Online COL 351 Major Exam

Timing: 1:00 PM - 2:30 PM

Important Guidelines:

1. This is a closed book exam. You cannot look at your notes or browse the internet.
2. You can directly reference an algorithm / theorem proved in the lectures. However, results proved in tutorials cannot be directly used in Major exam.
3. Your answer to each question must be formal and have a correctness proof.
4. Total marks are 30.
5. The scanned solution **must be** uploaded on Gradescope by 2:45 PM sharp. The late submission deadline is 2:50 PM, and NO submissions will be accepted later under any circumstance.

1. Short Questions [$3 + 3 + 2 = 8$ marks]

(a) A subset S of vertices in a graph is said to be *nice* if every two distinct vertices in it are adjacent. Consider the following problem: Given an undirected graph G and an integer k, does G contain a nice set of size k? Show that this problem is NP-complete. Clearly state **all** the properties/facts you use to prove your claim.

(b) Let $G = (V, E)$ be an undirected connected graph, $s \in V$ be a vertex, and T be a BFS tree of G rooted at s. Prove that G is acyclic iff T is also a DFS tree of G.

(c) Let $A = [a_1, \ldots, a_n]$ and $B = [b_1, \ldots, b_n]$ be two arrays of positive integers in range $[1, 100n]$. Prove that the set $Z = \{a + b \mid a \in A,\ b \in B\}$ is computable in $O(n \log n)$ time.

2. Divide and Conquer [$6 + 3 = 9$ marks]

(a) Let $C = [c_1, c_2, \ldots, c_n]$ be an array with all entries in range $[1, 100]$. Use 1(c) to obtain a divide and conquer algorithm to compute the set

$$S = \left\{C[i] + C[i+1] + \cdots + C[j] \;\middle|\; i \leq j \in [1, n]\right\}$$

Argue that the time-complexity of your algorithm follows the relation $T(n) = 2T(n/2) + O(n \log n)$.

(b) **Without** using Master's theorem prove that the recurrence $T(n) = 2T(n/2) + O(n \log n)$ satisfies the relation $T(n) = O(n \log^2 n)$. You can assume that $T(0), T(1)$ are $O(1)$.

3. Max flow Application [6 marks] The basic rule for blood donation are: A patient of blood group A can receive only blood of group A or O. A patient of blood group B can receive only blood of group B or O. A patient of blood group O can receive only blood of group O. A patient of blood group AB can receive blood of any group.

Let s_O, s_A, s_B, s_{AB} denote the supply in whole units of the different blood types in a hospital for the coming week. Assume that the hospital knows the projected demand for each blood type d_O, d_A, d_B, and d_{AB} for the coming week. Give a max-flow based algorithm to check if the supply would meet the projected demand.

4. Profit Optimization [7 marks] Consider a 2-D map with a river passing parallel to x-axis in east-west direction. There are n factories on the northern bank, namely, $a_1, \ldots, a_n$ and n suppliers on the southern bank, namely, $b_1, \ldots, b_n$. Further, it is given that the x-coordinate of a_i and b_i is i. You are given a bijective map $M : [1, n] \to [1, n]$ such that factory a_i requires to be connected to supplier $b_{M(i)}$.

Let $P : [1, n] \to \mathbb{R}^+$ be a function such that $P(i)$ denotes the profit of connecting a_i with $b_{M(i)}$. Your task is to connect factory and corresponding suppliers with non-crossing bridges so as to maximize the sum total profit. Design a polynomial time algorithm to achieve the same.

5. Question paper Scan Take a picture of exam paper on your computer and upload it.