Sampling: Probability 101

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$ echo "Data Science Institute"
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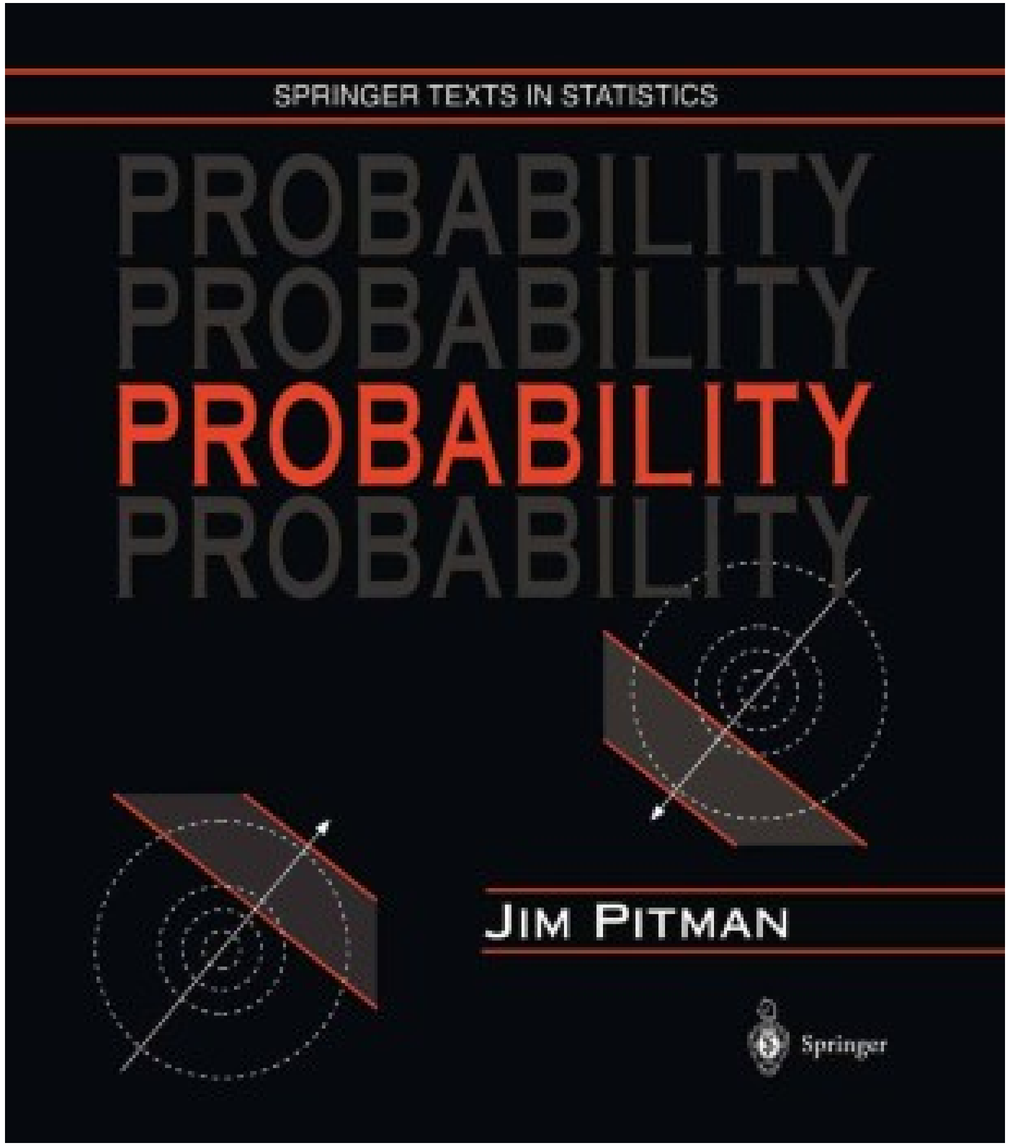
Learning Goals

- Introduce essential probability concepts
- Discuss statistical distributions (and why they matter!)
- Use examples to build statistical intuition

How do we calculate and interpret probabilities? What is a statistical distribution?

Key Texts

- Pitman, 1993, *Probability* , Springer, Chapters 1-3
- Image source: *Springer*



Before we start...

Intro to Probability

Definitions

- An **outcome space** is a set of all possible outcomes of some kind, often represented by Ω.
  - For example, Ω = {A, B, C,..., Z} is an outcome space containing all letters of the alphabet
- An **event** is a subset of an outcome space.
  - There are often many possible events for a specific outcome space
  - Possible events for Ω above could be vowels {A, E, I, O, U} or letters before E {A, B, C, D}
- A **probability** is a function of an event describing how likely it is to occur

Equally Likely Outcomes

- If all outcomes in a set Ω are equally likely, the probability of event *A* is the number of outcomes in *A* divided by the total number of outcomes,

> $$P(A) = \frac{\#A}{\#\Omega}$$

- $P(A)$ can be read as “the probability of A”.

Example: Rolling a die

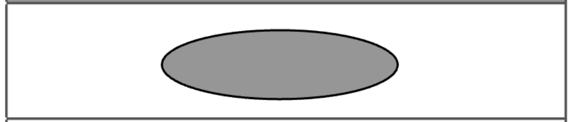
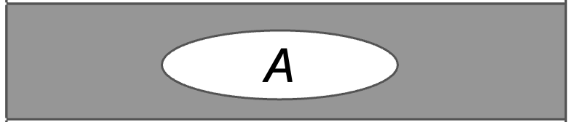
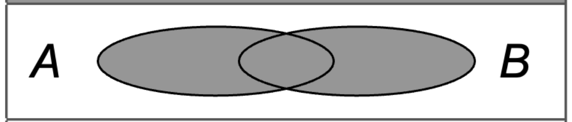
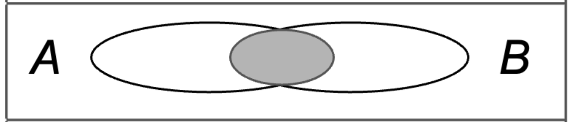
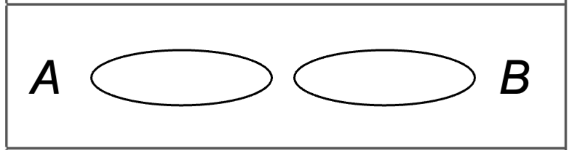
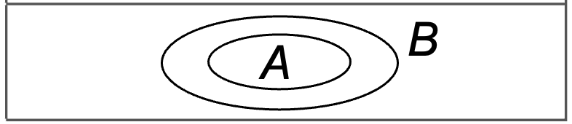
- For a six-sided die, the outcome space is,

> $$\Omega = \{1, 2, 3, 4, 5, 6\}$$

- Some possible events and their probabilities are,

Description	Event	Probability
An even number is rolled	A = {2, 4, 6}	3/6 = ½ = 0.5
A number less than 6 is rolled	B = {1, 2, 3, 4, 5}	⅚ = 0.833…
A 6 is rolled	C = {6}	⅙ = 0.166…

Sets and Events

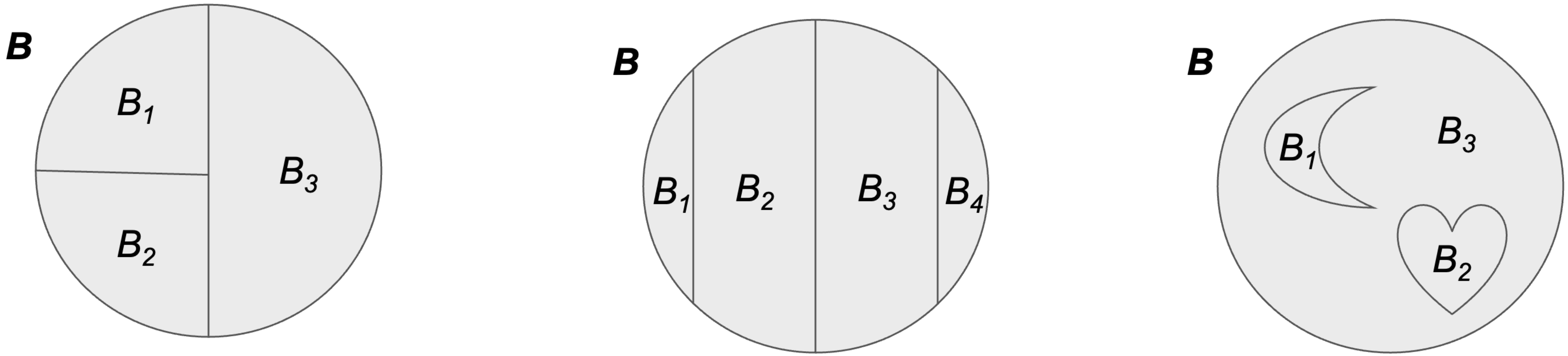
Event language	Set language	Set notation	Venn diagram
Outcome space	Universal set	$\boldsymbol{\Omega}$	
Event	Subset of $\boldsymbol{\Omega}$	*A, B, C*, etc.	
Impossible event	Empty set	$\emptyset$	
Not *A*, or the opposite of *A*	Complement of *A*	A^C	
Either *A* or *B* or both	Union of *A* and *B*	$A \cup B$	
Both *A* and *B*	Intersection of *A* and *B*	$A \cap B$	
A and *B* are mutually exclusive	*A* and *B* are disjoint	$A \cap B = \emptyset$	
If *A*, then *B*	*A* is a subset of *B*	$A \subseteq B$	

Partitions

- An event B is **partitioned** into n events $B_1, B_2, \ldots, B_n$ if

1. $B = B_1 \cup B_2 \cup \ldots \cup B_n$ – every outcome in *B* belongs to some event B_i , none are left out
2. $B_1, B_2, \ldots, B_n$ are **mutually exclusive** – if an outcome is in event B_i , it is not in any other event

Adapted from Pitman (1993), Figure 1



Rules of Probability

- For an event *B* over an outcome space Ω,
- **Non-negativity**: $P(B) \geq 0$
- **Addition**: If $B_1, B_2, \ldots, B_n$ is a partition of *B* , then

> $P(B) = P(B_1) + P(B_2) + \ldots + P(B_n)$

- **Total one**: $P(\Omega) = 1$

Example: Drawing cards

- Suppose you have a regular deck of cards. Let *B* represent the event “drawing a heart”. Let B_1 and B_2 be a partition of *B* , with B_1 = “drawing non-numeric heart card (J, Q, K, A)” and B_2 = “drawing a numeric heart card (2,…,10)”
- B_1 and B_2 is a valid partition, since all heart cards are either numeric or non numeric, and a card cannot be both a numeric card and a non-numeric card (mutual exclusivity)

> $$P(B) = 13/52 = 1/4 \geq 0$$
> $$P(B1) + P(B2) = 4/52 + 9/52 = 13/52 = 1/4 = P(B)$$
> $$P(\Omega) = P(\text{“draw any card in the deck”}) = 1$$

Conditional Probability

Conditional Probability

- ⚠️ **Conditional probability** can be described as the probability that event *A* will happen **given that event *B* has already happened** ⚠️
- The notation for conditional probability is $P(A|B)$
- The formula for conditional probability is,

> $$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- As long as $P(B) \neq 0$

Example: Rolling a die

- Suppose you roll a regular die, but haven't yet looked at the result. Let event A be "rolling a 4", and let event B be "rolling an even number".
- The probability that number you rolled is a 4 is,

> $$P(A) = \frac{1}{6}$$

- Now suppose I look at the die and tell you that the number you rolled is **even** . Given this new information, the probability that the number you rolled is a 4 is,

> $$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\text{"rolling a 4" and "rolling an even number"})}{P(\text{"rolling an even number"})} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

Independence

- If events A and B are **independent** , event A is not affected by the occurrence of event B.
- This can be described mathematically as,

> $$P(A|B) = P(A)$$

- From the formula for conditional probability, we can then derive the following formula for independent events,

> $$P(A \cap B) = P(A)P(B)$$

- The same applies for any number of disjoint events.

Random Variables

Random Variables

- **Random variables** are a way to describe a set of possible outcomes with a distribution of probabilities over the set of outcomes.
- Usually denoted with capital letters: X, Y, Z, etc.
- Random variables are similar to events
  - Events are a specific outcome or set of outcomes, while random variables describe possible outcomes and their various probabilities
  - An event “Number on the die is a five” or {5} or $X = 5$ is one possible outcome of the random variable X, “the number obtained by rolling a die”

Random Variables

- The **range** of a random variable is all of the values it could possibly take. This can be continuous ($0 \leq X \leq 10$) or discrete ($X \in \{1, 2, 3\}$).
- The distribution of random variable is determined by the probabilities of values within its range,

> $$P(X = x) \text{ for x } \in \text{ range of } X$$

Example: Rolling two dice

- Let X represent the sum of the faces showing on two rolled dice.

x	2	3	4	5	6	7	8	9	10	11	12
Possible outcomes	1+1	1+2, 2+1	1+3, 2+2, 3+1	1+4, 2+3, 3+2, 4+1	1+5, 2+4, 3+3, 4+2, 5+1	1+6, 2+5, 3+4, 4+3, 5+2, 6+1	2+6, 3+5, 4+4, 5+3, 6+2	3+6, 4+5, 5+4, 6+3	4+6, 5+5, 6+4	5+6, 6+5	6+6
P(X = x)	1/36	1/18	1/12	1/9	5/36	1/6	5/36	1/9	1/12	1/18	1/36

Indicator Variables

- **Indicator variables** , denoted I_A, are a specific type of random variable that take the value 0 or 1 to indicate the occurrence of a given event A .
- Some examples of indicator variables may be votes in a two-party election (with event A being a vote for a particular candidate), votes for or against a bill, satisfied versus not satisfied reviews for a product, etc.

Distributions

Distributions

- ⚠️ A **probability distribution** is a statistical function that describes the probabilities of all possible events in an outcome space. ⚠️
- Distributions can be **discrete** (if the outcome space is distinct events, like rolling a die) or **continuous** (if the outcome space is a range of values, like choosing any real number between 1 and 10).
- Some features of interest for distributions may be their mean, variance, mode, skew, etc.

Binomial Distribution

- The **binomial distribution** concerns sequences of events with two possible outcomes: **success** and **failure**.
- Success occurs with probability ***p*** , and failure occurs with probability ***q = 1 - p***. Trials defined this way are called **Bernoulli trials** .
- ⚠️ The binomial distribution helps determine the probability of getting ***k* successes in *n* independent trials (with replacement)** ⚠️

Binomial Distribution Formula

- For *n* independent trials with probability *p* of success and probability *q* = 1 - *p* of failure, the probability of ***k* success in *n* independent trials** (with replacement) is,

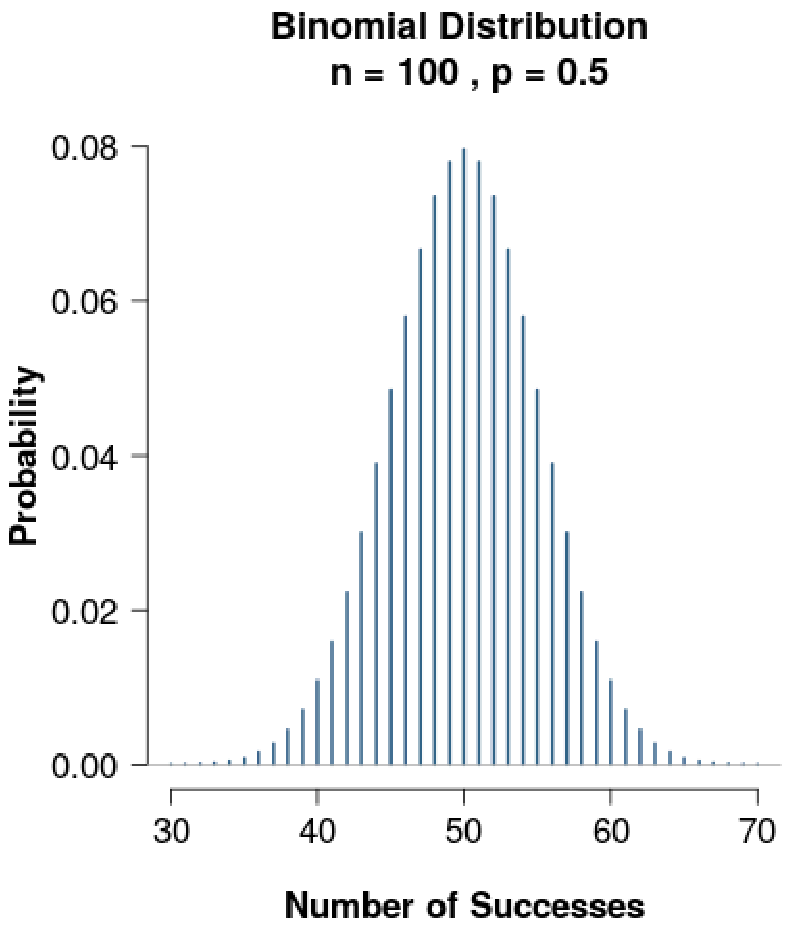
> $$P(\text{k success in n trials}) = \binom{n}{k} p^k q^{n-k}$$

- $\binom{n}{k}$ is called " *n* choose *k* " and describes the number of possible combinations of *k* successes and *n* - *k* failures:

> $$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Binomial Distribution

- For a large number of trials n , we expect the number of successes to be about np .
- For n = 100 and p = 0.5, the distribution of number of successes is centered around 50 (the most likely) and the total number of successes gets less likely as the numbers get farther from 50.



Example: Drawing cards

- Suppose you draw n = 5 cards from a standard deck, and your desired outcome is drawing a club. Then p = 13/52 = ¼ and q = 1 - p = 1 - ¼ = ¾.
- For k = 1 success, the possible combinations of cards drawn are:

> {➡️♣️⬅️,2,3,4,5},
> {1,➡️♣️⬅️,3,4,5},
> {1,2,➡️♣️⬅️,4,5},
> {1,2,3,➡️♣️⬅️,5},
> {1,2,3,4,➡️♣️⬅️}

- Mathematically this can be represented as,

$$\binom{5}{1} = \frac{5!}{1!(5-1)!} = \frac{5\cdot4\cdot3\cdot2\cdot1}{1(4\cdot3\cdot2\cdot1)} = 5$$

Example: Drawing cards

- Since the trials are independent, the probability of getting 1 success in 5 trials is the product of the probability of getting a club on one trial and the probability of getting non-clubs on four trials.

> $$P(\text{Club})P(\text{Non-club})P(\text{Non-club})P(\text{Non-club})P(\text{Non-club}) = \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{1^1}{4} \cdot \frac{3^4}{4} = 0.0791$$

- Putting the two calculations together, we have,

> $$P(1 \text{ club in } 5 \text{ draws}) = \binom{5}{1} \frac{1^1}{4} \cdot \frac{3^4}{4} = 5 \cdot 0.0791 = 0.396$$

Uniform Distribution

- The **uniform distribution** describes a situation in which every outcome on a certain set or interval is **equally likely** .
- This can be represented mathematically as,

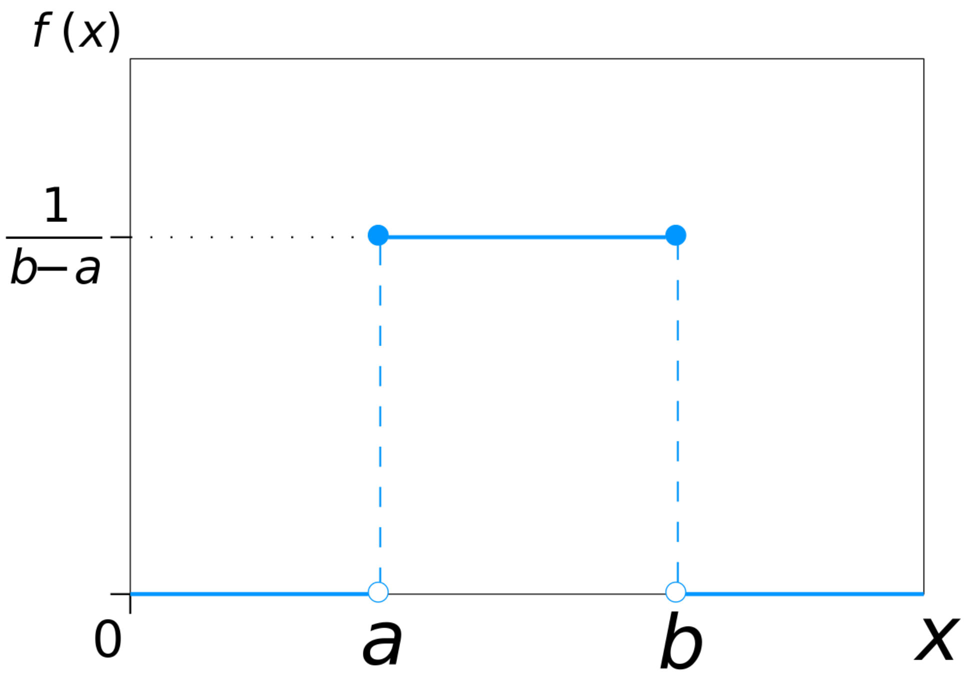
> $$P(X = x) = \begin{cases} \frac{1}{|\Omega|} & \text{if } x \in \Omega, \\ 0 & \text{otherwise} \end{cases}$$

or

> $$P(X = x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b, \\ 0 & \text{otherwise} \end{cases}$$

Uniform Distribution

- Examples
  - Rolling a die
  - Drawing any card from a normal deck
  - Choosing a random number between 1 and 100
  - Choosing a random student in a classroom



Poisson Distribution

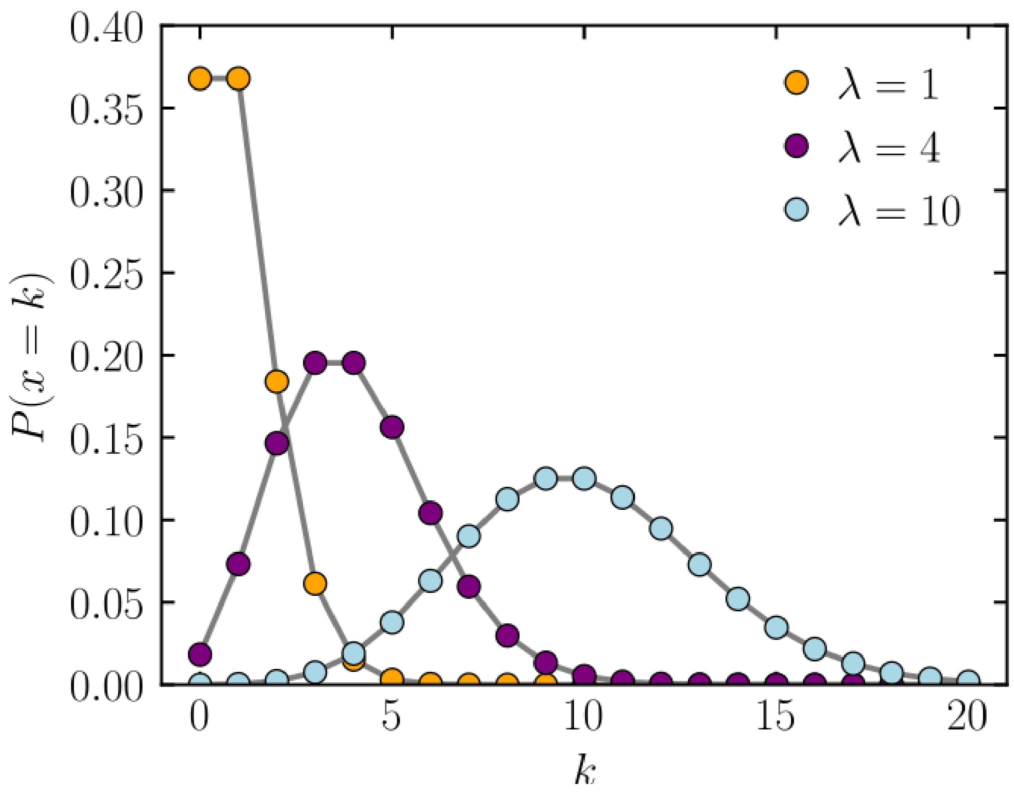
- The **Poisson distribution** is an approximation of the distribution of the **number *N* of occurrences of events of some kind**, when the events all have small probabilities and are independent.

> $$P(N = k) \approx \frac{e^{-\mu}\mu^k}{k!} \text{ for k} = 1,\, 2,\, \ldots$$

- The Poisson distribution is a discrete probability distribution.

Poisson Distribution

- Examples
  - Number of wins in n games of roulette for a gambler who bets on a single number each game
  - Number of rain drops that land on a particular area of a roof during a set time interval
  - Number of people who enter a store in a certain time interval



Normal Distribution

- The **normal distribution** is one of the most common and important distributions
- It is represented by the equation,

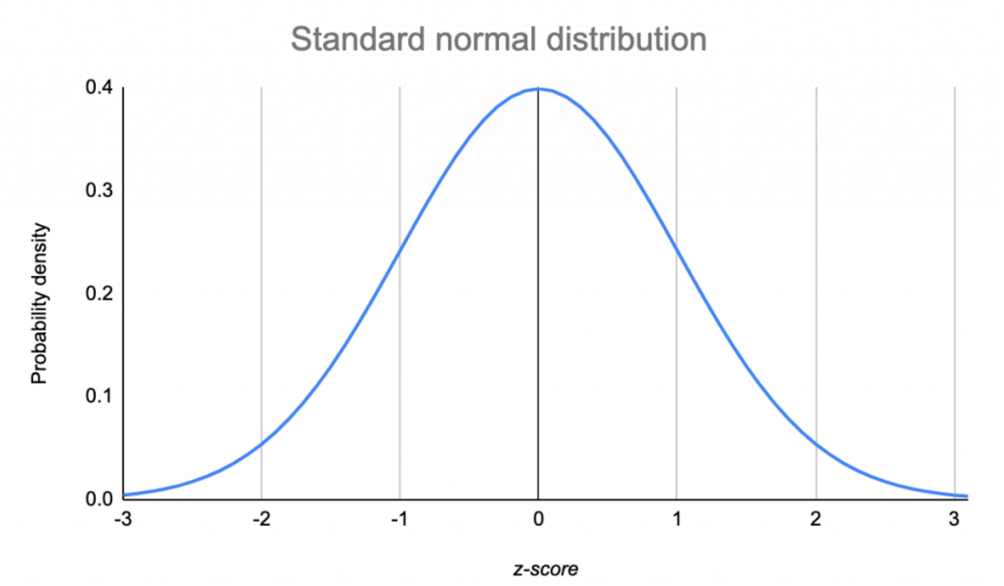
> $$P(X=x)=\frac{1}{\sqrt{2\pi\sigma}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- where μ is the mean of X and σ is the standard deviation.
- A random variable X following a normal distribution is often denoted,

> $$X\sim\mathcal{N}(\mu,\sigma^2)$$

Normal Distribution

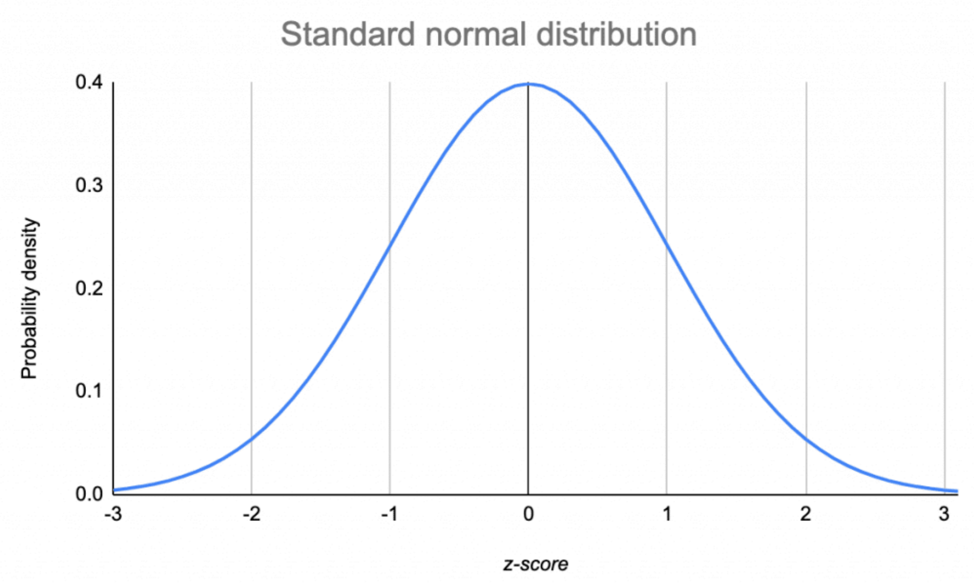
- The normal distribution is centered and symmetric about μ. σ describes the horizontal spread (how wide the distribution is).
- The normal distribution can be used to be use to approximate other distributions for easy calculations of probabilities



Standard Normal Distribution

- The standard normal distribution is a normal distribution with mean 0 and standard deviation 1.
- In general, a random variable X with a normal distribution can be standardized using the following formula,

> $$Z = \frac{X - \mu}{\sigma}$$



Example: Z-Scores

- Z-Scores represent the probability that a value is less than or equal to the value of a given standardized random variable.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
–3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
–3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
–3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
–3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
–3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
–2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
–2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
–2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
–2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
–2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
–2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
–2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
–2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
–2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
–2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
–1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
–1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
–1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
–1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
–1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
–1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
–1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
–1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
–1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
–1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
–0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
–0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
–0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
–0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
–0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
–0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
–0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
–0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
–0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
–0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

Expected Value

Expected Value

- The **expected value** or **expectation** of a random variable X is the mean of the distribution of X, denoted E(X) or μ. This is represented mathematically as,

> $$E(X) = \sum_{\text{Every x}} xP(X = x)$$

- The expected value is the average of all possible values of X weighted by their probabilities.
- The expected value of indicator variable I_A is the probability of event A,

> $$E(I_A) = P(A)$$

Example: Sampling a student

- Suppose you are randomly sampling a student from a school. There are 200 students each of ages 16, 17, and 18. Let random variable X represent the age of the student sampled.

- The expected age of the student selected is,

> $$E(X) = \sum_{\text{Every x}} xP(X = x) = 16(\frac{200}{600}) + 17(\frac{200}{600}) + 18(\frac{200}{600}) = 17$$

- Now suppose there are 100 students age 16, 200 students age 17, and 300 students age 18. The new expected age is,

> $$E(X) = \sum_{\text{Every x}} xP(X = x) = 16(\frac{100}{600}) + 17(\frac{200}{600}) + 18(\frac{300}{600}) = 17.333$$

Properties of Expectation

- Constants : The expectation of a constant random variable is its constant value

> $$E(c) = c$$

- Scalar multiplication : For random variable X multiplied by constant c,

> $$E(cX) = cE(X)$$

- Addition : The expectation of a sum of random variables is the sum of the expectations

> $$E(X + Y) = E(X) + E(Y)$$

Variance and Standard Deviation

Variance and Standard Deviation

- The **variance** of X, denoted Var(X), is the mean squared deviation of X from its expected value E(X),

> $$Var(x) = E([X - E(X)]^2) = E(X^2) - [E(X)]^2$$

- The **standard deviation** of X, denoted SD(X), is the square root of the variance of X:

> $$SD(X) = \sqrt{Var(X)}$$

Variance Properties

- **Addition**: for independent random variables X_1, X_2,..., X_n, the variance of their sum is,

> $$Var(X_1 + ... + X_n) = Var(X_1) + ... + Var(X_n)$$

- **Scalar multiplication**: for random variable X and scalar c,

> $$Var(cX) = c^2 Var(X)$$

Variance and Standard Deviation

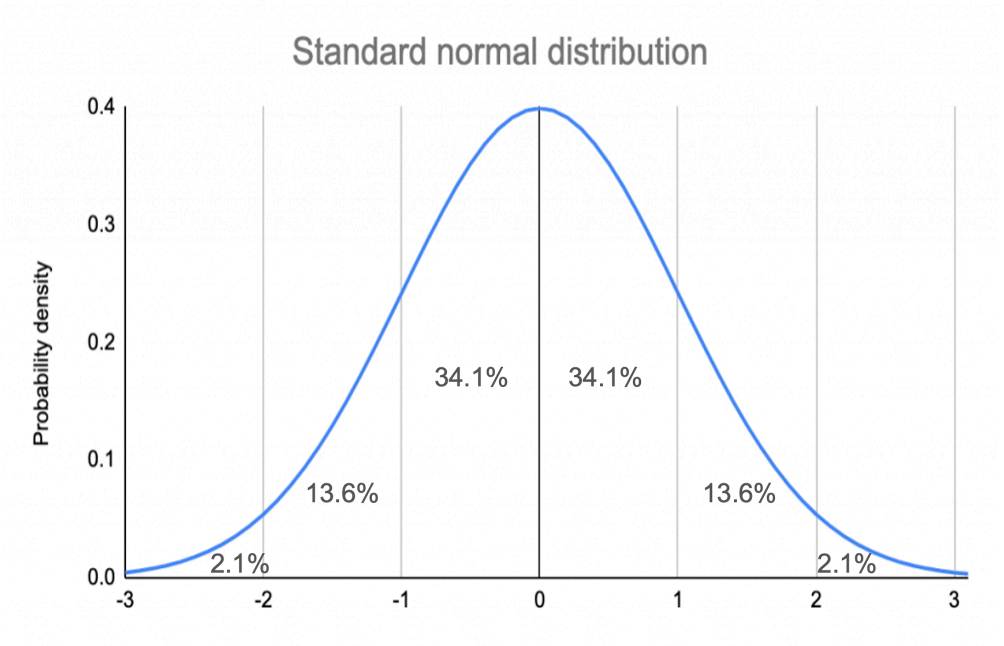
- Variance is often denoted σ^2, with SD denoted σ
- Variance and SD describe how spread out the distribution of a variable is
- SD is often easier to interpret since its units are the same as the mean
- ⚠️ **In general** ⚠️
  - For a random variable X with some distribution, you should expect the value of X to be around the expected value $\mathrm{E}(X)$, plus or minus a few times the standard deviation $\mathrm{SD}(X)$

Example: Normal Distribution

- On a normal distribution, ~68% of the probability density lies within one SD of the mean:

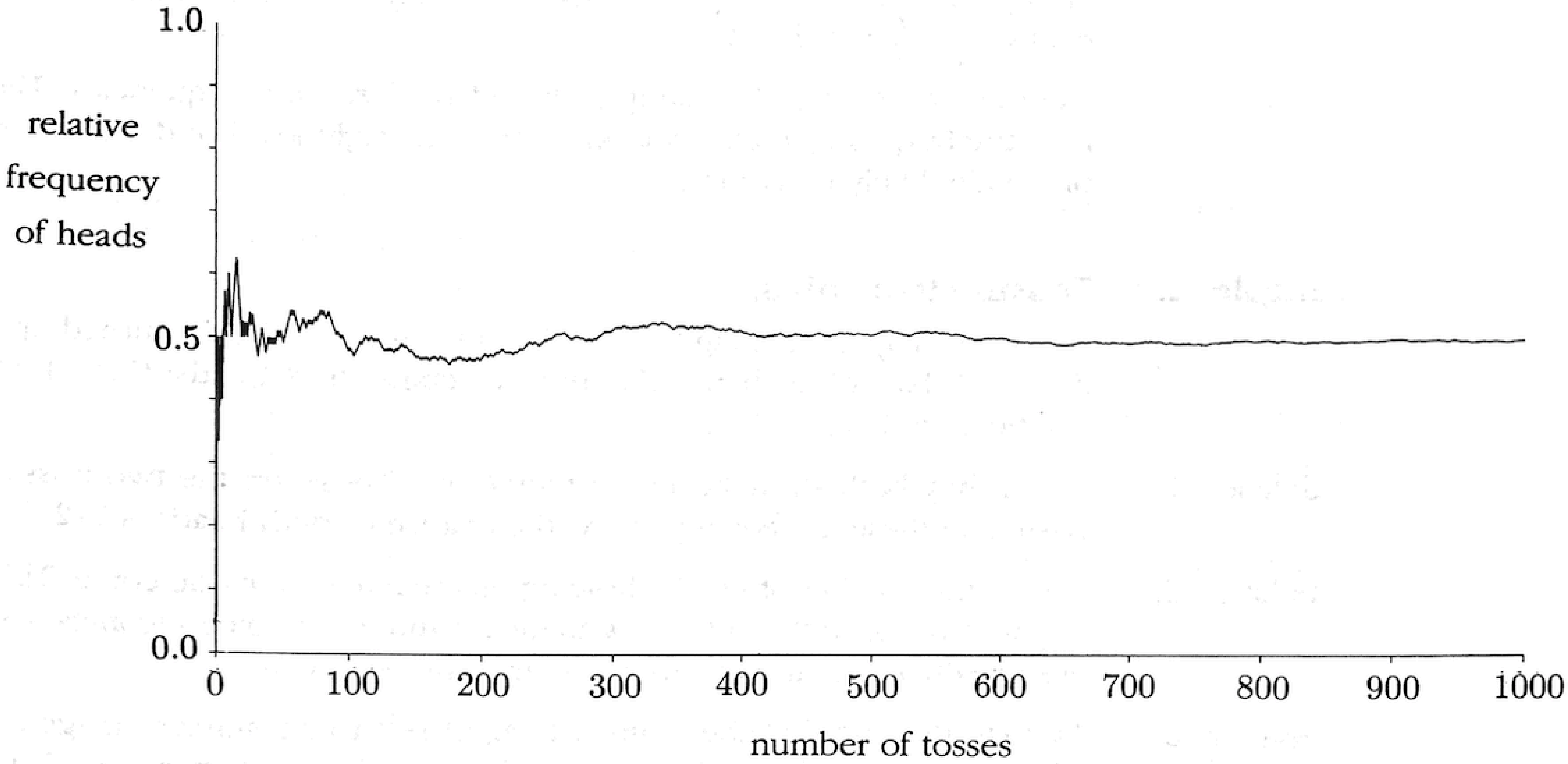
> $$P(\mu - \sigma \leq X \leq \mu + \sigma) \approx 0.68$$

- For the standard normal *N(0, 1)* , this means *X* is fairly likely to be between -1 and 1, and that 2.5 would be a very unlikely value of *X*.



Law of Large Numbers

Law of Large Numbers



Law of Large Numbers

- ⚠️ If the number of trials n is large, the proportion of successes in n independent trials will, with overwhelming probability, be very close to p , the probability of success on each trial ⚠️
- **Intuition for sampling:**
  - As the number of units sampled increases, the proportion of units that exhibits a certain trait will grow closer and closer to the true proportion of individuals in the population with that trait

Central Limit Theorem

Central Limit Theorem

- ⚠️ Let S_n = X_1 + X_2 + … + X_n be the sum of n independent random variables each with the same distribution. For large n, the distribution of S_n is approximately normal, with mean E(S_n) = $n\mu$ and variance Var(S_n) = ${\sigma^2}_n$, where μ = E(X_i) and σ^2 = Var(X_i). ⚠️
- This also holds true for,

> $$\bar{X}_n = \frac{S_n}{n}, \text{ with } E(\bar{X}_n) = \mu \text{ and } Var(\bar{X}_n) = \frac{\sigma^2}{n}$$

> $$\bar{X}_n \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$$

Who cares?

- Sample size and quantity affect how well our conclusions can represent our target population (LLN, CLT)
- Distributions and their assumptions affect our models and the ways that we calculate various statistics

Next

- Probability vs Non-probability sampling
- Sampling Types and Sample Design
- Observational Studies and Censuses