第4章积分变换方法

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问题引入

- 行波法用来求解**无界波动**问题
- 分离变量法主要用来求解一些**有界**问题
- 其他类型的**无界**问题用什么办法求解？

学习目的： 用积分变换法求解各种无界问题

主要内容：

- Fourier变换定义、性质；
- 用Fourier变换法求解偏微分方程的定解问题
- Laplace变换定义、性质；
- 用Laplace变换法求解偏微分方程的定解问题

§4.1用Fourier变换解题

一、Fourier变换

设函数$f(x)$在整个数轴上绝对可积，在任何有界闭区间上逐段光滑，则对任意实数x，函数$f(x)$所对应的$Fourier$积分必收敛于它在该点左右极限的平均值，即

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}f(t)e^{i\lambda t}\mathrm{d}t\right]e^{-i\lambda x}\mathrm{d}\lambda=\frac{f(x+0)+f(x-0)}{2},$$

$-\infty<x<+\infty$

定义

设函数$f(x)$在整个数轴上绝对可积，在任何有界闭区间上逐段光滑，

(1) $$F(\lambda)=\int_{-\infty}^{+\infty}f(x)e^{i\lambda x}\mathrm{d}x$$

称函数$F(\lambda)$ 为函数$f(x)$的*Fourier*变换 或 像函数，
记为 $F=\mathcal{F}[f]$;

(2) $$f(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}F(\lambda)e^{-i\lambda x}\mathrm{d}\lambda$$

称为函数$F(\lambda)$的*Fourier*逆变换 或 本函数，
记为 $f=\mathcal{F}^{-1}[F]$.

Fourier变换的性质

1°线性关系

若f,g 存在Fourier 变换，则对任意常数α,β，函数$\alpha f(x)+\beta g(x)$也存在Fourier 变换，且

$$\mathcal{F}[\alpha f+\beta g]=\alpha\mathcal{F}[f]+\beta\mathcal{F}[g].$$

2°频移特性和时移特性

设$\mathcal{F}[f(x)]=F(\lambda)$.

$$\mathcal{F}[f(x)e^{i\lambda_0 x}]=F(\lambda+\lambda_0).$$

$$\mathcal{F}^{-1}[F(\lambda)e^{-ix_0\lambda}]=f(x+x_0).$$

证明:

$$\mathcal{F}[f(x)e^{i\lambda_0 x}]=\int_{-\infty}^{+\infty}f(x)e^{i\lambda_0 x}e^{i\lambda x}\mathrm{d}x=\int_{-\infty}^{+\infty}f(x)e^{i(\lambda+\lambda_0)x}\mathrm{d}x$$

3° 本函数微分法

设$\mathcal{F}[f(x)] = F(\lambda)$, 若当$|x| \to +\infty$ 时, $f(x)$ 趋于零, 且$f'(x)$的Fourier 变换存在, 则$\mathcal{F}[f'(x)] = -i\lambda F(\lambda)$.

一般地, 若当$|x| \to +\infty$ 时, $f(x)$ 及前$k-1$ 阶导函数都趋于零, 并且$f^{(k)}(x)$ 的Fourier 变换存在, 则$\mathcal{F}[f^{(k)}(x)] = (-i\lambda)^k F(\lambda)$.

证明: 分部积分可以得到

$$
\begin{aligned}
\mathcal{F}[f'(x)] &= \int_{-\infty}^{+\infty} f'(x)e^{i\lambda x}\mathrm{d}x \\
&= f(x)e^{i\lambda x}\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} f(x)(i\lambda)e^{i\lambda x}\mathrm{d}x \\
&= -i\lambda \int_{-\infty}^{+\infty} f(x)e^{i\lambda x}\mathrm{d}x = -i\lambda F(\lambda).
\end{aligned}
$$

4°像函数微分法

若函数$f(x)$ 和$xf(x)$ 的Fourier 变换都存在，$\mathcal{F}[f(x)] = F(\lambda)$，则$f(x)$ 的Fourier 变换是可微的，且

$$F'(\lambda) = \mathcal{F}[ixf(x)].$$

证明： 因为

$$F(\lambda) = \int_{-\infty}^{+\infty} f(x)e^{i\lambda x}\mathrm{d}x,$$

利用求导与积分的交换性质，得

$$\begin{aligned} F'(\lambda) &= \int_{-\infty}^{+\infty} (ix)f(x)e^{i\lambda x}\mathrm{d}x \\ &= \mathcal{F}[ixf(x)]. \end{aligned}$$

5° 积分的Fourier变换

$\int_{x_0}^{x} f(t)\mathrm{d}t, f(x)$在$(-\infty,+\infty)$绝对可积，Fourier变换存在，则

$$\mathcal{F}\left[\int_{x_0}^{x} f(t)\mathrm{d}t\right] = -\frac{1}{i\lambda}\mathcal{F}[f(x)].$$

证明:设$g(x) = \displaystyle\int_{x_0}^{x} f(t)\mathrm{d}t,$

$$\text{则}\mathcal{F}[f(x)] = \mathcal{F}[g'(x)] = -i\lambda\mathcal{F}[g(x)].$$

$$\mathcal{F}\left[\int_{x_0}^{x} f(t)\mathrm{d}t\right] = -\frac{1}{i\lambda}\mathcal{F}[f(x)].$$

卷积 设函数$f(x)$ 和$g(x)$ 都在$(-\infty,+\infty)$ 上可积且平方可积. 称含参变量积分

$$f*g := \int_{-\infty}^{+\infty} f(x-t)g(t)\,dt$$

为f 与g 的卷积. 易知, 卷积有如下性质:

(1) 设函数$f(x)$ 和$g(x)$ 都在$(-\infty,+\infty)$ 上可积且平方可积, 则$f*g(x)$ 在$(-\infty,+\infty)$ 上绝对可积.

(2) 卷积满足通常乘积的三个性质:

$$f*g = g*f \qquad \text{(交换律)}$$
$$(f*g)*h = f*(g*h) \qquad \text{(结合律)}$$
$$(f+g)*h = f*h+g*h \qquad \text{(分配律)}$$

注, 在证明结合率时要交换两个无穷积分号的顺序.

6°卷积的Fourier 变换

设函数$f(x)$ 与$g(x)$ 在区间$(-\infty,+\infty)$ 上可积且平方可积, 则有

$$\mathcal{F}[f*g]=\mathcal{F}[f]\cdot\mathcal{F}[g].$$

证明: $\mathcal{F}[f*g]=\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}f(x-t)g(t)dt\right]e^{i\lambda x}dx$, 注意到$f(x)$ 与$g(x)$ 的绝对可积性, 可知积分次序是可交换的, 于是得到

$$\mathcal{F}[f*g]=\int_{-\infty}^{+\infty}g(t)\left[\int_{-\infty}^{+\infty}f(x-t)e^{i\lambda x}dx\right]dt.$$

作变量代换$x=t+\xi$, 就有

$$\begin{aligned}\mathcal{F}[f*g]&=\int_{-\infty}^{+\infty}g(t)\left[\int_{-\infty}^{+\infty}f(\xi)e^{i\lambda(t+\xi)}d\xi\right]dt\\&=\int_{-\infty}^{+\infty}g(t)e^{i\lambda t}dt\cdot\int_{-\infty}^{+\infty}f(\xi)e^{i\lambda\xi}d\xi=\mathcal{F}[f]\cdot\mathcal{F}[g].\end{aligned}$$

Example 1

求$e^{-ax^2}, xe^{-ax^2}(a>0)$的Fourier变换.

解: 设$f(x)=e^{-ax^2}, f'(x)=-2axf(x)$, 记$\mathcal{F}[f(x)]=F(\lambda)$ 由Fourier变换的性质, 得

$$-i\lambda F(\lambda)=\mathcal{F}[f'(x)]=\mathcal{F}[-2axf(x)]=2ai\mathcal{F}[ixf(x)]=2aiF'(\lambda),$$

$$F'(\lambda)=-\frac{\lambda}{2a}F(\lambda).$$

解此微分方程, 得 $F(\lambda)=Ce^{-\frac{\lambda^2}{4a}}$.

$$C=F(0)=\int_{-\infty}^{+\infty}e^{-a\xi^2}\,d\xi=\sqrt{\frac{\pi}{a}}.$$

$$\mathcal{F}[e^{-ax^2}]=\sqrt{\frac{\pi}{a}}e^{-\frac{\lambda^2}{4a}},\qquad \mathcal{F}[xe^{-ax^2}]=\frac{i\lambda}{2a}\sqrt{\frac{\pi}{a}}e^{-\frac{\lambda^2}{4a}}$$

Example 2

已知$\mathcal{F}[\varphi(x)] = G(\lambda)$, 求$\mathcal{F}^{-1}[G(\lambda)\cos a\lambda t], \mathcal{F}^{-1}[\dfrac{G(\lambda)}{a\lambda}\sin a\lambda t]$.

解:$\cos a\lambda t = \frac{1}{2}(e^{ia\lambda t} + e^{-ia\lambda t}), \qquad \sin a\lambda t = \frac{1}{2i}(e^{ia\lambda t} - e^{-ia\lambda t})$

$$\begin{aligned}\mathcal{F}^{-1}[G(\lambda)\cos a\lambda t] =& \frac{1}{2}\mathcal{F}^{-1}[G(\lambda)e^{ia\lambda t} + G(\lambda)e^{-ia\lambda t}]\\ =& \frac{1}{2}\left(\varphi(x-at) + \varphi(x+at)\right)\end{aligned}$$

由积分的Fourier变换$\mathcal{F}[\int_{x_0}^{x}\varphi(\xi)\mathrm{d}\xi] = -\dfrac{G(\lambda)}{i\lambda}$

$$\begin{aligned}\mathcal{F}^{-1}[\frac{G(\lambda)}{a\lambda}\sin a\lambda t] &= \mathcal{F}^{-1}\left[\frac{G(\lambda)}{2ia\lambda}(e^{ia\lambda t} - e^{-ia\lambda t})\right]\\ &= \frac{1}{2a}\left(-\int_{x_0}^{x-at}\varphi(\xi)\mathrm{d}\xi + \int_{x_0}^{x+at}\varphi(\xi)\mathrm{d}\xi\right)\\ &= \frac{1}{2a}\int_{x-at}^{x+at}\varphi(\xi)\mathrm{d}\xi\end{aligned}$$

三维Fourier变换

$$F(\lambda,\mu,\nu)=\iiint\limits_{\mathbb{R}^3} f(x,y,z)exp\left(i(\lambda x+\mu y+\nu z)\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

称为函数$f(x,y,z)$的Fourier变换，记为$\mathcal{F}[f]$.

$$f(x,y,z)=\frac{1}{(2\pi)^3}\iiint\limits_{\mathbb{R}^3} F(\lambda,\mu,\nu)exp\left(-i(\lambda x+\mu y+\nu z)\right)\mathrm{d}\lambda\mathrm{d}\mu\mathrm{d}\nu$$

称为$F(\lambda,\mu,\nu)$的Fourier逆变换，记为$\mathcal{F}^{-1}[F]$.

$$\mathcal{F}[\frac{\partial f}{\partial x}]=-i\lambda\mathcal{F}[f],\quad \mathcal{F}[\frac{\partial f}{\partial y}]=-i\mu\mathcal{F}[f],\quad \mathcal{F}[\frac{\partial f}{\partial z}]=-i\nu\mathcal{F}[f]$$

$$\mathcal{F}[\frac{\partial^2 f}{\partial x^2}]=(-i\lambda)^2\mathcal{F}[f]\quad \mathcal{F}[\frac{\partial^2 f}{\partial y^2}]=(-i\mu)^2\mathcal{F}[f]\quad \mathcal{F}[\frac{\partial^2 f}{\partial z^2}]=(-i\nu)^2\mathcal{F}[f]$$

Example 3

用Fourier变换法求解 $\begin{cases} u_{tt} = a^2 u_{xx}, & -\infty < x < +\infty, t > 0 \\ u(x,0) = \varphi(x), \ u_t(x,0) = \psi(x) \end{cases}$

解: 记$\overline{u}(\lambda,t) = \mathcal{F}[u(x,t)]$, $\mathcal{F}[u_{xx}(x,t)] = (-i\lambda)^2\overline{u} = -\lambda^2\overline{u}$,
对方程和定解条件作Fourier变换.

$$\begin{cases} \dfrac{\mathrm{d}^2\overline{u}}{\mathrm{d}t^2} + a^2\lambda^2\overline{u} = 0, & t > 0 \\ \overline{u}\Big|_{t=0} = \overline{\varphi}(\lambda), \dfrac{\mathrm{d}\overline{u}}{\mathrm{d}t}\Big|_{t=0} = \overline{\psi}(\lambda) \end{cases}$$

此定解问题的通解是 $\overline{u} = c_1 \cos a\lambda t + c_2 \sin a\lambda t$, 代入初始条件

$$\overline{u} = \overline{\varphi}(\lambda)\cos a\lambda t + \frac{\overline{\psi}(\lambda)}{a\lambda}\sin a\lambda t$$

作Fourier逆变换得

$$u = \frac{1}{2}[\varphi(x+at) + \varphi(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)\mathrm{d}\xi$$

Example 4

求解一维热传导方程Cauchy问题

$$\begin{cases} \dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2} + f(x,t), \qquad t > 0, -\infty < x < +\infty & \text{(1a)} \\ u(x,0) = \varphi(x) & \text{(1b)} \end{cases}$$

$f(x,t)$为二元函数，$\overline{f}(\lambda,t) = \mathcal{F}[f(x,t)]$ 表示对空间变量x作Fourier变换的像函数，此时t作为参数对待.

解： 对（1a）-（1b）作Fourier变换得

$$\begin{cases} \dfrac{\mathrm{d}\overline{u}}{\mathrm{d}t} + a^2\lambda^2\overline{u} = \overline{f}, \qquad t > 0 \\ \overline{u}(\lambda,0) = \overline{\varphi}(\lambda) \end{cases}$$

解此一阶线性常微分方程初值问题可得

$$\overline{u}(\lambda,t)=\overline{\varphi}(\lambda)e^{-a^2\lambda^2t}+\int_0^t\overline{f}(\lambda,\tau)e^{-a^2\lambda^2(t-\tau)}\mathrm{d}\tau$$

再进行Fourier逆变换即得$u(x,t)$.

$$\boxed{\mathcal{F}[e^{-bx^2}]=\sqrt{\frac{\pi}{b}}e^{-\frac{\lambda^2}{4b}}}\Longrightarrow\mathcal{F}^{-1}[e^{-a^2\lambda^2t}]=\frac{1}{\sqrt{4\pi a^2t}}e^{-\frac{x^2}{4a^2t}}$$

$$\begin{aligned}\mathcal{F}^{-1}[\overline{\varphi}(\lambda)e^{-a^2\lambda^2t}]&=\varphi(x)*\frac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2t}}\\&=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{+\infty}\varphi(\xi)e^{-\frac{(x-\xi)^2}{4a^2t}}\mathrm{d}\xi\end{aligned}$$

$$
\begin{aligned}
&\mathcal{F}^{-1}[\int_0^t \overline{f}(\lambda,\tau)e^{-a^2\lambda^2(t-\tau)}\mathrm{d}\tau] \\
=&\int_0^t \mathcal{F}^{-1}[\overline{f}(\lambda,\tau)e^{-a^2\lambda^2(t-\tau)}]\mathrm{d}\tau \\
=&\int_0^t f(x,\tau) * \left(\frac{1}{2a\sqrt{\pi(t-\tau)}}e^{-\frac{x^2}{4a^2(t-\tau)}}\right)\mathrm{d}\tau \\
=&\int_0^t \frac{1}{2a\sqrt{\pi(t-\tau)}}\int_{-\infty}^{+\infty} f(\xi,\tau)e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}}\mathrm{d}\xi\mathrm{d}\tau
\end{aligned}
$$

$$
\begin{aligned}
u(x,t) = &\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{+\infty}\varphi(\xi)e^{-\frac{(x-\xi)^2}{4a^2t}}\mathrm{d}\xi \\
&+\int_0^t \frac{1}{2a\sqrt{\pi(t-\tau)}}\int_{-\infty}^{+\infty} f(\xi,\tau)e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}}\mathrm{d}\xi\mathrm{d}\tau
\end{aligned}
$$

Example 5

求解定解问题 $\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + a^2 \dfrac{\partial^4 u}{\partial x^4} = 0, & t > 0,\ x \in (-\infty, +\infty) \\ u(x,0) = \varphi(x), \quad u_t(x,0) = a\psi''(x) \end{cases}$

解: 记$\overline{u}(\lambda, t) = \mathcal{F}[u(x,t)]$, $\mathcal{F}\left[\dfrac{\partial^4 u}{\partial x^4}\right] = (-i\lambda)^4 \overline{u} = \lambda^4 \overline{u}$, 对方程和定解条件作Fourier变换.

$$\begin{cases} \dfrac{\mathrm{d}^2 \overline{u}}{\mathrm{d}t^2} + a^2 \lambda^4 \overline{u} = 0, & t > 0 \\ \overline{u}\Big|_{t=0} = \overline{\varphi}(\lambda), \dfrac{\mathrm{d}\overline{u}}{\mathrm{d}t}\Big|_{t=0} = -a\lambda^2 \overline{\psi}(\lambda) \end{cases}$$

此定解问题的通解是 $\overline{u} = c_1 \cos a\lambda^2 t + c_2 \sin a\lambda^2 t$, 代入初始条件

$$\overline{u} = \overline{\varphi}(\lambda) \cos a\lambda^2 t - \overline{\psi}(\lambda) \sin a\lambda^2 t$$

$$\mathcal{F}^{-1}[\cos b\lambda^2 + i\sin b\lambda^2] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} exp(ib\lambda^2 - i\lambda x)\mathrm{d}\lambda$$

$$=\frac{1}{2\pi}exp\left(-i\frac{x^2}{4b}\right)\int_{-\infty}^{+\infty} exp[ib(\lambda - \frac{x}{2b})^2]\mathrm{d}\lambda$$

$$=\frac{1}{2\pi}exp\left(-i\frac{x^2}{4b}\right)\int_{-\infty}^{+\infty} exp(iby^2)\mathrm{d}y$$

$$=\frac{1}{2\pi}exp\left(-i\frac{x^2}{4b}\right)\left[\int_{-\infty}^{+\infty}\cos by^2\mathrm{d}y + i\int_{-\infty}^{+\infty}\sin by^2\mathrm{d}y\right]$$

由于 $\displaystyle\int_{-\infty}^{+\infty}\cos x^2\mathrm{d}x = \int_{-\infty}^{+\infty}\sin x^2\mathrm{d}x = \sqrt{\frac{\pi}{2}}$

$$\int_{-\infty}^{+\infty}\cos by^2\mathrm{d}y = \int_{-\infty}^{+\infty}\sin by^2\mathrm{d}y = \sqrt{\frac{\pi}{2b}}$$

所以

$$\mathcal{F}^{-1}[\cos a\lambda^2 t] = \frac{1}{2\sqrt{2\pi a t}}\left(\cos\frac{x^2}{4at} + \sin\frac{x^2}{4at}\right)$$

$$= \frac{1}{2\sqrt{\pi a t}}\cos\left(\frac{\pi}{4} - \frac{x^2}{4at}\right)$$

$$\mathcal{F}^{-1}[\sin a\lambda^2 t] = \frac{1}{2\sqrt{2\pi a t}}\left(\cos\frac{x^2}{4at} - \sin\frac{x^2}{4at}\right)$$

$$= \frac{1}{2\sqrt{\pi a t}}\sin\left(\frac{\pi}{4} - \frac{x^2}{4at}\right)$$

定解问题的解是

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}\left[\overline{\varphi}(\lambda)\cos a\lambda^2 t - \overline{\psi}(\lambda)\sin a\lambda^2 t\right] \\ &= \frac{1}{2\sqrt{\pi a t}}\left[\varphi(x) * \cos\left(\frac{\pi}{4} - \frac{x^2}{4at}\right) - \psi(x) * \sin\left(\frac{\pi}{4} - \frac{x^2}{4at}\right)\right] \end{aligned}$$

Example 6

求解定解问题的有界解.

$$\begin{cases} u_{xx} + u_{yy} = 0, & -\infty < x < \infty, y > 0 \\ u\big|_{y=0} = \varphi(x) \end{cases}$$

解:(1)设$\bar{u}(\lambda, y) = \mathcal{F}[u(x,y)] = \int_{-\infty}^{+\infty} u(x,y)e^{i\lambda x}\mathrm{d}x$, $\bar{\varphi}(\lambda) = \mathcal{F}[\varphi(x)]$,

问题转化为

$$\begin{cases} \dfrac{\mathrm{d}\bar{u}^2}{\mathrm{d}y^2} - \lambda^2\bar{u} = 0, & y > 0 \\ \bar{u}\big|_{y=0} = \bar{\varphi} \end{cases}$$

方程的通解$\bar{u} = C_1 e^{\lambda y} + C_2 e^{-\lambda y}$,

代入边界条件且考虑有界性条件,

$$\bar{u} = \bar{\varphi} g(\lambda, y), \quad g(\lambda, y) = \begin{cases} e^{\lambda y}, & \lambda < 0 \\ e^{-\lambda y}, & \lambda \geqslant 0 \end{cases}$$

(2)逆变换求原问题的解.

$$\begin{aligned}
\mathcal{F}^{-1}[g(\lambda, y)] &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} g(\lambda, y) e^{-i\lambda x} \mathrm{d}\lambda \\
&= \frac{1}{2\pi} \int_{-\infty}^{0} e^{\lambda y} e^{-i\lambda x} \mathrm{d}\lambda + \frac{1}{2\pi} \int_{0}^{\infty} e^{-\lambda y} e^{-i\lambda x} \mathrm{d}\lambda \\
&= \frac{1}{2\pi} \left(\frac{1}{y - ix} - \frac{1}{-y - ix} \right) = \frac{y}{\pi(x^2 + y^2)}
\end{aligned}$$

$$u(x, y) = \varphi(x) * \frac{y}{\pi(x^2 + y^2)} = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{\varphi(\xi)}{(x - \xi)^2 + y^2} \mathrm{d}\xi$$

Example 7

求解定解问题 $\begin{cases} \dfrac{\partial u}{\partial t} = a^2 \Delta_3 u, & t > 0,\ x, y, z \in (-\infty, +\infty) \\ u\big|_{t=0} = \varphi(x, y, z) \end{cases}$

解: 记$\overline{u}(\lambda, \mu, \nu, t) = \mathcal{F}[u(x, y, z, t)]$,
对方程和定解条件作三维Fourier变换.

$$\begin{cases} \dfrac{\mathrm{d}\overline{u}}{\mathrm{d}t} = a^2 \left[(-i\lambda)^2 + (-i\mu)^2 + (-i\nu)^2\right] \overline{u} = -a^2(\lambda^2 + \mu^2 + \nu^2)\overline{u} \\ \overline{u}\big|_{t=0} = \overline{\varphi}(\lambda, \mu, \nu) \end{cases}$$

此方程的解是 $\overline{u}(\lambda, \mu, \nu, t) = \overline{\varphi}(\lambda, \mu, \nu) e^{-a^2(\lambda^2+\mu^2+\nu^2)t}$

$$
\begin{aligned}
&\mathcal{F}^{-1}\left[e^{-a^2(\lambda^2+\mu^2+\nu^2)t}\right] \\
=&\frac{1}{(2\pi)^3}\iiint\limits_{\mathbb{R}^3} e^{-a^2(\lambda^2+\mu^2+\nu^2)t}e^{-i(\lambda x+\mu y+\nu z)}\mathrm{d}\lambda\mathrm{d}\mu\mathrm{d}\nu \\
=&\left(\frac{1}{2a\sqrt{\pi t}}\right)^3 exp\left(-\frac{x^2+y^2+z^2}{4a^2t}\right)
\end{aligned}
$$

$$
\begin{aligned}
u=&\left(\frac{1}{2a\sqrt{\pi t}}\right)^3 \\
&\cdot\iiint\limits_{\mathbb{R}^3}\varphi(\xi,\eta,\zeta)exp\left(-\frac{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}{4a^2t}\right)\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta
\end{aligned}
$$

正弦变换与余弦变换

设$f(x)$是定义在$(0,+\infty)$的函数，定义

(1) 余弦变换

$$\overline{f}_c(\lambda)=\int_0^{+\infty} f(t)\cos\lambda t\mathrm{d}t,$$

其逆变换公式为

$$f(x)=\frac{2}{\pi}\int_0^{+\infty}\overline{f}_c(\lambda)\cos\lambda x\mathrm{d}\lambda.$$

(2) 正弦变换

$$\overline{f}_s(\lambda)=\int_0^{+\infty} f(t)\sin\lambda t\mathrm{d}t,$$

其逆变换公式为

$$f(x)=\frac{2}{\pi}\int_0^{+\infty}\overline{f}_s(\lambda)\sin\lambda x\mathrm{d}\lambda.$$

以余弦变换为例：将$f(x)$作偶延拓$\widehat{f}(x)=\begin{cases} f(x), & x\geq 0 \\ f(-x), & x<0 \end{cases}$

(1) $\widehat{f}(x)$是偶函数，对其作Fourier变换

$$F(\lambda)=\int_{-\infty}^{+\infty}\widehat{f}(x)e^{i\lambda x}\mathrm{d}x=\int_{-\infty}^{+\infty}\widehat{f}(x)(\cos\lambda x+i\sin\lambda x)\mathrm{d}x$$
$$=2\int_{0}^{+\infty}f(x)\cos\lambda x\mathrm{d}x=2\overline{f}_c(\lambda)$$

(2) $F(\lambda)$也是偶函数，

$$\widehat{f}(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}F(\lambda)e^{-i\lambda x}\mathrm{d}\lambda$$
$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}2\overline{f}_c(\lambda)(\cos\lambda x-i\sin\lambda x)\mathrm{d}\lambda$$
$$=\frac{2}{\pi}\int_{0}^{+\infty}\overline{f}_c(\lambda)\cos\lambda x\mathrm{d}\lambda$$

对函数$u(x,t)$的偏导数作正弦变换：

$$\int_0^{+\infty}\frac{\partial u}{\partial t}\sin\lambda x\mathrm{d}x=\frac{\partial}{\partial t}\int_0^{+\infty}u\sin\lambda x\mathrm{d}x=\frac{\partial\overline{u}_s}{\partial t}$$

$$\int_0^{+\infty}\frac{\partial^2 u}{\partial x^2}\sin\lambda x\mathrm{d}x=\frac{\partial u}{\partial x}\sin\lambda x\Big|_0^{+\infty}-\lambda\int_0^{+\infty}\frac{\partial u}{\partial x}\cos\lambda x\mathrm{d}x$$

$$=\left[\frac{\partial u}{\partial x}\sin\lambda x-\lambda u\cos\lambda x\right]\Big|_0^{+\infty}-\lambda^2\int_0^{+\infty}u\sin\lambda x\mathrm{d}x$$

$$=\lambda u(0,t)-\lambda^2\overline{u}_s(\lambda)$$

- 半无界定解问题可以用正弦变换或余弦变换求解。
- 选取正弦变换还是余弦变换与边界条件及方程有关，如果方程中只出现x的二阶偏导，一般第一类边界条件用正弦变换，第二类边界条件用余弦变换。

Example 8

半无界杆的热传导问题 $\begin{cases} u_t - a^2 u_{xx} = 0, & 0 < x < +\infty,\ t > 0 \\ u(0,t) = u_0, & u(x,0) = 0 \end{cases}$

解: 设$\overline{u} = \mathcal{F}_s[u]$, 对定解问题作正弦变换.

$$\begin{cases} \overline{u}_t = -a^2\lambda^2\overline{u} + a^2\lambda u_0 \\ \overline{u}\big|_{t=0} = 0 \end{cases}$$

此常微分方程的通解是 $\overline{u} = Ce^{-a^2\lambda^2 t} + \dfrac{u_0}{\lambda}$.

代入初始条件可得 $\overline{u} = -\dfrac{u_0}{\lambda}e^{-a^2\lambda^2 t} + \dfrac{u_0}{\lambda}$

作正弦变换的逆变换

$$\begin{aligned} u(x,t) &= \frac{2}{\pi}\int_0^{+\infty} -\frac{u_0}{\lambda}e^{-a^2\lambda^2 t}\sin\lambda x + \frac{u_0}{\lambda}\sin\lambda x \mathrm{d}\lambda \\ &= u_0 - \frac{2}{\pi}\int_0^{+\infty}\frac{u_0}{\lambda}e^{-a^2\lambda^2 t}\sin\lambda x \mathrm{d}\lambda \triangleq u_0 - I(x,t) \end{aligned}$$

$$\frac{\partial I}{\partial x} = \frac{2u_0}{\pi}\int_0^{+\infty} e^{-a^2\lambda^2 t}\cos\lambda x \mathrm{d}\lambda = \frac{u_0}{a\sqrt{\pi t}}\exp(-\frac{x^2}{4a^2 t})$$

$I(0,t) = 0$, 所以,

$$u(x,t) = u_0 - \int_0^x \frac{u_0}{a\sqrt{\pi t}}\exp(-\frac{\xi^2}{4a^2 t})\mathrm{d}\xi$$

$$\boxed{\int_0^{+\infty}\frac{\sin x}{x}\mathrm{d}x = \frac{\pi}{2}} \qquad \boxed{\int_0^{+\infty} e^{-a\lambda^2}\cos b\lambda \mathrm{d}\lambda = \sqrt{\frac{\pi}{4a}}\exp\left(-\frac{b^2}{4a}\right)}$$

Example 9

利用余弦变换求解 $\begin{cases} u_t = a^2 u_{xx}, & x > 0, t > 0 \\ u(0,x) = 0, u_x(t,0) = Q \\ u(t,+\infty) = u_x(t,+\infty) = 0 \end{cases}$

解: 令 $\bar{u}(t,\lambda) = \mathcal{F}_c[u(x,t)] = \int_0^{+\infty} u\cos\lambda x \mathrm{d}x$, 则

$$\mathcal{F}_c[u_{xx}] = \int_0^{+\infty} u_{xx}(t,x)\cos\lambda x \mathrm{d}x = u_x\cos\lambda x\Big|_0^{+\infty} + \lambda\int_0^{+\infty} u_x\sin\lambda x \mathrm{d}$$

$$= -Q + \lambda u\sin\lambda x\Big|_0^{+\infty} - \lambda^2\int_0^{+\infty} u\cos\lambda x \mathrm{d}x = -Q - \lambda^2\bar{u}$$

方程化为 $\begin{cases} \dfrac{\mathrm{d}\bar{u}}{\mathrm{d}t} + a^2\lambda^2\bar{u} = -a^2 Q \\ \bar{u}(\lambda,0) = 0 \end{cases}$

此方程的解是 $\bar{u} = \dfrac{Q}{\lambda^2}\left[e^{-a^2\lambda^2 t} - 1\right] = -a^2 Q\int_0^t e^{-a^2\lambda^2\tau}\mathrm{d}\tau.$

最后作逆变换得

$$\begin{aligned} u(x,t) &= \frac{2}{\pi}\int_0^{+\infty} \bar{u}(\lambda t)\cos\lambda x \mathrm{d}\lambda \\ &= -\frac{2a^2Q}{\pi}\int_0^t \mathrm{d}\tau \int_0^{+\infty} e^{-a^2\lambda^2\tau}\cos\lambda x \mathrm{d}\lambda \\ &= -\frac{2a^2Q}{\pi}\int_0^t \frac{1}{2a}\sqrt{\frac{\pi}{\tau}}\exp\left(-\frac{x^2}{4a^2\tau}\right)\mathrm{d}\tau \\ &= -\frac{aQ}{\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{\pi}}\exp\left(-\frac{x^2}{4a^2\tau}\right)\mathrm{d}\tau \end{aligned}$$

作变量代换$y=\dfrac{x}{2a\sqrt{\tau}}$，则$\tau=\dfrac{x^2}{4a^2y^2}$，$\mathrm{d}\tau=-\dfrac{x^2}{2a^2y^3}\mathrm{d}y$

$$u(x,t) = -\frac{xQ}{\sqrt{\pi}}\int_{\frac{x}{2a\sqrt{t}}}^{+\infty}\frac{1}{y^2}e^{-y^2}\mathrm{d}y.$$

§4.2 用Laplace变换解题

一、复习Laplace变换

Definition 10

设函数$f(t)$当$t \geqslant 0$ 时有定义，且广义积分$\int_0^{+\infty} f(t)h(t)e^{-pt}\mathrm{d}t$在$p = \sigma + \mathrm{i}s$的某一区域内收敛，则由此积分确定的参数为$p$的函数

$$F(p) = \int_0^{+\infty} f(t)h(t)e^{-pt}\mathrm{d}t$$

叫做函数$f(t)$的拉普拉斯变换（简称拉氏变换），记作$F(p) = \mathcal{L}[f(t)]$，函数$F(p)$也可称为$f(t)$ 的像函数．而$f(t)$ 则称为$F(p)$的拉氏逆变换或本函数，记做$f(t) = \mathcal{L}^{-1}[F(p)]$.

$$h(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

函数$f(t)$满足
(1)$f(t)$在$(0,+\infty)$的任意有界区间内逐段光滑；
(2)$f(t)$是指数增长型函数，即存在常数$k>0, c\geqslant 0$，使得$|f(t)|\leqslant ke^{ct}$；
则像函数$F(p)=\mathcal{L}[f(t)]$在$\mathrm{Re}\ p>c$上有意义，且是解析函数。

Laplace变换的性质：

1.线性性质:

$$\mathcal{L}[\alpha f(t)+\beta g(t)]=\alpha\mathcal{L}[f(t)]+\beta\mathcal{L}[g(t)]$$

2.相似定理:

设$\mathcal{L}[f(t)]=F(p)$，则对任意常数$a>0$有$\mathcal{L}[f(at)]=\dfrac{1}{a}F(\dfrac{p}{a})$.

$$\mathcal{L}[f(at)]=\int_0^{+\infty}f(at)e^{-pt}\mathrm{d}t=\frac{1}{a}\int_0^{+\infty}f(u)e^{-\frac{p}{a}u}\mathrm{d}u=\frac{1}{a}F(\frac{p}{a}).$$

3.延滞性质:

设$\mathcal{L}[f(t)]=F(p)$,对于$a>0$,有$\mathcal{L}[f(t-a)]=e^{-ap}F(p)$.

证明:$\mathcal{L}[f(t-a)]=\int_0^{+\infty}f(t-a)e^{-pt}\mathrm{d}t=\int_{-a}^{+\infty}f(u)e^{-p(u+a)}\mathrm{d}u$

$$=e^{-ap}\int_{-a}^{+\infty}f(u)e^{-pu}\mathrm{d}u=e^{-ap}F(p).$$

4.平移性质:

设$\mathcal{L}[f(t)]=F(p)$, 则对任意复常数λ有

$$\mathcal{L}[e^{\lambda t}f(t)]=F(p-\lambda).$$

证明:

$$\mathcal{L}[e^{\lambda t}f(t)]=\int_0^{+\infty}e^{\lambda t}f(t)e^{-pt}\mathrm{d}t=\int_0^{+\infty}f(t)e^{-(p-\lambda)t}\mathrm{d}u=F(p-\lambda).$$

5.本函数微分:

设$\mathcal{L}[f(t)]=F(p)$，则

$$\mathcal{L}[f'(t)]=pF(p)-f(0+0).$$

$$\mathcal{L}[f^{(n)}(t)]=p^nF(p)-p^{n-1}f(0+0)-p^{n-2}f'(0+0)-\cdots-f^{(n-1)}(0+0).$$

证明:由分部积分法

$$\int_0^{+\infty}f'(t)e^{-pt}\mathrm{d}t=\int_0^{+\infty}e^{-pt}\mathrm{d}f(t)$$

$$=f(t)e^{-pt}\Big|_0^{+\infty}+p\int_0^{+\infty}f(t)e^{-pt}\mathrm{d}t=pF(p)-f(0+0).$$

递推可得

$$\mathcal{L}[f^{(n)}(t)]=p^nF(p)-p^{n-1}f(0+0)-\cdots-pf^{(n-2)}(0+0)-f^{(n-1)}(0+0).$$

6.像函数微分法:

设$\mathcal{L}[f(t)]=F(p)$, 则$F'(p)=\mathcal{L}[-tf(t)]$.

$$F^{(n)}(p)=\mathcal{L}[(-1)^n t^n f(t)]\text{或}\mathcal{L}[t^n f(t)]=(-1)^n F^{(n)}(p).$$

证明: 由$|f(t)|<Ke^{\alpha t}$, α,K是正常数，对于任意$\sigma>\alpha$

$$\lim_{t\to+\infty}\frac{tf(t)}{e^{\sigma t}}=\lim_{t\to+\infty}\frac{t}{e^{(\sigma-\alpha)t}}\cdot\frac{f(t)}{e^{\alpha t}}=0.$$

所以$\dfrac{|tf(t)|}{e^{\sigma t}}$有界，即存在$K_1$使得$|f(t)|<K_1e^{\sigma t}$.

由σ的任意性，$\mathcal{L}[tf(t)]$在$Real(p)>\alpha$存在，也在此区域内闭一致收敛。

$$F'(p)=\int_0^{+\infty}-tf(t)e^{-pt}\mathrm{d}t=\mathcal{L}[-tf(t)].$$

归纳可得

$$F^{(n)}(p)=\mathcal{L}[(-1)^n t^n f(t)].$$

7.本函数积分法:

设$\mathcal{L}[f(t)]=F(p)$，则$\mathcal{L}\left[\int_0^t f(s)\mathrm{d}s\right]=\dfrac{F(p)}{p}$.

证明:记$g(t)=\int_0^t f(u)\mathrm{d}u$.则$g'(t)=f(t)$,

$$\mathcal{L}[g'(t)]=p\mathcal{L}[g(t)]-g(0)=p\mathcal{L}[g(t)].$$

又成立　$\mathcal{L}[g'(t)]=\mathcal{L}[f(t)]=F(p)$,所以

$$\mathcal{L}\left[\int_0^t f(u)\mathrm{d}u\right]=\frac{F(p)}{p}.$$

8.卷积性质:

作Laplace变换的函数$f(x)$,在$x<0$时，定义为0

$$f(x)*g(x)=\int_0^x f(x-\xi)g(\xi)\mathrm{d}\xi$$

$$\mathcal{L}[f*g]=\mathcal{L}[f]\mathcal{L}[g]$$

证明:

$$\begin{aligned}
\mathcal{L}[f*g] &= \int_0^{+\infty}\left(\int_0^t f(t-\tau)g(\tau)\mathrm{d}\tau\right)e^{-pt}\mathrm{d}t \\
&= \int_0^{+\infty}\left(\int_\tau^{+\infty} f(t-\tau)g(\tau)e^{-pt}\mathrm{d}t\right)\mathrm{d}\tau \\
&= \int_0^{+\infty} g(\tau)\left(\int_0^{+\infty} f(s)e^{-p(s+\tau)}\mathrm{d}s\right)\mathrm{d}\tau \\
&= \int_0^{+\infty} g(\tau)e^{-p\tau}\mathrm{d}\tau\int_0^{+\infty} f(s)e^{-ps}\mathrm{d}s=\mathcal{L}[f]\mathcal{L}[g]
\end{aligned}$$

设$\mathcal{L}[f]=F(p)$, $\mathcal{L}[g]=G(p)$,则

$$\mathcal{L}^{-1}[F(p)G(p)]=f(x)*g(x)=\int_0^x f(x-\xi)g(\xi)\mathrm{d}\xi$$

逆变换计算: 设$\mathcal{L}[f]=F(p)$, 在$f(x)$的连续点处

$$f(x)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}F(p)e^{px}\mathrm{d}p$$

根据留数定理

$$f(x)=\sum Res\left[F(p)e^{px},p\right],\quad p\text{取遍所有极点}$$

f(t)	F(p)	f(t)	F(p)
1	$\frac{1}{p}$	$t^n (n \geqslant 1)$	$\frac{n!}{p^{n+1}}$
t^α	$\frac{\Gamma(\alpha+1)}{p^{\alpha+1}}$	$e^{\lambda t}$	$\frac{1}{p-\lambda}$
$\sin \omega t$	$\frac{\omega}{p^2+\omega^2}$	$\cos \omega t$	$\frac{p}{p^2+\omega^2}$
$e^{-\lambda t} \sin \omega t$	$\frac{\omega}{(p+\lambda)^2+\omega^2}$	$e^{-\lambda t} \cos \omega t$	$\frac{p+\lambda}{(p+\lambda)^2+\omega^2}$
$\sinh \omega t$	$\frac{\omega}{p^2-\omega^2}$	$\cosh \omega t$	$\frac{p}{p^2-\omega^2}$

Example 11

求解热传导问题 $\begin{cases} u_t - a^2 u_{xx} = 0 & x > 0,\ t > 0 \\ u(0,t) = f(t),\ u\big|_{x\to+\infty} = 0 \\ u(x,0) = 0 \end{cases}$

分析：Laplace变换在$(0,+\infty)$进行，但本函数微分公式需要初始条件，一般对变量t 进行.

解：(1)选择变量t作Laplace变换, 记$\widehat{u} = \mathcal{L}[u(x,t)]$.

$$\begin{cases} p\widehat{u}(x,p) - u(x,0) = a^2 \dfrac{\mathrm{d}^2\widehat{u}}{\mathrm{d}x^2} \\ \widehat{u}(0,p) = \widehat{f}(p),\ \widehat{u}\big|_{x\to+\infty} = 0 \end{cases}$$

(2)求解像函数的常微分方程 $\dfrac{\mathrm{d}^2\widehat{u}}{\mathrm{d}x^2} = \dfrac{p}{a^2}\widehat{u}$

通解是$\widehat{u} = c_1 e^{\frac{\sqrt{p}}{a}x} + c_2 e^{-\frac{\sqrt{p}}{a}x}$
$\widehat{u}\Big|_{x\to+\infty} = 0 \Longrightarrow c_1 = 0$
$\widehat{u}(0,p) = \widehat{f}(p) \Longrightarrow c_2 = \widehat{f}(p)$
$u(x,t)$的像函数为 $\widehat{u}(x,p) = \widehat{f}(p)e^{-\frac{\sqrt{p}}{a}x}$.

(3)求$u(x,t)$

$$
\begin{aligned}
u(x,t) &= f(t) * \mathcal{L}^{-1}[e^{-\frac{\sqrt{p}}{a}x}] = f(t) * \frac{x}{2a\sqrt{\pi}} t^{-\frac{3}{2}} e^{-\frac{x^2}{4a^2 t}} \\
&= \frac{x}{2a\sqrt{\pi}} \int_0^t f(\tau)(t-\tau)^{-\frac{3}{2}} e^{-\frac{x^2}{4a^2(t-\tau)}} \mathrm{d}\tau
\end{aligned}
$$

Example 12

求解有界热传导问题

$$\begin{cases} u_t = a^2 u_{xx}, \quad 0 < x < 1,\ t > 0 \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = \sin 2\pi x \end{cases}$$

解： 1°. 对方程作Laplace变换，记$\widehat{u}(p,x) = \int_0^{+\infty} u(t,x)e^{-pt}\mathrm{d}t$

$$\begin{cases} a^2 \dfrac{\mathrm{d}^2\widehat{u}}{\mathrm{d}x^2} - p\widehat{u} + \sin 2\pi x = 0 \\ \widehat{u}(p,0) = \widehat{u}(p,1) = 0 \end{cases}$$

2°. 此方程的通解是$\widehat{u}(p,x) = c_1 e^{\frac{\sqrt{p}}{a}x} + c_2 e^{-\frac{\sqrt{p}}{a}x} + \dfrac{\sin 2\pi x}{p + 4\pi^2 a^2}$

代入边界条件得$c_1 = c_2 = 0$

3° $u(x,t) = \mathcal{L}^{-1}\left[\dfrac{\sin 2\pi x}{p + 4\pi^2 a^2}\right] = \exp(-4\pi^2 a^2 t)\sin 2\pi x$

Example 13

求解半无界波动问题 一根半无界弦一端固定，另一端自由，求弦在外力$f(t)=\cos\omega t$作用下的振动，初位移和初速度为零.

$$\begin{cases} u_{tt}-a^2u_{xx}=\cos\omega t, \quad x>0,\ t>0 \\ u(x,0)=0,\ u_t(x,0)=0 \\ u(0,t)=0,\ \lim\limits_{x\to+\infty} u_x(x,t)=0 \end{cases}$$

解: 1° 记$\widehat{u}(p,t)=\mathcal{L}[u(x,t)]$, $\mathcal{L}[\cos\omega t]=\dfrac{p}{p^2+\omega^2}=\widehat{f}(p)$,

对方程和定解条件作L 变换

$$\begin{cases} \dfrac{\mathrm{d}^2\widehat{u}}{\mathrm{d}x^2}-\dfrac{p^2}{a^2}\widehat{u}=-\dfrac{\widehat{f}(p)}{a^2} \\ \widehat{u}\big|_{x=0}=0,\ \dfrac{\mathrm{d}\widehat{u}}{\mathrm{d}x}\Big|_{x\to\infty}=0 \end{cases}$$

2°方程的通解是 $\widehat{u}=c_1e^{\frac{p}{a}x}+c_2e^{-\frac{p}{a}x}+\dfrac{\widehat{f}(p)}{p^2}$

代入初始条件：

$$x\to\infty,\frac{\mathrm{d}\widehat{u}}{\mathrm{d}x}\to 0\to c_1=0$$

$$x=0\widehat{u}=0\to c_2=-\frac{\widehat{f}(p)}{p^2}$$

像函数为$\widehat{u}=-\dfrac{\widehat{f}(p)}{p^2}e^{-\frac{p}{a}x}+\dfrac{\widehat{f}(p)}{p^2}$.

3°求逆变换

$$\mathcal{L}^{-1}[\frac{\widehat{f}(p)}{p^2}] = \mathcal{L}^{-1}[\frac{1}{(p^2+\omega^2)p}]$$
$$= \frac{1}{\omega^2}\mathcal{L}^{-1}[\frac{1}{p} - \frac{p}{p^2+\omega^2}] = \frac{1}{\omega^2}(1-\cos\omega t)$$

$$\mathcal{L}^{-1}[\frac{-\widehat{f}(p)}{p^2}e^{-\frac{p}{a}x}] = \begin{cases} -\frac{1}{\omega^2}(1-\cos\omega(t-\frac{x}{a})) & t > \frac{x}{a} \\ 0 & t \le \frac{x}{a} \end{cases}$$

$$u(x,t) = \begin{cases} \frac{2}{\omega^2}\left[\sin^2\frac{\omega t}{2} - \sin^2\frac{\omega(t-\frac{x}{a})}{2}\right] & t > \frac{x}{a} \\ \frac{2}{\omega^2}\sin^2\frac{\omega t}{2} & t \le \frac{x}{a} \end{cases}$$

Example 14

$$\begin{cases} u_{tt} = a^2 u_{xx} \qquad t > 0, 0 < x < l \\ u(0,t) = 0,\ \ u_x(l,t) = A\sin\omega t \\ u(x,0) = 0,\ \ u_t(x,0) = 0 \end{cases}$$

设$\bar{u} = L[u(x,t)] = \int_0^{+\infty} u(x,t)e^{-pt}\mathrm{d}t$, 对定解问题作Laplace变换

$$\begin{cases} p^2\bar{u} = a^2\bar{u}_{xx} \\ \bar{u}(0,p) = 0, \bar{u}_x(l,p) = \dfrac{A\omega}{p^2+\omega^2} \end{cases}$$

此方程的通解是$\bar{u}C\cosh\frac{p}{a}x + D\sinh\frac{p}{a}x$

代入边界条件得$\bar{u}(x,p) = \dfrac{Aa\omega}{p(p^2+\omega^2)\cosh\frac{lp}{a}}\sinh\frac{p}{a}x.$

$$u(x,t) = L^{-1}[\bar{u}(x,p)] = \sum Res\left[\bar{u}(x,p)e^{pt}, p\right]$$

$\bar{u}(x,p)e^{pt}$的极点有$\pm i\omega, \pm i\omega_k = i\dfrac{(2k-1)\pi a}{2l}, k = 1, 2, \cdots$

$$Res[\bar{u}(x,p)e^{pt}, i\omega] = \left.\frac{Aa\omega \sinh \frac{px}{a}}{p(p+i\omega)\cosh \frac{lp}{a}} e^{pt}\right|_{p=i\omega} = \frac{Aa \sin \frac{\omega x}{a}}{2i\omega \cos \frac{\omega l}{a}} e^{i\omega t}$$

$$Res[\bar{u}(x,p)e^{pt}, -i\omega] = \left.\frac{Aa\omega \sinh \frac{px}{a}}{p(p-i\omega)\cosh \frac{lp}{a}} e^{pt}\right|_{p=-i\omega} = \frac{Aa \sin \frac{\omega x}{a}}{-2i\omega \cos \frac{\omega l}{a}} e^{-i\omega t}$$

$$Res[\bar{u}(x,p)e^{pt}, i\omega] + Res[\bar{u}(x,p)e^{pt}, -i\omega] = \frac{Aa}{\omega\cos\frac{\omega l}{a}}\sin\frac{\omega x}{a}\sin\omega t$$

$$Res[\bar{u}(x,p)e^{pt}, i\omega_k] = \left.\frac{Aa\omega\sinh\frac{px}{a}}{p(p^2+\omega^2)(\cosh\frac{lp}{a})'}e^{pt}\right|_{p=i\omega_k}$$
$$=\frac{Aa\omega\sin\frac{\omega_k x}{a}}{i\omega_k(-\omega_k^2+\omega^2)\frac{l}{a}\sin\frac{(2k-1)\pi}{2}}e^{i\omega_k t}$$

$$Res[\bar{u}(x,p)e^{pt}, i\omega_k] = \left.\frac{Aa\omega\sinh\frac{px}{a}}{p(p^2+\omega^2)(\cosh\frac{lp}{a})'}e^{pt}\right|_{p=i\omega_k}$$
$$=\frac{Aa\omega\sin\frac{\omega_k x}{a}}{-i\omega_k(-\omega_k^2+\omega^2)\frac{l}{a}\sin\frac{(2k-1)\pi}{2}}e^{-i\omega_k t}$$

$$Res[\bar{u}(x,p)e^{pt}, i\omega_k] + Res[\bar{u}(x,p)e^{pt}, -i\omega_k]$$

$$=\frac{(-1)^{k-1}16Aa\omega l^2}{(2k-1)\pi(4l^2\omega^2 - a^2(2k-1)^2\pi^2)}\sin\frac{(2k-1)\pi x}{2l}\sin\frac{(2k-1)\pi at}{2l}$$

所以定解问题的解是

$$u(x,t) = \frac{Aa}{\omega\cos\frac{\omega l}{a}}\sin\frac{\omega x}{a}\sin\omega t$$

$$+\frac{16Aa\omega l^2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\sin\frac{(2k-1)\pi x}{2l}\sin\frac{(2k-1)\pi at}{2l}}{(2k-1)(4l^2\omega^2 - a^2(2k-1)^2\pi^2)}.$$

Example 15

$$\begin{cases} u_{tt} = \Delta_3 u, \qquad t > 0, r > 0 \\ u\big|_{r=0}\text{有界}, (r = \sqrt{x^2+y^2+z^2}) \\ u\big|_{t=0} = 0,\ u_t\big|_{t=0} = (1+r^2)^{-2} \end{cases}$$

解: 初始条件与θ, φ无关，设$u = u(r,t)$,

$$\begin{cases} u_{tt} = a^2 \left(u_{rr} + \dfrac{2}{r} u_r \right) \\ u\big|_{r=0}\text{有界}, (r = \sqrt{x^2+y^2+z^2}) \\ u\big|_{t=0} = 0,\ u_t\big|_{t=0} = (1+r^2)^{-2} \end{cases}$$

令$v = ru$,则$u_r = \dfrac{1}{r}v_r - \dfrac{1}{r^2}v,\ u_{rr} = \dfrac{1}{r}v_{rr} - \dfrac{2}{r^2}v_r + \dfrac{2}{r^3}v$,

方程化为$\begin{cases} v_{tt} = a^2 v_{rr} \\ v\big|_{r=0} = 0 \\ v\big|_{t=0} = 0, v_t\big|_{t=0} = \dfrac{r}{(1+r^2)^2} \end{cases}$.

对方程作Laplace变换，记$\bar{v}=L[v(r,t)]$

$$\bar{v}_{rr}-\frac{p^2}{a^2}\bar{v}=-\frac{r}{a^2(1+r^2)^2}$$

根据线性常微分方程理论，$\bar{v}=c_1e^{\frac{pr}{a}}+c_2e^{-\frac{pr}{a}}+\bar{v}^*$.
设$\bar{v}^*=c_1(r)e^{\frac{pr}{a}}+c_2(r)e^{-\frac{pr}{a}}$, $c_1(r),c_2(r)$满足

$$\begin{cases} c_1'e^{\frac{pr}{a}}+c_2'e^{-\frac{pr}{a}}=0\\ c_1'\frac{p}{a}e^{\frac{pr}{a}}-c_2'\frac{p}{a}e^{-\frac{pr}{a}}=-\dfrac{r}{a^2(1+r^2)^2}\end{cases}$$

解得 $c_1'=-\dfrac{re^{-\frac{pr}{a}}}{2ap(1+r^2)^2}$, $c_2'=\dfrac{re^{\frac{pr}{a}}}{2ap(1+r^2)^2}$, 取

$$c_1=\frac{1}{2ap}\int_r^{+\infty}\frac{\xi e^{-\frac{p\xi}{a}}}{(1+\xi^2)^2}\mathrm{d}\xi,\ c_2=\frac{1}{2ap}\int_{-\infty}^{r}\frac{\xi e^{\frac{p\xi}{a}}}{(1+\xi^2)^2}\mathrm{d}\xi$$

由于$r=0$时$v=0$, 所以$c_1=0$, $r\to\infty$时v有界，得$c_2=0$, 问题的解是

$$\bar{v}=\frac{1}{2ap}\int_{r}^{+\infty}\frac{\xi e^{-\frac{p(\xi-r)}{a}}}{(1+\xi^2)^2}\mathrm{d}\xi+\frac{1}{2ap}\int_{-\infty}^{r}\frac{\xi e^{\frac{p(\xi-r)}{a}}}{(1+\xi^2)^2}\mathrm{d}\xi.$$

下面进行逆变换求$v(r,t)$.

由$L^{-1}\left[\frac{1}{p}\right]=h(t)=\begin{cases}1, & t\geq 0\\ 0, & t<0\end{cases}$,

所以$L^{-1}\left[\frac{e^{-p\tau}}{p}\right]=h(t-\tau)=\begin{cases}1, & t\geq \tau\\ 0, & t<\tau\end{cases}$

$$L^{-1}\left[\frac{e^{-\frac{p(\xi-r)}{a}}}{p}\right]=h(t-\frac{\xi-r}{a})=\begin{cases}1, & \xi\leq at+r\\ 0, & \xi>at+r\end{cases}$$

$$L^{-1}\left[\frac{1}{2ap}\int_{r}^{+\infty}\frac{\xi e^{-\frac{p(\xi-r)}{a}}}{(1+\xi^2)^2}\mathrm{d}\xi\right]$$

$$=\frac{1}{2a}\int_{r}^{r+at}\frac{\xi}{(1+\xi^2)^2}\mathrm{d}\xi=\frac{1}{4a}\left(\frac{1}{1+r^2}-\frac{1}{1+(r+at)^2}\right)$$

同理可得

$$L^{-1}\left[\frac{1}{2ap}\int_{-\infty}^{r}\frac{\xi e^{\frac{p(\xi-r)}{a}}}{(1+\xi^2)^2}\mathrm{d}\xi\right]$$

$$=\frac{1}{2a}\int_{r-at}^{r}\frac{\xi}{(1+\xi^2)^2}\mathrm{d}\xi=\frac{1}{4a}\left(\frac{1}{1+(r+at)^2}-\frac{1}{1+r^2}\right)$$

$$u(r,t)=\frac{t}{(1+(r-at)^2)(1+(r+at)^2)}.$$

总结

用积分变换法解题步骤

1. 对方程和定解条件(关于某个变量)取变换;
2. 解变换后的像函数的常微或代数方程的定解问题;
3. 求像函数的逆变换(反演)即得原定解问题的解.

积分变换选择原则

1. 自变量取值区间与积分变换区间一致;
2. 估计未知函数及定解条件中函数的性质，积分变换需存在;
3. 函数$u(x,t)$与其偏导数在积分变换下有简单的代数关系;
4. 积分变换中所需的特殊函数值由定解条件给出.

优点：

- 1.减少了自变量个数，将偏微方程化为常微方程，常微方程化为代数方程求解，使问题大为简化;
- 2.不必考虑方程(边界条件)的是否为齐次，都采用一种固定的步骤求解，易于掌握。

缺点：

- Fourier变换： 对函数要求苛刻(绝对可积)
- Laplace变换： 逆变换计算困难.