(u\nabla v)\mathrm{d}V$

由于$\nabla\cdot(u\nabla v) = \nabla u\cdot\nabla v + u\Delta v$

$$\iint\limits_S (u\nabla v)\cdot \vec{n}\mathrm{d}S = \iiint\limits_\Omega \nabla u\cdot\nabla v + u\Delta v\mathrm{d}V \tag{1}$$

同理可证

$$\iint\limits_S (v\nabla u)\cdot \vec{n}\mathrm{d}S = \iiint\limits_\Omega \nabla v\cdot\nabla u + v\Delta u\mathrm{d}V \tag{2}$$

(1),(2)两式相减可得Green第二公式.

## 数学工具II

Green第二公式：

$$\iint\limits_S \left( u\frac{\partial v}{\partial \vec{n}} - v\frac{\partial u}{\partial \vec{n}} \right) \mathrm{d}S = \iiint\limits_\Omega u\Delta v - v\Delta u \mathrm{d}V$$

$\vec{n}$是曲面$S$的单位外法向量.

## 二维情形Green公式

$D$是平面逐段光滑封闭曲线$L$围成的有界区域，$u(x,y), v(x,y)$有二阶连续偏导数，$\vec{n}$ 是边界曲线的单位外法向量，

$$\oint_L u\frac{\partial v}{\partial \vec{n}}\mathrm{d}l = \iint\limits_D u\Delta_2 v + \nabla u\cdot\nabla v \mathrm{d}x\mathrm{d}y$$

$$\oint_L u\frac{\partial v}{\partial \vec{n}} - v\frac{\partial u}{\partial \vec{n}}\mathrm{d}l = \iint\limits_D u\Delta_2 v - v\Delta_2 u \mathrm{d}x\mathrm{d}y$$

(1) 问题引入

(2) 无边界场势方程的基本解

(3) 有边界场势问题的基本解

(4) 镜像法求基本解

(5) 分离变量法求基本解

# §5.2场势方程的边值问题

## 问题引入

$$\Delta_3 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f(x, y, z), \qquad (x, y, z) \in \mathbb{R}^3$$

记$(x, y, z) = M$,

$$f(M) = \delta(M) * f(M) = \iiint\limits_{\mathbb{R}^3} \delta(M - M_0) f(M_0) \mathrm{d}M_0.$$

连续源可以看做强度为$f(M_0)$的点源的叠加

只要求出一个点电荷产生的的场，再将所有点电荷产生的场叠加起来就得到方程的解.

求解无界势场方程 $\Delta_3 u(M) = f(M)$

先解出 $\Delta_3 U(M, M_0) = \delta(M - M_0)$,

令$u(M) = \iiint\limits_{\mathbb{R}^3} U(M, M_0) f(M_0) \mathrm{d}M_0$, 则$u(M)$是所求方程的解.

**验证:**

$$\Delta_3 u = \iiint\limits_{\mathbb{R}^3} \Delta_3 U(M, M_0) f(M_0) \mathrm{d}M_0$$
$$= \iiint\limits_{\mathbb{R}^3} \delta(M - M_0) f(M_0) \mathrm{d}M_0 = f(M)$$

说明：此问题是无界问题，不受边界影响，位于原点$(0,0,0)$的点源在点$M_0$处产生的势函数与位于点$M_0$的点源在原点产生的势函数相等。所以问题问题转化为

$\Delta_3 U(M) = \delta(M)$——基本解 $\quad u(M) = \iiint\limits_{\mathbb{R}^3} U(M - M_0) f(M_0) \mathrm{d}M_0$

## Theorem 6

$f(M)$是连续函数，$U(M)$满足方程$LU = \delta(M)$,则

$$u = U * f = \iiint\limits_{\mathbb{R}^3} U(M - M_0) f(M_0) \mathrm{d}M_0$$

满足非齐次方程 $Lu = f(M)$. 其中$L$是线性偏微分算符。
$U$称为方程$Lu = f(M)$的基本解

**证明:**$u = U * f = \iiint\limits_{\mathbb{R}^3} U(M - M_0) f(M_0) \mathrm{d}M_0$代入方程:

$$Lu = L \iiint\limits_{\mathbb{R}^3} U(M - M_0) f(M_0) \mathrm{d}M_0 = \iiint\limits_{\mathbb{R}^3} LU(M - M_0) f(M_0) \mathrm{d}M_0$$

$$= \iiint\limits_{\mathbb{R}^3} \delta(M - M_0) f(M_0) \mathrm{d}M_0 = f(M)$$

这样就证明了$u = U * f$满足方程$Lu = f(M)$.

**Example 7**

求方程$y' + ay = f(x)$的基本解.

**解:** 基本解满足方程 $\boxed{U' + aU = \delta(x)}$

此一阶线性微分方程的解是

$$U = e^{-\int_0^x a\mathrm{d}\tau}\left(\int_{-\infty}^{x} e^{\int_0^\tau a\mathrm{d}s}\delta(\tau)\mathrm{d}\tau + C\right)$$

取 $U = e^{-ax}\int_{-\infty}^{x} e^{a\tau}\delta(\tau)\mathrm{d}\tau = e^{-ax}h(x)$

注: 如不特意申明，求基本解只需求出方程的一个特解即可，一般选取简单的形式.

## Example 8

求势场方程$\Delta_3 u = f(M)$的基本解.

**解:** 求解$\Delta_3 u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = \delta(x, y, z)$

对方程作三维Fourier变换，记$\widehat{u} = \mathcal{F}[u]$

$$(-i\lambda)^2 \widehat{u} + (-i\mu)^2 \widehat{u} + (-i\nu)^2 \widehat{u} = 1$$

所以$\widehat{u} = -\dfrac{1}{\lambda^2 + \mu^2 + \nu^2} \widehat{=} -\dfrac{1}{\rho^2}$.

$$u = \mathcal{F}^{-1}[\widehat{u}] = -\frac{1}{(2\pi)^3} \iiint\limits_{\mathbb{R}^3} \frac{1}{\lambda^2 + \mu^2 + \nu^2} e^{-i(\lambda x + \mu y + \nu z)} \mathrm{d}\lambda \mathrm{d}\mu \mathrm{d}\nu$$

记$\vec{\rho} = (\lambda, \mu, \nu), \vec{r} = (x, y, z)$, $\rho^2 = \lambda^2 + \mu^2 + \nu^2$, $r^2 = x^2 + y^2 + z^2$, $\theta$为$\vec{\rho}, \vec{r}$的夹角.

由对称性，不妨把$\nu$轴取为向径$\vec{r}(x,y,z)$的方向.做球坐标代换$\lambda=\rho\sin\theta\cos\varphi, \mu=\rho\sin\theta\sin\varphi, \nu=\rho\cos\theta$

$$\begin{aligned}
u &= -\frac{1}{(2\pi)^3}\iiint\limits_{\mathbb{R}^3}\frac{1}{\lambda^2+\mu^2+\nu^2}e^{-i(\lambda x+\mu y+\nu z)}\mathrm{d}\lambda\mathrm{d}\mu\mathrm{d}\nu \\
&= -\frac{1}{(2\pi)^3}\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\mathrm{d}\vartheta\int_0^{+\infty}e^{-i\rho r\cos\vartheta}\sin\vartheta\mathrm{d}\rho \\
&= -\frac{1}{(2\pi)^2}\int_0^{+\infty}\frac{1}{i\rho r}e^{-i\rho r\cos\vartheta}\Big|_0^{\pi}\mathrm{d}\rho \\
&= -\frac{1}{2\pi^2 r}\int_0^{+\infty}\frac{\sin\rho r}{\rho}\mathrm{d}\rho \\
&= -\frac{1}{4\pi r}
\end{aligned}$$

场势方程的基本解是$\boxed{U(x,y,z) = -\frac{1}{4\pi r} = -\frac{1}{4\pi\sqrt{x^2+y^2+z^2}}}$

## Example 9

求二维Poisson方程无界问题的基本解.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

解：基本解满足方程$\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \delta(x, y)$.

作极坐标变换得：

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = \delta(r)$$

由于点源位于原点，问题的解与$\theta$无关，当$r > 0$时，$u$满足

$$r^2 u_{rr} + r u_r = 0$$

此方程为Euler方程，作变量代换$r = e^t$, 方程化为$u_{tt} = 0$,

所以$u = A + Bt = A + B\ln r$. 又因$u(0) = \infty$，所以取$A = 0$.

下面确定$B$的值：取圆心在原点，半径为$\varepsilon$的圆盘$D_\varepsilon$.

$$\iint\limits_{D_\varepsilon} \Delta_2 u \mathrm{d}x\mathrm{d}y = \iint\limits_{D_\varepsilon} \delta(x,y)\mathrm{d}x\mathrm{d}y = 1$$

根据第二型曲线积分Green公式

$$\iint\limits_{D_\varepsilon} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\mathrm{d}x\mathrm{d}y = \int\limits_{\partial D_\varepsilon} \frac{\partial u}{\partial \vec{n}}\mathrm{d}l$$

$$= \int\limits_{\partial D_\varepsilon} \frac{\partial u}{\partial r}\mathrm{d}l = \int\limits_{\partial D_\varepsilon} \frac{B}{r}\mathrm{d}l = 2\pi B = 1$$

所以$B = \dfrac{1}{2\pi}$, 二维无界Poisson方程的基本解是

$$\boxed{u = \frac{1}{2\pi}\ln r = \frac{1}{2\pi}\ln(\sqrt{x^2+y^2})}$$

## Example 10

求下列基本解方程.

$(1)a^2u_{xx} + b^2u_{yy} = \delta(x,y), (a,b>0)$　　$(2)\Delta_3(\Delta_3 u) = \delta(x,y,z)$

**解:** (1)作变量代换$s = \frac{x}{a}$, $t = \frac{y}{b}$,

则$u_{xx} = u_{ss}\frac{1}{a^2}$, $u_{yy} = u_{tt}\frac{1}{b^2}$

则方程化为

$$u_{ss} + u_{tt} = \delta(as, bt) = \frac{1}{ab}\delta(s,t)$$

由二维Laplace方程基本解的结论

$$u = \frac{1}{ab}\frac{1}{2\pi}\ln\sqrt{s^2+t^2} = \frac{1}{4\pi ab}\ln\left((\frac{x}{a})^2 + (\frac{y}{b})^2\right)$$

**解**:(2) 记$\Delta_3 u = w$, 则方程化为$\Delta_3 w = \delta(x,y,z)$

由已知结论$w = -\frac{1}{4\pi r}$, $\quad (r = \sqrt{x^2+y^2+z^2})$.

所以 $\quad \Delta_3 u = -\frac{1}{4\pi r}$,

等式右边只与$r$有关，根据问题的对称性,设$u = u(r)$

$$\Delta_3 u = \frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}u}{\mathrm{d}r} = -\frac{1}{4\pi r}$$

令$r = e^t$ 方程化为$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + \frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{e^t}{4\pi}$

可得到一个特解$u = -\frac{e^t}{8\pi} = -\frac{r}{8\pi}$

(1) 问题引入

(2) 无边界场势方程的基本解

(3) 有边界场势问题的基本解

(4) 镜像法求基本解

(5) 分离变量法求基本解

## 第一边值问题

$$\begin{cases} \Delta_3 u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = -f(M), & M(x,y,z) \in V \\ u\big|_{\partial V} = \varphi(M), & M \in \partial V = S \end{cases}$$

说明：

- 思路与无边界空间类似，将空间电荷分布看成点源的叠加.
- 由于边界条件的制约，在边界上会产生感生电荷的分布.
- 问题是如何通过点电荷电势的叠加找到满足约束条件的解。

## 第一边值问题

$$\begin{cases} \Delta_3 u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = -f(M), & M(x,y,z) \in V \\ u\big|_{\partial V} = \varphi(M), & M \in \partial V = S \end{cases}$$

根据线性方程的叠加原理,设$u = u_1 + u_2$,分别满足非齐次方程和边界条件.

$$(I)\begin{cases} \Delta u_1 = -f(M), & M(x,y,z) \in V \\ u_1\big|_{\partial V} = 0, & M \in \partial V = S \end{cases}$$

$$(II)\begin{cases} \Delta u_2 = 0, & M(x,y,z) \in V \\ u_2\big|_{\partial V} = \varphi(M), & M \in \partial V = S \end{cases}$$

$$(I)\begin{cases}\Delta u_1 = -f(M), & M(x,y,z)\in V\\ u_1\big|_{\partial V} = 0, & M\in\partial V = S\end{cases}$$

**定义:**

定解问题

$$\begin{cases}\Delta G(M,M_0) = -\delta(x-\xi,y-\eta,z-\zeta), & M(x,y,z)\in V\\ G\big|_{\partial V} = 0, & M\in\partial V = S\end{cases}$$

的解$G(M,M_0)$称为Poisson方程第一边值问题的基本解或Green函数. 其中$M(x,y,z)$称为场点，$M_0(\xi,\eta,\zeta)$称为源点，$G(M,M_0)$表示$M_0$点的点电荷在点$M$处的电势.

求出基本解$G(x,y,z;\xi,\eta,\zeta) = G(M, M_0)$,则

$$u_1 = \iiint\limits_V G(M, M_0) f(M_0) \mathrm{d}M_0$$

$$\begin{aligned}\Delta u_1 &= \iiint\limits_V \Delta G(M, M_0) f(M_0) \mathrm{d}M_0 \\ &= -\iiint\limits_V \delta(M - M_0) f(M_0) \mathrm{d}M_0 = -f(M)\end{aligned}$$

$$u_1(M)\Big|_{\partial V} = \iiint\limits_V G(M, M_0)\Big|_{\partial V} f(M_0) \mathrm{d}M_0 = \iiint\limits_V 0 \mathrm{d}V = 0.$$

显然，$u_1$满足问题$(I)$.

$$\begin{cases} \Delta u_2 = 0, & M(x,y,z) \in V \\ u_2\big|_{\partial V} = \varphi(M), & M \in \partial V = S \end{cases}$$

由Green第二公式

$$\begin{aligned} u_2(M) &= \iiint\limits_V u_2(M_0)\delta(M - M_0)\mathrm{d}M_0 \\ &= -\iiint\limits_V u_2(M_0)\Delta G(M, M_0)\mathrm{d}M_0 \\ &= \iiint\limits_V G(M, M_0)\Delta u_2(M_0) - u_2(M_0)\Delta G(M, M_0)\mathrm{d}M_0 \end{aligned}$$

$\Delta G(M; M_0)$对变量$M(x,y,z)$作偏导运算，与Gauss公式要求不一致.

是否成立$G(M, M_0) = G(M_0, M)$? 若成立，可以运用Gauss公式将内部积分转化为边界上的积分.

**Theorem 11 (对称性（倒易性）)**

设$G(M;M_0)$为场位方程第一边值问题的Green函数，则对任意$M_1, M_2 \in V$，有$G(M_1;M_2) = G(M_2;M_1)$

**证明:** 由Green函数定义，$G(M;M_1), G(M;M_2)$分别满足

$$\begin{cases} \Delta G(M;M_1) = -\delta(M-M_1) \\ G(M;M_1)\big|_{\partial V} = 0 \end{cases} \qquad \begin{cases} \Delta G(M;M_2) = -\delta(M-M_2) \\ G(M;M_2)\big|_{\partial V} = 0 \end{cases}$$

$$\begin{aligned} & G(M_2;M_1) - G(M_1;M_2) \\ = & \iiint_V G(M;M_1)\delta(M-M_2)\mathrm{d}M - \iiint_V G(M;M_2)\delta(M-M_1)\mathrm{d}M \\ = & -\iiint_V G(M;M_1)\Delta G(M;M_2) - G(M;M_2)\Delta G(M;M_1)\mathrm{d}M \\ = & -\iint_{\partial V} G(M;M_1)\frac{\partial G(M;M_2)}{\partial \vec{n}} - G(M;M_2)\frac{\partial G(M;M_1)}{\partial \vec{n}}\mathrm{d}S = 0 \end{aligned}$$

**Theorem 12 (Poisson公式)**

设$G(M, M_0)$是场位方程第一类边值问题的*Green*函数，则方程

$$\begin{cases} \Delta u = -f(M), & M(x,y,z) \in V \\ u\big|_{\partial V} = \varphi(M), & M \in \partial V = S \end{cases}$$

的解是

$$u(M) = \iiint\limits_V f(M_0) G(M, M_0) \mathrm{d}M_0 - \iint\limits_{\partial V} \varphi(M_0) \frac{\partial G}{\partial \vec{n}_0} \mathrm{d}S_0$$

**证明:** 由Green第二公式

$$\iiint\limits_V u(M) \Delta G(M, M_0) - G(M, M_0) \Delta u(M) \mathrm{d}M = \iint\limits_{\partial V} u \frac{\partial G}{\partial \vec{n}} - G \frac{\partial u}{\partial \vec{n}} \mathrm{d}S$$

其中$\vec{n}$是$\partial V$的单位外法向量.

代入方程和边界条件得

$$\iiint\limits_V -u(M)\delta(M-M_0)+G(M,M_0)f(M)\mathrm{d}M=\iint\limits_{\partial V}\varphi(M)\frac{\partial G}{\partial \vec{n}}\mathrm{d}S$$

所以

$$u(M_0)=\iiint\limits_V G(M,M_0)f(M)\mathrm{d}M-\iint\limits_{\partial V}\varphi(M)\frac{\partial G}{\partial \vec{n}}\mathrm{d}S$$

或写成

$$\begin{aligned}u(M)&=\iiint\limits_V G(M_0,M)f(M_0)\mathrm{d}M_0-\iint\limits_{\partial V}\varphi(M_0)\frac{\partial G(M_0,M)}{\partial \vec{n}_0}\mathrm{d}S_0\\&=\iiint\limits_V G(M,M_0)f(M_0)\mathrm{d}M_0-\iint\limits_{\partial V}\varphi(M_0)\frac{\partial G(M,M_0)}{\partial \vec{n}_0}\mathrm{d}S_0\end{aligned}$$

其他类型边界条件的场位方程的Green函数$G(M, M_0)$满足的定解问题是：

$$\begin{cases} \Delta G(M, M_0) = -\delta(M - M_0), & M(x, y, z), M_0(\xi, \eta, \zeta) \in V \\ \left(\alpha G + \beta \frac{\partial G}{\partial \vec{n}}\right)\Big|_{\partial V} = 0, & M \in \partial V = S \end{cases}$$

$\beta = 0$是第一型边界条件.
$\alpha = 0$是第二型边界条件,此时Green函数不存在.
$\alpha\beta \neq 0$是第三类边界条件，方程的解是

$$\begin{aligned} u(M) &= \iiint\limits_V f(M_0)G(M, M_0)\mathrm{d}M_0 - \frac{1}{\alpha}\oiint\limits_{\partial V} \varphi(M_0)\frac{\partial G}{\partial \vec{n}_0}\mathrm{d}S_0 \\ &= \iiint\limits_V f(M_0)G(M, M_0)\mathrm{d}M_0 + \frac{1}{\beta}\oiint\limits_{\partial V} \varphi(M_0)G(M, M_0)\mathrm{d}S_0 \end{aligned}$$

对于二维场位方程$\begin{cases} \Delta_2 u = -f(x,y), & (x,y) \in D \\ u\big|_{\partial D} = \varphi(x,y) \end{cases}$

基本解满足$\begin{cases} \Delta_2 G = -\delta(x-\xi, y-\eta), & (x,y),(\xi,\eta) \in D \\ G\big|_{\partial D} = 0 \end{cases}$

二维场位方程的解是

$$u(M_0) = \iint\limits_D G(M, M_0) f(M) \mathrm{d}M - \int_{\partial D} \varphi(M) \frac{\partial G}{\partial \vec{n}} \mathrm{d}l$$

或写成

$$u(M) = \iint\limits_D G(M, M_0) f(M_0) \mathrm{d}M_0 - \int_{\partial D} \varphi(M_0) \frac{\partial G}{\partial \vec{n}_0} \mathrm{d}l_0$$

(1) 问题引入

(2) 无边界场势方程的基本解

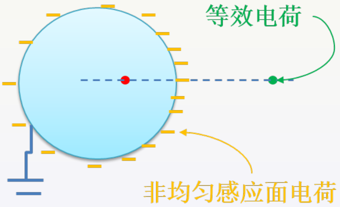
(3) 有边界场势问题的基本解

(4) 镜像法求基本解

(5) 分离变量法求基本解

$$\begin{cases} \Delta G(M, M_0) = -\delta(M - M_0) \\ G\big|_{\partial V} = 0 \end{cases} \qquad M, M_0 \in V$$

在点$M_0$放置一个点电荷时，边界上将产生感生电荷，共同作用使边界电势为0.


边界面不均匀感生电荷产生的电势不易求解，用一个等效的虚设电荷代替，使得边界电势为0, 此方法称为镜像法.

(1) 镜像法基本原理：用放置在所求场域之外的假想电荷（镜像电荷）等效的替代导体表面（或介质分界面）上的感应电荷（或极化电荷）对场分布的影响

(2) 镜像法目的：将复杂的边值问题转化为无边界的均匀介质问题.

若镜像电荷的引入满足：电势函数满足原方程与边界条件，根据边值问题解的唯一性，解是正确的。

镜像电荷的确定应遵循以下两条原则：

- 所有的镜像电荷必须位于所求的场域以外的空间中；
- 镜像电荷的个数位置及电荷量的大小由满足场域边界上的边界条件来确定。

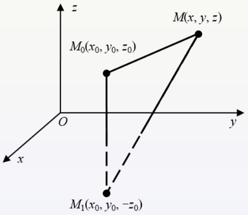
# 一、三维空间Poisson方程基本解的例子

## 半空间的Green函数

$$\begin{cases} \Delta G(M, M_0) = -\delta(M - M_0), & z > 0 \\ G = 0, & z = 0 \end{cases}$$

点$M_0$处电量为$\varepsilon$电荷产生的电势为

$$U_0 = \frac{1}{4\pi r(M, M_0)}$$
$$= \frac{1}{4\pi\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}}$$




在边界$z = 0$

$$U_0(x, y, 0) = \frac{1}{4\pi\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z_0)^2}}$$

在$M_0$关于平面$z=0$的对称点$M_1(x_0, y_0, -z_0)$处电量为$-\varepsilon$的电荷产生电场为

$$U_1 = -\frac{1}{4\pi r(M, M_1)} = -\frac{1}{4\pi\sqrt{(x-x_0)^2+(y-y_0)^2+(z+z_0)^2}}$$

$$U_1(x, y, 0) = -\frac{1}{4\pi\sqrt{(x-x_0)^2+(y-y_0)^2+(z_0)^2}}$$

所以半空间的Green函数为

$$G(M, M_0) = \frac{1}{4\pi r(M, M_0)} - \frac{1}{4\pi r(M, M_1)}$$

满足

$$\begin{cases} \Delta G(M, M_0) = -\delta(M-M_0), & z>0 \\ G=0, & z=0 \end{cases}$$

## Example 13

$$\begin{cases} \Delta_3 u = -f(x,y,z), & z > 0 \\ u\big|_{z=0} = \varphi(x,y), & z = 0 \end{cases}$$

**解:**

$$u(x,y,z) = \iiint\limits_{z_0>0} f(x_0,y_0,z_0)G(M,M_0)\mathrm{d}M_0 - \iint\limits_{\mathbb{R}^2} \varphi(x_0,y_0)\frac{\partial G(M,M_0)}{\partial \vec{n}_0}\mathrm{d}x_0\mathrm{d}y_0$$

$$
\begin{aligned}
\frac{\partial G(M,M_0)}{\partial \vec{n}_0} &= -\left.\frac{\partial G(M,M_0)}{\partial z_0}\right|_{z_0=0} \\
&= -\left[\frac{z-z_0}{4\pi r(M,M_0)^3} - \frac{z+z_0}{4\pi r(M,M_1)^3}\right]\Bigg|_{z_0=0} \\
&= -\frac{z}{2\pi\left((x-x_0)^2+(y-y_0)^2+z^2\right)^{\frac{3}{2}}}
\end{aligned}
$$

**注：** 边值问题$\begin{cases}\Delta_3 u = 0, & z>0 \\ u\big|_{z=0} = \varphi(x,y), & \end{cases}$ 的解是

$$
u(x,y,z) = \iint\limits_{\mathbb{R}^2} \frac{\varphi(\xi,\eta)z}{2\pi\left((x-\xi)^2+(y-\eta)^2+z^2\right)^{\frac{3}{2}}}\mathrm{d}\xi\mathrm{d}\eta
$$

## Example 14

求平面$x+y+z=0$上方空间的Poisson方程第一边值问题的Green函数.

**解:** 设$M(x,y,z)$, $M_0(\xi,\eta,\zeta)$, Green函数满足

$$\begin{cases} \Delta_3 G(M;M_0) = -\delta(M_0), \qquad x+y+z>0 \\ G(M;M_0)\big|_{x+y+z=0} = 0 \end{cases}$$

设$M_0$关于平面$x+y+z=0$的对称点是$M_1(a,b,c)$, 根据几何意义$M_1M_0//(1,1,1)$,且中点在平面上。

$$\begin{cases} \dfrac{\xi-a}{1} = \dfrac{\eta-b}{1} = \dfrac{\zeta-c}{1} \\ (\xi+a)+(\eta+b)+(\zeta+c)=0 \end{cases}$$

解得对称点坐标是$M_1(\dfrac{\xi-2\eta-2\zeta}{3}, \dfrac{\eta-2\xi-2\zeta}{3}, \dfrac{\zeta-2\xi-2\eta}{3})$.

由镜像法，Green函数是

$$\begin{aligned} G(M;M_0) =& \frac{1}{4\pi r(M;M_0)} - \frac{1}{4\pi r(M;M_1)} \\ =& \frac{1}{4\pi}\left(\frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}\right. \\ & \left. - \frac{1}{\sqrt{(x-\frac{\xi-2\eta-2\zeta}{3})^2+(y-\frac{\eta-2\xi-2\zeta}{3})^2+(z-\frac{\zeta-2\xi-2\eta}{3})^2}}\right) \end{aligned}$$

$$\begin{cases} \Delta_3 u(M;M_0) = -f(x,y,z), & x+y+z>0 \\ u(M;M_0)\big|_{x+y+z=0} = \varphi(x,y,z) \end{cases}$$

$$u(x,y,z) = \iiint\limits_{\xi+\eta+\zeta>0} f(M_0)G(M;M_0)\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta$$

$$- \iint\limits_{\xi+\eta+\zeta=0} \varphi(\xi,\eta,\zeta)\frac{\partial G(M;M_0)}{\partial \vec{n}_0}\mathrm{d}S$$

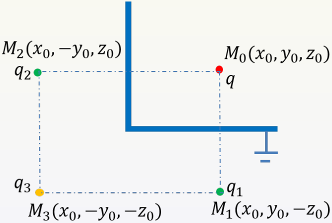
其中 $\dfrac{\partial G(M;M_0)}{\partial \vec{n}_0} = -\dfrac{1}{\sqrt{3}}\left(\dfrac{\partial G(M;M_0)}{\partial \xi} + \dfrac{\partial G(M;M_0)}{\partial \eta} + \dfrac{\partial G(M;M_0)}{\partial \zeta}\right)$

## Example 15

求$\frac{1}{4}$空间$y > 0, z > 0$的Green函数

$$\begin{cases} \Delta_3 u = -f(x,y,z), & y > 0, z > 0 \\ u\Big|_{z=0} = \varphi(x,y), & z = 0, y = 0 \end{cases}$$

要保证导体平面上电势为零，区域外需设置三个电荷




$$q = -q_1 = -q_2 = q_3$$

$$G(M,M_0) = \frac{1}{4\pi r(M,M_0)} - \frac{1}{4\pi r(M,M_1)} - \frac{1}{4\pi r(M,M_2)} + \frac{1}{4\pi r(M,M_3)}$$

## Example 16

$$\begin{cases} 9u_{xx} + 4u_{yy} + u_{zz} = 0, & x > 0 \\ u\big|_{x=0} = \varphi(y, z) \end{cases}$$

**解:** 令$a = \frac{1}{3}x, b = \frac{1}{2}y, c = z$ 则$u_{xx} = \frac{1}{9}u_{aa}, u_{yy} = \frac{1}{4}u_{bb}, u_{zz} = u_{cc}$,

方程化为
$$\begin{cases} u_{aa} + u_{bb} + u_{cc} = 0 & a > 0 \\ u\big|_{a=0} = \varphi(2b, c) \end{cases}$$

记场点$M(a,b,c)$, 源点$M_0(\xi,\eta,\zeta)$, 像点$M_1(-\xi,\eta,\zeta)$,此方程的Green函数满足

$$\begin{cases} G(M; M_0) = -\delta(M_0), & a > 0 \\ G(M; M_0)\big|_{a=0} = 0 \end{cases}$$

$$G(M;M_0) = \frac{1}{4\pi}\left(\frac{1}{\sqrt{(a-\xi)^2+(b-\eta)^2+(c-\zeta)^2}}\right.$$
$$\left.-\frac{1}{\sqrt{(a+\xi)^2+(b-\eta)^2+(c-\zeta)^2}}\right)$$

$$\left.\frac{\partial G(M;M_0)}{\partial \vec{n}}\right|_{\xi=0} = -\left.\frac{\partial G(M;M_0)}{\partial \xi}\right|_{\xi=0}$$
$$= -\frac{1}{2\pi}\frac{a}{\sqrt{a^2+(b-\eta)^2+(c-\zeta)^2}}$$

$$
\begin{aligned}
u(a,b,c) &= -\iint\limits_{\xi=0} \frac{\partial G(M;M_0)}{\partial \vec{n}} \mathrm{d}S \\
&= \frac{a}{2\pi} \iint\limits_{\mathbb{R}^2} \frac{1}{\sqrt{a^2+(b-\eta)^2+(c-\zeta)^2}} \mathrm{d}\eta \mathrm{d}\zeta
\end{aligned}
$$

原方程的解是

$$
u(x,y,z) = \frac{x}{6\pi} \iint\limits_{\mathbb{R}^2} \frac{1}{\sqrt{\frac{x^2}{9}+(\frac{y}{2}-\eta)^2+(z-\zeta)^2}} \mathrm{d}\eta \mathrm{d}\zeta
$$

## Example 17

求半径为$R$球域内部Poisson方程第一边值问题的Green函数.

**解:**球域内Green函数满足

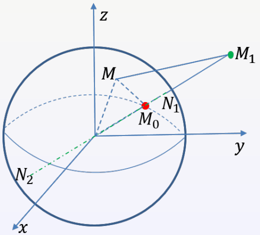
$$\begin{cases} \Delta_3 G(M, M_0) = -\delta(M - M_0), & 0 \le r, \rho < R \\ G\big|_{r=R} = 0 \end{cases}$$

其中$r = \sqrt{x^2 + y^2 + z^2}$, $\rho = \sqrt{x_0^2 + y_0^2 + z_0^2}$

球是轴对称区域,

虚设电荷$\varepsilon q$应在$OM_0$延长线上点$M_1$.

设$|OM_1| = d$, $|OM_0| = \rho$, $d, q$待定.



$OM_0$延长线与球面交于$N_1$，$N_2$两点，根据虚设电荷要求

$$G(N_1) = \frac{1}{4\pi|M_0N_1|} + \frac{q}{4\pi|M_1N_1|} = \frac{1}{4\pi(R-\rho)} + \frac{q}{4\pi(d-R)} = 0$$

$$G(N_2) = \frac{1}{4\pi|M_0N_2|} + \frac{q}{4\pi|M_1N_2|} = \frac{1}{4\pi(R+\rho)} + \frac{q}{4\pi(d+R)} = 0$$

解得　$d = \frac{R^2}{\rho}, q = -\frac{R}{\rho}$

所以Green函数是$G(M,M_0) = \frac{1}{4\pi}\left(\frac{1}{r(M,M_0)} - \frac{R}{\rho r(M,M_1)}\right)$

下面计算具体表达式，用球坐标表示：$M_0(\rho,\ \theta_0,\ \varphi_0)$，

$M_1(\frac{R^2}{\rho},\ \theta_0,\ \varphi_0)$，$M(r,\ \theta,\ \varphi)$

$$
\begin{aligned}
& r(M, M_0) \\
= & \left[(r\sin\theta\cos\varphi - \rho\sin\theta_0\cos\varphi_0)^2 + (r\sin\theta\sin\varphi - \rho\sin\theta_0\sin\varphi_0)^2\right. \\
& \left. +(r\cos\theta - \rho\cos\theta_0)^2\right]^{\frac{1}{2}} \\
= & \left[r^2 + \rho^2 - 2r\rho(\sin\theta\sin\theta_0\cos(\varphi-\varphi_0) - \cos\theta\cos\theta_0)\right]^{\frac{1}{2}} \\
= & \left[r^2 + \rho^2 - 2r\rho\cos\psi\right]^{\frac{1}{2}} \quad \psi\text{为}OM_0\text{与}OM\text{ 的夹角}.
\end{aligned}
$$

$$
\begin{aligned}
r(M, M_1) &= \left[r^2 + \left(\frac{R^2}{\rho}\right)^2 - 2r\frac{R^2}{\rho}\cos\psi\right]^{\frac{1}{2}} \\
&= \frac{1}{\rho}\left[r^2\rho^2 + R^4 - 2r\rho R^2\cos\psi\right]^{\frac{1}{2}}
\end{aligned}
$$

$$
G(M, M_0) = \frac{1}{4\pi}\left[\frac{1}{\sqrt{r^2+\rho^2-2r\rho\cos\psi}} - \frac{R}{\sqrt{r^2\rho^2+R^4-2r\rho R^2\cos\psi}}\right]
$$

$$\begin{cases} \Delta U = -f(r,\theta,\varphi), & r < R \\ U\big|_{r=R} = g(\theta,\varphi) \end{cases}$$

$$U(r,\theta,\varphi) = \iiint\limits_{\rho<R} f(\rho,\theta_0,\varphi_0) G(r,\theta,\varphi;\rho,\theta_0,\varphi_0) \mathrm{d}\rho \mathrm{d}\theta \mathrm{d}\varphi$$

$$- \iint\limits_{\rho=R} g(\theta_0,\varphi_0) \frac{\partial G}{\partial \vec{n}_0} \mathrm{d}S_0$$

$$\frac{\partial G}{\partial \vec{n}_0}\bigg|_{\rho=R} = \frac{\partial G}{\partial \rho}\bigg|_{\rho=R}$$

$$= \frac{1}{4\pi} \left[ \frac{-(\rho - r\cos\psi)}{(r^2+\rho^2-2r\rho\cos\psi)^{\frac{3}{2}}} - \frac{-(\rho r^2 - rR^2\cos\psi)}{R(r^2\rho^2+R^4-2r\rho R^2\cos\psi)^{\frac{3}{2}}} \right]_{\rho=R}$$

$$= \frac{1}{4\pi R} \frac{r^2-R^2}{(r^2+R^2-2rR\cos\psi)^{\frac{3}{2}}}$$

# 二、二维Poisson方程基本解的例子

**Example 18**

上半平面的Green函数$\begin{cases} \Delta_2 G(M, M_0) = -\delta(x - x_0, y - y_0), \quad y > 0 \\ G\big|_{y=0} = 0 \end{cases}$.

**解:**$M_0$点电量为$\varepsilon$电荷产生的场为

$$U_0(M, M_0) = \frac{1}{2\pi} \ln \frac{1}{r(M, M_0)} = \frac{1}{4\pi} \ln \frac{1}{(x - x_0)^2 + (y - y_0)^2}$$

像点$M_1(x_0, -y_0)$处电量$-\varepsilon$电荷产生的场是

$$U_1(M, M_1) = -\frac{1}{2\pi} \ln \frac{1}{r(M, M_1)} = -\frac{1}{4\pi} \ln \frac{1}{(x - x_0)^2 + (y + y_0)^2}$$

$$G(M, M_0) = \frac{1}{4\pi} \ln \frac{(x - x_0)^2 + (y + y_0)^2}{(x - x_0)^2 + (y - y_0)^2}$$

$$\begin{cases} \Delta_2 u(x,y) = -f(x,y), & y > 0 \\ u\big|_{y=0} = \varphi(x) \end{cases}$$

**解:**

$$u(x,y) = \iint\limits_{y_0>0} f(x_0,y_0)G(x,y;x_0,y_0)\mathrm{d}x_0\mathrm{d}y_0 - \int_{y_0=0} \varphi(x_0,y_0)\frac{\partial G}{\partial \vec{n}_0}\mathrm{d}l$$

$$\frac{\partial G}{\partial \vec{n}_0}\bigg|_{y_0=0} = -\frac{\partial G}{\partial y_0}\bigg|_{y_0=0} = -\frac{1}{\pi}\cdot\frac{y}{(x-x_0)^2+y^2}$$

$$u(x,y) = \iint\limits_{y_0>0} f(x_0,y_0)\frac{1}{4\pi}\ln\frac{(x-x_0)^2+(y+y_0)^2}{(x-x_0)^2+(y-y_0)^2}\mathrm{d}x_0\mathrm{d}y_0 + \int_{-\infty}^{+\infty}\varphi(x_0)\frac{1}{\pi}\cdot\frac{y}{(x-x_0)^2+y^2}\mathrm{d}x_0$$

## Example 19

(1)求平面区域$D$的第一边值问题Green函数，其中$D$为$y = |x|$上方区域

(2)利用Green函数求解$\begin{cases} u_{xx} + u_{yy} = 0, & (x, y) \in D^\circ \\ u\big|_{\partial D} = \varphi(x) \end{cases}$.

**解:**(1) 设源点$M_0(\xi, \eta)$, 镜像点$M_1(\eta, \xi), M_2(-\eta, -\xi), M_3(-\xi, -\eta)$, 场点$M(x, y)$. 利用镜像法可得Green函数

$$\begin{aligned} G(M; M_0) &= \frac{1}{2\pi} \ln \frac{1}{r(M, M_0)} - \frac{1}{2\pi} \ln \frac{1}{r(M, M_1)} \\ &\quad - \frac{1}{2\pi} \ln \frac{1}{r(M, M_2)} + \frac{1}{2\pi} \ln \frac{1}{r(M, M_3)} \\ &= \frac{1}{4\pi} \ln \frac{[(x-\eta)^2 + (y-\xi)^2][(x+\eta)^2 + (y+\xi)^2]}{[(x-\xi)^2 + (y-\eta)^2][(x+\xi)^2 + (y+\eta)^2]} \end{aligned}$$

(2)

$$\frac{\partial G}{\partial \xi} = \frac{1}{4\pi}\left[\frac{-2(y-\xi)}{(x-\eta)^2+(y-\xi)^2} + \frac{2(y+\xi)}{(x+\eta)^2+(y+\xi)^2} \right.$$
$$\left. + \frac{2(x-\xi)}{(x-\xi)^2+(y-\eta)^2} + \frac{-2(x+\xi)}{(x+\xi)^2+(y+\eta)^2}\right]$$

$$\frac{\partial G}{\partial \eta} = \frac{1}{4\pi}\left[\frac{-2(x-\eta)}{(x-\eta)^2+(y-\xi)^2} + \frac{2(x+\eta)}{(x+\eta)^2+(y+\xi)^2} \right.$$
$$\left. + \frac{2(y-\eta)}{(x-\xi)^2+(y-\eta)^2} + \frac{-2(y+\eta)}{(x+\xi)^2+(y+\eta)^2}\right]$$

在边界$y = x$上，外法向量$\vec{n} = \dfrac{1}{\sqrt{2}}(1, -1)$.

$$\frac{\partial G}{\partial \vec{n}}\Big|_{\eta=\xi} = \frac{1}{\sqrt{2}}\left(\frac{\partial G}{\partial \xi} - \frac{\partial G}{\partial \eta}\right)\Big|_{\eta=\xi}$$
$$= \frac{4}{\sqrt{2}\pi}\frac{\xi(x^2-y^2)}{[(x-\xi)^2+(y-\xi)^2][(x+\xi)^2+(y+\xi)^2]}$$

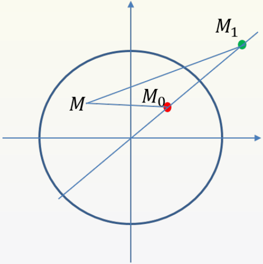
在边界$y = -x$上，外法向量$\vec{n} = \dfrac{1}{\sqrt{2}}(-1, -1)$.

$$\frac{\partial G}{\partial \vec{n}}\Big|_{\eta=-\xi} = \frac{1}{\sqrt{2}}\left(\frac{\partial G}{\partial \xi} - \frac{\partial G}{\partial \eta}\right)\Big|_{\eta=-\xi}$$
$$= \frac{-4}{\sqrt{2}\pi}\frac{\xi(x^2-y^2)}{[(x+\xi)^2+(y-\xi)^2][(x-\xi)^2+(y+\xi)^2]}$$

$$
\begin{aligned}
u(x,y) &= -\int\limits_{\eta=\xi} \frac{\partial G}{\partial \vec{n}}\varphi(\xi)\mathrm{d}s - \int\limits_{\eta=-\xi} \frac{\partial G}{\partial \vec{n}}\varphi(\xi)\mathrm{d}s \\
&= \frac{4}{\pi}\int_0^{+\infty} \frac{\xi(x^2-y^2)\varphi(\xi)}{[(x-\xi)^2+(y-\xi)^2][(x+\xi)^2+(y+\xi)^2]}\mathrm{d}\xi \\
&- \frac{4}{\pi}\int_{-\infty}^{0} \frac{\xi(x^2-y^2)\varphi(\xi)}{[(x+\xi)^2+(y-\xi)^2][(x-\xi)^2+(y+\xi)^2]}\mathrm{d}\xi
\end{aligned}
$$

## Example 20

求圆内的Green函数 $\begin{cases} \Delta_2 G = -\delta(x - x_0, y - y_0), & r < R \\ G\big|_{r=R} = 0 \end{cases}$.




圆是轴对称区域，

虚设电荷$-\varepsilon$应在$OM_0$延长线上点$M_1$.

设$|OM_1| = d$, $|OM_0| = \rho$, 与球域类似,$d = \frac{R^2}{\rho}$

两个电荷对应的势函数为

$$u_0 = \frac{1}{2\pi} \ln \frac{1}{r(M, M_0)}, \quad u_1 = -\frac{1}{2\pi} \ln \frac{1}{r(M, M_1)}$$

$$u_0 + u_1 = \frac{1}{2\pi} \ln \frac{r(M, M_1)}{r(M, M_0)} = \frac{1}{2\pi} \ln \frac{R}{\rho}$$

满足方程的两个解$u_0, u_1$加上一个常数后仍满足方程，取常数使得边界圆周上势函数为0.

$$G(M, M_0) = u_0 + u_1 - \frac{1}{2\pi} \ln \frac{R}{\rho} = \frac{1}{2\pi} \left[ \ln \frac{1}{r(M, M_0)} - \ln \frac{R}{\rho r(M, M_1)} \right]$$

取极坐标$M_0(\rho, \theta_0), M_1(\frac{R^2}{\rho}, \theta_0), M(r, \theta), \psi = \theta - \theta_0$

$$r(M, M_0) = \sqrt{r^2 + \rho^2 - 2r\rho\cos\psi}$$

$$r(M, M_1) = \sqrt{r^2 + \left(\frac{R^2}{\rho}\right)^2 - 2r\left(\frac{R^2}{\rho}\right)\cos\psi}$$

$$\begin{cases} \Delta_2 u = -f(r,\theta), & r < R \\ u\big|_{r=R} = g(\theta) \end{cases}$$

$$G(M, M_0) = \frac{1}{4\pi} \ln \frac{r^2\rho^2 + R^4 - 2r\rho R^2 \cos\psi}{R^2(r^2 + \rho^2 - 2r\rho\cos\psi)}$$

$$\left.\frac{\partial G}{\partial \vec{n}_0}\right|_{\rho=R} = \left.\frac{\partial G}{\partial \rho}\right|_{\rho=R} = \frac{1}{2\pi} \frac{r^2 - R^2}{(r^2 + R^2 - 2rR\cos\psi)}$$

$$u(r,\theta) = \iint\limits_{r<R} f(\rho,\theta_0) \frac{1}{4\pi} \ln \frac{r^2\rho^2 + R^4 - 2r\rho R^2\cos\psi}{R^2(r^2+\rho^2-2r\rho\cos\psi)} \mathrm{d}\rho \mathrm{d}\theta$$

$$- \frac{1}{2\pi} \int_0^{2\pi} \frac{(r^2 - R^2) g(\theta_0)}{r^2 + R^2 - 2rR\cos(\theta - \theta_0)} \mathrm{d}\theta_0$$

(1) 问题引入

(2) 无边界场势方程的基本解

(3) 有边界场势问题的基本解

(4) 镜像法求基本解

(5) 分离变量法求基本解

# 用分离变量法求Green函数

**Example 21**

求解矩形域Dirichlet问题的Green函数。

$$\begin{cases} \Delta_2 G(M, M_0) = -\delta(M - M_0), \quad M, M_0 \in (0, a) \times (0, b) \\ G\big|_{x=0} = G\big|_{x=a} = G\big|_{y=0} = G\big|_{y=b} = 0 \end{cases}$$

**解:** $1^\circ$ 考虑同一齐次边界条件下固有值问题

$$\begin{cases} \Delta_2 U + \lambda U = 0, \qquad 0 < x < a, \ 0 < y < b \\ U\big|_{x=0} = U\big|_{x=a} = U\big|_{y=0} = U\big|_{y=b} = 0 \end{cases}$$

令$U = X(x)Y(y)$代入方程得

$$(I) \begin{cases} X'' + \mu X = 0, \quad 0 < x < a \\ X(0) = X(a) = 0 \end{cases} \qquad (II) \begin{cases} Y'' + \nu Y = 0, \quad 0 < y < b \\ Y(0) = Y(b) = 0 \end{cases}$$

$\mu + \nu = \lambda$, 解得固有值和固有函数

$$\mu_n = \left(\frac{n\pi}{a}\right)^2, \quad X_n = \sin\frac{n\pi x}{a}, \quad n = 1, 2, \cdots$$
$$\nu_m = \left(\frac{m\pi}{b}\right)^2, \quad Y_m = \sin\frac{m\pi y}{b}, \quad m = 1, 2, \cdots$$

$$\lambda_{mn} = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2, \quad U_{mn} = \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}$$

$2^\circ$ 将$G(x, y)$按$U_{mn}$作广义Fourier展开

设$G(x, y) = \sum\limits_{n,m=1}^{\infty} C_{nm}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}$

$$\begin{aligned}\Delta_2 G = &-\sum_{n,m=1}^{\infty} C_{nm}\left(\left(\frac{n\pi}{a}\right)^2+\left(\frac{n\pi}{a}\right)^2\right)\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\\ &=-\delta(x-x_0,y-y_0)\end{aligned}$$

$$\begin{aligned}&C_{nm}\left(\left(\frac{n\pi}{a}\right)^2+\left(\frac{n\pi}{a}\right)^2\right)\\ =&\frac{1}{\|U_{nm}(x,y)\|^2}\int_0^a \mathrm{d}x\int_0^b \delta(x-x_0,y-y_0)\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\mathrm{d}y\\ =&\frac{4}{ab}\sin\frac{n\pi x_0}{a}\sin\frac{m\pi y_0}{b}\end{aligned}$$

$$G(x,y;x_0,y_0)=\frac{4ab}{\pi^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sin\frac{n\pi x_0}{a}\sin\frac{m\pi y_0}{b}}{(nb)^2+(am)^2}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}$$

# §5 基本解和解的积分表达式

一、 $\delta$函数

二、 场势方程的边值问题

三、 $u_t = Lu$型方程Cauchy问题的基本解

四、 $u_{tt} = Lu$型方程Cauchy问题的基本解

$$\begin{cases} \frac{\partial u}{\partial t} = Lu + f(t, M), \quad t > 0, M \in \mathbb{R}^n, n = 1, 2, 3 \\ u\big|_{t=0} = \varphi(M) \end{cases} \tag{3}$$

其中$L$是关于$x, y, z$的线性偏微分算符

**Definition 22**

$$\begin{cases} \frac{\partial U}{\partial t} = LU, \quad t > 0, M \in \mathbb{R}^n, n = 1, 2, 3 \\ U\big|_{t=0} = \delta(M) \end{cases} \tag{4}$$

的解称为方程（3）的基本解

## Theorem 23

设$\varphi(M)$，$f(t,M)$是连续函数，且$U(t,M)*\varphi(M)$，$U(t,M)*f(t,M)$ 存在，则方程（3）的解为

$$u(t,x,y,z) = U(t,M)*\varphi(M) + \int_0^t U(t-\tau,M)*f(\tau,M)\mathrm{d}\tau$$

$$= \iiint_{-\infty}^{+\infty} U(t,x-\xi,y-\eta,z-\zeta)\varphi(\xi,\eta,\zeta)\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta$$

$$+ \int_0^t \left[\iiint_{-\infty}^{+\infty} U(t-\tau,x-\xi,y-\eta,z-\zeta)f(\tau,\xi,\eta,\zeta)\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta\right]\mathrm{d}\tau$$

**证明:** 令$u_1 = U(t,M) * \varphi(M)$, $u_2 = \int_0^t U(t-\tau, M) * f(\tau, M)\mathrm{d}\tau$.

$$
\begin{aligned}
\frac{\partial u_1}{\partial t} = &\iiint\limits_{\mathbb{R}^3} \frac{\partial}{\partial t} U(t, M - M_0) * \varphi(M_0)\mathrm{d}M_0 \\
=&U_t(t,M) * \varphi(M) = [LU(t,M)] * \varphi(M) \\
=&L[U(t,M) * \varphi(M)] = Lu_1
\end{aligned}
$$

$$
u_1\big|_{t=0} = U(0,M) * \varphi(M) = \delta(M) * \varphi(M) = \varphi(M)
$$

所以$u_1$满足方程

$$
\begin{cases}
\frac{\partial u_1}{\partial t} = Lu_1, \quad t > 0, M \in \mathbb{R}^n, n = 1,2,3 \\
u_1\big|_{t=0} = \varphi(M)
\end{cases}
$$

$$
\begin{aligned}
\frac{\partial u_2}{\partial t} &= \frac{\partial}{\partial t}\int_0^t U(t-\tau, M) * f(\tau, M)\mathrm{d}\tau \\
&= \int_0^t \frac{\partial}{\partial t}U(t-\tau, M) * f(\tau, M)\mathrm{d}\tau + U(0, M) * f(t, M) \\
&= \int_0^t LU(t-\tau, M) * f(\tau, M)\mathrm{d}\tau + \delta(M) * f(t, M) \\
&= L\int_0^t U(t-\tau, M) * f(\tau, M)\mathrm{d}\tau + f(t, M) \\
&= Lu_2 + f(t, M)
\end{aligned}
$$

由叠加原理可知

$$
U(t, M) * \varphi(M) + \int_0^t U(t-\tau, M) * f(\tau, M)\mathrm{d}\tau
$$

是所求定解问题的解.

## Example 24

用基本解方法求解

$$\begin{cases} u_t = a^2 u_{xx} + f(x,t) & t > 0, -\infty < x < +\infty \\ u|_{t=0} = \varphi(x) \end{cases}$$

**解**:(1)基本解满足

$$\begin{cases} U_t = a^2 U_{xx} & t > 0, -\infty < x < +\infty \\ U|_{t=0} = \delta(x) \end{cases}$$

用Fourier变换求解:设$\bar{U} = \mathcal{F}[U] = \displaystyle\int_{-\infty}^{+\infty} U(x,t)e^{i\lambda x}\mathrm{d}x$.

$$\mathcal{F}[U_{xx}] = -\lambda^2\bar{U}, \mathcal{F}[U_t] = \frac{\mathrm{d}\bar{U}}{\mathrm{d}t}, \mathcal{F}[\delta(x)] = 1$$

对基本解问题作Fouier变换$\begin{cases} \dfrac{\mathrm{d}\bar{U}}{\mathrm{d}t} = -a^2\lambda^2\bar{U} \\ \bar{U}|_{t=0} = 1 \end{cases}$，解得$\bar{U} = e^{-a^2\lambda^2 t}$.

$$U(x,t) = \mathcal{F}^{-1}[e^{-a^2\lambda^2 t}] = \frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2 t}}$$

(2)根据用基本解方法求解$u_t = Lu$型方程的基本公式可得

$$\begin{aligned} u(x,t) &= U(x,t) * \varphi(x) + \int_0^t U(x,t-\tau) * f(x,\tau)\mathrm{d}\tau \\ &= \int_{-\infty}^{+\infty} \frac{1}{2a\sqrt{\pi t}} e^{-\frac{(x-\xi)^2}{4a^2 t}} \varphi(\xi)\mathrm{d}\xi \\ &+ \int_0^t \mathrm{d}\tau \int_{-\infty}^{+\infty} \frac{1}{2a\sqrt{\pi(t-\tau)}} e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}} f(\xi,\tau)\mathrm{d}\xi \end{aligned}$$

## Example 25

求三维热传导方程Cauchy问题的基本解.

$$\begin{cases} u_t = a^2\Delta_3 u, \quad t > 0, -\infty < x, y, z < +\infty \\ u\big|_{t=0} = \delta(x, y, z) \end{cases}$$

**解**:记

$$\widehat{u}(t, \lambda, \mu, \nu) = \mathcal{F}[u(t, x, y, z)] = \iiint\limits_{\mathbb{R}^3} u(t, x, y, z) e^{i(\lambda x + \mu y + \nu z)} \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

对定解问题作Fourier变换得,

$$\begin{cases} \dfrac{\mathrm{d}\widehat{u}}{\mathrm{d}t} = -a^2(\lambda^2 + \mu^2 + \nu^2)\widehat{u}(t, \lambda, \mu, \nu) \\ \widehat{u}\big|_{t=0} = 1 \end{cases}$$

所以 $\widehat{u}(t, \lambda, \mu, \nu) = e^{-a^2\rho^2 t}$, 其中$\rho^2 = \lambda^2 + \mu^2 + \nu^2$.

作Fourier逆变换

$$u(t,x,y,z) = \frac{1}{(2\pi)^3} \iiint\limits_{\mathbb{R}^3} e^{-a^2\rho^2 t} e^{-i(\lambda x+\mu y+\nu z)} \mathrm{d}\lambda \mathrm{d}\mu \mathrm{d}\nu$$

$$= \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} e^{-a^2\lambda^2 t - i\lambda x} \mathrm{d}\lambda \int_{-\infty}^{+\infty} e^{-a^2\mu^2 t - i\mu y} \mathrm{d}\mu$$

$$\int_{-\infty}^{+\infty} e^{-a^2\nu^2 t - i\nu z} \mathrm{d}\nu$$

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-a^2\lambda^2 t + i\lambda x} \mathrm{d}\lambda = \frac{1}{2a\sqrt{\pi t}} exp(-\frac{x^2}{4a^2 t})$$

$$u(t,x,y,z) = \frac{1}{(2a\sqrt{\pi t})^3} exp\left(-\frac{x^2+y^2+z^2}{4a^2 t}\right)$$

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta_3 u + f(t, x, y, z), \quad t > 0, M \in \mathbb{R}^n, n = 1, 2, 3 \\ u\Big|_{t=0} = \varphi(x, y, z) \end{cases}$$

的解为

$$u(t, x, y, z) = \varphi * U + \int_0^t f(\tau, M) * U(t-\tau, M)\mathrm{d}\tau$$

$$= \frac{1}{(2a\sqrt{\pi t})^3} \iiint\limits_{\mathbb{R}^3} \varphi(\xi, \eta, \zeta)\exp\left(-\frac{r_1^2}{4a^2 t}\right) \mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta$$

$$+ \int_0^t \frac{1}{(2a\sqrt{\pi(t-\tau)})^3} \left[\iiint\limits_{\mathbb{R}^3} f(\tau, \xi, \eta, \zeta)\exp\left(-\frac{r_1^2}{4a^2(t-\tau)}\right) \mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta\right] \mathrm{d}\tau$$

其中 $r_1^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2.$

# §5 基本解和解的积分表达式

一、 $\delta$函数

二、 场势方程的边值问题

三、 $u_t = Lu$型方程Cauchy问题的基本解

四、 $u_{tt} = Lu$型方程**Cauchy**问题的基本解

$$(\mathbf{I})\begin{cases}\dfrac{\partial^2 u}{\partial t^2} = Lu + f(t,M), \quad (M\in\mathbb{R}^3)\\ u\big|_{t=0} = \varphi(M), u_t\big|_{t=0} = \psi(M)\end{cases}$$

$L$是关于$x,y,z$的常系数线性偏微分算符，$M=(x,y,z)$是空间中点的坐标

## Definition 26

$\begin{cases}\dfrac{\partial^2 U}{\partial t^2} = LU\\ U\big|_{t=0} = 0, U_t\big|_{t=0} = \delta(M)\end{cases}$ 的解称为问题$(\mathbf{I})$ 的基本解

要求解问题$(\mathbf{I})$，根据叠加原理，要分成三个分成分别求解.

$$(\mathbf{I_1}) \quad \begin{cases} \dfrac{\partial^2 u_1}{\partial t^2} = Lu_1 \\ u_1\Big|_{t=0} = 0, \quad \dfrac{\partial u_1}{\partial t}\Big|_{t=0} = \psi(M) \end{cases}$$

$$(\mathbf{I_2}) \quad \begin{cases} \dfrac{\partial^2 u_2}{\partial t^2} = Lu_2 \\ u_2\Big|_{t=0} = \varphi(M), \quad \dfrac{\partial u_2}{\partial t}\Big|_{t=0} = 0 \end{cases}$$

$$(\mathbf{I_3}) \quad \begin{cases} \dfrac{\partial^2 u_3}{\partial t^2} = Lu_3 + f(t, M) \\ u_3\Big|_{t=0} = 0, \quad \dfrac{\partial u_3}{\partial t}\Big|_{t=0} = 0 \end{cases}$$

$u_1 + u_2 + u_3$是方程$(\mathbf{I})$的解.

设$U(t,M)$是基本解，$M_0=(\xi,\eta,\zeta)$，$M=(x,y,z)$，则有

$$u_1 = U(t,M) * \psi(M) = \iiint\limits_{\mathbb{R}^3} U(t, M - M_0)\psi(M_0)\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta$$

$$\begin{aligned}
\frac{\partial^2 u_1}{\partial t^2} &= \iiint\limits_{\mathbb{R}^3} \frac{\partial^2 U(t, M - M_0)}{\partial t^2}\psi(M_0)\mathrm{d}M_0 \\
&= \iiint\limits_{\mathbb{R}^3} LU(t, M - M_0)\psi(M_0)\mathrm{d}M_0 \\
&= L\iiint\limits_{\mathbb{R}^3} U(t, M - M_0)\psi(M_0)\mathrm{d}M_0 \\
&= LU(t,M) * \psi(M) = Lu_1(t,M)
\end{aligned}$$

$$u_1(0,M) = \iiint\limits_{\mathbb{R}^3} U(0, M - M_0)\psi(M_0)\mathrm{d}M_0 = 0$$

$$\left.\frac{\partial u_1}{\partial t}\right| \quad = \left.\frac{\partial U}{\partial t}\right| \quad * \psi(M) = \delta(M) * \psi(M) = \psi(M)$$

设$U(t,M)$是基本解，则$u_2 = \frac{\partial}{\partial t}\left(U(t,M) * \varphi(M)\right)$

$$\begin{aligned}\frac{\partial^2 u_2}{\partial t^2} &= \frac{\partial^3}{\partial t^3}\left[U(t,M) * \varphi(M)\right] = \frac{\partial}{\partial t}\left[\frac{\partial^2 U(t,M)}{\partial t^2} * \varphi(M)\right] \\ &= \frac{\partial}{\partial t}\left[LU(t,M) * \varphi(M)\right] = L\left[\frac{\partial U(t,M)}{\partial t} * \varphi(M)\right] = Lu_2(t,M)\end{aligned}$$

$$u_2(0,M) = \left.\frac{\partial U}{\partial t}\right|_{t=0} * \varphi(M) = \delta(M) * \varphi(M) = \varphi(M)$$

$$\left.\frac{\partial u_2}{\partial t}\right|_{t=0} = \left.\frac{\partial^2 U}{\partial t^2}\right|_{t=0} * \varphi(M) = LU\big|_{t=0} * \varphi(M) = 0$$

设$U(t, M)$是基本解，则$u_3 = \int_0^t U(t-\tau, M) * f(\tau, M)\mathrm{d}\tau$

设$V$满足

$$\begin{cases} \frac{\partial^2 V}{\partial t^2} = LV, t > \tau \\ V\big|_{t=\tau} = 0, \quad \frac{\partial V}{\partial t}\big|_{t=\tau} = f(\tau, M) \end{cases}$$

此方程的解是 $V(t, M; \tau) = U(t-\tau, M) * f(\tau, M)$.

由齐次化原理可知: $u_3 = \int_0^t V(t, M; \tau)\mathrm{d}\tau$

**Theorem 27**

方程($\mathbf{I}$)的解是

$$u = U(t, M) * \psi(M) + \frac{\partial}{\partial t}(U(t, M) * \varphi(M)) + \int_0^t U(t-\tau, M) * f(\tau, M)\mathrm{d}\tau$$

## Example 28

求三维波动方程的基本解

$$\begin{cases} \dfrac{\partial^2 U}{\partial t^2} = \Delta U \quad (t > 0, (x, y, z) \in \mathbb{R}^3 \\ U(0, x, y, z) = 0 \\ U_t(0, x, y, z) = \delta(x, y, z) \end{cases}$$

**解:** 作Fourier变换,记

$$\overline{U}(t, \lambda, \mu, \nu) = \iiint\limits_{\mathbb{R}^3} U(t, x, y, z) \exp(i(\lambda x + \mu y + \nu z)) \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

可得 $$\begin{cases} \dfrac{\mathrm{d}^2\overline{U}}{\mathrm{d}t^2} = -a^2\rho^2\overline{U}, \quad (\rho^2 = \lambda^2 + \mu^2 + \nu^2) \\ \overline{U}(0, \lambda, \mu, \nu) = 0, \quad \overline{U}_t(0, \lambda, \mu, \nu) = 1 \end{cases}.$$

此方程的解是$\overline{U} = \dfrac{\sin a\rho t}{a\rho}$

用逆变换求基本解。

$$U(t,x,y,z) = \frac{1}{(2\pi)^3}\iiint\limits_{\mathbb{R}^3}\frac{\sin a\rho t}{a\rho}\exp(-i(\lambda x+\mu y+\nu z))\mathrm{d}\lambda\mathrm{d}\mu\mathrm{d}\nu$$

作变量代换$\lambda = \rho\sin\theta\cos\varphi,\ \mu = \rho\sin\theta\sin\varphi,\ \nu = \rho\cos\theta$

$$\lambda x+\mu y+\nu z = \vec{\rho}\cdot\vec{r} = \rho r\cos\theta$$

$$\begin{aligned}
&U(t,x,y,z)\\
=&\frac{1}{(2\pi)^3}\int_0^{+\infty}\frac{\sin a\rho t}{a\rho}\rho^2\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\exp(-i\rho r\cos\theta)\sin\theta\mathrm{d}\theta\\
=&\frac{1}{2\pi^2 ar}\int_0^{+\infty}\sin a\rho t\sin\rho r\mathrm{d}\rho\\
=&\frac{1}{8\pi^2 ar}\int_{-\infty}^{+\infty}[\cos\rho(r-at)-\cos\rho(r+at)]\mathrm{d}\rho\\
=&\frac{1}{4\pi ar}[\delta(r-at)-\delta(r+at)] = \frac{1}{4\pi ar}\delta(r-at)
\end{aligned}$$

## Example 29

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2\Delta_3 u + f(t,x,y,z), \quad (x,y,z)\in\mathbb{R}^3, t>0 \\ u(0,x,y,z) = \varphi(x,y,z),\ u_t(0,x,y.z) = \psi(x,y,z) \end{cases}$$

**解:** 记$M = (x,y,z)$,

$$u(t,M) = U(t,M)*\psi(M) + \frac{\partial}{\partial t}[U(t,M)*\varphi(M)] + \int_0^t U(t-\tau,M)*f(\tau,M)\mathrm{d}\tau$$

$$U(t,M)*\psi(M) = \frac{1}{4\pi a}\iiint\limits_{\mathbb{R}^3}\frac{\delta(r-at)}{r}\psi(\xi,\eta,\zeta)\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta$$

其中，$r = \sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}$, 作变量代换

$$\xi = x + r\sin\alpha\cos\beta, \eta = y + r\sin\alpha\sin\beta, \zeta = z + r\cos\alpha$$

$$
\begin{aligned}
U(t,M)*\psi(M) &= \frac{1}{4\pi a}\int_0^{+\infty}\frac{\delta(r-at)}{r}\mathrm{d}r\iint\limits_{S_r}\psi(\xi,\eta,\zeta)\mathrm{d}S \\
&= \frac{t}{4\pi a^2t^2}\iint\limits_{S_{at}}\psi(\xi,\eta,\zeta)\mathrm{d}S \overset{\triangle}{=} tM_{at}(\psi)
\end{aligned}
$$

$$
\frac{\partial}{\partial t}[U(t,M)*\varphi(M)] = \frac{\partial}{\partial t}[tM_{at}(\varphi)]
$$

$$
\int_0^t U(t-\tau,M)*f(\tau,M)\mathrm{d}\tau = \int_0^t (t-\tau)M_{a(t-\tau)}(f(\tau,M))\mathrm{d}\tau
$$

$$
= \int_0^t\left[\frac{1}{4\pi a^2(t-\tau)}\iint\limits_{S_{a(t-\tau)}} f(\tau,\xi,\eta,\zeta)\mathrm{d}S\right]\mathrm{d}\tau
$$

$$
= \frac{1}{4\pi a^2}\int_0^t\left[\iint\limits_{S_r} f(t-\frac{r}{a},\xi,\eta,\zeta)\mathrm{d}S\right]\mathrm{d}r
$$

# 二维波动方程解法-降维法

**求解二维波动方程的Cauchy问题**

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \Delta_2 u, \quad (t > 0, -\infty < x, y < +\infty) \\ u(0, x, y) = \varphi(x, y), \ u_t(0, x, y) = \psi(x, y) \end{cases}$$

此问题可以看作三维波动方程

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \Delta_3 u, \quad (t > 0, -\infty < x, y, z < +\infty) \\ u(0, x, y) = \varphi(x, y, 0), \ u_t(0, x, y) = \psi(x, y, 0) \end{cases}$$

即自变量限制在$z = 0$ 平面的特殊情形.

方程的解是

$$u(t, x, y) = \frac{1}{4\pi a^2 t} \iint\limits_{S_{at}} \psi(\xi, \eta) \mathrm{d}S + \frac{\partial}{\partial t} \left[ \frac{1}{4\pi a^2 t} \iint\limits_{S_{at}} \varphi(\xi, \eta) \mathrm{d}S \right]$$

$S_{at}$是以$(x,y,0)$为中心，半径为$at$的球面. 球面方程

$$\zeta = \pm\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}$$

面积微元$\mathrm{d}S = \sqrt{1+\zeta_\xi^2+\zeta_\eta^2}\mathrm{d}\xi\mathrm{d}\eta = \frac{at\mathrm{d}\xi\mathrm{d}\eta}{\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}}$

$$u(t,x,y) = \frac{1}{2\pi a}\iint\limits_{D_{at}} \frac{\psi(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta}{\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}} + \frac{1}{2\pi a}\frac{\partial}{\partial t}\left[\iint\limits_{D_{at}} \frac{\varphi(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta}{\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}}\right]$$

$D_{at}:\ (\xi-x)^2+(y-\eta)^2 < a^2t^2$

二维波动方程的基本解：

$$\begin{cases} \frac{\partial^2 U}{\partial t^2} = a^2 \Delta_2 U, \quad (t > 0, -\infty < x, y < +\infty) \\ U(0, x, y) = 0, \ U_t(0, x, y) = \delta(\xi, \eta) \end{cases}$$

$$U(t, x, y) = \frac{1}{2\pi a} \iint\limits_{D_{at}} \frac{\delta(\xi, \eta) \mathrm{d}\xi \mathrm{d}\eta}{\sqrt{(at)^2 - (\xi - x)^2 - (\eta - y)^2}}$$

$$= \begin{cases} \frac{1}{2\pi a} \frac{1}{\sqrt{(at)^2 - x^2 - y^2}}, & (x^2 + y^2 \leq a^2 t^2) \\ 0 & (x^2 + y^2 > a^2 t^2) \end{cases}$$

用降维法解$\begin{cases} u_{tt} = a^2 u_{xx}, \quad t > 0, -\infty < x < +\infty \\ u\big|_{t=0} = \varphi(x),\ u_t\big|_{t=0} = \psi(x) \end{cases}$

在二维波动方程的解中限定$y = 0$

$$\begin{aligned} u(t,x) =& \frac{1}{2\pi a} \iint\limits_{D_{at}} \frac{\psi(\xi)\mathrm{d}\xi\mathrm{d}\eta}{\sqrt{(at)^2 - (\xi - x)^2 - \eta^2}} \\ &+ \frac{1}{2\pi a}\frac{\partial}{\partial t}\left[\iint\limits_{D_{at}} \frac{\varphi(\xi)\mathrm{d}\xi\mathrm{d}\eta}{\sqrt{(at)^2 - (\xi - x)^2 - \eta^2}}\right] \\ =& \frac{1}{2\pi a}\int_{x-at}^{x+at} \mathrm{d}\xi \int_{-\sqrt{a^2t^2-(\xi-x)^2}}^{\sqrt{a^2t^2-(\xi-x)^2}} \frac{\psi(\xi)\mathrm{d}\eta}{\sqrt{(at)^2 - (\xi - x)^2 - \eta^2}} \\ &+ \frac{1}{2\pi a}\frac{\partial}{\partial t}\left[\int_{x-at}^{x+at} \mathrm{d}\xi \int_{-\sqrt{a^2t^2-(\xi-x)^2}}^{\sqrt{a^2t^2-(\xi-x)^2}} \frac{\varphi(\xi)\mathrm{d}\eta}{\sqrt{(at)^2 - (\xi - x)^2 - \eta^2}}\right] \end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\mathrm{d}\xi+\frac{1}{2a}\frac{\partial}{\partial t}\left[\int_{x-at}^{x+at}\varphi(\xi)\mathrm{d}\xi\right]\\
=&\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\mathrm{d}\xi+\frac{1}{2}\left(\varphi(x+at)+\varphi(x-at)\right)
\end{aligned}
$$