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Categorical Foundations for Lambek Encoding

Technical Article

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Kyiv 2016

Contents

1 Category Theory

1.1 Programs and Functions

Category theory is widely used as an instrument for mathematicians for software analisys. Category theory could be treated as an abstract algebra of functions. Let's define an Category formally: **Category** consists of two lists: the one is morphisms (arrows) and the second is objects (domains and codomains of arrows) along with assoociative operation of composition and unit morphism that exists for all objects in category.

The formation axoims of objects and arrows are not given here and autopostulating yet. Formation axoims will be introduced during exponential definition. Objects A and B of an arrow $f : A \to B$ are called **domain** and **codomain** respectively.

Intro axioms – associativity of composition and left/right unit arrow compisitions show that categories are actually typed monoids, which consist of morphisms and operation of composition. There are many languages to show the semantic of categories such as commutative diagrams and string diagrams however here we define here in proof-theoretic manner:

$$\frac{\Gamma \vdash f : A \to B \qquad \Gamma \vdash g : B \to C}{\Gamma \vdash g \circ f : A \to C}$$

$$\frac{}{\Gamma \vdash id_A : A \to A}$$

$$\frac{\Gamma \vdash f : B \to A \qquad \Gamma \vdash g : C \to B \qquad \Gamma \vdash h : D \to C}{\Gamma \vdash (f \circ g) \circ h = f \circ (g \circ h) : D \to A}$$

$$\frac{\Gamma \vdash f : A \to B}{\Gamma \vdash f \circ id_A = f : A \to B}$$

$$\frac{\Gamma \vdash f : A \to B}{\Gamma \vdash id_B \circ f = f : A \to B}$$

Composition shows an ability to connect the result space of the previous evaluation (codomain) and the arguments space of the next evaluation (domain). Composition is fundamental property of morphisms that allows us to chain evaluations.

1. $A : *$
2. $A : * \ , \ B : * \implies f : A \to B$
3. $f : B \to C \ , \ g : A \to B \implies f \circ g : A \to C$
4. $(f \circ g) \circ h = f \circ (g \circ h)$
5. $A \implies id : A \to A$
6. $f \circ id = f$
7. $id \circ f = f$

1.2 Algebraic Types and Cartesian Categories

After composition operation of construction of new objects with morphisms we introduce operation of construction cartesian product of two objects A and B of a given category along with morphism product $< f, g >$ with a common domain, that is needed for full definition of cartesian product of $A \times B$.

This is an internal language of cartesian category, in which for all two selected objects there is an object of cartesian product (sum) of two objects along with its $\bot$ terminal (or $\top$ coterminal) type. Exe languages is always equiped with product and sum types.

Product has two eliminators π with an common domain, which are also called projections of an product. The sum has eliminators i with an common codomain. Note that eliminators π and i are isomorphic, that is $\pi \circ \sigma = \sigma \circ \pi = id$.

$$\frac{\Gamma\ x : A \times B}{\Gamma \vdash \pi_1\ : A \times B \to A; \Gamma \vdash \pi_2\ : A \times B \to B}$$

$$\frac{}{\Gamma \vdash\ \bot}$$

$$\frac{\Gamma \vdash\ a : A \qquad \Gamma \vdash\ b : B}{\Gamma \vdash\ (a, b) : A \times B}$$

$$\frac{\Gamma \vdash\ a : A \qquad \Gamma \vdash\ b : B}{\Gamma \vdash a \mid b : A \otimes B}$$

$$\frac{}{\Gamma \vdash\ \top}$$

$$\frac{\Gamma\ x : A \otimes B}{\Gamma \vdash ß_1 : A \to A \otimes B; \Gamma \vdash ß_2 : B \to A \otimes B}$$

The $\bot$ type in Haskell is used as **undefined** type (empty sum component presented in all types), that is why Hask category is not based on cartesian closed but CPO [**?**]. The $\bot$ type has no values. The $\top$ type is known as unit type or zero tuple () often used as an default argument for function with zero arguments. Also we include here an axiom of morphism product which is given during full definition of product using commutative diagram. This axiom is needed for applicative programming in categorical abstract machine. Also consider co-version of this axiom for $[f, g] : B + C \to A$ morphism sums.

$$\frac{\Gamma \vdash\ f : A \to B \qquad \Gamma \vdash\ g : A \to C \qquad \Gamma \vdash\ B \times C}{\Gamma \vdash\ \langle f, g \rangle : A \to B \times C}$$

$$\pi_1 \circ \langle f, g \rangle = f$$
$$\pi_2 \circ \langle f, g \rangle = g$$
$$\langle f \circ \pi_1, f \circ \pi_2 \rangle = f$$
$$\langle f, g \rangle \circ h = \langle f \circ h, g \circ h \rangle$$
$$\langle \pi_1, \pi_2 \rangle = id$$

1.3 Exponential, λ-calculus and Cartesian Closed Categories

Being an internal language of cartesian closed category, lambda calculus except variables and constants provides two operations of abstraction and applications which defines complete evaluation language with higher order functions, recursion and corecursion, etc.

To explain functions from the categorical point of vew we need to define categorica exponential $f : A^B$, which are analogue to functions $f : A \to B$. As we already defined the products and terminals we could define an exponentials with three axioms of function construction, one eliminator of application with apply a function to its argument and axiom of currying the function of two arguments to function of one argument.

$$\frac{\Gamma x : A \vdash M : B}{\Gamma \vdash \lambda\ x\ .\ M : A \to B}$$

$$\frac{\Gamma\ f : A \to B \qquad \Gamma\ a : A}{\Gamma \vdash apply\ f\ a\ : (A \to B) \times A \to B}$$

$$\frac{\Gamma \vdash f : A \times B \to C}{\Gamma \vdash curry\ f : A \to (B \to C)}$$

$$apply \circ \langle (curry\ f) \circ \pi_1, \pi_2 \rangle = f$$
$$curry\ apply \circ \langle g \circ \pi_1, \pi_2 \rangle) = g$$
$$apply \circ \langle curry\ f, g \rangle = f \circ \langle id, g \rangle$$
$$(curry\ f) \circ g = curry\ (f \circ \langle g \circ \pi_1, \pi_2 \rangle)$$
$$curry\ apply = id$$

1.4 λ-language

$$\text{Objects} : \bot \mid\ \to\ \mid \times$$
$$\text{Morphisms} : id \mid f \circ g \mid \langle f, g \rangle \mid apply \mid \lambda \mid curry$$

1.5 Functors, Λ-calculus

Functor comes as a notion of morphisms in categories whose objects are categories. Functors preserve compositions of arrows and identities, otherwise it would be impossible to deal with categories. One level up is notion of morphism between categories whose objects are Functors, such morphisms are called natural transformations. Here we need only functor definition which is needed as general type declarations.

$$\frac{\Gamma \vdash f\ :\ A \to B}{\Gamma \vdash F\ f\ :\ (A \to B) \to (F\ a \to F\ b)}$$

$$\frac{\Gamma \vdash id_A\ : A \to A}{\Gamma \vdash F\ id_A\ =\ id_{FA}\ :\ F\ A \to F\ A}$$

$$\frac{\Gamma \vdash f\ : B \to C,\ g : A \to B}{\Gamma \vdash F\ f \circ F\ g\ =\ F(f \circ g)\ :\ F\ A \to F\ C}$$

We start thinking of functors on dealing with typed theories, because functors usually could be seen as higher order type con

1.6 Algebras

F-Algebras gives us a categorical understanding recursive types. Let $F : C \to C$ be an endofunctor on category C. An F-algebra is a pair (C, ϕ), where C is an object and $\phi \ : F\ C \to C$ an arrow in the category C. The object C is the carrier and the functor F is the signature of the algebra. Reversing arrows gives us F-Coalgebra.

$$\begin{array}{ccc} F\ C & \xrightarrow{\varphi} & C \\ {\scriptstyle F\ f}\downarrow & & \downarrow{\scriptstyle f} \\ F\ D & \xrightarrow{\psi} & D \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{\varphi} & F\ C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F\ f} \\ D & \xrightarrow{\psi} & F\ D \end{array}$$

$$f \circ \varphi = \psi \circ F\ f \qquad \psi \circ f = F\ f \circ \varphi$$

1.7 Initial Algebras

A F-algebra $(\mu F, in)$ is the initial F-algebra if for any F-algebra (C, φ) there exists a unique arrow $(\!|\varphi|\!) : \mu F \to C$ where $f = (\!|\varphi|\!)$ and is called catamorphism. Similar a F-coalgebra $(\nu F, out)$ is the terminal F-coalgebra if for any F-coalgebra (C, φ) there exists unique arrow $[\![\varphi]\!] : C \to \nu F$ where $f = [\![\varphi]\!]$

$$\begin{array}{ccc} F\ \mu F & \xrightarrow{in} & \mu F \\ {\scriptstyle F\ (\!|\varphi|\!)}\downarrow & & \downarrow{\scriptstyle (\!|\varphi|\!)} \\ FC & \xrightarrow{\varphi} & C \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{\phi} & F\ C \\ {\scriptstyle [\![\varphi]\!]}\downarrow & & \downarrow{\scriptstyle F\ [\![\varphi]\!]} \\ \nu F & \xrightarrow{out} & F\nu F \end{array}$$

$$f \circ in = \varphi \circ F\ f \equiv f = (\!|\varphi|\!) \qquad out \circ f = F\ f \circ \varphi \equiv f = [\![\varphi]\!]$$

1.8 Recursive Types

As was shown by Wadler [**?**] we could deal with recusrive equations having three axioms: one $fix : (A \to A) \to A$ fixedpoint axiom, and axioms $in : F\ T \to T$ and $out : T \to F\ T$ of recursion direction. We need to define fixed point as axiom because we can't define recursive axioms. This axioms also needs functor axiom defined earlier.

$$\frac{\Gamma \vdash M : F\ (\mu\ F)}{\Gamma \vdash in_{\mu F}\ M : \mu\ F} \qquad \frac{\Gamma \vdash M : A \to A}{\Gamma \vdash fix\ M : A}$$

$$\frac{\Gamma \vdash M : \mu F}{\Gamma \vdash out_{\mu F}\ M : F\ (\mu\ F)}$$

2 Type Theory

2.1 Intuitionistic Type Theory

Using Build Yourself Type Teory approach sooner or later you should decide the pallete of inductive structures. Such in Coq abstract algebra framework was built upon polymorphic records [**?**] rather that type classes engine like it is used in Agda and Idris. However Idris still lack of polymorphic records and coinductive types. Lean also lack of coinductive structures but has powerful non-recursive polymorphic records which are used in Lean HoTT library.

As was show by Spephan Kaes [**?**], one of strategy of type clasess engine implementataion is using polymorphic structures which allows us do deal with persistent structures on a low theoretical level. Moreover this style of coding is completly compatible with Erlang records which are used to model KVS and N2O hierarchies.

2.2 Universes

$$\frac{i : Nat}{Type_i} \quad \text{(sorts)}$$

$$\frac{i : Nat}{Type_i : Type_{i+1}} \quad \text{(axioms)}$$

$$\frac{i : Nat, \quad j : Nat}{Type_i \to Type_j : Type_{max(i,j)}} \quad \text{(rules)}$$

2.3 Dependent Types

$$\frac{\pi_1 : A \quad u : A \vdash \pi_2 : B}{[\pi_1/u]\ \pi_2 : B} \quad \text{(subst)}$$

$$\frac{x : A \vdash B : Type}{\Pi\ (x : A) \to B : Type} \quad \text{(Π-formation)}$$

$$\frac{x : A \vdash b : B}{\lambda\ (x : A) \to b : \Pi\ (x : A) \to B} \quad \text{(λ-intro)}$$

$$\frac{f : (\Pi\ (x : A) \to B) \quad a : A}{f\ a : B\ [a/x]} \quad \text{(}App\text{-elimination)}$$

$$\frac{x : A \vdash b : B \quad a : A}{(\lambda\ (x : A) \to b)\ a = b\ [a/x] : B\ [a/x]} \quad \text{(}\beta\text{-computation)}$$

2.4 Equality Types

$$\frac{x : A \quad b : A \quad A : Type}{Id(A, a, b) : Type} \qquad (Id\text{-formation})$$

$$\frac{a : A}{refl(A, a) : Id(A, a, a)} \qquad (Id\text{-intro})$$

$$\frac{p : Id(a, b) \quad x, y : A \quad u : Id(x, y) \vdash E : Type \quad x : A \vdash d : E\ [x/y,\ refl(x)/u]}{J(a, b, p, (x, y, u)\ d) : E\ [a/x,\ b/y,\ p/u]} \qquad (J\text{-elimination})$$

$$\frac{a, x, y : A, \quad u : Id(x, y) \vdash E : Type \quad x : A \vdash d : E\ [x/y,\ refl(x)/u]}{J(a, a, refl(a), (x, y, u)\ d) = d\ [a/x] : E\ [a/y,\ refl(a)/u]} \qquad (Id\text{-computation})$$

2.5 Inductive Types

$$\frac{A : Type \quad x : A \quad B(x) : Type}{W(x : A) \to B(x) : Type} \qquad (W\text{-formation})$$

$$\frac{a : A \quad t : B(a) \to W}{sup(a, t) : W} \qquad (W\text{-intro})$$

$$\frac{w : W \vdash C(w) : Type \quad x : A,\ u : B(x) \to W, \quad v : \Pi(y : B(x)) \to C(u(y)) \vdash c(x, u, v) : C(sup(x, u))}{w : W \vdash wrec(w, c) : C(w)} \qquad (W\text{-elimination})$$

$$\frac{w : W \vdash C(w) : Type \quad x : A,\ u : B(x) \to W, \quad v : \Pi(y : B(x)) \to C(u(y)) \vdash c(x, u, v) : C(sup(x, u))}{x : A, \quad u : B(x) \to W \vdash wrec(sup(x, u), c) \ = \ c(x, u, \lambda(y : B(x)) \to wrec(u(y), c)) : C(sup(x, u))} \qquad (W\text{-computation})$$

2.6 Recursor

There is a belief that recursor (non-dependent eliminator) in Type Theory is a weaker property than induction principle (dependent eliminator). At the same time from category theory we know that Universal Property defines the object uniquely. In the case of initial object in the category of algebras, the initiality could be defined by recursor. That means that all properties of algebra follow from its initiality, as a case it is possible to get the recursor from induction. There is a sensitive moment here, all categorical constructions are being formulated with defined equality on morphisms, in type theory the equality is built-in type that could have extended properties. Simplify we could say that we can get recursor from induction without equality, and with proper equality we could get induction from recursor.

2.7 Fibrations

Mechanism of getting induction principle from equality is based on the presentation of dependent types through fibrations. Hereby dependent type $(D : B \to Type)$ is defined as $(p : Sigma\ B\ P \to B)$ which projects dependent pair to the first field. In topology such approach is called fibration. To the other direction for a given morphism $(p : E \to B)$ which we understands as fibration with projection p, we could get its dependent type as $(D : B \to Type)$ by calculation in every point $(b : B)$ its image of projection p by using equality on elements of B. In type theory besides depndent pair $Sigma$ also used the dependent product Pi. In encoding of dependent types with fibrations there is a correspondance between elements of dependent and morphism-fibrations for projection p: such $(s : B \to E)$ that $s * p = I$. The example of this implementaion could be seen in

EXE[1] OM[2]

2.8 Induction

The input for induction is a predicate — dependent type encoded with $(p : E \to B)$. Induction needed additional information for predicate. The type of induction is defined by set of inductive constructors. Induction is just a statement that on E we have the structure of F-algebra of inductive type. Now we could apply recursor to E getting the map $(I \to E)$ from initial object which in fact the section (fibre bundle) of fibration and thus defines the dependent function which is a proved value of induction principle. The example for $Bool$ could be found in

EXE [3] OM [4]

Summarizing we encode types of source lambda calculus with objects of selected category, dependent types with fibrations, dependent function as fibrations, inductive types as limits of identity functors on category of F-algebras.

[1] https://github.com/groupoid/exe/blob/master/prelude/macro.new/Mini.macro
[2] https://github.com/groupoid/om/blob/master/priv/posets/sec2all
[3] https://github.com/groupoid/exe/blob/master/prelude/macro.new/Data.Bool.macro
[4] https://github.com/groupoid/om/blob/master/priv/posets/Data/Bool/induc