Tutorial-4 [PHN-624]

Q1: The energy eigen-values of the Hamiltonian, $H = \frac{-\hbar^2}{2m}\nabla^2 + \frac{1}{2}mw^2r^2 + C\vec{l}.\vec{s} + D\vec{l}^2$, for the independent particle model can be written as-

$$E = (N + \frac{3}{2})\hbar w + C\langle\vec{l}.\vec{s}\rangle + D\langle\vec{l}^2\rangle, \tag{1}$$

where,

$$\langle\vec{l}.\vec{s}\rangle = \begin{cases} l/2, & \text{for } j = l + 1/2 \\ -(l+1)/2, & \text{for } j = l - 1/2, \end{cases} \tag{2}$$

and

$$\langle\vec{l}^2\rangle = l(l+1). \tag{3}$$

Using the values $C = -0.1\hbar w$, and $D = -0.0225\hbar w$, write a code for calculating eigen energies for the N values (starting from zero). Also, draw the energy diagram for $N = 0$ to 4. [Hint: Take $\hbar w = 1$]

Q2: The Hamiltonian for the anisotropic harmonic oscillator can be written as-

$$H = \frac{-\hbar^2}{2m}\nabla^2 + \frac{1}{2}m(w_x^2x^2 + w_y^2y^2 + w_z^2z^2). \tag{4}$$

We can write the energy eigen values $E = (n_x + \frac{1}{2})w_x + (n_y + \frac{1}{2})w_y + (n_z + \frac{1}{2})w_z$ as

$$E(n_\rho, n_z) = \hbar w_0[N + \frac{3}{2} - \frac{1}{3}\delta_{\text{osc}}(2n_z - n_\rho)], \tag{5}$$

where, $N = n_x + n_y + n_z = n_\rho + n_z$, $w_x = w_y = w_\rho$, $\delta_{\text{osc}} = (w_\rho - w_z)/w_0$, and $w_0 = \frac{1}{3}(2w_\rho + w_z)$.

(i) Take the value $\hbar w_0 = 1$, and plot a diagram for the variation of energy $E(n_\rho, n_z)$ with the increment of N with respect to the deformation parameter, δ_{osc}. [Hint: Take the values of δ_{osc} in between the range -1 to 1.]

(ii) Plot the same diagram for $\hbar w_0 = 41A^{-1/3}[f(\delta_{\text{osc}})]$ MeV, where $f(\delta_{\text{osc}}) = [(1 + (2/3)\delta_{\text{osc}})^2(1 - (4/3)\delta_{\text{osc}})]^{-1/6}$. [Take A=80]

Q3: The two-body matrix elements (TBMEs) for the Surface-delta interaction (SDI) in the JT scheme can be written as

$$\begin{aligned}\langle j_aj_b|V^{SDI}(1,2)|j_cj_d\rangle_{JT} = (-1)^{n_a+n_b+n_c+n_d}\frac{A_T}{2(2J+1)}\sqrt{\frac{(2j_a+1)(2j_b+1)(2j_c+1)(2j_d+1)}{(1+\delta_{ab})(1+\delta_{cd})}} \\ \Big\{(-1)^{j_b+j_d+l_b+l_d}\langle j_b\frac{-1}{2}j_a\frac{1}{2}|J0\rangle\langle j_d\frac{-1}{2}j_c\frac{1}{2}|J0\rangle[1-(-1)^{l_a+l_b+J+T}] \\ -\langle j_b\frac{1}{2}j_a\frac{1}{2}|J1\rangle\langle j_d\frac{1}{2}j_c\frac{1}{2}|J1\rangle[1+(-1)^T]\Big\}.\end{aligned} \tag{6}$$

Here, $A_T = A_T^{'}C(R_0)$.

Write a code to print the TBMEs for the SDI in the $0p$- and $1s0d$-shells (Take $A_T = 1$). We have also shown a Table for the two-body matrix elements $\langle ab; JT|V^{SDI}|cd; JT\rangle$ of Surface delta interaction for $0s0p$-shells in Fig. 1. Here, the orbitals are numbered as $1 = 0s_{1/2}$, $2 = 0p_{3/2}$, and $3 = 0p_{1/2}$.

abcd	JT	⟨V_{SDI}⟩	JT	⟨V_{SDI}⟩	JT	⟨V_{SDI}⟩	JT	⟨V_{SDI}⟩
1111	01	-1.0000	10	-1.0000				
1122	01	1.4142	10	1.0541				
1123	10	-1.3333						
1133	01	1.0000	10	-0.3333				
1212	10	-0.6667	11	-1.3333	20	-2.0000	21	0
1213	10	0.9428	11	-0.9428				
1313	00	-2.0000	01	0	10	-1.3333	11	-0.6667
2222	01	-2.0000	10	-1.2000	21	-0.4000	30	-1.2000
2223	10	1.2649	21	-0.5657				
2233	01	-1.4142	10	0.6325				
2323	10	-2.0000	11	0	20	-1.2000	21	-0.8000
2333	10	0						
3333	01	-1.0000	10	-1.0000				

FIG. 1: Two-body matrix elements $\langle ab; JT|V^{SDI}|cd; JT\rangle$ with $A_T = 1$ for the $0s - 0p$ shells.

Q4: The total Hamiltonian (H) of the Nilsson model can be written as

$$H = H_0^0 + H_\delta + C\vec{l}.\vec{s} + D\vec{l}^2. \tag{7}$$

Here, $H_0^0 = \frac{1}{2}\hbar w_0[-\nabla^2 + r^2]$, and $H_\delta = -\delta\hbar w_0 \frac{4}{3}\sqrt{\frac{\pi}{5}} r^2 Y_{20}$.
The matrix elements of H_0^0 and $\vec{l}^2$ are diagonal and can be written as

$$\langle N'l'\Lambda'\Sigma'|H_0^0|Nl\Lambda\Sigma\rangle = (N + \frac{3}{2})\hbar w_0 \delta_{N'N}\delta_{l'l}\delta_{\Lambda'\Lambda}\delta_{\Sigma'\Sigma}, \tag{8}$$

and,

$$\langle N'l'\Lambda'\Sigma'|\vec{l}^2|Nl\Lambda\Sigma\rangle = l(l+1)\delta_{N'N}\delta_{l'l}\delta_{\Lambda'\Lambda}\delta_{\Sigma'\Sigma}. \tag{9}$$

The matrix elements for $\vec{l}.\vec{s}$, r^2, and Y_{20} terms can be written as

$$\langle\Lambda'\Sigma'|\vec{l}.\vec{s}|\Lambda\Sigma\rangle = \frac{1}{2}\sqrt{(l-\Lambda)(l+\Lambda+1)}\delta_{\Lambda'\Lambda+1}\delta_{\Sigma'\Sigma-1} + \frac{1}{2}\sqrt{(l+\Lambda)(l-\Lambda+1)}\delta_{\Lambda'\Lambda-1}\delta_{\Sigma'\Sigma+1} + \Lambda\Sigma\delta_{\Lambda'\Lambda}\delta_{\Sigma'\Sigma}, \tag{10}$$

$$\langle l'|r^2|l\rangle = \sqrt{(N-l+2)(N+l+1)}\delta_{l'l-2} + (N+\frac{3}{2})\delta_{l'l} + \sqrt{(N-l)(N+l+3)}\delta_{l'l+2}, \tag{11}$$

and,

$$\langle l'\Lambda'|Y_{20}|l\Lambda\rangle = \sqrt{\frac{5(2l+1)}{4\pi(2l'+1)}}\langle l\Lambda 20|l'\Lambda'\rangle\langle l020|l'0\rangle \tag{12}$$

(i) Take $C = -2\kappa\hbar w_0^0$, and $D = C\mu/2$, where $\kappa = 0.05$, and μ varies as 0.0, 0.0, 0.0, 0.35, 0.625, 0.63, 0.448, 0.434 from $N = 0$ to $N = 7$. Plot a diagram for single-particle energies calculated from Hamiltonian (H) as a function of deformation δ (varies from -0.3 to 0.3). [Hint: $\hbar w_0^0 = \frac{\hbar w_0}{f(\delta)}$. Take $\hbar w_0$, and $f(\delta)$ as in Question-2(ii).]

(ii) Add a term $H_{\text{crank}} = -hw_0 * \omega * \Omega$, in the total Hamiltonian (H), and plot another diagram for single-particle energies calculated from Hamiltonian (H) as a function of 'ω' for deformation $\delta = 0.0$ and mass number $A = 20$. [**Hint:** $\Omega = \Lambda + \Sigma$. Vary the '$\omega$' from 0.0 to 0.2.]

Q5: The single-particle occupation number can be expressed in terms of Fermi-Dirac distribution as follows

$$n_i = \frac{1}{1 + exp(\frac{e_i - \lambda}{KT})}, \tag{13}$$

where e_i is the single-particle energy of the i^{th} state. T represents the temperature and K stands for the Boltzmann's constant. λ is the chemical potential which guarantees the particle number conservation through the constraint

$$N_p = \sum_i^{\infty} n_i. \tag{14}$$

Here, N_p denotes the total number of particles.

(i) Using the above equations and assuming that two fermions can be filled into a state, calculate and plot the single-particle occupation number (n_i) with respect to the single-particle energies (e_i) for different values of KT from 0.5 to 5.0 MeV with a interval of 0.5 MeV, corresponding to the ${}^{84}_{40}\mathrm{Zr}_{44}$. [Hint: Take the deformation, $\delta = 0.2$ and for N_p, take either proton number 40 or neutron number 44.]

(ii) The single-particle entropy is given by

$$s_i = -[n_i \ln n_i + (1 - n_i)\ln(1 - n_i)], \tag{15}$$

and hence the total entropy of the system is written as

$$S = \sum_i^{\infty} s_i. \tag{16}$$

Plot a diagram s_i versus e_i for different values of KT from 0.5 to 5.0 MeV with a interval of 0.5 MeV.

(iii) The free energy is given by

$$F = E^T - TS, \tag{17}$$

where, the internal energy E^T is given by

$$E^T = \sum_{i=1}^{N_p} e_i n_i. \tag{18}$$

Plot the free energy (F) with respect to the KT by considering $K = 1$.

Q6: The self-consistent transcendental equation for the effective mass of nuclear/neutron matter can be given as

$$M^* = M - \frac{g_s^2}{m_s^2}\frac{\gamma M^*}{4\pi^2}\Big[k_F E_F - M^{*2}\ln\Big(\frac{k_F + E_F}{M^*}\Big)\Big], \tag{19}$$

where, $E_F = (k_F^2 + M^{*2})^{1/2}$. Here, the spin-isospin degenracy $\gamma = 4$ (for nuclear matter) and $\gamma = 2$ (for neutron matter). k_F is the fermi wave vector.

(i) Taking the values $M = m_s = 939$ MeV/c^2; $g_s^2 = 267.1$, draw a curve for M^*/M with respect to $k_F/\hbar c$, for both the nuclear matter and neutron matter. Here, $\hbar c = 197.3269$ [Hint: Vary k_F from $0.1 \times \hbar c$ to $5.0 \times \hbar c$.]

(ii) The energy density of the system can be written as

$$E = \frac{g_v^2}{2m_v^2}\rho_B^2 + \frac{m_s^2}{2g_s^2}(M - M^*)^2 + \frac{\gamma}{(2\pi)^3}\int_0^{k_F} d^3k(k^2 + M^{*2})^{1/2}, \tag{20}$$

or,

$$E = \frac{g_v^2}{2m_v^2}\rho_B^2 + \frac{m_s^2}{2g_s^2}(M - M^*)^2 + \frac{\gamma}{(4\pi)^2}\Big[k_F E_F(K_F^2 + E_F^2) - M^{*4}\ln\Big(\frac{k_F + E_F}{M^*}\Big)\Big], \tag{21}$$

where,

$$\rho = \frac{\gamma}{(2\pi)^3}\int_0^{k_F} d^3k = \frac{\gamma}{6\pi^2}k_F^3. \tag{22}$$

Taking the values of $g_v^2 = 195.9$; $g_s^2 = 267.1$; $m_v = m_s = M = 939$ MeV$/c^2$, draw a curve for $\frac{E}{\rho_B} - M$ with respect to $k_F/\hbar c$, for both the nuclear matter and neutron matter.

(iii) Similarly, we can write the expression for pressure as

$$P = \frac{g_v^2}{2m_v^2}\rho_B^2 - \frac{m_s^2}{2g_s^2}(M - M^*)^2 + \frac{1}{3}\frac{\gamma}{(2\pi)^3}\int_0^{k_F} d^3k\frac{k^2}{(k^2 + M^{*2})^{1/2}}, \tag{23}$$

or,

$$P = \frac{g_v^2}{2m_v^2}\rho_B^2 - \frac{m_s^2}{2g_s^2}(M - M^*)^2 + \frac{1}{3}\frac{\gamma}{(4\pi)^2}\Big[k_F E_F(2k_F^2 - 3M^{*2}) + 3M^{*4}\ln\Big(\frac{k_F + E_F}{M^*}\Big)\Big]. \tag{24}$$

Taking the values of $g_v^2 = 195.9$; $g_s^2 = 267.1$; $m_v = m_s = M = 939$ MeV$/c^2$, draw a curve for $P/\hbar c^3$ with respect to $k_F/\hbar c$, for both the nuclear matter and neutron matter.

(iv) Draw the above three curves also by taking the mass of neutral scalar meson (σ) as $m_s = 500$ MeV/c^2 and the mass of neutral vector meson (ω) as $m_v = 782$ MeV/c^2.