

$$1. \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_x^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\sigma_y^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\sigma_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$-i\sigma_x\sigma_y\sigma_z = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= -i \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = I$$

$$[\sigma_x, \sigma_y] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = 2i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2i\sigma_z$$

$$[\sigma_y, \sigma_z] = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = 2i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2i\sigma_x$$

$$[\sigma_z, \sigma_x] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 2i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = 2i\sigma_y$$

$$\{\sigma_x, \sigma_y\} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = 0$$

$$\{\sigma_y, \sigma_z\} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = 0$$

$$\{\sigma_z, \sigma_x\} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 0$$

$$2. (\vec{\sigma} \cdot \vec{a})(\vec{\sigma} \cdot \vec{b})$$

$$= (a_x \sigma_x + a_y \sigma_y + a_z \sigma_z)(b_x \sigma_x + b_y \sigma_y + b_z \sigma_z)$$

$$= a_x b_x \sigma_x^2 + a_x a_y \sigma_x \sigma_y + a_x a_z \sigma_x \sigma_z$$

$$+ a_y b_x \sigma_y \sigma_x + a_y b_y \sigma_y^2 + a_y b_z \sigma_y \sigma_z \\ + a_z b_x \sigma_z \sigma_x + a_z b_y \sigma_z \sigma_y + a_z b_z \sigma_z^2$$

根据 1, 有 $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I \stackrel{\text{or}}{=} 1$

$$[\sigma_x, \sigma_y] = 2i\sigma_z \quad \{\sigma_x, \sigma_y\} = 0$$

$$\Rightarrow \sigma_x \sigma_y = \frac{[\sigma_x, \sigma_y] + \{\sigma_x, \sigma_y\}}{2} = i\sigma_z$$

$$\sigma_y \sigma_z = \frac{[\sigma_y, \sigma_z] + \{\sigma_y, \sigma_z\}}{2} = i\sigma_x$$

$$\sigma_z \sigma_x = \frac{[\sigma_z, \sigma_x] + \{\sigma_z, \sigma_x\}}{2} = i\sigma_y$$

代入上式, 有:

$$(\vec{\sigma} \cdot \vec{a})(\vec{\sigma} \cdot \vec{b}) = a_x b_x + a_y b_y + a_z b_z$$

$$+ i\sigma_z(a_x b_y - a_y b_x)$$

$$+ i\sigma_y(a_z b_x - a_x b_z)$$

$$+ i\sigma_x(a_y b_z - a_z b_y)$$

$$= \vec{a} \cdot \vec{b} + i\vec{\sigma} \cdot (\vec{a} \times \vec{b})$$