11-21 作业

1. 总体 X 的均值为

$$EX = 2\theta(1-\theta) + 2\theta^2 + 3(1-2\theta) = 3 - 4\theta$$
$$\Rightarrow \quad \theta = \frac{3 - EX}{4}$$

样本均值 $\bar{X} = \frac{53\times1+16\times2+21\times3}{100} = 1.48$, 所以 θ 的矩估计为 $\widehat{\theta} = \frac{3-\bar{X}}{4} = 0.38$.

3. (3) X 服从负二项分布 $Nb(2,\theta)$, 期望为 $2/\theta$ (可直接利用结论), 具体计算如下:

$$\begin{aligned}
EX &= \sum_{x=2}^{\infty} x(x-1)\theta^2(1-\theta)^{x-2} = \theta^2\left(\sum_{x=2}^{\infty} x(x-1)(1-\theta)^{x-2}\right) \\
&= \theta^2\left(\sum_{x=2}^{\infty} x(1-\theta)^{x-1}\right)' = \theta^2\left(\sum_{x=2}^{\infty}(1-\theta)^{x}\right)'' = \theta^2\left(\frac{(1-\theta)^2}{1-(1-\theta)}\right)'' \\
&= \theta^2\left(\frac{1}{\theta}\right)'' = \theta^2\left(\frac{2}{\theta^3}\right) = \frac{2}{\theta} \\
\Rightarrow \theta &= \frac{2}{EX}
\end{aligned}$$

所以, θ 的矩估计为 $\widehat{\theta} = 2/\bar{X}$.

(4) 求 X 的期望及二阶矩 (仅由期望无法解出):

$$\begin{aligned}
E(X) &= \sum_{x=1}^{\infty} x \cdot \left[\frac{-\theta^x}{(x\ln(1-\theta))}\right] = -\frac{1}{\ln(1-\theta)}\sum_{x=1}^{\infty}\theta^x = -\frac{\theta}{(1-\theta)\ln(1-\theta)} \\
E(X^2) &= \sum_{x=1}^{\infty} x^2 \cdot \left[\frac{-\theta^x}{(x\ln(1-\theta))}\right] = -\frac{1}{\ln(1-\theta)}\sum_{x=1}^{\infty}x\theta^x = -\frac{\theta}{(1-\theta)^2\ln(1-\theta)} \\
\Rightarrow \quad \theta &= 1 - E(X)/E(X^2)
\end{aligned}$$

所以,θ 的矩估计为 $1 - \bar{X}/a_2$, 其中 $a_2 = \sum_{i=1}^{n} X_i^2/n$ 为二阶原点矩。

4. (2) X 的期望为

$$EX = \int_0^1 x(\theta+1)x^\theta dx = \int_0^1 (\theta+1)x^{\theta+1} = \frac{\theta+1}{\theta+2}x^{\theta+2}\Big|_0^1 = 1 - \frac{1}{\theta+1}$$
$$\Rightarrow \theta = \frac{1}{1-EX} - 2$$

所以, θ 的矩估计为 $\hat{\theta} = \frac{1}{1-\bar{X}} - 2 = \frac{2\bar{X}-1}{1-\bar{X}}$.

(4) X 的期望为

$$EX = \int_c^\infty x\theta c^\theta x^{-(\theta+1)} dx = \int_c^\infty \theta c^\theta x^{-\theta} dx = \frac{\theta c^\theta}{-\theta+1} x^{-\theta+1}\Big|_c^\infty = \frac{\theta c}{\theta - 1}$$

$$\Rightarrow \theta = \frac{EX}{EX - c}$$

所以, θ 的矩估计为 $\hat{\theta} = \frac{\bar{X}}{\bar{X}-c}$.

(6) X 的期望为

$$EX = \int_0^\infty x \frac{\theta^2}{x^3} e^{-\frac{\theta}{x}} dx = \int_0^\infty \frac{\theta^2}{x^2} e^{-\frac{\theta}{x}} = \theta e^{-\frac{\theta}{x}}\Big|_0^\infty = \theta$$

所以, θ 的矩估计为 $\hat{\theta} = \bar{X}$.

25. (3) $p(x;\theta) = (x-1)\theta^2(1-\theta)^{x-2}, x = 2, 3, \cdots; 0 < \theta < 1$. 则似然函数为

$$L(\theta) = \theta^{2n}(1-\theta)^{\sum_{i=1}^n x_i - 2n} \prod_{i=1}^n (x_i - 1)$$

对数似然函数, 并将其关于 θ 求导并令为 0 ,

$$\ln L(\theta) = 2n \ln\theta + \left(\sum_{i=1}^n x_i - 2n\right) \ln(1-\theta) + C$$

$$\frac{\partial \ln L(\theta)}{\partial\theta} = \frac{2n}{\theta} - \frac{\sum_{i=1}^n x_i - 2n}{1-\theta} = 0$$

$$\Rightarrow \theta = \frac{2n}{\sum_{i=1}^n x_i}$$

所以, θ 的极大似然估计为 $\widehat{\theta} = \frac{2n}{\sum_{i=1}^n X_i} = \frac{2}{\bar{X}}$.

(4)$p(x;\theta) = -\frac{1}{\ln(1-\theta)}\frac{\theta^x}{x}, x = 1, 2, \cdots$. 则似然函数为

$$L(\theta) = [-\ln(1-\theta)]^{-n} \theta^{\sum_{i=1}^n x_i} \prod_{i=1}^n x_i^{-1}.$$

对数似然函数, 并将其关于 θ 求导并令为 0 ,

$$\ln L(\theta) = -n\ln(-\ln(1-\theta)) + \ln\theta \sum_{i=1}^n x_i + C$$

$$\frac{\partial \ln L(\theta)}{\partial\theta} = \frac{\sum_{i=1}^n x_i}{\theta} + \frac{n}{(1-\theta)\ln(1-\theta)} = 0$$

$$\Rightarrow \frac{(1-\theta)}{\theta} \ln(1-\theta) = -\frac{n}{\sum_{i=1}^n x_i}$$

方程无法给出显式解, θ 的极大似然估计为上述方程的解。或表示为

$$\widehat{\theta} = \arg\max_{0<\theta<1} L(\theta) = \arg\max_{0<\theta<1} \ln L(\theta).$$

26. (2) $f(x,\theta)=\begin{cases}(\theta+1)x^{\theta}, & 0<x<1,\theta>0\\ 0 & \text{其他.}\end{cases}$ 似然函数为 $L(\theta)=(\theta+1)^n\prod_{i=1}^n x_i^{\theta}$.
对数似然函数为

$$\ln L(\theta)=n\ln(\theta+1)+\theta\sum_{i=1}^n \ln x_i$$

将其关于 θ 求导并令为 0 ,

$$\frac{\partial \ln L(\theta)}{\partial\theta}=\frac{n}{\theta+1}+\sum_{i=1}^n \ln x_i$$
$$\Rightarrow \theta=-\frac{n}{\sum_{i=1}^n \ln x_i}-1$$

所以, θ 的极大似然估计为 $\widehat{\theta}=-\frac{n}{\sum_{i=1}^n \ln X_i}-1$.

(4) $f(x,\theta)=\begin{cases}\theta c^{\theta}x^{-(\theta+1)}, & x>c,\theta>1\\ 0 & \text{其他.}\end{cases}$ 似然函数为 $L(\theta)=\theta^n c^{n\theta}x^{-n(\theta+1)}$.
对数似然函数为

$$\ln L(\theta)=n\ln\theta+n\theta\ln c-(\theta+1)\sum_{i=1}^n \ln x_i$$

将其关于 θ 求导并令为 0 ,

$$\frac{\partial \ln L(\theta)}{\partial\theta}=\frac{n}{\theta}+n\ln c-\sum_{i=1}^n \ln x_i$$
$$\Rightarrow \theta=\frac{n}{\sum_{i=1}^n \ln x_i-n\ln c}$$

所以, θ 的极大似然估计为 $\widehat{\theta}=\frac{n}{\sum_{i=1}^n \ln X_i-n\ln c}$.

(6) $p(x;\theta)=\begin{cases}\frac{\theta^2}{x^3}e^{-\theta/x}, & x>0,\theta>0\\ 0 & \text{其他.}\end{cases}$ 似然函数为 $L(\theta)=\theta^{2n}e^{-\theta\sum_{i=1}^n x_i^{-1}}\prod_{i=1}^n x_i^{-3}$.
对数似然函数为

$$\ln L(\theta)=2n\ln\theta-\theta\sum_{i=1}^n x_i^{-1}+C$$

将其关于 θ 求导并令为 0 ,

$$\frac{\partial \ln L(\theta)}{\partial\theta}=\frac{2n}{\theta}-\sum_{i=1}^n x_i^{-1}$$
$$\Rightarrow \theta=\frac{2n}{\sum_{i=1}^n x_i^{-1}}$$

所以, θ 的极大似然估计为 $\widehat{\theta}=\frac{2n}{\sum_{i=1}^n X_i^{-1}}$.