1D Fokker-Planck eqn: $\dfrac{\partial P(x,t)}{\partial t} - D_0\beta U'(x)\dfrac{\partial P(x,t)}{\partial x} = D_0\dfrac{\partial^2 P(x,t)}{\partial x^2} + D_0\beta U''(x)P(x,t)$

replace the coefficient. $a(x) = -D_0\beta U'(x)$, $\partial = D_0$, $b(x) = D_0\beta U''(x) = -\dfrac{\partial a(x)}{\partial x}$.

and change the form to general one-dimensional convection-diffusion-reaction eqn.

$$\frac{\partial u}{\partial t} + a(x)\frac{\partial u}{\partial x} = \partial\frac{\partial^2 u}{\partial x^2} + b(x)u \quad \text{or} \quad u_t + a(x)u_x = \partial u_{xx} + b(x)u$$

Aproximate $\left.\dfrac{\partial u}{\partial t}\right|_i^n = \dfrac{(u_i^{n+1} - u_i^n)}{h_t}$, $a(x_i) = a_i$, $b(x_i) = b_i$, $u\big|_i^n = u_i^n$

$$\left.\frac{\partial u}{\partial x}\right|_i^n = (1-\phi)\left[\frac{(1-\gamma)(u_i^n - u_{i-1}^n) + \gamma(u_{i+1}^n - u_i^n)}{h_x}\right] + \phi\left[\frac{(1-\gamma)(u_i^{n+1} - u_{i-1}^{n+1}) + \gamma(u_{i+1}^{n+1} - u_i^{n+1})}{h_x}\right]$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_i^n = (1-\phi)\left[\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h_x^2}\right] + \phi\left[\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h_x^2}\right]$$

u_{i-1}^{n+1}, u_i^{n+1}, u_{i+1}^{n+1}

$\sim u_{i-1}^n$, u_i^n, u_{i+1}^n

$$\frac{1}{h_t}(u_i^{n+1} - u_i^n) + \frac{a_i}{h_x}\Big[(1-\phi)(1-\gamma)(u_i^n - u_{i-1}^n) + (1-\phi)\gamma(u_{i+1}^n - u_i^n) + \phi(1-\gamma)(u_i^{n+1} - u_{i-1}^{n+1}) + \phi\gamma(u_{i+1}^{n+1} - u_i^{n+1})\Big] = \partial(1-\phi)\left[\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h_x^2}\right] + \partial\phi\left[\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h_x^2}\right] + b_i\phi u_i^{n+1} + b_i(1-\phi)u_i^n$$

let $C_i = \dfrac{a_i h_t}{h_x}$, $S = \dfrac{\partial h_t}{h_x^2}$

$$\left[-C_i\phi(1-\gamma) - S\phi\right]u_{i-1}^{n+1} + \left[1 - b_i\phi h_t + C_i\phi(1-\gamma) - C_i\phi\gamma + 2S\phi\right]u_i^{n+1}$$
$$+ \left[C_i\phi\gamma - S\phi\right]u_{i+1}^{n+1} = \left[C_i(1-\phi)(1-\gamma) + S(1-\phi)\right]u_{i-1}^n + \left[1 - C_i(1-\phi)(1-\gamma) + C_i(1-\phi)\gamma - 2S(1-\phi)\right.$$
$$\left. + b_i(1-\phi)h_t\right]u_i^n + \left[-C_i(1-\phi)\gamma + S(1-\phi)\right]u_{i+1}^n$$

$A_0 = -\phi[S + C_i(1-\gamma)]$, $A_1 = 1 - \phi[b_i h_t + C_i(2\gamma - 1) - 2S]$, $A_2 = \phi[C_i\gamma - S]$

$A_3 = (1-\phi)[C_i(1-\gamma) + S]$, $A_4 = 1 + (1-\phi)[b_i h_t + C_i(2\gamma - 1) - 2S]$, $A_5 = (1-\phi)[S - C_i\gamma]$

$\Rightarrow A_0 u_{i-1}^{n+1} + A_1 u_i^{n+1} + A_2 u_{i+1}^{n+1} = A_3 u_{i-1}^n + A_4 u_i^n + A_5 u_{i+1}^n$

Boundary conditions.

(1) reflecting boundary at position a: (Assume $x_0 = a$)

$$\frac{\partial u}{\partial t} = \partial\frac{\partial^2 u}{\partial x^2} - a(x)\frac{\partial u}{\partial x} - \frac{\partial a(x)}{\partial x}u = \frac{\partial}{\partial x}\left[\partial\frac{\partial u}{\partial x} - a(x)u\right] \Rightarrow \partial\frac{\partial u}{\partial x} - a(x)u = 0$$

$$\Rightarrow \frac{u_1 - u_{-1}}{2h_x} - \frac{a_0}{\partial}u_0 = 0 \Rightarrow u_{-1} = u_1 - \frac{2a_0 h_x}{\partial}u_0 \ \& \ A_0 u_{-1}^{n+1} + A_1 u_0^{n+1} + A_2 u_1^{n+1} = A_3 u_{-1}^n + A_4 u_0^n + A_5 u_1^n$$

$$u_{-1}^{n+1} = u_1^{n+1} - \frac{2a_0 h_x}{\partial}u_0^{n+1} \quad \& \quad u_{-1}^n = u_1^n - \frac{2a_0 h_x}{\partial}u_0^n$$

$$A_0\left(u_1^{n+1}-\frac{2a_0h_x}{a}u_0^{n+1}\right)+A_1u_0^{n+1}+A_2u_1^{n+1}=A_3\left(u_1^n-\frac{2a_0h_x}{a}u_0^n\right)+A_4u_0^n+A_5u_1^n$$

$$\left(A_1-\frac{2a_0h_x}{a}A_0\right)u_0^{n+1}+(A_0+A_2)u_1^{n+1}=\left(A_4-\frac{2a_0h_x}{a}A_3\right)u_0^n+(A_3+A_5)u_1^n$$

(2) absorbing boundary at position b. (Assume $x_m=b$)

$u_m=0$, $\Rightarrow u_m^n=0,\ u_m^{n+1}=0$

last row of the assemble matrix: (only need to be row $m-1$)

$$A_0u_{m-2}^{n+1}+A_1u_{m-1}^{n+1}=A_3u_{m-2}^n+A_4u_{m-1}^n$$

$\Rightarrow P_h\vec{u}^{n+1}=Q_h\vec{u}^n$:

$$\begin{pmatrix} A_1-\frac{2a_0h_x}{a}A_0 & A_0+A_2 & & & \\ A_0 & A_1 & A_2 & & \\ & A_0 & A_1 & A_2 & \\ & & \ddots & \ddots & \ddots \\ & & A_0 & A_1 & A_2 \\ & & & A_0 & A_1 \end{pmatrix}\begin{pmatrix} u_0^{n+1}\\ u_1^{n+1}\\ u_2^{n+1}\\ \vdots\\ u_{m-2}^{n+1}\\ u_{m-1}^{n+1}\end{pmatrix}=$$

$$\begin{pmatrix} A_4-\frac{2a_0h_x}{a}A_3 & A_3+A_5 & & & \\ A_3 & A_4 & A_5 & & \\ & A_3 & A_4 & A_5 & \\ & & \ddots & \ddots & \ddots \\ & & A_3 & A_4 & A_5 \\ & & & A_3 & A_4 \end{pmatrix}\begin{pmatrix} u_0^{n}\\ u_1^{n}\\ u_2^{n}\\ \vdots\\ u_{m-2}^{n}\\ u_{m-1}^{n}\end{pmatrix}$$

PS: reflecting boundary at position b. ($x_m=b$, $u(x_m)=u_m$, $a(x_m)=a_m$)

with ghost point u_{m+1} & $a\frac{\partial u}{\partial x}-a(x)u=0 \Rightarrow a\frac{u_{m+1}-u_{m-1}}{2h_x}-a_mu_m=0$

$\Rightarrow u_{m+1}=u_{m-1}+\frac{2a_mh_x}{a}u_m$, with $A_0u_{m-1}^{n+1}+A_1u_m^{n+1}+A_2u_{m+1}^{n+1}=A_3u_{m-1}^n+A_4u_m^n+A_5u_{m+1}^n$

$$(A_0+A_2)u_{m-1}^{n+1}+\left(A_1+\frac{2a_mh_x}{a}A_2\right)u_m^{n+1}=(A_3+A_5)u_{m-1}^n+\left(A_4+\frac{2a_mh_x}{a}A_5\right)u_m^n$$

last two raws:

$$\begin{pmatrix} \ddots & \ddots & \ddots \\ A_0 & A_1 & A_2 \\ & A_0+A_2 & A_1+\frac{2a_mh_x}{a}A_2 \end{pmatrix}\begin{pmatrix}\vdots\\ u_{m-1}\\ u_m\end{pmatrix}=\begin{pmatrix} \ddots & \ddots & \ddots \\ A_3 & A_4 & A_5 \\ & A_3+A_5 & A_4+\frac{2a_mh_x}{a}A_5 \end{pmatrix}$$