Intermediate Language with Dependent Types and Strong Normalization for Erlang/OTP applications.

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1 Introduction

1.1 History

LISP. Untyped lambda calculus was discovered as an inner language of the space at origin (Curry, Church, 1932). This language was manifested as LISP (McCarthy, 1958) that was built upon: cons, nil, eq, atom, car, cdr, lambda, apply and id. It was parts of inductive types lately known as inductive type constructors. Still untyped lambda calculus is used as an extraction target for many provers (Idris, F*), and also manifests in different domain languages (JavaScript, Erlang).

ML/LCF. Further teardown of inner space language was ML language, founded merely on algebraic datatypes and algebra on higher terms rather than categorical semantic. Lately it was fixed with categorical methods in CPL (Hagino, 1987) and Charity (Cockett, 1992). Milner, assisted by Morris and Newey designed Meta Language for the purpose of builing LCF in early 70-s. LCF was a predecessor family of automated math provers: HOL88, HOL90, HOL98 and HOL/Isabelle which is now built using Poly/ML.

Fully Automated Provers. In that period during 80-90s other automated math systems were appeared: AUTOMATH (de Bruijn, 1967), Mizar (Trybulec, 1989), PVS (Owre, Rushby, Shankar, 1995), ACL2 (Boyer, Kaufmann, Moore, 1996) and Otter (McCune, 1996).

MLTT. Contemporary provers (built upon consistent Martin-Löf Type Theory, 1972) like Agda, Coq, Lean, F*, Idris are based on Barendregt and Coquand' Calculus of Constructions with different flavours of inifinity universe hierarchies and Calculus of Inductive Constructions for modeling polynomial functors of well-founded trees. Some of them are automated and some are trying to be and general purpose programming languages with proving facilities, like Idris, Coq (coq.io), Agda (M-Alonso).

Cubical Type Theory The further extensions of MLTT theory is a Homotopy Types needed for reasoning about abstract shapes and multidimentional path types. The Cubical approach was chosen in favour of Globular and Simplical due to computability of Voevodsky's univalence axiom.

1.2 Background

From the partical point of view there are exists two approaches. One involve the usage of two languages: meta language for your models and separate language for prover, like in **HOL**, **Andromeda** or **NuPRL**, **TLA+**, **Twelf**. The second approach propose embedding models into a unified language. And certified programs are extrated as proofs from that models. This is a land of MLTT provers.

2 Motivation and Vision

From PTS to HTS. We want to have flexible detachable layers on top of PTS core. Then Sigma for proving. Then well-founded trees or polynomial functors as known as data and record. Then higher path types, interval arithmetic, glue and comp for HIT. Each layers is driven by differenth math, the common in only the method – category theory.

Extensible Language Design. Encoding of inductive types is based on categorical semantic of compilation to PTS. All other syntax constructions are inductive definitions, plugged into the stream parser. AST of the PTS language is also defined in terms of inductive constructions and thus allowed in the macros. The language of polynomial functors (data and record) and core language of the process calculus (spawn, receive and send) are just macrosystem over Om language, its syntax extensions.

Changable Encodings. In pure CoC we have only arrows, so all inductive type encodings would be Church-encoding variations. Most extended nowadays is Church-Boehm-Berrarducci encoding, which dedicated to inductive types. Another well known are Scott (lazyness), Parigot (lazyness and constant-time iterators) and CPS (continuations) encodings. However most of them require variations of Fixpoint types.

Proved Categorical Semantic. There was modeled a math model (using higher-order categorical logic) of encoding, which calculates (co)limits in a cathegory of (co)algebras built with given set of (de)constructors. We call such encoding in honour of Lambek lemma that leeds us to the equality of (co)initial object and (co)limit in the categories of (co)algebras. Such encoding works with dependent types and its consistency is proved in Lean model.

General Architecture. This article covers only the central part of of the proving system, the **OM** intermediate language with strong normalization properties and extraction to **Erlang/OTP** bytecode. Also we describe briefly the prototype of **EXE** language enriched with inductive constructions.
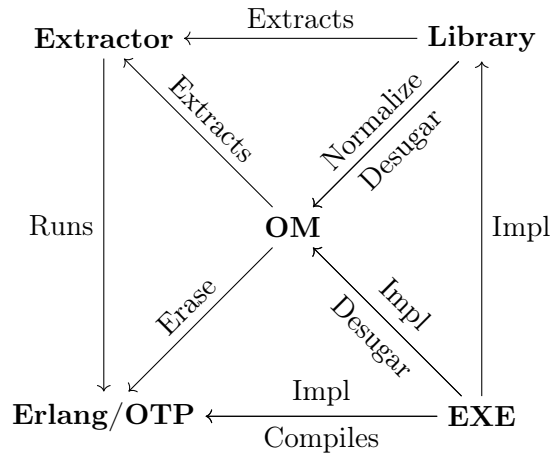
3 Pure Type System as Intermediate Language

The Om language is a dependently typed lambda calculus, an extension of Barendregt' and Coquand Calculus of Constructions with predicative hierarchy of indexed universes. There is no fixpoint axiom needed for the definition of infinity term dependance.

All terms respect ranking *Axioms* inside sequence of universes *Sorts* and complexity of the dependent term is equal maximum complexity of term and its dependency *Rules*. The type system is completely described by the following PTS notation (due to Barendregt):

$$\begin{cases} Sorts = Type.\{i\}, \ i : Nat \\ Axioms = Type.\{i\} : Type.\{inc \ i\} \\ Rules = Type.\{i\} \rightsquigarrow Type.\{j\} : Type.\{max \ i \ j\} \end{cases}$$

An intermediate Om language is based on Henk [6] languages described first by Erik Meijer and Simon Peyton Jones in 1997. Leter on in 2015 Morte impementation of Henk design appeared in Haskell, using Boem-Berrarducci encoding of non-recursive lambda terms. It is based only on one type constructor Π, its special case λ and theirs eliminators: *apply* and *curry*, infinity number of universes, and one computation rule called β-reduction. The design of Om language resemble Henk and Morte both design and implementation. This language indended to be small, concise, easy provable and able to produce verifiable peace of code that can be distributed over the networks, compiled at target with safe trusted linkage.

3.1 BNF

Om syntax is compatible with λC Coquand's Calculus of Constructions presented in Morte and Henk languages. However it has extension in a part of specifying universe index as a **Nat** number.

```
<> ::= #option
 I ::= #identifier
 U ::= * < #number >
 O ::= U
     | I | ( O ) | O O | O → O
     | λ ( I : O ) → O
     | ∀ ( I : O ) → O
```

Equivalent tree encoding for parsed terms is following:

```
data pts = star (n: nat)
         | var (n: name)
         | app (f a: pts)
         | lambda (x: name) (d c: pts)
         | pi (x: name) (d c: pts)
```

3.2 Universes

The OM language is a higher-order dependently typed lambda calculus, an extension of Coquand's Calculus of Constructions with the predicative/impredicative hierarchy of indexed universes. This extension is motivated avoiding paradoxes in dependent theory. Also there is no fixpoint axiom needed for the definition of infinity term dependance.

```
U_0 : U_1 : U_2 : U_3 : ...

U_0 --- propositions
U_1 --- values and sets
U_2 --- types
U_3 --- sorts
```

$$\frac{o : Nat}{Type_o} \tag{S}$$

3.3 Predicative Universes

All terms obey the A ranking inside the sequence of S universes, and the complexity R of the dependent term is equal to a maximum of the term's complexity and its dependency. The universes system is completely described by the following PTS notation (due to Barendregt):

```
S   (n : nat) = U n
A_1 (n m : nat) = U n : U m where m > n    - cumulative
R_1 (m n : nat) = U m → U n : U (max m n) - predicative
```

Note that predicative universes are incompatible with Church lambda term encoding. You can switch predicative vs impredicative uninverses by typecheker parameter.

$$\frac{i : Nat, j : Nat, i < j}{Type_i : Type_j} \tag{A_1}$$

$$\frac{i : Nat, j : Nat}{Type_i \to Type_j : Type_{max(i,j)}} \tag{R_1}$$

3.4 Impredicative Universes

Propositional contractible bottom space is the only available extension to predicative hierarchy that not leads to inconsistency. However there is another option to have infinite impredicative hierarchy.

```
A_2 (n : nat) = U n : U (n + 1)    - non-cumulative
R_2 (m n : nat) = U m → U n : U n - impredicative
```

$$\frac{i : Nat}{Type_i : Type_{i+1}} \tag{A_2}$$

$$\frac{i : Nat, \quad j : Nat}{Type_i \to Type_j : Type_j} \tag{R_2}$$

3.5 Single Axiom Language

This langauge is called one axiom language (or pure) as eliminator and introduction adjoint functors inferred from type formation rule. The only computation rule of Pi type is called beta-reduction.

$$\frac{x : A \vdash B : Type}{\Pi \ (x : A) \to B : Type} \qquad (\Pi\text{-formation})$$

$$\frac{x : A \vdash b : B}{\lambda \ (x : A) \to b : \Pi \ (x : A) \to B} \qquad (\lambda\text{-intro})$$

$$\frac{f : (\Pi \ (x : A) \to B) \quad a : A}{f \ a : B \ [a/x]} \qquad (App\text{-elimination})$$

$$\frac{x : A \vdash b : B \quad a : A}{(\lambda \ (x : A) \to b) \ a = b \ [a/x] : B \ [a/x]} \qquad (\beta\text{-computation})$$

This language could be embedded in itself and used as Logical Framework for the Pi type:

```
record Pi (A: Type) :=
      (intro:  (A → Type) → Type)
      (lambda: (B: A → Type) → pi A B → intro B)
      (app:    (B: A → Type) → intro B → pi A B)
      (applam: (B: A → Type) (f: pi A B) → (a: A) →
               Path (B a) ((app B (lambda B f)) a) (f a))
      (lamapp: (B: A → Type) (p: intro B) →
               Path (intro B) (lambda B (λ (a:A) → app B p a)) p)
```

We extend the $PTS\infty$ with remote AST node wich means remote file loading from trusted storage, anyway this will be checked by typechecker. We deny recursion over remote node. We also add index to var for simplified de Bruin indexes.

```
data om = star              (n: nat)
        | var    (n: name) (n: nat)
        | remote (n: name) (n: nat)
        | app               (f a: pts)
        | lambda (x: name)   (d c: pts)
        | arrow              (d c: pts)
        | pi     (x: name)   (d c: pts)
```

3.6 Hierarchy

H returns the target Universe of B term dependendance on A. Thereare two dependance rules known as predicative and impredicative which returns max universe or universe of last term respectively.

```
dep A B impredicative → B
dep A B predicative   → max A B

h Arg Out → dep Arg Out om:hierarchy(impredicative)
```

3.7 Universes

Star returns the number of Universe. If argument is not Universe it returns $\{error, \_\}$.

```
star (:star,N) → N
star _         → (:error, "*")
```

3.8 Functions

Func returns true if the argument is functional space. Otherwise returns $\{error, \_\}$.

```
func ((:forall,),(I,O)) → true
func T                  → (:error,(:forall,T))
```

3.9 Variables

Var returns true if the var N is defined in dictionary B. Otherwise returns $\{error, \_\}$.

```
var N B            → var N B (proplists:is_defined N B)
var N B true       → true
var N B false      → (:error,("free var",N,proplists:get_keys(B)))
```

3.10 Shift

Shift renames var N in N

```
sh (:var,(N,I)),N,P) when I>=P   → (var,(N,I+1))
sh ((:forall,(N,0)),(I,O)),N,P) → ((:forall,(N,0)),sh I N P,sh O N P+1)
sh ((:lambda,(N,0)),(I,O)),N,P) → ((:lambda,(N,0)),sh I N P,sh O N P+1)
sh (Q,(L,R),N,P)                 → (Q,sh L N P,sh R N P)
sh (T,N,P)                       → T
```

3.11 Substitution

```
sub Term Name Value                   → sub Term Name Value 0
sub (:arrow,        (I,O)) N V L → (:arrow,        sub I N V L,sub O N V L);
sub ((:forall,(N,0)),(I,O)) N V L → ((:forall,(N,0)),sub I N V L,sub O N(sh V N 0)L+1)
sub ((:forall,(F,X)),(I,O)) N V L → ((:forall,(F,X)),sub I N V L,sub O N(sh V F 0)L)
sub ((:lambda,(N,0)),(I,O)) N V L → ((:lambda,(N,0)),sub I N V L,sub O N(sh V N 0)L+1)
sub ((:lambda,(F,X)),(I,O)) N V L → ((:lambda,(F,X)),sub I N V L,sub O N(sh V F 0)L)
sub (:app,          (F,A)) N V L → (:app,sub F N V L,sub A N V L)
sub (:var,          (N,L)) N V L → V
sub (:var,          (N,I)) N V L when I>L → (:var,(N,I-1))
sub T                      _ _ _ → T.
```

3.12 Type Checker

For sure in pure system we should be careful with **:remote** AST node. Remote AST nodes like #**List/Cons or** #**List/map** are a remote link to files. So using trick one should desire circular dependency over **:remote**. This is denied in the system. Same notes apply to normalization and definitional equality.

```
type (:star,N)                  _ → (:star,N+1)
type (:var,(N,I))               D → let true = var N D in keyget N D I
type (:remote,N)                D → cache type N D
type (:arrow,(I,O))             D → (:star,h(star(type I D)),star(type O D))
type ((:forall,(N,0)),(I,O)) D → (:star,h(star(type I D)),star(type O [(N,norm I)|D]))
type ((:lambda,(N,0)),(I,O)) D → let star (type I D),
                                      NI = norm I in ((:forall,(N,0)),(NI,type(O,[(N,NI)|D])))
type (:app,(F,A))               D → let T = type(F,D),
                                      true = func T,
                                      ((:forall,(N,0)),(I,O)) = T,
                                      Q = type A D,
                                      true = eq I Q in norm (subst O N A)
```

3.13 Normalization

Normalization perform substitutions on applications to functions searching over all function spaces, performing recursive normalization over the lambda and pi nodes.

```
norm :none                      → :none
norm :any                       → :any
norm (:app,(F,A))               → case norm F of
                                      ((:lambda,(N,0)),(I,O)) → norm (subst O N A)
                                                      NF → (:app,(NF,norm A)) end
norm (:remote,N)                → cache (norm N [])
norm (:arrow,       (I,O)) → ((:forall,("_",0)),(norm I,norm O))
norm ((:forall,(N,0)),(I,O)) → ((:forall,(N,0)),  (norm I,norm O))
norm ((:lambda,(N,0)),(I,O)) → ((:lambda,(N,0)),  (norm I,norm O))
norm T                          → T
```

3.14 Definitional Equality

Definitional Equality simple ensures the erlang term equality.

```
eq ((:forall,("_",0)), X) (:arrow,Y)      → eq X Y
eq (:app,(F1,A1))          (:app,(F2,A2)) → let true = eq F1 F2 in eq A1 A2
eq (:star,N)               (:star,N)      → true
eq (:var,(N,I))            (:var,(N,I))   → true
eq (:remote,N)             (:remote,N)    → true
eq ((:farall,(N1,0)),(I1,O1))
   ((:forall,(N2,0)),(I2,O2)) →
   let true = eq I1 I2 in eq O1 (subst (shift O2 N1 0) N2 (:var,(N1,0)) 0)
eq ((:lambda,(N1,0)),(I1,O1))
   ((:lambda,(N2,0)),(I2,O2)) →
   let true = eq I1 I2 in eq O1 (subst (shift O2 N1 0) N2 (:var,(N1,0)) 0)
eq (A,B)                          → (:error,(:eq,A,B))
```

4 Language Usage

This work expects to compile to limited target platforms. For now Erlang, Haskell and LLVM are awaiting. Erlang version is expected to be useful both on LING and BEAM Erlang virtual machines. This language allows you to define trusted operations in System F and extract this routines to Erlang/OTP platform and plug as trusted resourses. As example we also provide infinite coinductive process creation and inductive shell that linked to Erlang/OTP IO functions directly.

IO protocol We can construct in pure type system the state machine based on (co)-free monads driven by IO protocols. Assume String is equal List of Nat (as it is in Erlang natively), and three external constructors: getLine, putLine and pure. We need to put correspondent implementations on host platform as parameters to perform the actual IO.

```
String: Type = List Nat
data IO: Type =
     (getLine: (String -> IO) -> IO)
     (putLine: String -> IO)
     (pure: () -> IO)
```

4.1 Recursive Example

```
((#IO/replicateM #Nat/Five)
 (((((#IO/[>>=] #IO/data) #Unit/@) #IO/getLine) #IO/putLine))
```

Erlang Inductive Bindings.

```
pure() ->
    fun(IO) -> IO end.

getLine() ->
    fun(IO) -> fun(_) ->
        L = ch:list(io:get_line("> ")),
        ch:ap(IO,[L]) end end.

putLine() ->
    fun (S) -> fun(IO) ->
        io:format(": "),
        io:put_chars(ch:unlist(S)),
        ch:ap(IO,[S]) end end.

rec() ->
    ap('Morte':recursive(),
        [getLine(),putLine(),pure(),list([])]).
```

Here is example of Erlang/OTP shell running recursive example.

```
> om:rec().
> 1
: 1
> 2
: 2
> 3
: 3
> 4
: 4
> 5
: 5
#Fun<List.28.113171260>
```

4.2 Corecursive Example

```
( \ (r: *1)
 -> ( (((#IOI/MkIO r) (#Maybe/@ #IOI/data)) (#Maybe/Nothing #IOI/data))
    ( \ (m: (#Maybe/@ #IOI/data))
     -> (((((#Maybe/maybe #IOI/data) m) ((#IOI/F r) (#Maybe/@ #IOI/data)))
          ( \ (str: #IOI/data)
           -> ((((#IOI/putLine r) (#Maybe/@ #IOI/data)) str)
                (#Maybe/Nothing #IOI/data))))
        (((#IOI/getLine r) (#Maybe/@ #IOI/data))
         (#Maybe/Just #IOI/data))))))
```

Erlang Coinductive Bindings.

```
copure() ->
    fun (_) -> fun(IO) -> IO end end.

cogetLine() ->
    fun(IO) -> fun(_) ->
        L = ch:list(io:get_line("> ")),
        ch:ap(IO,[L]) end end.

coputLine() ->
    fun (S) -> fun(IO) ->
        X = ch:unlist(S),
        io:format(": "), io:put_chars(X),
        case X of "0\n" -> list([]);
                    _ -> corec() end end end.

corec() ->
    ap('Morte':corecursive(),
        [copure(),cogetLine(),coputLine(),copure(),list([])]).

> om:corec().
> 1
: 1
> 0
: 0
#Fun<List.3.113171260>
```

4.3 Sigma Type

Here we will show you some examples of Om Language usage. The PTS system is extremely powerful even without Sigma type. But we can encode **Sigma** type similar how we encode **Prod** tuple pair in Boehm encoding. Let's formulate Sigma type as inductive type:

```
data Sigma (A: Type) (P: A -> Type): Type =
    (intro: (x:A) (y:P x) -> Sigma A P)

-- Sigma/@
   \ (A: *)
-> \ (P: A -> *)
-> \/ (Exists: *)
-> \/ (Intro: \/ (x: A) -> P x -> Exists)
-> Exists

-- Sigma/Intro
   \ (A: *)
-> \ (P: A -> *)
-> \ (x: A)
-> \ (y: P x)
-> \ (Exists: *)
-> \ (Intro: \/ (z: A) -> P z -> Exists)
-> Intro x y

-- Sigma/fst
   \ (A: *)
-> \ (B: A -> *)
-> \ (S: #Sigma/@ A B)
-> S A ( \ (x: A) -> \ (y: B) -> x)

-- Sigma/snd
   \ (A: *)
-> \ (B: A -> *)
-> \ (S: #Sigma/@ A B)
-> S A ( \ (x: A) -> \ (y: B) -> y)
```

For using **Sigma** type for Logic purposes one should change the home Universe of the type to **Prop**. Here it is:

```
data Sigma (A: Prop) (P: A -> Prop): Prop =
    (intro: (x:A) (y:P x) -> Sigma A P)
```

4.4 Equality Type

Another example of expressivness is Equality type a la Martin-Löf.

```
data Equ (A: Type): A -> A -> Type :=
    (refl (a: A): Equ A a a)

-- Equ/@
   \ (A: *)
-> \ (x: A)
-> \ (y: A)
-> \/ (Equ: A -> A -> *)
-> \/ (Refl: \/ (z: A) -> Equ z z)
-> Equ x y

-- Equ/Refl
   \ (A: *)
-> \ (x: A)
-> \ (Equ: A -> A -> *)
-> \ (Refl: \/ (z: A) -> Equ z z)
-> Refl x
```

5 Language with Inductive Types

Dealing in Pure Type System setting limits the expressiveness of the language. So called Calculus of Inductive constructions are used as a top language on top of PTS to construct proofs and write programs. He we will show you a sketch of such inductive language which indendent to be a language extension to PTS system.

Exe is a general purpose functional language with Π and Σ types, recursive algebraic types, higher order functions, corecursion, free monad for effects encoding. It compiles to a small MLTT core of dependent type system with inductive types and equality.

It also has an *Id* type wich is useful for proving things, *Let* sytax sugar, case analisys over inductive types and also a *pi*-Calculus primitives: *spawn*, *send* and *receive* which are native mappings to Erlang/OTP.

5.1 BNF

```
<> ::= #option
[] ::= #list
 | ::= #sum
 1 ::= #unit
 I ::= #identifier

 U ::= Type < #nat >
 T ::= 1 | ( I : O ) T
 F ::= 1 | I : O = O , F
 B ::= 1 | [ | I [ I ] → O ]
 L ::= 1 | I T
 O ::= I | ( O ) |
       U | O → O
         | O O
         | fun ( I : O ) → O
         | fst O
         | snd O
         | id O O O
         | transport O O O O O
         | ( I : O ) * O
         | ( I : O ) → O
         | data   I T : O := T
         | record I T : O := T
         | let F in O
         | case O B
         | receive B
         | spawn O O
         | send O O
```

5.2 AST

The model in Cubical and Coq of the Exe language is available at infinity[1] repository of groupoid organization.

```
data tele (A: U)   = emp | tel (n: name) (b: A) (t: tele A)
data branch (A: U) =        br (n: name) (args: list name) (term: A)
data label (A: U)  =       lab (n: name) (t: tele A)
data ind
    = star                      (n: nat)
    | var    (n: name)          (i: nat)
    | app             (f a: ind)
    | lambda (x: name) (d c: ind)
    | pi      (x: name) (d c: ind)
    | sigma  (n: name) (a b: ind)
    | arrow            (d c: ind)
    | pair            (a b: ind)
    | fst             (p:   ind)
    | snd             (p:   ind)
    | id              (a b: ind)
    | idpair          (a b: ind)
    | idelim          (a b c d e: ind)
    | data_ (n: name) (t: tele ind) (labels:   list (label ind))
    | case   (n: name) (t: ind)      (branches: list (branch ind))
    | ctor   (n: name)               (args:     list ind)
```

The Erlang version of parser encoded with OTP library **yecc** which implements LALR-1 grammar generator. This version resembles the model and slightly based on cubical type checker by Mortberg et all.

```
mod   -> 'module' id 'where' imp dec        : {module,'$2','$3','$4','$5'}.
imp   -> skip imp                           : '$2'.
imp   -> '$empty'                           : [].
imp   -> 'import' id imp                    : [{import,'$2'}|'$3'].
tele  -> '$empty'                           : [].
tele  -> '(' exp ':' exp ')' tele           : {tele,uncons('$2'),'$4','$6'}.
exp   -> app                                : '$1'.
exp   -> exp arrow exp                       : {arrow,'$1','$3'}.
exp   -> '(' exp ')'                        : '$2'.
exp   -> lam '(' exp ':' exp ')' arrow exp : {lam,uncons('$3'),'$5','$8'}.
exp   -> '(' exp ':' exp ')' arrow exp      : {pi,uncons('$2'),'$4','$7'}.
exp   -> '(' id ':' exp ')' '*' exp         : {sigma,'$2','$4','$7'}.
exp   -> id                                 : '$1'.
app   -> exp exp                            : {app,'$1','$2'}.
dec   -> '$empty'                           : [].
dec   -> def skip dec                       : ['$1'|'$3'].
def   -> 'data' id tele '=' sum             : {data,'$2','$3','$5'}.
def   -> id tele ':' exp '=' exp            : {def,'$1','$2','$4','$6'}.
sum   -> '$empty'                           : [].
sum   -> id tele                            : {ctor,'$1','$2'}.
sum   -> id tele '|' sum                    : [{ctor,'$1','$2'}|'$4'].
```

---

[1]https://github.com/groupoid/infinity/tree/master/priv

5.3 Inductive Types

There are two types of recursion: one is least fixed point (as $F_A\ X\ =\ 1 + A \times X$ or $F_A\ X\ =\ A + X \times X$), in other words the recursion with a base (terminated with a bounded value), lists and trees are examples of such recursive structures (so we call induction recursive sums); and the second is greatest fixed point or recursion withour a base (as $F_A\ X\ =\ A \times X$) — such kind of recursion on infinite lists (codata, streams, coinductive types) we can call recursive products.

5.4 Polynomial Functors

Least fixed point trees are called well-founded trees and encode polynomial functors.

Natural Numbers: $\mu\ X \to 1 + X$
List A: $\mu\ X \to 1 + A \times X$
Lambda calculus: $\mu\ X \to 1 + X \times X + X$
Stream: $\nu\ X \to A \times X$
Potentialy Infinite List A: $\nu\ X \to 1 + A \times X$
Finite Tree: $\mu\ X \to \mu\ Y \to 1 + X \times Y = \mu\ X = List\ X$

As we know there are several ways to appear for variable in recursive algebraic type. Least fixpoint are known as an recursive expressions that have a base of recursion Both recursive and corecursive datatypes could be encoded using Boem-Berarducci encoding as an non-recursive definitions of folds that include in indentity signature all the constructor components of (co)inductive type.

5.5 Lists

The data type of lists over a given set A can be represented as the initial algebra $(\mu L_A, in)$ of the functor $L_A(X) = 1 + (A \times X)$. Denote $\mu L_A = List(A)$. The constructor functions $nil : 1 \to List(A)$ and $cons : A \times List(A) \to List(A)$ are defined by $nil = in \circ inl$ and $cons = in \circ inr$, so $in = [nil, cons]$. Given any two functions $c : 1 \to C$ and $h : A \times C \to C$, the catamorphism $f = ([c, h]) : List(A) \to C$ is the unique solution of the equation system:

$$\begin{cases} f \circ nil = c \\ f \circ cons = h \circ (id \times f) \end{cases}$$

where $f = foldr(c, h)$. Having this the initial algebra is presented with functor $\mu(1 + A \times X)$ and morphisms sum $[1 \to List(A), A \times List(A) \to List(A)]$ as catamorphism. Using this encdoding the base library of List will have following form:

$$\begin{cases} foldr = (\![f \circ nil, h]\!), \, f \circ cons = h \circ (id \times f) \\ len = (\![zero, \lambda \, a \, n \rightarrow succ \, n]\!) \\ (++) = \lambda \, xs \, ys \rightarrow (\![\lambda(x) \rightarrow ys, cons]\!)(xs) \\ map = \lambda \, f \rightarrow (\![nil, cons \circ (f \times id)]\!) \end{cases}$$

```
data list: (A: *) → * :=
    (nil: list A)
    (cons: A → list A → list A)
```

$$\begin{cases} list = \lambda \, ctor \rightarrow \lambda \, cons \rightarrow \lambda \, nil \rightarrow ctor \\ cons = \lambda \, x \, \rightarrow \lambda \, xs \rightarrow \lambda \, list \rightarrow \lambda \, cons \rightarrow \, \lambda \, nil \rightarrow cons \, x \, (xs \, list \, cons \, nil) \\ nil = \lambda \, list \rightarrow \lambda \, cons \rightarrow \lambda \, nil \rightarrow nil \end{cases}$$

```
-- List/@
   \ (A : *)
-> \/ (List: *)
-> \/ (Cons: \/ (Head: A) -> \/ (Tail: List) -> List)
-> \/ (Nil: List)
-> List

-- List/Cons
   \ (A: *)
-> \ (Head: A)
-> \ (Tail:
      \/ (List: *)
   -> \/ (Cons:
         \/ (Head: A)
      -> \/ (Tail: List)
      -> List)
   -> \/ (Nil: List)
   -> List)
-> \ (List: *)
-> \ (Cons:
      \/ (Head: A)
   -> \/ (Tail: List)
   -> List)
-> \ (Nil: List)
-> Cons Head (Tail List Cons Nil)

-- List/Nil
   \ (A: *)
-> \ (List: *)
-> \ (Cons:
      \/ (Head: A)
   -> \/ (Tail: List)
   -> List)
-> \ (Nil: List)
-> Nil
```

```
record lists: (A B: *) :=
        (len: list A → integer)
        ((++): list A → list A → list A)
        (map: (A → B) → (list A → list B))
        (filter: (A → bool) → (list A → list A))
```

$$\begin{cases} len = foldr \ (\lambda \ x \ n \to succ \ n) \ 0 \\ (++) = \lambda \ ys \to foldr \ cons \ ys \\ map = \lambda \ f \to foldr \ (\lambda x \ xs \to cons \ (f \ x) \ xs) \ nil \\ filter = \lambda \ p \to foldr \ (\lambda x \ xs \to if \ p \ x \ then \ cons \ x \ xs \ else \ xs) \ nil \\ foldl = \lambda \ f \ v \ xs = foldr \ (\lambda \ xg \to \ (\lambda \to g \ (f \ a \ x))) \ id \ xs \ v \end{cases}$$

5.6 Normal Forms

Here is example of List/map generic function.

Lists/map

```
λ (a:  *) → λ (b:  *) → λ (f:  a → b) → λ (xs:  ∀ (List:  *) →
∀ (Cons:  ∀ (head:  a) → ∀ (tail:  List) → List) → ∀ (Nil:  List)
→ List) → xs (∀ (List:  *) → ∀ (Cons:  ∀ (head:  b) → ∀ (tail:  List)
→ List) → ∀ (Nil:  List) → List) (λ (head:  a) → λ (tail:  ∀ (List:
*) → ∀ (Cons:  ∀ (head:  b) → ∀ (tail:  List) → List) → ∀ (Nil:
List) → List) → λ (List:  *) → λ (Cons:  ∀ (head:  b) → ∀ (tail:
List) → List) → λ (Nil:  List) → Cons (f head) (tail List Cons Nil))
(λ (List:  *) → λ (Cons:  ∀ (head:  b) → ∀ (tail:  List) → List)
→ λ (Nil:  List) → Nil)
```

5.7 Prelude Base Library

The base library is modeled in cubical type checker.

```
data Nat: Type :=
    (Zero: Unit → Nat)
    (Succ: Nat → Nat)

data List (A: Type) : Type :=
    (Nil: Unit → List A)
    (Cons: A → List A → List A)

record list: Type :=
    (len: List A → integer)
    ((++): List A → List A → List A)
    (map: (A,B: Type) (A → B) → (List A → List B))
    (filter: (A → bool) → (List A → List A))

record String: List Nat := List.Nil

data IO: Type :=
    (getLine: (String → IO) → IO)
    (putLint: String → IO)
    (pure: () → IO)

record IO: Type :=
    (data: String)
    ([>>=]: ...)

record Morte: Type :=
    (recursive: IO.replicateM Nat.Five
                (IO.[>>=] IO.data Unit IO.getLine IO.putLine))
```

5.8 Compiler Passes

The underlying OM typechecker and compiler is a target language for EXE general purpose language. The overal size of OM language with extractor to Erlang is 263 lines of code.

TOKEN	54 LOC	Handcoded Tokenizer
PARSER	81 LOC	Parser
NORMAL	60 LOC	Term normalization and typechecking
ERASE	36 LOC	Delete information about types
EXTRACT	32 LOC	Extract Erlang Code

We benchmarked the unrolling of inductive list type in Church encoding extracted with OM with native erlang **lists:foldl**.

```
Pack/Unpack 1 000 000 Inductive Nat:    776407 us
Pack/Unpack 1 000 000 ErlangOTP List:   148084 us
Pack/Unpack 1 000 000 Inductive List: 1036461 us
```

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