10-24 作业

2. 证明：(1)X 为非负整值随机变量，

$$\begin{aligned}
E(X) &= \sum_{k=1}^{\infty} kP(X=k) = \sum_{k=1}^{\infty}\sum_{n=1}^{k} P(X=k) \\
&= \sum_{n=1}^{\infty}\sum_{k=n}^{\infty} P(X=k) = \sum_{n=1}^{\infty} P(X \geq n) \text{ (第一个“=”成立)} \\
&= \sum_{n=1}^{\infty} P(X > n-1) = \sum_{n=0}^{\infty} P(X > n) \text{ (第二个“=”成立)}
\end{aligned}$$

(2)X 为非负连续型随机变量且 $X \sim F$, 设对应概率密度为 $f(\cdot)$.

$$\begin{aligned}
E(X) &= \int_0^{\infty} xf(x)dx = \int_0^{\infty} \Big(\int_0^x f(x)\ dt \Big)\ dx \\
&= \int_0^{\infty} \Big(\int_t^{\infty} f(x)\ dx \Big)\ dt \\
&= \int_0^{\infty} F(x)\Big|_t^{\infty}\ dt = \int_0^{\infty} 1 - F(t)dt \\
&= \int_0^{\infty} 1 - F(x)dx
\end{aligned}$$

(3)X 为非负随机变量，

$$\begin{aligned}
E(X) &= E\Big[\int_0^X 1dx \Big] = E\Big[\int_0^{\infty} I_{(X>x)}\ dx \Big] \\
&= \int_0^{\infty} E(I_{(X>x)})dx = \int_0^{\infty} P(X>x)dx \\
&= \int_0^{\infty} 1 - F(x)dx
\end{aligned}$$

3. 记 $\phi(x)$ 为标准正态分布的密度函数，则 X 的密度函数为
$f(x) = 0.5\phi(x) + 0.25 \cdot \phi\left(\frac{x-4}{2}\right)$, 则

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} xf(x)dx = 0.5\int_{-\infty}^{\infty} x\phi(x)dx + 0.25\int_{-\infty}^{\infty} x\phi\left(\frac{x-4}{2}\right)dx \\
&= 0.25\int_{-\infty}^{\infty} (2y+4)\phi(y)d(2y+4) = 0.5\int_{-\infty}^{\infty} 2y\phi(y)dy + 0.5\int_{-\infty}^{\infty} 4\phi(y)dy \\
&= 2\int_{-\infty}^{\infty} \phi(y)dy = 2
\end{aligned}$$

4. (1)X 服从 Reyleigh 分布，

$$
\begin{aligned}
EX &= \int_0^{\infty} x \cdot \frac{x}{\sigma^2} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{\sigma^2} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} dx \\
&= \frac{1}{\sigma}\sqrt{\frac{\pi}{2}} \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} dx \\
&= \frac{1}{\sigma}\sqrt{\frac{\pi}{2}} \cdot \sigma^2 \\
&= \sigma\sqrt{\frac{\pi}{2}}
\end{aligned}
$$

(2) X 服从 Beta 分布，

$$
\begin{aligned}
EX &= \int_0^1 x \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} dx \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{\alpha-1+1}(1-x)^{\beta-1} dx \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} \\
&= \frac{\alpha}{\alpha+\beta}
\end{aligned}
$$

(3)X 服从 Weibull 分布，

$$
\begin{aligned}
EX &= \int_0^{\infty} x \cdot \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} \exp\left\{-\left(\frac{x}{\lambda}\right)^k\right\} dx \\
&= \lambda \int_0^{\infty} \left[\left(\frac{x}{\lambda}\right)^k\right]^{\frac{1}{k}} \exp\left\{-\left(\frac{x}{\lambda}\right)^k\right\} d\left(\frac{x}{\lambda}\right)^k \\
&= \lambda\Gamma\left(1+\frac{1}{k}\right)
\end{aligned}
$$

7. (1) 记 $X_i = \begin{cases} 1, \text{第 i 个盒子为空} \\ 0, \text{第 i 个盒非空.} \end{cases}$ ，$i = 1, ..., n$

则空盒子总数为 $Y = \sum_{i=1}^{n} X_i$，因为 $E(X_i) = P(X_i = 1) = (1 - \frac{1}{n})^n$，所以

$$
E(Y) = \sum_{i=1}^{n} E(X_i) = n\big(1 - 1/n\big)^n.
$$

(2) $n \to \infty$ 时, 空盒的平均比例为

$$\lim_{n\to\infty} \frac{n\,(1-1/n)^n}{n} = \lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n = \lim_{n\to\infty} \left(1-\frac{1}{n}\right)^{-n\cdot(-1)} = e^{-1}$$

8. $X = n, n+1, \ldots$，记 Y_j 为抽到 i-1 种卡后，抽到新卡所需的次数，则 $X_n = \sum\limits_{j=1}^{n} Y_j$,

$$P(Y_j = k) = \frac{n-j+1}{n} \cdot (\frac{j-1}{n})^{k-1},$$

$$EY_j = \frac{n-j+1}{n} \sum_{k=1}^{\infty} k \left(\frac{j-1}{n}\right)^{k-1} = \frac{n-j+1}{n} \cdot \frac{n^2}{(n-j+1)^2} = \frac{n}{n-j+1},$$

所以

$$EX_n = E\sum_{j=1}^{n} Y_j = \sum_{j=1}^{n} EY_j = \sum_{j=1}^{n} \frac{n}{n-j+1} = \sum_{k=1}^{n} \frac{n}{k}.$$

(1)$n = 12$时,

$$EX_n = \sum_{k=1}^{n} \frac{n}{k} = 12 \sum_{k=1}^{12} \frac{1}{k} \approx 37.24.$$

(2)

$$\lim_{n\to\infty} \frac{EX_n}{n \ln n} = \lim_{n\to\infty} \frac{n \sum\limits_{k=1}^{n} \frac{1}{k}}{n \ln n} = \lim_{n\to\infty} \frac{\sum\limits_{k=1}^{n} \frac{1}{k}}{\ln n} = 1$$

15. $X \sim U(-\pi/2, \pi/2)$, 则 $f(x) = \begin{cases} 1/\pi, & -\pi/2 < x < \pi/2, \\ 0, & \text{其他}. \end{cases}$.

由 $\sin x$ 为奇函数, $\cos x$ 为偶函数，($x \cos x$ 为奇函数, 得：

$$E(\sin X) = \int_{-\pi/2}^{\pi/2} \frac{\sin x}{\pi} dx = 0;$$

$$E(\cos X) = \int_{-\pi/2}^{\pi/2} \frac{\cos x}{\pi} dx = 2\int_{0}^{\pi/2} \frac{\cos x}{\pi} dx = \frac{2}{\pi};$$

$$E(X\cos X) = \int_{-\pi/2}^{\pi/2} \frac{x \cos x}{\pi} dx = 0.$$