11-28 作业

30. 因为

$$\int f(x;a,b)dx = \frac{\sqrt{2\pi}}{\sqrt{2}b} c \int \frac{1}{\sqrt{2\pi}} e^{-\frac{(x+a/b)^2}{1/b^2}} dx = \frac{c\sqrt{\pi}}{b} = 1$$

所以 $c = \frac{b}{\sqrt{\pi}}$, 即 $X \sim N(-a/b, 1/2b^2)$。似然函数及对数似然为

$$L(a,b) = \prod_{i=1}^{n} f(x_i;a,b) = \left(\frac{b}{\sqrt{\pi}}\right)^n \exp\left\{-\sum_{i=1}^{n}(a+bx_i)^2\right\},$$

$$\ln L(a,b) = -\frac{n}{2}\ln\pi + n\ln b - \sum_{i=1}^{n}(a+bx_i)^2$$

对数似然方程组为

$$\frac{\partial \ln L(a,b)}{\partial a} = -2\sum_{i=1}^{n}(a+bx_i) = 0$$

$$\frac{\partial \ln L(a,b)}{\partial b} = \frac{n}{b} - 2\sum_{i=1}^{n} x_i(a+bx_i) = 0$$

解得

$$a = -b\overline{x}$$

$$b^2 = \frac{n}{2[\sum_{i=1}^{n} x_i^2 - n(\overline{x})^2]}$$

所以 $\hat{b} = \sqrt{\frac{n}{2[\sum_{i=1}^{n} X_i^2 - n(\overline{X})^2]}}$, $\hat{a} = -\hat{b}\overline{x} = -\bar{x}\sqrt{\frac{n}{2[\sum_{i=1}^{n} X_i^2 - n(\overline{X})^2]}}$.

32. $X \sim Exp(\lambda)$, $E(X) = 1/\lambda$, 所以 λ 的矩估计为 $\lambda_M = 1/\bar{X}$.

$$L(\lambda) = \prod_{i=1}^{n} f(x_i) = \lambda^n e^{-\sum\limits_{i=1}^{n}\lambda x_i}, \quad x_i > 0,$$

令

$$\frac{\partial lnL(\lambda)}{\partial\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i = 0,$$

λ 的极大似然估计为 $\hat{\lambda}_L = 1/\bar{X}$，

$$P(\lambda < X \le 2\lambda) = \int_{\lambda}^{2\lambda} \lambda e^{-\lambda x} dx = e^{-\lambda^2} - e^{-2\lambda^2}$$

所以

$$\hat{P}_M = \hat{P}_L = e^{-1/(\overline{X})^2} - e^{-2/(\overline{X})^2}.$$

39. (1) X 的一阶矩及二阶矩为

$$EX = \int_0^\infty \frac{2x^2}{\theta} e^{-\frac{x^2}{\theta}} dx = \int_0^\infty \frac{\sqrt{\theta}}{2\sqrt{t}} \frac{\theta t}{\theta} e^{-t} dt = \frac{\sqrt{\theta}}{2} \int_0^\infty t^{\frac{1}{2}} e^{-t} dt = \frac{\sqrt{\theta}}{2} \Gamma(\frac{3}{2}) = \frac{\sqrt{\theta\pi}}{2}$$

$$EX^2 = \int_0^\infty \frac{2x^3}{\theta} e^{-\frac{x^2}{\theta}} dx = \int_0^\infty \frac{\sqrt{\theta}}{2\sqrt{t}} \frac{2\theta^{\frac{3}{2}} t^{\frac{3}{2}}}{\theta} e^{-t} dt = \theta \int_0^\infty t e^{-t} dt = \theta\Gamma(2) = \theta.$$

(2) 似然函数为

$$L(\theta) = \prod_{i=1}^n f(x_i) = \frac{2^n \prod\limits_{i=1}^n x_i}{\theta^n} e^{-\frac{\sum\limits_{i=1}^n x_i}{\theta}}, \quad x_i \geq 0$$

$$\ln L(\theta) = n \ln 2 + \sum_{i=1}^n \ln x_i - n \ln \theta - \frac{1}{\theta} \sum_{i=1}^n x_i^2, \quad x_i \geq 0,$$

令

$$\frac{\partial \ln L(\theta)}{\partial \theta} = -\frac{n}{\theta} + \frac{1}{\theta} \sum_{i=1}^n x_i^2 = 0$$

所以

$$\hat{\theta} = \sum_{i=1}^n X_i^2 / n$$

(3) 求 X^2 的方差：

$$EX^4 = \int_0^\infty \frac{2x^5}{\theta} e^{-\frac{x^2}{\theta}} dx = \int_0^\infty \frac{\sqrt{\theta}}{2\sqrt{t}} \frac{2(\theta t)^{\frac{5}{2}}}{\theta} e^{-t} dt = \theta^2 \int_0^\infty t^2 e^{-t} dt = \theta^2 \Gamma(3) = 4\theta^2$$

所以

$$\mathrm{Var}(X^2) = EX^4 - (EX^2)^2 = 3\theta^2$$

由切比雪夫不等式可知，对于任意的 $\epsilon > 0$，有

$$P(|\hat{\theta} - \theta| \geq \epsilon) \leq \frac{\mathrm{Var}(X^2)}{n\epsilon^3} = \frac{3\theta^2}{n\epsilon^3},$$

当 $n \to \infty$ 时，$P(|\hat{\theta} - \theta| \geq \epsilon) \to 0$，所以存在 $a = \theta$，使得 $\hat{\theta}$ 依概率收敛到 a。

(或直接利用大数定律，令 $Y := X^2, Y_1, Y_2, \cdots, Y_n$ $i.i.d.$, 则 $\overline{Y} \xrightarrow{P} E(Y)$, 即 $\hat{\theta} \xrightarrow{P} \theta$.)

41. (1) 似然函数为

$$L(\theta) = \prod_{i=1}^n f(x_i) = \frac{1}{(c-1)^n \theta^n}, \quad \theta < x_{(1)} \leq ... \leq x_{(n)} < c\theta$$

显然，θ 越大，$L(\theta)$ 越小，所以

$$\hat{\theta}_L = X_{(n)}/c.$$

(2) 因为 $EX = \frac{(c+1)\theta}{2}$，所以

$$\hat{\theta}_M = \frac{2\bar{X}}{c+1}.$$

又因为 $E\frac{2\bar{X}}{(c+1)} = \frac{2}{(c+1)}EX = \theta$，所以矩估计是一个无偏估计。

44. (1)X 的期望

$$EX = \int_\theta^\infty \frac{x}{\sigma}e^{-\frac{x-\theta}{\sigma}}dx = \int_0^\infty \frac{t+\theta}{\sigma}e^{-\frac{t}{\sigma}}dt = \sigma+\theta,$$

所以矩估计

$$\hat{\theta}_1 = \bar{X} - \sigma.$$

似然函数为

$$L(\theta) = \prod_{i=1}^n f(x_i) = \frac{1}{\sigma^n}e^{-\frac{\sum\limits_{i=1}^n (x_i-\theta)}{\sigma}}, \quad \theta < x_{(1)} \leq ... \leq x_{(n)}.$$

显然，当 θ 增加时，$L(\theta)$ 也随之增加。所以最大似然估计

$$\hat{\theta}_2 = X_{(1)}.$$

(2)$E\hat{\theta}_1 = E\bar{X} - \sigma = EX - \sigma = \theta$，所以 $\hat{\theta}_1$ 是一个无偏估计。

令 $Y = X_{(1)}$，所以

$$P(Y \leq y) = 1 - P(X_1 > y, ..., X_n > y) = 1 - \left(\int_y^\infty f(x)dx\right)^n = 1 - e^{-\frac{n(y-\theta)}{\sigma}}, \quad y > \theta$$

$$EY = \int_\theta^\infty ydF(y) = \int_\theta^\infty \frac{ny}{\sigma}e^{-\frac{n(y-\theta)}{\sigma}}dy = \int_0^\infty (\theta + \frac{t}{n})\frac{1}{\sigma}e^{-\frac{t}{\sigma}}dt = \theta + \frac{\sigma}{n}$$

所以 $\hat{\theta}_2$ 不是无偏估计，可修正为 $\widetilde{\theta}_2 = X_{(1)} - \frac{\sigma}{n}$。

(3) 修正之后两估计量都是无偏估计，比较两者方差即可。

$$EX^2 = \int_\theta^\infty \frac{x^2}{\sigma}e^{-\frac{x-\theta}{\sigma}}dx = \int_0^\theta \frac{(t+\theta)^2}{\sigma}e^{-\frac{t}{\sigma}}dt = 2\sigma^2 + 2\theta\sigma + \theta^2,$$

$$\mathrm{Var}(\hat{\theta}_1) = \frac{1}{n}\mathrm{Var}(X) = \frac{1}{n}\left(EX^2 - (EX)^2\right) = \frac{\sigma^2}{n}.$$

又有

$$EY^2 = \int_\theta^\infty \frac{ny^2}{\sigma}e^{-\frac{n(y-\theta)}{\sigma}}dy = \int_0^\infty (\theta + \frac{t}{n})^2\frac{1}{\sigma}e^{-\frac{t}{\sigma}}dt = \theta^2 + \frac{2\theta\sigma}{n} + \frac{2\sigma^2}{n^2}$$

$$\mathrm{Var}(\widetilde{\theta}_2) = \mathrm{Var}(Y) = EY^2 - (EY)^2 = \frac{\sigma^2}{n^2} < \frac{\sigma^2}{n}$$

所以 $\widetilde{\theta}_2$ 更优。

52. (1) 由题意知, 似然函数为

$$L(\mu)=\prod_{i=1}^{n} e^{-(x_i-\mu)} I\left(x_i \geq \mu\right)=e^{-\sum_{i=1}^{n}(x_i-\mu)} I\left(x_{(1)} \geq \mu\right),$$

要使 $L(\mu)$ 达到最大, 首先示性函数取值应为 1 , 其次 $e^{-\sum_{i=1}^{n}(x_i-\mu)}$ 尽可能大, 所以 μ 取值应尽可能大, 但示性函数为 1 确定了 $\mu \leq x_{(1)}$, 由此 μ 的极大似然估计 $\hat{\mu}^*=X_{(1)}$. 由最小值的分布结论可知, $X_{(1)}$ 的密度函数为

$$f_1(x)=\begin{cases} n(1-F(x))^{n-1} f(x)=n e^{-n(x-\mu)}, & x \geq \mu, \\ 0, & \text{其他}. \end{cases}$$

$$E\left(X_{(1)}\right)=\int_{\mu}^{\infty} x \cdot n e^{-n(x-\mu)} dx=\int_{0}^{\infty}(y+\mu) \cdot n e^{-n y} dy=\mu+\frac{1}{n}$$

所以 $\hat{\mu}^*=X_{(1)}$ 不是 μ 的无偏估计. 修正之后的无偏估计 $\hat{\mu}^{**}=X_{(1)}-\frac{1}{n}$.
(2) X 的期望为

$$E(X)=\int_{\mu}^{\infty} x \cdot e^{-(x-\mu)} dx=\int_{0}^{\infty}(y+\mu) \cdot e^{-y} dy=\mu+1.$$

记 $\bar{X}=\frac{1}{n} \sum_{i=1}^{n} X_i$, 所以 μ 的矩估计 $\hat{\mu}=\bar{X}-1$,, 且

$$E(\hat{\mu})=E(\bar{X})-1=E(X)-1=\mu,$$

$\hat{\mu}$ 是 μ 的无偏估计.
(3) $\hat{\mu}^{**}$ 及 $\hat{\mu}$ 都是 μ 的无偏估计, 比较两者方差

$$\begin{aligned} \operatorname{Var}\left(\hat{\mu}^{**}\right) &= \operatorname{Var}\left(X_{(1)}-\frac{1}{n}\right)=\operatorname{Var}\left(X_{(1)}\right) \\ &= \int_{\mu}^{\infty} x^2 \cdot n e^{-n(x-\mu)} dx-\left(\mu+\frac{1}{n}\right)^2=\frac{1}{n^2} \\ \operatorname{Var}(\hat{\mu}) &= \operatorname{Var}(\bar{X}-1)=\operatorname{Var}(\bar{X})=\frac{1}{n} \operatorname{Var}(X) \\ &= \frac{1}{n}\left[\int_{\mu}^{\infty} x^2 \cdot e^{-(x-\mu)} dx-(\mu+1)^2\right]=\frac{1}{n} \end{aligned}$$

所以 $\hat{\mu}^{**}$ 更有效.

注：不难发现 $(X-\mu) \sim Exp(1)$, $(X_{(1)}-\mu) \sim Exp(n)$. 由指数分布的期望方差结论，可直接得到 $E(X)=\mu+1, \operatorname{Var}(X)=1, E(X_{(1)})=\mu+\frac{1}{n}, \operatorname{Var}(X_{(1)})=\frac{1}{n^2}$. 此方法更快.