10-31 作业

14. 令 $Y=\ln X$, 所以对 $\forall x>0$,

$$P(X\le x)=P(\ln X\le \ln x)=P(Y\le \ln x)=\Phi\left(\frac{\ln x-\mu}{\sigma}\right),$$

所以

$$p(x)=\begin{cases}\frac{1}{\sqrt{2\pi}\sigma x}e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}, & x>0,\\ 0, & \text{其他}.\end{cases}$$

$$\begin{aligned}EX&=\int_0^\infty x\frac{1}{\sqrt{2\pi}\sigma x}e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}dx=\int e^{t\sigma+\mu}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt\\&=e^{\frac{\mu^2}{2}+\mu}\int\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2-2t\sigma+\sigma^2}{2}}dt=e^{\frac{\mu^2}{2}+\mu}\\EX^2&=\int_0^\infty x^2\frac{1}{\sqrt{2\pi}\sigma x}e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}dx=\int_{-\infty}^\infty e^{2(t\sigma)+\mu}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt\\&=e^{2\mu+2\sigma^2}\int_{-\infty}^\infty\frac{1}{\sqrt{2\pi}}e^{-\frac{(t-2\sigma)^2}{2}}dt=e^{2\mu+2\sigma^2}\end{aligned}$$

$$\Rightarrow\quad Var(X)=EX^2-(EX)^2=e^{\sigma^2+2\mu}(e^{\sigma^2}-1)$$

17. 已知 X 的密度函数, 有

$$\begin{aligned}&E[\min\{|X|,1\}]=\int_{|x|>1}f(x)dx+\int_{|x|\le 1}|x|\cdot f(x)dx\\&=2\int_1^\infty\frac{1}{\pi(1+x^2)}dx+2\int_0^1\frac{x}{\pi(1+x^2)}dx(\text{ 对称性})\\&=\left.\frac{2\arctan x}{\pi}\right|_1^\infty+\int_0^1\frac{d(1+x^2)}{\pi(1+x^2)}=\frac{1}{2}+\left.\frac{1}{\pi}\ln\left(1+x^2\right)\right|_0^1\\&=\frac{1}{2}+\frac{\ln 2}{\pi}\end{aligned}$$

19. $(X,Y)\sim N(1,1,0.5,0.5,0.5)$, 则 (X,Y) 的联合密度为

$$f(x,y)=\frac{2}{\sqrt{3}\pi}\exp\left\{-\frac{2}{3}\left[2(x-1)^2-2(x-1)(y-1)+2(y-1)^2\right]\right\}.$$

(1) $Z=|X-Y|$, 可以先求 $X-Y$ 的密度。令 $\left\{\begin{array}{l}S=X-Y\\T=Y\end{array}\right.$ 则 $\left\{\begin{array}{l}X=S+T\\Y=T\end{array}\right.$,

对应的雅可比行列式为

$$J = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1.$$

所以 (S,T) 的联合密度为

$$\begin{aligned} f_{ST}(s,t) &= f(s+t,t) \cdot |J| \\ &= \frac{2}{\sqrt{3}\pi} \exp\left\{-\frac{2}{3}\left[2(s+t-1)^2 - 2(s+t-1)(t-1) + 2(t-1)^2\right]\right\} \\ &= \frac{2}{\sqrt{3}\pi} \exp\left\{-\frac{4}{3}s^2 - \frac{4}{3}s(t-1) - \frac{4}{3}(t-1)^2\right\}, -\infty < s,t < \infty. \end{aligned}$$

则 U 的边际密度为

$$\begin{aligned} f_S(s) &= \int_{-\infty}^{\infty} \frac{2}{\sqrt{3}\pi} \exp\left\{-\frac{4}{3}s^2 - \frac{4}{3}s(t-1) - \frac{4}{3}(t-1)^2\right\} dt \\ &= \int_{-\infty}^{\infty} \frac{2}{\sqrt{3}\pi} \exp\left\{-\frac{4}{3}\left[\left(t + \frac{s-2}{2}\right)^2 + + \frac{3}{4}s^2\right]\right\} dt \\ &= \frac{1}{\sqrt{\pi}} e^{-s^2} \int_{-\infty}^{\infty} \frac{2}{\sqrt{3\pi}} \exp\left\{-\frac{\left(t+\frac{s-2}{2}\right)^2}{3/4}\right\} dt \\ &= \frac{1}{\sqrt{\pi}} e^{-s^2}, \quad -\infty < s < \infty \end{aligned}$$

对于 $Z = |S|, P(Z \le z) = P(-z \le S \le z) = 2\int_0^z \frac{1}{\sqrt{\pi}} e^{-s^2} ds, (z > 0)$, 可得 Z 的密度函数为

$$f_Z(z) = \frac{2}{\sqrt{\pi}} e^{-z^2}, z > 0.$$

期望为

$$E(Z) = \int_0^{\infty} z \cdot \frac{2}{\sqrt{\pi}} e^{-z^2} dz = -\frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-z^2} d\left(-z^2\right) = \frac{1}{\sqrt{\pi}}.$$

(2) $U = \max(X,Y), V = \min(X,Y)$, 则有 $\begin{cases} U + V = X + Y \\ U - V = |X - Y| \end{cases}$ 由期望的性质有:

$$\begin{cases} E(U+V) = E(X+Y) \\ E(U-V) = E|X-Y| \end{cases} \Rightarrow \begin{cases} E(U) + E(V) = E(X) + E(Y) = 2 \\ E(U) - E(V) = EZ = \frac{1}{\sqrt{\pi}} \end{cases}$$

有

$$\begin{cases} E(U) = 1 + \frac{1}{2\sqrt{\pi}} \\ E(V) = 1 - \frac{1}{2\sqrt{\pi}} \end{cases}$$

20. 由题意易知, 对于一球, 在末落入第 1 个盒子的条件下, 落入第 2 个盒子的概率为 $\frac{p_2}{1-p_1}$. 由此可得, 给定 X_1 时, $X_2 \mid X_1 = k \sim B\left(m-k, \frac{p_2}{1-p_1}\right)$, 所以

$$E\left(X_2 \mid X_1 = k\right) = (m-k)\frac{p_2}{1-p_1},$$
$$\operatorname{Var}\left(X_2 \mid X_1 = k\right) = (m-k)\frac{p_2}{1-p_1}\left(1-\frac{p_2}{1-p_1}\right)$$

(2) 由题意易知: $X_k \sim b\left(m, p_k\right)$, 所以

$$E\left(X_1+X_2\right) = EX_1 + EX_2 = m\left(p_1+p_2\right).$$

且可将 $X_1+\ldots+X_k$ 看做落入前 k 个盒子的总球数, 每个小球独立地以概率 p_k 落入第 k 个盒子, 则 $X_1+\ldots+X_k \sim b\left(m, p_1+\ldots+p_k\right), k=1,\ldots,n$, 所以

$$\operatorname{Var}\left(X_1+\ldots+X_k\right) = m\left(\sum_{i=1}^{k} p_i\right)\left(1-\sum_{i=1}^{k} p_i\right)$$

27. (1)$X_1 \sim b(n,p)$，所以

$$M_{X_1}(t) = E(e^{tX_1}) = \sum_{k=1}^{n} e^{tk}\binom{n}{k}p^k(1-p)^{n-k} = \sum_{k=1}^{n}\binom{n}{k}(pe^t)^k(1-p)^{n-k} = (pe^t+1-p)^n$$

(2)$X_2 \sim P(\lambda)$，所以

$$M_{X_2}(t) = E(e^{tX_2}) = \sum_{k=0}^{\infty} e^{tk}\frac{\lambda^k}{k!}e^{-\lambda} = \sum_{k=0}^{\infty}\frac{(\lambda e^t)^k}{k!}e^{-\lambda} = e^{-\lambda}e^{\lambda e^t} = e^{\lambda(e^t-1)}$$

(3)$X_3 \sim EXP(\lambda)$，所以

$$M_{X_3}(t) = E(e^{tX_3}) = \int_0^{\infty} e^{tx_3}\lambda e^{-\lambda x_3}dx_3 = \int_0^{\infty}\lambda e^{-(\lambda-t)x_3}dx_3 = \frac{\lambda}{\lambda-t}(t<\lambda).$$

(4)$X_4 \sim N(\mu,\sigma^2)$，所以

$$\begin{aligned} M_{X_4}(t) = E(e^{tX_4}) &= \int e^{tx_4}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x_4-\mu)^2}{2\sigma^2}}dx_4 = \int e^{ty\sigma+t\mu}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{y^2}{2}}\sigma dy \\ &= e^{\frac{t^2\sigma^2}{2}+t\mu}\int\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2-2ty\sigma+t^2\sigma^2}{2}}dy \\ &= e^{\frac{t^2\sigma^2}{2}+t\mu} \end{aligned}$$