Sample

Friday, 3 September 2021

QUESTION 2, TUTORIAL 3

Fix a vertex s in G. We will present an algorithm to compute smallest cycle containing s.

Algorithm (Find minimum cycle containing *s*):

1. Compute $T = BFS(s)$.
2. Let $v_1, v_2, \ldots, v_k$ be children of s in T.
3. For ($i = 1$ *to* k): Scan vertices in $T(v_i)$ and for each $x \in T(v_i)$, set $L[x] = v_i$.

   /*Note: For any $a, b \in V(G)$, $s = LCA(a, b)$ iff $L[a] \neq L[b]$ */
4. $Q = \{(a, b) \in E(G) \setminus E(T) \mid L[a] \neq L[b]\}$.
5. $(x_0, y_0) =$ The edge in G for which $dist(s, x_0) + dist(s, y_0)$ is minimum.
6. Output $C_0 = treepath(s, x_0) :: (x_0, y_0) :: treepath(y_0, s)$.

Claim 1: For any edge $(x, y) \in Q$, $C = treepath(s, x) :: (x, y) :: treepath(y, s)$ is a cycle
Proof: Recall Q comprises of those non-tree edges *(a,b)* for which $s = LCA(a, b)$.
Thus, C is a closed walk where no vertex is repeated. Hence C is a cycle.

Claim 2: Any cycle containing v must contain an edge from set Q.
Proof: Let C be a cycle containing v. Without loss of generality assume z_1, z_2 are neighbors of v in C. Let (x, y) be first edge on segment $C[z_1, z_2]$ such that $x \in T(z_1)$ and $y \notin T(z_1)$. Such

an edge must exists as $z_2 \notin T(z_1)$. Thus, (x, y) is an edge lying in Q. Hence proved.

Claim 3: Let $(x, y) \in Q$ be an edge that lie on some smallest cycle of s, say C. Then, $|C| = 1 + dist(s, x) + dist(s, y)$.

Proof: $|C[s, x]| \geq dist(s, x)$ and $|C[s, y]| \geq dist(s, y)$. Therefore,

$$|C| \geq 1 + dist(s, x) + dist(s, y).$$

Now, by Claim 1, $treepath(s, x) :: (x, y) :: treepath(y, s)$ is a cycle. Since C is cycle of minimum size. We have,

$$|C| \leq 1 + dist(s, x) + dist(s, y).$$

The claim follows by above two inequalities.

Correctness of Algorithm to find minimum length cycle containing *s*:

By Claim 1, C_0 is a cycle, and by Claim 3 and definition of (x_0, y_0), we get that $|C_0| \leq |C|$. This proves that above algorithm correctly computes a smallest cycle containing v.

<u>Finding minimum length cycle in G:</u>

The time complexity to find a minimum length cycle containing a given vertex is $O(m)$.
We can apply the same algorithm to each vertex of the graph to compute minimum length cycle.
So, we get that a cycle of minimum length in an undirected graph can be computed in $O(mn)$ time.

Lecture 23

Tuesday, 5 October 2021 9:59 AM

Given : Two polynomials $A(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$

$B(x) = b_0 + b_1 x + b_2 x^2 + \dots + b_n x^n$

with
- degree $\leq n$
- coeff – integers

Find : $C(x) = A(x) \cdot B(x)$

$= C_0 + C_1 x + C_2 x^2 + \dots + C_{2n} x^{2n}$

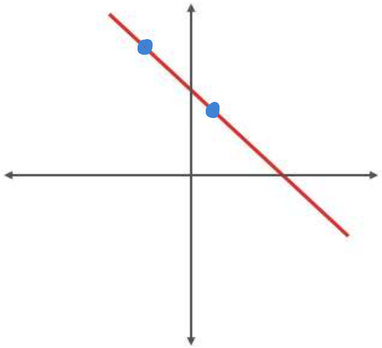
$C_i = a_0 b_i + a_1 b_{i-1} + \dots + a_i b_0$ } Time $= O(i)$

(coeff of x^i)

$\Downarrow$

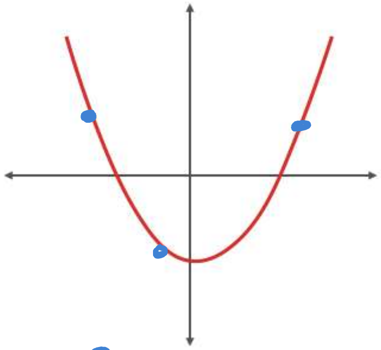
$\boxed{\text{Time to find } C(x) \text{ will be } O(n^2).}$

$A(x) = a_0 + a_1 x$



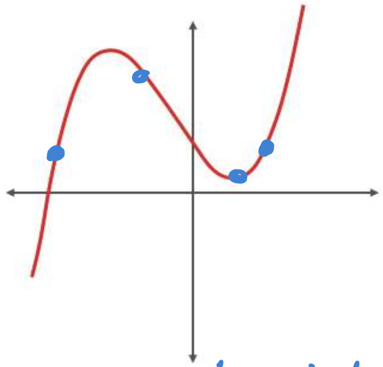
2 points

$A(x) = a_0 + a_1 x + a_2 x^2$



3 points give uniqueness

$A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$



4 points

Alternate Representation of Polynomials of deg $\leq n$ $\rightarrow$ Evaluation at $(n+1)$ points

Lemma: Given $(n+1)$ pairs (x_0, y_0) (x_1, y_1) ... (x_n, y_n), then there is a unique polynomial of degree at most n (say A) s.t. $y_i = A(x_i)$ $\quad 0 \leq i \leq n$

Proof:

$P_1(x)$, $P_2(x)$

$Q = P_1 - P_2$

$Q(x_0) = 0$
$Q(x_1) = 0$
$\vdots$
$Q(x_n) = 0$

$\deg(Q) \leq n$
evaluation of Q at $\underline{n+1}$ points is zero

H.W.

Why we are looking at alternate representation?

Efficient way to compute Product $S = \{x_1, x_2 \ldots x_{2n+1}\}$

$A(x)$ & $B(x)$ $(\deg \le n)$

$C(x) = A(x) \cdot B(x)$ $(\deg \le 2n)$

GOAL

$\boxed{\text{STEP 1}}$ $O(n \log n)$ time

$\boxed{\text{STEP 3}}$ $\boxed{\text{Interpolation}}$ $O(n \log n)$ time

$A(x_1)$ $\quad$ $B(x_1)$

$A(x_2)$ $\quad$ $B(x_2)$

$\vdots$ $\quad$ $\vdots$

$A(x_{2n+1})$ $\quad$ $B(x_{2n+1})$

$\boxed{\text{Evaluating } A, B \text{ at } (2n+1) \text{ points}}$

$\xrightarrow[\text{STEP 2}]{O(n)}$

$C(x_1) = A(x_1) \cdot B(x_1)$

$C(x_2) = A(x_2) * B(x_2)$

$\vdots$

$C(x_{2n+1})$

$\boxed{\text{Pair wise product}}$

Point wise Evaluation: Given a polynomial of $\deg \le n$, find its evaluation at some set 'S' of $n+1$ points.

Assume $N = n+1$

$$A(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

(Trivial $O(n|S|)$)

Assume $N = n+1$ is power of 2

$$\underbrace{A(x)}_{|S|=N} = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n \qquad (O(n|S|))$$

$\deg < N$

$$= \underbrace{\left(a_0 + a_2 x^2 + a_4 x^4 + \dots\right)} + x \underbrace{\left(a_1 + a_3 x^2 + a_5 x^4 + \dots\right)}$$

$$= A_{even}(\underbrace{x^2}_{|S^2| = N/2}) + x A_{odd}(\underbrace{x^2}_{|S^2| = N/2})$$

$\deg(A_{even}), \deg(A_{odd}) \le \lfloor n/2 \rfloor < N/2$

$$\boxed{S^i := \{ z^i \mid z \in S \}}$$

Good choice of S - for each $x \in S$ ensure $(-x) \in S$

Can we say: $T(N) = 2T(N/2) + O(N)$?

Only if $|S| = N$

$|S^2| = N/2$

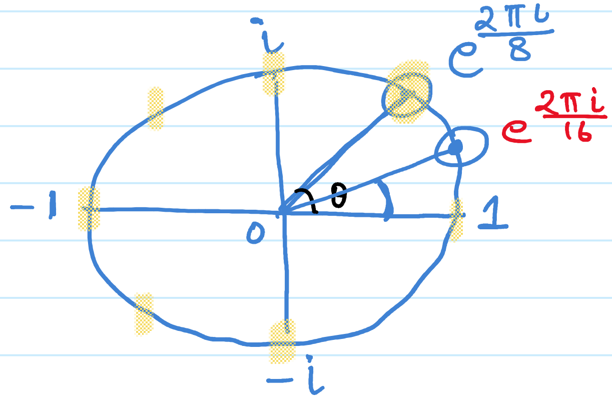
$|S^4| = N/4$

$$|S^2| = N/2$$
$$|S^4| = N/4$$
$$\vdots$$
$$|S^N| = 1 \longrightarrow S = \{ x^N = 1 \mid x \in \mathbb{C} \}$$

(N^{th} root of unity.)

We say x is N^{th} PRIMITIVE root of unity if $x^N = 1$
$x^i \neq 1 \quad 1 \leq i \leq N-1$

$$\underbrace{e^{\frac{2\pi i}{N}} = \cos\left(\frac{2\pi}{N}\right) + i \sin\left(\frac{2\pi}{N}\right)}_{N^{th} \text{ root of unity.}}$$



$$\boxed{S = \left\{ \underset{\omega}{e^{\frac{2\pi i}{N}}},\ \underset{\omega^2}{e^{\frac{2\pi i}{N}(2)}},\ \dots,\ \underset{\omega^{N-1}}{e^{\frac{2\pi i}{N}(N-1)}},\ \underset{\omega^N}{1} \right\}}$$

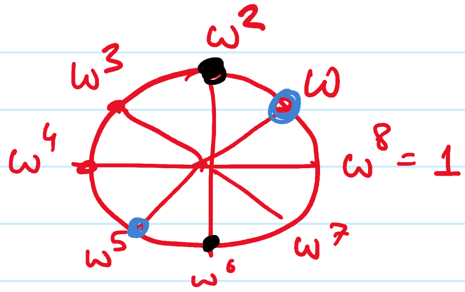
$N = 8$

$$S = \{\omega, \omega^2, \dots \omega^7, \omega^8 = 1\}$$

$$S^2 = \{\omega^2, \omega^4, \omega^6, \omega^8\}$$

$$S^4 = \{\omega^4, \omega^8\}$$

$$S^8 = \{\omega^8 = 1\}$$



$$A(x) = A_{even}(x^2) + x\, A_{odd}(x^2)$$

You can choose a set S of size N s.t. $|S| = N$

$|S^2| = N/2$

$|S^4| = N/4$

$|S^2| = N/2$
$|S^4| = N/4$
$\vdots$

$$\underbrace{T(N)}_{\text{evaluating poly of deg} < N \text{ at } N \text{ points}} = 2\,T(N/2) + O(N)$$

$$T(N) = O(N \log N)$$

Step 1 is DONE