Financial Engineering Project:

SPX Options Calibration - Affine Jump-diffusion Model

Author: Dantong Liu 刘丹彤

Email: dantongliu1999@outlook.com

LinkedIn: http://linkedin.com/in/dantong-liu-273271311

This project is based on the Affine Jump-diffusion Model (Bakshi, Cao & Chen(1997) and Bates(2006)) which is derived from Heston Model (Heston(1993)) and adding jumps in price process(SVJY model).

$$dX_t = \left(r - \bar{q} - \bar{\lambda}\mu_j - \frac{1}{2}V_t\right)\mathrm{dt} + \sqrt{V_t}\, dW_t^s + J_t\, dP_t$$

$$dV_t = \kappa_v(\bar{v} - V_t)\,\mathrm{dt} + \sigma_v\, dW_t^v$$

To find out more theoretical details, please see **Section 3.**

Summary:

- **Section 1**: Removed options with negative bid prices, bid-ask spreads less than $5, missing prices, deep out-of-the-money, and deep in-the-money options.
- **Section 2**: Estimated the SPX index dividend yield and spot rate (risk-free rate) using put-call parity. Calculated implied volatility from SPX option market prices.
- **Section 3**: Explained the stochastic differential equation for the Affine Jump Diffusion model.
- **Section 4**: Prepared the two-part characteristic function: one for the Heston process and one for the pure jump process.
- **Section 5**: Presented the Call option pricing formula (Lewis' approach using inverse Fourier transform) and the objective function for parameter calibration (Average Relative Pricing Error, ARPE).
- **Section 6**: Conducted two-step calibration based on the two-part characteristic function. First, calibrated the pure jump process parameters by assuming the volatility of the price process was the minimum implied volatility of the most out-of-the-money option using the Interior-point algorithm (local optimization). Second, calibrated the Heston process parameters using Sequential Quadratic Programming (SQP), Interior-point, and Particle Swarm Optimization (PSO) algorithms (global optimization). Achieved an ARPE of 1.16, consistent with Kokholm (2016). Calibrating the Heston process parameters was particularly challenging.
- **Section 7**: Calculated SPX option model prices using the calibrated parameters and compared implied volatility surfaces between model prices and market prices for all strike prices and maturities.
- **Section 8**: Used Monte Carlo simulation to illustrate the SPX index path and its stochastic variance path.
- **Functions.**

Reference:

- *Bates, D. S. (2006). Maximum likelihood estimation of latent affine processes. The Review of Financial Studies, 19(3), 909-965.*
- *Bakshi, G., Cao, C., & Chen, Z. (1997). Empirical performance of alternative option pricing models. The Journal of finance, 52(5), 2003-2049.*
- *Duffie, D., Pan, J., & Singleton, K. (2000). Transform analysis and asset pricing for affine jump‐diffusions. Econometrica, 68(6), 1343-1376.*
- *Heston, S. L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. The review of financial studies, 6(2), 327-343.*
- *Lewis, A. (2001) A Simple Option Formula for General Jump-Di7usion and other Exponential Lévy Processes, Envision Financial Systems and OptionCity.net, California.*
- *Kokholm, T. (2016). Pricing and hedging of derivatives in contagious markets. Journal of Banking & Finance, 66, 19-34.*
- *Schoutens, W., Simons, E., & Tistaert, J. (2003). A perfect calibration! Now what?. The best of Wilmott, 281.*


```
clear;  clc;

% load data
data = readtable('SPX.csv');

% variables
market_price = data.market_price;   % market price = 0.5 * (best_bid + best offer)
maturity = data.maturity;           % maturity =exdate - today(net working days),
T = data.tao;                       % tao = maturity/252
strike = data.strike;               % K

S0 = 3094.04;   % underlying SPX index closing price
```

1. Clean Data (remove data who unsatisfy below restrictions)

- bid price > 0
- moneyness 0.8 < K/S0 < 1.2
- bid-ask spread < $5
- no missing put and call prices

```
% clean data
BidAskSpread = data.('best_bid') - data.('best_offer');

condition1 = data.('best_bid') > 0;         % Bid price > 0
condition2 = BidAskSpread < 5;              % Bid-ask spread < $5
condition3 = ~isnan(market_price);          % No missing prices

% conditions
combinedConditions = condition1 & condition2 & condition3;
filteredData = data(combinedConditions, :);
```

```matlab
% split Call_data & Put_data
Call_data = filteredData(strcmp(filteredData.cp_flag, 'C'), ...
    {'market_price', 'strike', 'tao', 'maturity', 'cp_flag'});
Put_data = filteredData(strcmp(filteredData.cp_flag, 'P'), ...
    {'market_price', 'strike', 'tao', 'maturity', 'cp_flag'});

% 4 shortest maturities
short_maturity = [28, 48, 73, 93];

OTM4_Call = [];
OTM4_Put = [];

% 8 most out-of-money options
for i = 1:length(short_maturity)
    current_maturity = short_maturity(i);
    current_Call = Call_data(Call_data.maturity == current_maturity, :);
    current_Call.OTM = current_Call.strike - S0;

    if height(current_Call) >= 4
        [~, call_idx] = maxk(current_Call.OTM, 4);
    else
        call_idx = 1:height(current_Call);
    end
    OTM4_Call = [OTM4_Call; current_Call(call_idx, :)];

    current_Put = Put_data(Put_data.maturity == current_maturity, :);
    current_Put.OTM = current_Put.strike - S0;

    if height(current_Put) >= 4
        [~, Put_idx] = maxk(current_Put.OTM, 4);
    else
        Put_idx = 1:height(current_Put);
    end
    OTM4_Put = [OTM4_Put; current_Put(Put_idx, :)];
end

OTM4_Options = [OTM4_Call; OTM4_Put];
OTM4_Options = sortrows(OTM4_Options, {'maturity', 'strike'});
% save
writetable(OTM4_Options, 'OTM4_options.csv');

% merge
Call_data.Properties.VariableNames{'market_price'} = 'Call_market';
Call_data = removevars(Call_data, {'cp_flag'});
Put_data.Properties.VariableNames{'market_price'} = 'Put_market';
Put_data = removevars(Put_data, {'cp_flag'});
new_data = outerjoin(Call_data, Put_data, 'Keys', {'tao', ...
    'maturity', 'strike'}, 'MergeKeys', true);

new_data.moneyness = new_data.strike ./ S0;
```

```
% moneyness 0.8 < K/S0 < 1.2
condition4 = new_data.moneyness > 0.6 & new_data.moneyness < 1.4;
new_data = new_data(condition4, :);

% drop NAN
new_data = rmmissing(new_data);

% save csv
writetable(new_data, 'new_data.csv');
```

```
clear;  clc;

% load data
data = readtable('new_data.csv');
disp(data(1:5, :));
```

```
    Call_market    strike      tao      maturity    Put_market    moneyness
    ___________    ______    _______    ________    __________    _________

      1041.2        2050     0.11111       28          0.075       0.66256
      1016.3        2075     0.11111       28          0.075       0.67064
       981.4        2110     0.11111       28          0.075       0.68196
       971.4        2120     0.11111       28          0.075       0.68519
      966.45        2125     0.11111       28            0.1        0.6868
```

2. Dividend Yield q and Spot Rates r_{τ_i}

Minimize the objective function:

$$\sum_{i=1}^{m}\sum_{j=1}^{2n}\left(P\left(\tau_i, K_j^i\right) - P^{\mathrm{PC}}\left(C\left(\tau_i, K_j^i\right), q, r_{\tau_i}\right)\right)^2$$

where

$$P^{\mathrm{PC}}\left(C\left(\tau_i, K_j^i\right), q, r_{\tau_i}\right) = C\left(\tau_i, K_j^i\right) + K_j^i e^{-r_{\tau_i}\tau_i} - S_0\, e^{-q\tau_i}$$

n denote the number of option pairs on each side of ATM (e.g. n = 5), and PC denote put-call parity.

```
n = 5;          % ATM option pairs
S0 = 3094.04;   % underlying SPX index closing price

% each maturity
unique_maturities = unique(data.tao);      % noted annulized tao
selected_data = table();

for i = 1:length(unique_maturities)
    T = unique_maturities(i);
    current_data = data(data.tao == T, :);

    % collect ATM options
    calls = current_data(current_data.strike > S0, :);
```

```
    if ~isempty(calls)
        calls_sorted = sortrows(calls, 'strike', 'ascend');
        selected_data = [selected_data; calls_sorted(1:min(n, height(calls_sorted)), :)];
    end

    puts = current_data(current_data.strike < S0, :);
    if ~isempty(puts)
        puts_sorted = sortrows(puts, 'strike', 'descend');
        selected_data = [selected_data; puts_sorted(1:min(n, height(puts_sorted)), :)];
    end
end

% for each maturity
unique_selected_maturities = unique(selected_data.tao);
M = length(unique_selected_maturities);

% initial guess
% [q, r1, r2, ..., rM]
initial_q = 0.02;                % dividend yield q
initial_r = 0.01 * ones(M, 1);   % spot rate r
initial_guess = [initial_q; initial_r];

% optimzation - no contraints
options = optimoptions('fminunc', 'Display', 'iter', 'Algorithm', ...
    'quasi-newton', 'MaxFunctionEvaluations', 1e4, 'MaxIterations', 1e4);
[estimated_params, fval] = fminunc(@(params) objective_function_q_r(params, ...
    selected_data, unique_selected_maturities, S0), initial_guess, options);
```

```
                                                        First-order 
 Iteration  Func-count       f(x)        Step-size       optimality
     0          14          111547                        2.24e+07
     1          70         2632.88     3.58243e-10        1.23e+06
     2          84         1980.64               1        9.88e+05
     3          98         583.813               1        3.67e+05
     4         112         521.841               1         3.2e+05
     5         126         307.802               1        1.03e+05
     6         140         295.506               1        9.26e+04
     7         154         205.859               1         1.1e+05
     8         168         169.355               1        8.11e+04
     9         182         150.724               1        7.05e+04
    10         196         143.405               1        6.41e+04
    11         210         130.279               1        7.04e+04
    12         224         112.671               1        8.44e+04
    13         238         96.6925               1        5.78e+04
    14         252         90.1393               1        3.53e+04
    15         266          88.018               1        3.39e+04
    16         280         85.9467               1        3.13e+04
    17         294         81.5976               1        3.99e+04
    18         308         75.2075               1        4.66e+04
    19         322         69.3247               1        3.25e+04
                                                        First-order 
 Iteration  Func-count       f(x)        Step-size       optimality
    20         336         66.6285               1        2.49e+04
    21         350         65.5747               1        2.35e+04
    22         364         64.4553               1        1.91e+04
    23         378         62.4179               1        2.45e+04
```

```
    24         392         59.9025              1        2.46e+04
    25         406         58.0769              1        1.54e+04
    26         420         57.2478              1        1.75e+04
    27         434         56.6244              1        1.61e+04
    28         448         55.4229              1        1.87e+04
    29         462         53.1134              1        2.84e+04
    30         476         49.7963              1        3.05e+04
    31         490         47.1906              1        1.91e+04
    32         504         46.2387              1        1.25e+04
    33         518         45.9658              1        1.21e+04
    34         532         45.6912              1        1.05e+04
    35         546         45.1524              1        1.18e+04
    36         560          44.467              1        1.18e+04
    37         574          43.948              1        7.53e+03
    38         588         43.7207              1        9.09e+03
    39         602         43.5704              1        9.04e+03
                                                        First-order
 Iteration  Func-count       f(x)        Step-size       optimality
    40         616         43.2829              1         8.3e+03
    41         630         42.6607              1        1.29e+04
    42         644         41.5306              1        1.67e+04
    43         658         40.2143              1        1.46e+04
    44         672         39.4792              1        6.97e+03
    45         686         39.3145              1        5.94e+03
    46         700         39.2831              1         5.7e+03
    47         714         39.2432              1        5.39e+03
    48         728         39.1348              1        4.72e+03
    49         742         38.8943              1        6.54e+03
    50         756         38.4226              1        8.44e+03
    51         770         37.8071              1        7.99e+03
    52         784         37.3931              1        4.55e+03
    53         798         37.2787              1        3.75e+03
    54         812         37.2573              1        3.67e+03
    55         826         37.2369              1        3.57e+03
    56         840         37.1767              1        3.35e+03
    57         854         37.0334              1        4.03e+03
    58         868         36.6781              1        5.91e+03
    59         882         35.9244              1        7.93e+03
                                                        First-order
 Iteration  Func-count       f(x)        Step-size       optimality
    60         896         34.6953              1        8.55e+03
    61         910         33.5348              1        5.95e+03
    62         924         33.0639              1        2.91e+03
    63         938         32.9963              1        2.86e+03
    64         952         32.9893              1        2.79e+03
    65         966         32.9812              1        2.72e+03
    66         980         32.9556              1         2.6e+03
    67         994         32.8923              1         2.4e+03
    68        1008         32.7221              1         2.1e+03
    69        1022         32.2843              1        3.17e+03
    70        1036         31.1668              1         4.8e+03
    71        1050         28.4693              1        7.07e+03
    72        1064         22.6972              1        9.47e+03
    73        1078         13.3404              1        1.01e+04
    74        1092         4.49259              1        6.82e+03
    75        1106        0.960051              1         2.2e+03
    76        1120        0.498564              1             209
    77        1134        0.483133              1              19

Local minimum found.

Optimization completed because the size of the gradient is less than
the value of the optimality tolerance.
```

```
<stopping criteria details>
```

```
% results
q_estimated = estimated_params(1);
r_estimated = estimated_params(2:end);

r_table = table(unique_selected_maturities, r_estimated, 'VariableNames', ...
    {'tao', 'Spot_Rate'});
writetable(r_table, 'r_table.csv');

% print
fprintf('Estimated dividend yield (q): %.4f\n', q_estimated);
```

```
Estimated dividend yield (q): 0.0182
```

```
disp('Estimated spot rate (r) for each maturity:');
```

```
Estimated spot rate (r) for each maturity:
```

```
disp(r_table);
```

```
      tao       Spot_Rate
    _______    _________

    0.11111    0.015879
    0.19048    0.021671
    0.28968    0.020289
    0.36905    0.019598
    0.44841    0.020045
    0.62698    0.018984
    0.88492    0.018483
    0.96429    0.018821
     1.0635    0.018387
     1.1429    0.018069
     1.6587    0.017832
     2.1746    0.017673
```

```
% merge
mergeddata = join(data, r_table, 'Keys', 'tao');
writetable(mergeddata, 'merged_data.csv');

data = readtable('OTM4_options.csv');
data = join(data, r_table, 'Keys', 'tao');
writetable(data, 'OTM4_options.csv');
```

```
% calculate market price implied volatility
clear;  clc;
mergeddata = readtable('merged_data.csv');

C_market_price = mergeddata.Call_market;
P_market_price = mergeddata.Put_market;
K = mergeddata.strike;
T = mergeddata.tao;
r = mergeddata.Spot_Rate;
```

```
q_estimated = 0.0182;
S0 = 3094.04;

N = length(K);
IV_C_MK = zeros(N, 1);
IV_P_MK = zeros(N, 1);

for i = 1:N
    K_i = K(i);
    T_i = T(i);
    r_i = r(i) - q_estimated;
    C_market_price_i = C_market_price(i);
    IV_C_MK(i) = bsm_implied_vol_call(C_market_price_i, S0, K_i, r_i, T_i, q_estimated);
    P_market_price_i = P_market_price(i);
    IV_P_MK(i) = bsm_implied_vol_put(P_market_price_i, S0, K_i, r_i, T_i, q_estimated);
end
mergeddata.IV_C_MK = IV_C_MK;
mergeddata.IV_P_MK = IV_P_MK;
writetable(mergeddata, 'merged_data.csv');
```

```
clear;  clc;
data = readtable('OTM4_options.csv');

market_price = data.market_price;
K = data.strike;
T = data.tao;
r = data.Spot_Rate;
cp_flag = data.cp_flag;
q_estimated = 0.0182;
S0 = 3094.04;

N = length(K);
IV_MK = zeros(N, 1);

for i = 1:N
    K_i = K(i);
    T_i = T(i);
    r_i = r(i) - q_estimated;
    cp_flag_i = cp_flag{i};
    market_price_i = market_price(i);
    if cp_flag_i == 'C'
        IV_MK(i) = bsm_implied_vol_call(market_price_i, S0, K_i, r_i, T_i, q_estimated);
    else
        IV_MK(i) = bsm_implied_vol_put(market_price_i, S0, K_i, r_i, T_i, q_estimated);
    end
end

data.IV_MK = IV_MK;
writetable(data, 'OTM4_options.csv');
```

3. Affine Jump-diffusion Model: Stochastic Differential Equation (SDE)

Stochastic volatility with independent jumps in log price process (Bakshi, Cao & Chen(1997)). In more generalized circumstance, jumps also may occur in Variance process(like the SVJJ model in Duffie, Pan & Singleton(2000)), even affected by other securities (like the Contagion model in Kokholm(2016)). But today, I only consider jumps in log price process.

$$dX_t = \left(r - \overline{q} - \overline{\lambda}\mu_j - \frac{1}{2}V_t\right)\mathrm{dt} + \sqrt{V_t}\, dW_t^s + J_t\, dP_t$$

$$dV_t = \kappa_\upsilon(\overline{\upsilon} - V_t)\,\mathrm{dt} + \sigma_\upsilon\, dW_t^v$$

Notations

- $X_t \;=\; \ln S_t$ underlying SPX index at time t
- V_t Variance of SPX index at time t (stochastic volatility)
- r spot rate
- $\overline{q}$ constant dividend yield
- κ_υ speed of mean reversion
- $\overline{\upsilon}$ long-term average level of variance
- σ_υ volatility of variance
- W_t^s and W_t^v are correlated Standard Brownian Motion with coefficient $\overline{\rho}$
- dt an infinitesimal increment of time
- J_t normally distributed jump size with mean μ_j and variance σ_j^2
- dP_t Poisson process with intensity $\overline{\lambda}$

4. Characteristic Function: Exponential-affine Form (calibration method from Kokholm(2016))

Split the characteristic function into 2 parts: $\Psi(u,\tau) = \psi_1(u,\tau) * \psi_2(u,\tau)$ At time 0, $\tau = T - t = T$.

$$\begin{aligned}\psi^i(u,t,T) &= \mathbb{E}\left[e^{\mathrm{i}u\left(-\frac{1}{2}\int_0^T V_s^i ds + \int_0^T \sqrt{V_s^i} dW_s^i\right)} |\mathcal{F}_t\right]\mathbb{E}\left[e^{\mathrm{i}u\left(-\xi_i\int_0^T \lambda_s^i ds + J_T^i\right)} |\mathcal{F}_t\right] \\ &= \psi_1^i(u,t,T)\psi_2^i(u,t,T),\end{aligned} \tag{11}$$

4.1) stochastic volatility process (Heston process)

2.3.1. Characteristic function of the Heston process

For the first part, $\psi_1^i(u,t,T)$ is equal to the characteristic function in the Heston (1993) model written here in the form given in Schoutens et al. (2004) and Gatheral (2006)

$$\psi_1^i(u,t,T) = \exp\left\{C_i(u,t,T)\overline{V}_i + D_i(u,t,T)V_t^i\right\} \times \exp\left\{iu\underbrace{\left(-\frac{1}{2}\int_0^t V_s^i ds + \int_0^t \sqrt{V_s^i}dW_s^i\right)}_{S_0(t)}\right\}, \quad (12)$$

where

$$C_i(u,t,T) = \frac{\kappa_i}{\eta_i^2}\left((m_i - d_i)(T-t) - 2\log\frac{1-g_i e^{-d_i(T-t)}}{1-g_i}\right) \quad (13)$$

$$D_i(u,t,T) = \frac{1}{\eta_i^2}(m_i - d_i)\frac{1-e^{-d_i(T-t)}}{1-g_i e^{-d_i(T-t)}}, \quad (14)$$

and

$$m_i = \kappa_i - \eta_i\rho_i ui, \quad d_i = \sqrt{m_i^2 + \eta_i^2(ui + u^2)}, \quad g_i = \frac{m_i - d_i}{m_i + d_i}.$$

4.2) pure jump process

First, calibrate pure jump process parameters by below characteristic function, then calibrate stochastic volatility process parameters by former Heston process.

In the first step we assume the log-asset returns follow jump diffusive processes with

$$dX_t^i = \left(-\frac{1}{2}I_i^2 - \lambda_t^i\xi_i\right)dt + I_i dW_t^i + dJ_t^i, \quad (32)$$

where I_i equals the minimum implied volatility in the set of options on index i. The joint asset dynamics are calibrated to the four most out-of-the-money call and put options for the four shortest maturities for the four indexes, a total of 128 prices when all quotes are available. Including the diffusion in the dynamics raises the model implied volatility to the level I_i. A calibration of the pure jump model with no diffusion would result in extreme values for the jump parameters and leave little freedom to calibrate the Heston dynamics to the individual surfaces in the next step.

$$\psi_2(u,\tau) = \exp\left(\overline{\lambda}\tau\left(e^{iu\mu_j - \frac{1}{2}u^2\sigma_j^2} - 1\right) + iu\ln S_0 - \frac{1}{2}I^2u^2\tau + iu\tau\left(-\frac{1}{2}I^2 - \overline{\lambda}\left(e^{\mu_j + \frac{1}{2}\sigma_j^2} - 1\right)\right)\right)$$

5. Option Pricing Formula: Lewis' Approach (Inverse Fourier Transform) (method from Kokholm(2016))

$$C(K, S_0, \tau) = S_0 - \frac{\sqrt{S_0 K}\, e^{-\frac{1}{2}(r-\bar{q})\tau}}{\pi} \int_0^{\infty} \frac{\Re\left(e^{-\mathrm{iu}\left(\log\left(\frac{S_0}{K}\right)+(r-\bar{q})\tau\right)} \Psi\left(u - \frac{i}{2}\right)\right)}{\left(u^2 + \frac{1}{4}\right)} \mathrm{du}$$

The pure jump parameters calibration was done by minimizing the sum-squared pricing error (SSE), defined as the sum of the squared differences between the observed and the modeled option prices across all strikes and maturities.

$$\mathrm{SSE} = \mathrm{sum}\left((\mathrm{IV}^{\mathrm{market}} - \mathrm{IV}^{\mathrm{model}})^2\right)$$

The Heston process parameters calibration was done by minimizing the Average Relative Pricing Error(ARPE), defined as below:

$$\mathrm{ARPE} = \frac{1}{\#\{\mathrm{options}\}} \sum_{\mathrm{options}} \frac{|\, I_{Calibration} - I_{True} \,|}{I_{True}}. \qquad (37)$$

6. Calibration

6.1) First step: calibrate the pure jump parameters by 4.2) characteristic function and 5) Lewis' Fourier Transform Approach.

- $\bar{\lambda} = 1.1004$ constant jump intensity (Poisson process)
- $\mu_j = -0.0612$ mean of jump size (normally distributed)
- $\sigma_j = 0.0573$ volatility of jump size (normally distributed)

```
clear;  clc;

% load data
rawdata = readtable('OTM4_options.csv');
C_data = rawdata(strcmp(rawdata.cp_flag, 'C'), :);
C_market_price = C_data.market_price;
K = C_data.strike;
T = C_data.tao;
r = C_data.Spot_Rate;
IV_C_MK = C_data.IV_MK;
q_estimated = 0.0182;
S0 = 3094.04;

% minimum implied volatility
IV_market = min(IV_C_MK)
```

```
IV_market = 0.1036
```

```
% [lambda_bar, sigma_j, mu_j]
initial_params = [2, 0.1, -0.3];

% boundary
```

```
lb = [0.001, 0.0001, -1];
ub = [10, 1, 0];

% calibrate
options = optimset('Display', 'iter');
calibrated_params = fmincon(@(params) objectiveFunction_purejump(params, ...
    K, S0, T, r, q_estimated, IV_market, IV_C_MK), ...
    initial_params, [], [], [], [], lb, ub, [], options);
```

```
                                            First-order      Norm of
 Iter F-count            f(x)  Feasibility   optimality         step
    0       4    2.236420e+00    0.000e+00    2.618e+00
    1       8    3.796254e-02    0.000e+00    9.841e+00    8.979e-01
    2      12    3.992845e-02    0.000e+00    5.726e-01    7.032e-02
    3      17    1.595036e-01    0.000e+00    1.413e+00    1.396e-01
    4      21    9.754898e-02    0.000e+00    2.491e-01    2.460e-01

Local minimum possible. Constraints satisfied.

fmincon stopped because the size of the current step is less than
the value of the step size tolerance and constraints are
satisfied to within the value of the constraint tolerance.

<stopping criteria details>
```

```
disp('Calibrated Pure Jump Parameters:');
```

```
Calibrated Pure Jump Parameters:
```

```
% [lambda_bar, sigma_j, mu_j]
disp(calibrated_params);
```

```
    1.1004    0.0573   -0.0612
```

6.2) Second step: calibrate the Heston process parameters by combining 4.1) with 4.2) characteristic functions and 5) Lewis' Fourier Transform Approach.

```
clear;  clc;
calibrated_params = [1.1004, 0.0573, -0.0612];
% load data
data = readtable('merged_data.csv');
disp(data(1:5, :));
```

Call_market	strike	tao	maturity	Put_market	moneyness	Spot_Rate	IV_C_MK	IV_P_MK
1041.2	2050	0.11111	28	0.075	0.66256	0.015879	0.60569	0.39083
1016.3	2075	0.11111	28	0.075	0.67064	0.015879	0.59121	0.37999
981.4	2110	0.11111	28	0.075	0.68196	0.015879	0.57176	0.365
971.4	2120	0.11111	28	0.075	0.68519	0.015879	0.5656	0.36076
966.45	2125	0.11111	28	0.1	0.6868	0.015879	0.5636	0.36725

```
Call_market = data.Call_market;
IV_C_MK = data.IV_C_MK;
K = data.strike;
T = data.tao;
r = data.Spot_Rate;
```

```
q_estimated = 0.0182;
S0 = 3094.04;   % underlying SPX index closing price
```

- Because of the highly dimensional complex structure of characteristic function, parameters of Affine Jump-diffusion model are very uneasy to calibrate. First, I tried to use SQP Algorithm (local optimization) to find the parameters in a relatively large range around empirical initial guess. You can see the calibrated result is very easy to fall into the wrong local optimal solution.

```
% parameters = [kappa_v, sigma_v, rho_bar, v_bar, v0]
initial_heston_params = [1, 1.5, -0.5, 0.02, 0.02];

% boundary
lb = [0.1, 0.5, -0.99, 0.01, 0.01];
ub = [3,    2,     0,   0.1,  0.1];

% Sequential quadratic programming (SQP)
options = optimoptions('fmincon', 'Display', 'iter', 'Algorithm', 'sqp', ...
                       'MaxFunctionEvaluations', 1e4, 'MaxIterations', 1e4);

[optimal_params, fval] = fmincon(@(params) objective_function_hestonjump(params, ...
    IV_C_MK, K, T, S0, r, q_estimated, calibrated_params), ...
    initial_heston_params, [], [], [], [], lb, ub, [], options);
```

```
 Iter  Func-count            Fval   Feasibility   Step Length       Norm of   First-order
                                                                       step    optimality
    0           6    1.160528e+00    0.000e+00     1.000e+00     0.000e+00     3.789e-02
    1          12    1.159391e+00    0.000e+00     1.000e+00     3.800e-02     3.944e-02
    2          18    1.155999e+00    0.000e+00     1.000e+00     1.326e-01     1.199e-02
    3          24    1.155987e+00    0.000e+00     1.000e+00     4.570e-03     1.093e-02
    4          30    1.155954e+00    0.000e+00     1.000e+00     1.151e-02     1.185e-02
    5          36    1.155837e+00    0.000e+00     1.000e+00     4.412e-02     2.494e-02
    6          42    1.155765e+00    0.000e+00     1.000e+00     2.715e-02     3.206e-03
    7          49    1.155764e+00    0.000e+00     7.000e-01     1.697e-03     2.322e-03
    8          56    1.155763e+00    0.000e+00     7.000e-01     3.069e-04     3.929e-03
    9          67    1.155760e+00    0.000e+00     1.681e-01     4.787e-05     2.242e-03
   10          83    1.155760e+00    0.000e+00     3.323e-03     9.732e-07     2.943e-04

Local minimum possible. Constraints satisfied.

fmincon stopped because the size of the current step is less than
the value of the step size tolerance and constraints are
satisfied to within the value of the constraint tolerance.

<stopping criteria details>
Calibrated Heston+Jump parameters:
```

```
% [kappa_v, sigma_v, rho_bar, v_bar, v0]
disp(optimal_params);
```

```
    0.9977    1.5017   -0.4899    0.3000    0.2000
```

- Second, I tried a more narrow range and change to Interior-point Algorithm, the calibrated v_bar and v0 are also very close to their upper boundaries. ARPE is around 1.161 that is close to the ARPE value in Kokholm(2016).

```
% local optimization: interior-point

% parameters = [kappa_v, sigma_v, rho_bar, v_bar, v0]
initial_heston_params = [1, 1.5, -0.5, 0.02, 0.02];

% boundary
lb = [0.5, 0.5, -0.99, 0.01, 0.01];
ub = [5,    2,     0,  0.03, 0.03];

options = optimoptions('fmincon', 'Display', 'iter', 'Algorithm', 'interior-point', ...
                       'MaxFunctionEvaluations', 1e4, 'MaxIterations', 1e4);

[optimal_params, fval] = fmincon(@(params) objective_function_hestonjump(params, ...
    IV_C_MK, K, T, S0, r, q_estimated, calibrated_params), ...
    initial_heston_params, [], [], [], [], lb, ub, [], options);
```

```
                                            First-order      Norm of
 Iter F-count            f(x)  Feasibility   optimality         step
    0       6    1.161303e+00    0.000e+00    1.310e-02
    1      12    1.161294e+00    0.000e+00    3.257e-03    2.001e-03
    2      18    1.161295e+00    0.000e+00    8.582e-03    9.958e-03
    3      24    1.161248e+00    0.000e+00    9.237e-03    3.568e-02
    4      30    1.161228e+00    0.000e+00    9.047e-03    5.676e-02
    5      36    1.161156e+00    0.000e+00    5.890e-03    1.536e-01
    6      42    1.161291e+00    0.000e+00    7.506e-03    2.455e-01
    7      48    1.161318e+00    0.000e+00    1.107e-02    3.622e-01
    8      59    1.161277e+00    0.000e+00    1.628e-03    2.666e-01
    9      67    1.161190e+00    0.000e+00    5.812e-03    2.429e-02
   10      78    1.161177e+00    0.000e+00    5.502e-04    1.377e-03
   11      84    1.161123e+00    0.000e+00    5.278e-03    2.888e-02
   12      90    1.161022e+00    0.000e+00    6.046e-03    2.170e-02
   13      99    1.161010e+00    0.000e+00    5.332e-04    2.377e-04
   14     121    1.161010e+00    0.000e+00    1.763e-03    5.761e-06
   15     127    1.161005e+00    0.000e+00    9.773e-03    1.234e-02
   16     133    1.160991e+00    0.000e+00    6.941e-03    1.203e-02
   17     139    1.160992e+00    0.000e+00    7.668e-03    1.408e-03
   18     145    1.160989e+00    0.000e+00    4.191e-03    1.653e-03
   19     151    1.160988e+00    0.000e+00    3.532e-03    9.676e-04
   20     158    1.160986e+00    0.000e+00    7.032e-03    9.483e-04
   21     164    1.160983e+00    0.000e+00    6.601e-03    6.566e-04
   22     170    1.160964e+00    0.000e+00    6.599e-03    3.496e-04
   23     177    1.160812e+00    0.000e+00    1.189e-02    9.488e-04
   24     193    1.160812e+00    0.000e+00    4.927e-04    2.574e-04
   25     201    1.160812e+00    0.000e+00    7.350e-03    5.179e-04
   26     218    1.160812e+00    0.000e+00    3.633e-04    6.676e-05
   27     226    1.160812e+00    0.000e+00    6.638e-03    6.089e-06
   28     240    1.160812e+00    0.000e+00    2.150e-04    5.840e-06
   29     247    1.160812e+00    0.000e+00    1.190e-03    6.906e-07
   30     258    1.160812e+00    0.000e+00    2.017e-04    7.028e-06

                                            First-order      Norm of
 Iter F-count            f(x)  Feasibility   optimality         step
   31     266    1.160812e+00    0.000e+00    2.460e-03    3.115e-07
```

```
Local minimum possible. Constraints satisfied.

fmincon stopped because the size of the current step is less than
the value of the step size tolerance and constraints are
satisfied to within the value of the constraint tolerance.

<stopping criteria details>
Calibrated Heston+Jump parameters:
```

```
% [kappa_v, sigma_v, rho_bar, v_bar, v0]
disp(optimal_params);
```

```
    1.9524    1.0113   -0.4483    0.0298    0.0299
```

- Third, I tried the Global Optimization Algorithm (Particle Swarm Optimization). Noted, the Global Optimization Algorithm spent much longer time to calculate, so I only set 20 MaxStallIterations and 30 SwarmSize but it still spent 4 hours to get the results.

```
% global optimization: Particle Swarm Optimization (PSO)

% parameters = [kappa_v, sigma_v, rho_bar, v_bar, v0]
initial_heston_params = [1, 1.5, -0.5, 0.02, 0.02];

% boundary
lb = [0.5, 0.5, -0.9, 0.01, 0.01];
ub = [5,    2,     0, 0.03, 0.03];

options = optimoptions('particleswarm', 'Display', 'iter', ...
                       'MaxIterations', 1e4, 'MaxStallIterations', 20, ...
                       'SwarmSize', 30);

nvars = length(initial_heston_params);

[optimal_params, fval] = particleswarm(@(params) objective_function_hestonjump( ...
    params, IV_C_MK, K, T, S0, r, q_estimated, calibrated_params), nvars, lb, ub, options);
```

```
                                 Best            Mean     Stall
Iteration     f-count            f(x)            f(x)    Iterations
    0             30            1.161           1.161        0
    1             60            1.161           1.161        0
    2             90            1.161           1.161        0
    3            120            1.161           1.161        0
    4            150            1.161           1.161        1
    5            180            1.161           1.161        0
    6            210            1.161           1.161        0
    7            240             1.16           1.161        0
    8            270             1.16           1.161        1
    9            300             1.16           1.161        2
   10            330             1.16           1.161        3
   11            360             1.16           1.161        4
   12            390             1.16           1.161        5
   13            420             1.16           1.161        6
   14            450             1.16           1.161        7
```

```
    15              480              1.16            1.161        8
    16              510              1.16            1.161        9
    17              540              1.16            1.161       10
    18              570              1.16            1.161       11
    19              600              1.16            1.161       12
    20              630              1.16            1.161       13
    21              660              1.16            1.161       14
    22              690              1.16            1.161       15
    23              720              1.16            1.161       16
    24              750              1.16            1.161       17
    25              780              1.16            1.161       18
    26              810              1.16            1.161       19
Optimization ended: relative change in the objective value
over the last OPTIONS.MaxStallIterations iterations is less than OPTIONS.FunctionTolerance.
Calibrated Pure Heston parameters:
```

```
% [kappa_v, sigma_v, rho_bar, v_bar, v0]
disp(optimal_params);
```

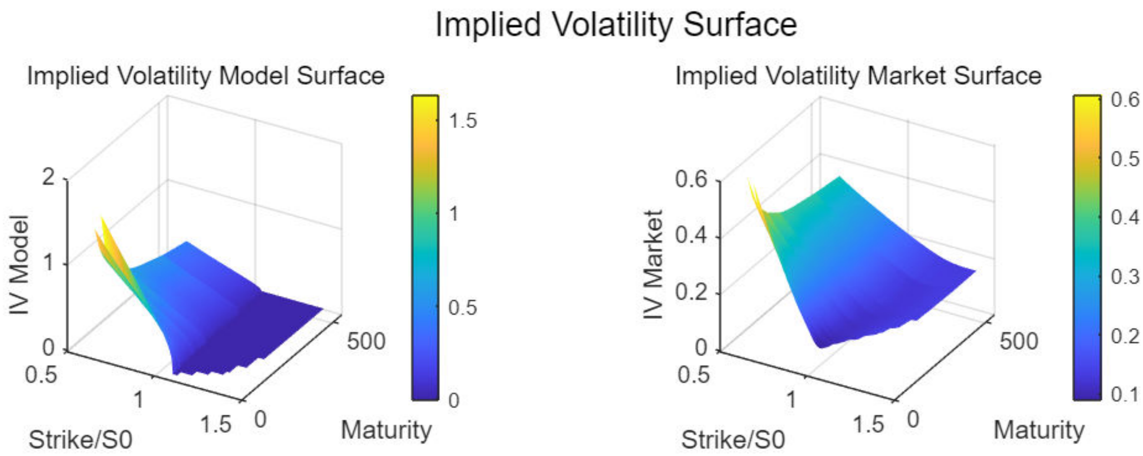
```
    0.5000    2.0000   -0.6916    0.0286    0.0300
```

7. Implied Volatility Surface

Even though the calibrated parameters are not completely correct, I still applied them to illustrate the implied volatility surface. The implied volatilities derived from model price is much higher than implied volatilities from market price, and they almost equals zero if the option strike prices are higher than spot price S0 (out-of-money call options). These strange implied volatilities may be explained by the wrong calibrated parameters.

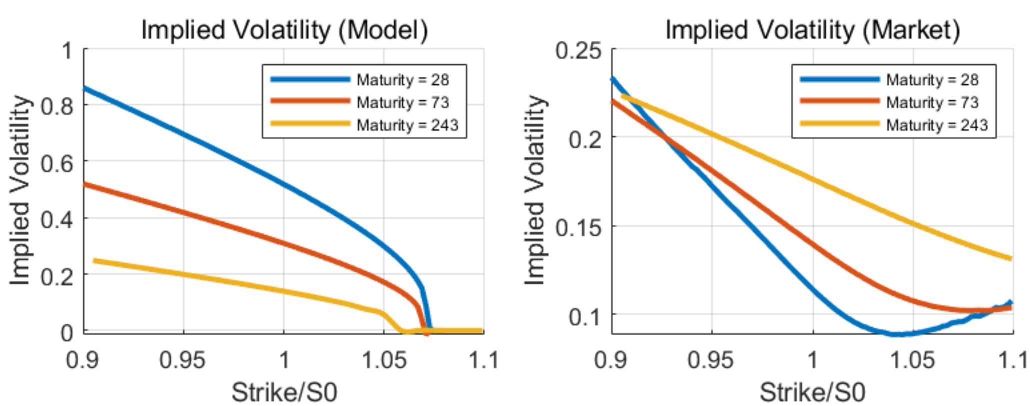
```
numRows = height(data); % 行数
Call_model = zeros(numRows, 1); % 预分配为列向量
IV_C_MD = zeros(numRows, 1);

% the local optimization results (by Interior-point Algorithm)
optimal_params = [1.9524, 1.0113, -0.4483, 0.0298, 0.0299];
calibrated_params = [1.1004, 0.0573, -0.0612];

% calculate Affine Jump-diffusion model price and implied volatility
for i = 1:numRows
    K_i = K(i);
    T_i = T(i);
    r_i = r(i);
    Call_model(i) = hestonjump_Lewis_call(S0, data.strike(i), data.tao(i), ...
        data.Spot_Rate(i), q_estimated, optimal_params(1), optimal_params(2), ...
        optimal_params(3), optimal_params(4), optimal_params(5), ...
        calibrated_params(1), calibrated_params(2), calibrated_params(3));
    IV_C_MD(i) = bsm_implied_vol_call(Call_model(i), S0, K_i, r_i, T_i, q_estimated);
end

data.CallModel = Call_model;
data.IV_C_MD = IV_C_MD;
writetable(data, 'IV_data.csv');
```

```
ivdata = readtable('IV_data.csv');

S0 = 3094.04;

x = ivdata.strike / S0;
y = ivdata.maturity;
plotIV_C_MD = ivdata.IV_C_MD;
plotIV_C_MK = ivdata.IV_C_MK;

[X, Y] = meshgrid(unique(x), unique(y));

IV_model_interp = griddata(x, y, plotIV_C_MD, X, Y, 'cubic');
IV_market_interp = griddata(x, y, plotIV_C_MK, X, Y, 'cubic');

figure('Position', [100, 100, 800, 250]);

% IV_model
subplot(1, 2, 1);
surf(X, Y, IV_model_interp, 'EdgeColor', 'none');
xlabel('Strike/S0');
ylabel('Maturity');
zlabel('IV Model');
title('Implied Volatility Model Surface');
colorbar;
view(30, 30);

% IV_market
subplot(1, 2, 2);
surf(X, Y, IV_market_interp, 'EdgeColor', 'none');
xlabel('Strike/S0');
ylabel('Maturity');
zlabel('IV Market');
title('Implied Volatility Market Surface');
colorbar;
view(30, 30);

sgtitle('Implied Volatility Surface');
```




```
% implied volatility

x = ivdata.strike / S0;
y_maturity = ivdata.maturity;
plotIV_C_MD = ivdata.IV_C_MD;
plotIV_C_MK = ivdata.IV_C_MK;

% only maturity 1 month, 3 month, 12 month
unique_maturities = [28, 73, 243];

colors = lines(length(unique_maturities));
figure('Position', [100, 100, 600, 200]);

% IV_model
subplot(1, 2, 1);
hold on;

for idx = 1:length(unique_maturities)
    current_maturity = unique_maturities(idx);

    x_current = x(y_maturity == current_maturity);
    iv_current = plotIV_C_MD(y_maturity == current_maturity);

    valid_indices = (x_current > 0.9) & (x_current < 1.1);    % strike price around S0
    x_filtered = x_current(valid_indices);
    iv_filtered = iv_current(valid_indices);

    xq = linspace(min(x_filtered), max(x_filtered), 100);
    ivq = interp1(x_filtered, iv_filtered, xq, 'spline');

    plot(xq, ivq, 'Color', colors(idx, :), 'LineWidth', 2, 'DisplayName', ...
        sprintf('Maturity = %d', current_maturity));
end

xlabel('Strike/S0');
ylabel('Implied Volatility');
```

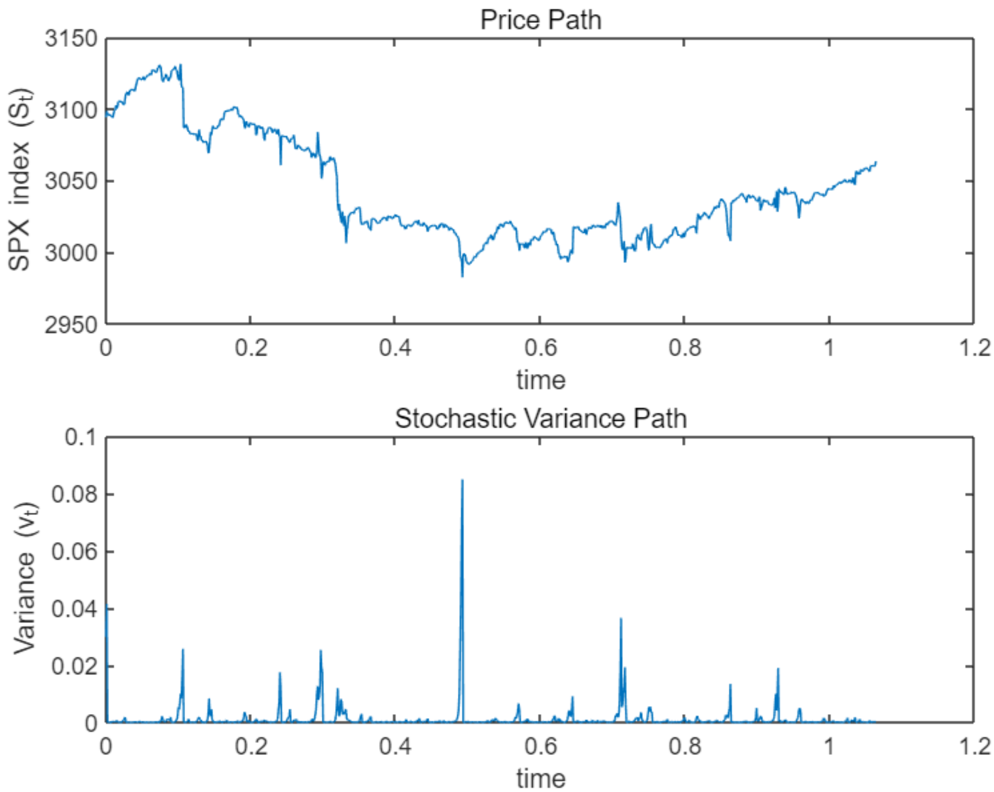
```
title('Implied Volatility (Model)');
legend('show');
set(legend, 'FontSize', 6);
grid on;
hold off;

% IV_market
subplot(1, 2, 2);
hold on;

for idx = 1:length(unique_maturities)
    current_maturity = unique_maturities(idx);

    x_current = x(y_maturity == current_maturity);
    iv_current = plotIV_C_MK(y_maturity == current_maturity);

    valid_indices = (x_current > 0.9) & (x_current < 1.1);    % strike price around S0
    x_filtered = x_current(valid_indices);
    iv_filtered = iv_current(valid_indices);

    xq = linspace(min(x_filtered), max(x_filtered), 100);
    ivq = interp1(x_filtered, iv_filtered, xq, 'spline');

    plot(xq, ivq, 'Color', colors(idx, :), 'LineWidth', 2, 'DisplayName', ...
        sprintf('Maturity = %d', current_maturity));
end

xlabel('Strike/S0');
ylabel('Implied Volatility');
title('Implied Volatility (Market)');
legend('show');
set(legend, 'FontSize', 6);
grid on;
hold off;
```



8. Monte Carlo Simulation: Price and Variance paths

```
clear;  clc;
```

```
% the local optimization results (by Interior-point Algorithm)
optimal_params = [1.9524, 1.0113, -0.4483, 0.0298, 0.0299];
calibrated_params = [1.1004, 0.0573, -0.0612];

S0 = 3094.04;
q_estimated = 0.0182;
r = 0.0184;           % 268 days spot rate
T = 1.0635;           % 268 days maturity
kappa_v = optimal_params(1);
sigma_v = optimal_params(2);
rho_bar = optimal_params(3);
v_bar = optimal_params(4);
v0 = optimal_params(5);
lambda_bar = calibrated_params(1);
sigma_j = calibrated_params(2);
mu_j = calibrated_params(3);
num_paths = 1;
num_steps = 1000;

% Monte Carlo Simulation
[S_paths, v_paths, time_vector] = simulate_heston_jump(S0, v0, r, q_estimated, ...
    T, kappa_v, v_bar, sigma_v, rho_bar, lambda_bar, mu_j, sigma_j, num_paths, num_steps);

figure;
subplot(2, 1, 1);
plot(time_vector, S_paths(1, :)');  % price
xlabel('time');
ylabel('SPX index (S_t)');
title('Price Path');

subplot(2, 1, 2);
plot(time_vector, v_paths(1, :)');  % variance
xlabel('time');
ylabel('Variance (v_t)');
title('Stochastic Variance Path');
```




```
function C = purejump_Lewis_call(S0, K, T, r, q_estimated, lambda, mu_j, sigma_j, IV_market)
    i = 1i;   % imaginary unit

    f = @(u) real(purejump_CF(u-0.5i, T, r, q_estimated, lambda, sigma_j, mu_j, ...
        IV_market) .* exp(i*u*(log(S0/K)+(r-q_estimated)*T))) ./ (u^2 + 0.25);
    integrand = integral(f, 0.001, 100, 'ArrayValued', true);

    % Lewis' Approach
    C = S0 * exp(-q_estimated * T) - sqrt(S0*K)*exp(-0.5*(r+q_estimated)*T) * ( ...
        (1/ pi) * integrand);
end

function C = hestonjump_Lewis_call(S0, K, T, r, q_estimated, kappa_v, sigma_v, rho_bar, ...
    v_bar, v0, lambda_bar, sigma_j, mu_j)
    i = 1i;   % imaginary unit

    f = @(u) real(heston_CF(u-0.5i, S0, T, r, q_estimated, kappa_v, sigma_v, rho_bar, ...
        v_bar, v0, lambda_bar, sigma_j, mu_j) .* exp(i*u*(log(S0/K)+(r-q_estimated)*T) ...
        ) ./ (u^2 + 0.25));
    integrand = integral(f, 0.001, 100, 'ArrayValued', true);

    % Lewis' Approach
    C = S0 * exp(-q_estimated * T) - sqrt(S0*K)*exp(-0.5*(r-q_estimated)*T) * ( ...
```

```
        (1/ pi) * integrand);
end

function phi = purejump_CF(u, T, r, q_estimated, lambda, sigma_j, mu_j, IV_market)
    jumpterm = exp(-lambda*mu_j*1i*u*T + ...
        lambda*T*((1+mu_j)^(1i*u) * exp(0.5*sigma_j^2*1i*u*(1i*u-1)) - 1));
    drift = 1i .* u .* T .* ( r- q_estimated + (-0.5) * IV_market^2);
    diffusion = -0.5 * IV_market^2 * T .* u.^2;

    phi = exp(drift + diffusion) * jumpterm;
end

function phi_heston = heston_CF(u, S0, T, r, q_estimated, kappa_v, sigma_v, rho_bar, ...
    v_bar, v0, lambda_bar, sigma_j, mu_j)

    m = kappa_v - sigma_v * rho_bar * u * 1i;
    d = sqrt(m^2 + sigma_v^2 * (u*1i + u^2));
    g = (m-d)/(m+d);
    C = kappa_v / sigma_v^2 *( (m-d)*T - 2*log( (1-g*exp(-d*T))/(1-g) ) );
    D = 1 / sigma_v^2 * (m-d) *(1-exp(-d*T)) / (1-g*exp(-d*T));
    S0_t = 1i.*u.*(log(S0)+(r-q_estimated)*T);
    jumpterm = exp(-lambda_bar*mu_j*1i*u*T + lambda_bar*T*((1+mu_j)^(1i*u) * ...
        exp(0.5*sigma_j^2*1i*u*(1i*u-1)) - 1));

    phi_heston = exp( C*v_bar + D*v0 + S0_t) * jumpterm;
end

function sse = objectiveFunction_purejump(params, K, S0, T, r, q_estimated, ...
    IV_market, IV_C_MK)
    lambda = params(1);
    mu_j = params(2);
    sigma_j = params(3);

    N = length(K);
    C_model_price = zeros(1, N);
    sse = 0;

    for i = 1:N
        K_i = K(i);
        T_i = T(i);
        r_i = r(i);
        C_model_price(i) = purejump_Lewis_call(S0, K_i, T_i, r_i, q_estimated, ...
            lambda, mu_j, sigma_j, IV_market);
        IV_C_MK_i = IV_C_MK(i);
        IV_C_MD_i = bsm_implied_vol_call(C_model_price(i), S0, K_i, r_i, T_i, ...
            q_estimated);

        sse = sse + ((IV_C_MK_i - IV_C_MD_i).^2);
    end
end
```

```matlab
% Average Relative Pricing Error (ARPE)
function ARPE = objective_function_hestonjump(params, IV_C_MK, strike, T, S0, r, ...
    q_estimated, calibrated_params)
    kappa_v = params(1);
    sigma_v = params(2);
    rho_bar = params(3);
    v_bar = params(4);
    v0 = params(5);
    lambda_bar = calibrated_params(1);
    sigma_j = calibrated_params(2);
    mu_j = calibrated_params(3);

    N = length(IV_C_MK);
    C_model_price = zeros(N, 1);
    obj = 0;

    for i = 1:N
        K_i = strike(i);
        T_i = T(i);
        r_i = r(i);
        C_model_price(i) = hestonjump_Lewis_call(S0, K_i, T_i, r_i, q_estimated, ...
            kappa_v, sigma_v, rho_bar, v_bar, v0, lambda_bar, sigma_j, mu_j);
        IV_C_MK_i = IV_C_MK(i);
        IV_C_MD_i = bsm_implied_vol_call(C_model_price(i), S0, K_i, r_i, T_i, ...
            q_estimated);

        obj = obj + abs( IV_C_MD_i - IV_C_MK_i )/IV_C_MK_i;
    end
    ARPE = obj / N;
end

function obj = objective_function_q_r(params, selected_data, unique_maturities, S0)
    q = params(1);
    r = params(2:end);

    maturity_r_map = containers.Map(unique_maturities, r);

    % minimize
    obj = 0;

    % loop
    for i = 1:height(selected_data)
        option = selected_data(i, :);
        C_market = option.Call_market;
        P_market = option.Put_market;
        T = option.tao;

        if isKey(maturity_r_map, T)
            r_T = maturity_r_map(T);
```

```
        else
            continue;
        end
        P_PC = C_market - S0 * exp(-q * T) + option.strike * exp(-r_T * T);

        obj = obj + (P_market - P_PC)^2;
    end
end

function implied_vol = bsm_implied_vol_call(C_market, S0, K, r, T, q)
    tol = 1e-6;
    max_iter = 1000;
    sigma_low = 1e-6;
    sigma_high = 3;
    iter = 0;

    while iter < max_iter
        iter = iter + 1;
        sigma_mid = (sigma_low + sigma_high) / 2;
        C_mid = bsm_call_price(S0, K, r, T, sigma_mid, q);

        if abs(C_mid - C_market) < tol
            implied_vol = sigma_mid;
            return;
        elseif C_mid > C_market
            sigma_high = sigma_mid;
        else
            sigma_low = sigma_mid;
        end
    end
    implied_vol = sigma_mid;
end

function C = bsm_call_price(S0, K, r, T, sigma, q)
    d1 = (log(S0 / K) + (r - q + 0.5 * sigma^2) * T) / (sigma * sqrt(T));
    d2 = d1 - sigma * sqrt(T);
    C = S0 * exp(-q * T) * normcdf(d1) - K * exp(-r * T) * normcdf(d2);
end

function implied_vol = bsm_implied_vol_put(P_market, S0, K, r, T, q)
    tol = 1e-6;
    max_iter = 1000;
    sigma_low = 1e-6;
    sigma_high = 3;
    iter = 0;

    while iter < max_iter
        iter = iter + 1;
        sigma_mid = (sigma_low + sigma_high) / 2;
        P_mid = bsm_put_price(S0, K, r, T, sigma_mid, q);
```

```
        if abs(P_mid - P_market) < tol
            implied_vol = sigma_mid;
            return;
        elseif P_mid > P_market
            sigma_high = sigma_mid;
        else
            sigma_low = sigma_mid;
        end
    end

    implied_vol = sigma_mid;
end

function P = bsm_put_price(S0, K, r, T, sigma, q)
    d1 = (log(S0 / K) + (r - q + 0.5 * sigma^2) * T) / (sigma * sqrt(T));
    d2 = d1 - sigma * sqrt(T);
    P = K * exp(-r * T) * normcdf(-d2) - S0 * exp(-q * T) * normcdf(-d1);
end

function [S_paths, v_paths, time_vector] = simulate_heston_jump(S0, v0, r, ...
    q_estimated, T, kappa, v_bar, sigma_v, rho, lambda, mu_j, sigma_j, ...
    num_paths, num_steps)
    dt = T / num_steps;
    time_vector = 0:dt:T;

    S_paths = zeros(num_paths, num_steps + 1);
    v_paths = zeros(num_paths, num_steps + 1);
    S_paths(:, 1) = S0;
    v_paths(:, 1) = v0;
    dW1 = randn(num_paths, num_steps);
    dW2 = randn(num_paths, num_steps);
    J = exp(mu_j + sigma_j * randn(num_paths, num_steps));
    N = poissrnd(lambda * dt, [num_paths, num_steps]);

    for t = 2:num_steps+1
        dW1_t = dW1(:, t-1);
        dW2_t = rho * dW1_t + sqrt(1 - rho^2) * dW2(:, t-1);

        v_paths(:, t) = v_paths(:, t-1) + kappa * (v_bar - ...
            v_paths(:, t-1)) * dt + sigma_v * sqrt(v_paths(:, t-1) .* max( ...
            v_paths(:, t-1), 0)) .* dW2_t;

        v_paths(:, t) = max(v_paths(:, t), 0);

        S_paths(:, t) = S_paths(:, t-1) .* exp((r - q_estimated - 0.5 * v_paths(:, t-1) ...
            ) * dt + sqrt(v_paths(:, t-1)) .* dW1_t * sqrt(dt)) .* (1 + ...
            J(:, t-1) .* N(:, t-1));
    end
end
```