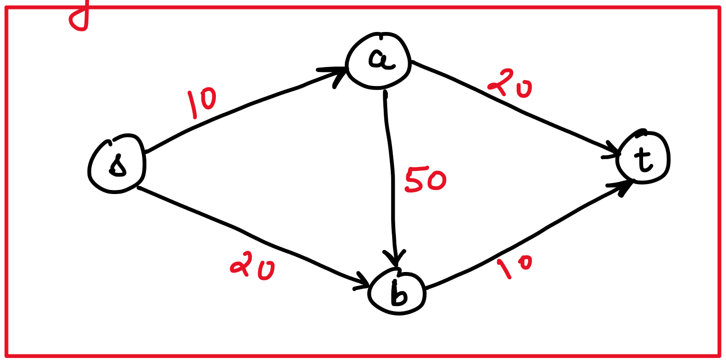
Lecture 30

Tuesday, 26 October 2021 10:00 AM

Max-Flow Problem

Given:
- A directed graph $G=(V,E)$, and each edge e has a capacity $c(e) \geq 0$.
- A source-destination pair (s,t).
- capacity of in-edges of s = capacity of out-edges of t $= 0$

Eg:



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Constraints / (Def^n of flow f):

① **Capacity:** $\forall e \in E, \quad f(e) \in [0, c(e)]$

② **Conservation of flow:** $\forall x \neq s,t, \quad f_{out}(x) = f_{in}(x)$ (out-flow = in-flow)

i.e,

$$\sum_{(x,y)} f(x,y) = \sum_{(z,x)} f(z,x)$$

$\in$ OUT-EDGES(x) $\qquad$ $\in$ IN-EDGES(x)

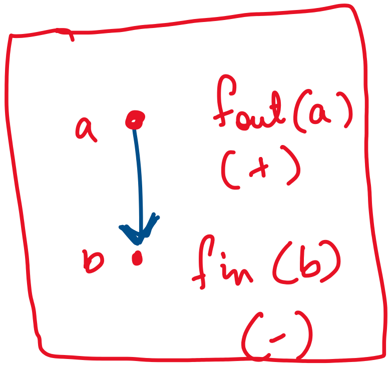
Value of flow f:

$$val(f) := \sum_{(s,x)\in E} f(s,x) = \underbrace{f_{out}(s)}_{①} \quad \text{OR} \quad \underbrace{f_{in}(t)}_{②}$$

Why ① = ② ?

$$① \quad f_{out}(s) = f_{out}(s) - f_{in}(s)$$

$$= \sum_{x \neq t} f_{out}(x) - f_{in}(x)$$



$$= \sum_{x\neq t}\left(\sum_{(x,y)\in out(x)} f(x,y) - \sum_{(z,x)\in in(x)} f(z,x) \right)$$

$$= \sum_{x\in V}\left(\sum_{(x,y)\in OUT(x)} f(x,y) - \sum_{(z,x)\in IN(x)} f(z,x) \right) - \Big(\underbrace{\sum_{(t,y)\in OUT(t)} f(t,\ldots}_{= 0}$$

$$= \sum_{e\in E} f(e) - \sum_{e\in E} f(e) - 0 + f_{in}(t)$$

$$= f_{in}(t)$$

Eg: $\quad c(x,y) = 50$

Residual graph G_f w.r.t. some flow f in G

$x \xrightarrow{f(x,y)=20} y$

For each edge $(x,y) \in E(G)$:

$x \xrightarrow{30} y$

(i) Include (x,y) in G_f and set $c_r(x,y) = c(x,y) - f(x,y)$ $\quad c_r(x,y) = 30$

also mark this edge as FORWARD edge.

$y \xrightarrow{20} x$

(ii) Include (y,x) in G_f and set $c_r(y,x) = f(x,y)$

also mark this edge as BACKWARD edge.

Ford-Fulkerson for Max-Flow

① Set $f(e)=0 \quad \forall e$. Compute G_f.

② While ($\exists$ $s \rightsquigarrow t$ path in G_f)

(i) $P \leftarrow$ s-t path in G_f $\quad \} \leftarrow O(m+n)$

(ii) $c = \text{Min}\{ \ldots$

(ii) $\delta = \min \{ c_r(e) \mid e \in P \}$

(iii) <u>For each $(x,y) \in P$</u> :

If (x,y) is FWD : $f(x,y) = f(x,y) + \delta$ ↑

Else : $f(x,y) = f(x,y) - \delta$ ↓

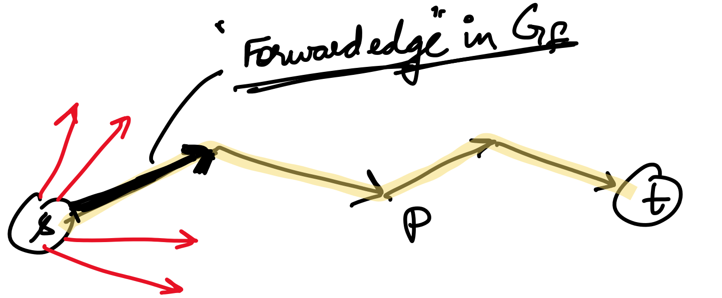
(iv) Update G_f

(3) Return "f".

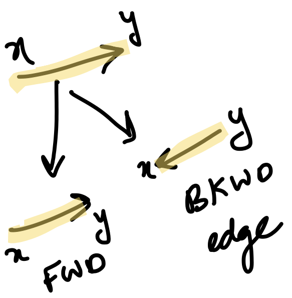
$\boxed{\text{Claim 1}}$: Value of flow f always increases in each iteration.

Proof :

Backward edges going out of s in G_f $\subseteq$ $\underbrace{\text{In-edges}_{G}(s)}_{\text{'zero-capacity'}}$



Residual graph "G_f" from last round.



$\boxed{\text{Corollary Claim 1}}$

- If $\forall e \in E$, $c(e)$ is an integer ≥ 1.
- No of iterations of while loop $\leq$ value of (s,t)-max-flow

$\Rightarrow$ Time complexity $= O((m+n) * \text{value of } (s,t)\text{-max-flow})$

Correctness

$\boxed{\text{Cuts}}$

- If $A \subseteq V$ s.t. $s \in A$, $t \in \bar{A}$, then

$$\text{cut}(A, \bar{A}) = \underbrace{\{(x,y) \in E \mid x \in A, y \in \bar{A}\}}_{\text{set of edges is called } (s,t)\text{-cut}}$$

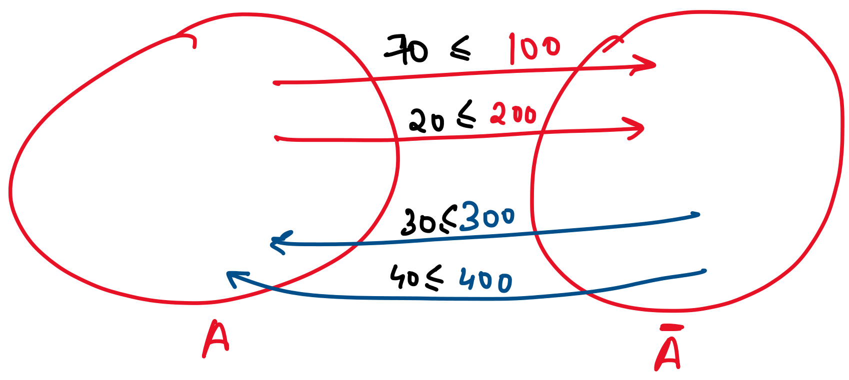
- Capacity of cut

$$c(A, \bar{A}) = \sum_{(x,y) \in \text{cut}(A,\bar{A})} c(x,y)$$

- $$f_{out}(A) := \sum_{(x,y) \in cut(A,\bar{A})} f(x,y)$$

$$f_{in}(A) = \sum_{\substack{(x,y) \in E \\ x \in \bar{A} \\ y \in A}} f(x,y)$$

Eg.



$$C(A,\bar{A}) = 100 + 200$$

$$f_{out}(A) = 70 + 20$$

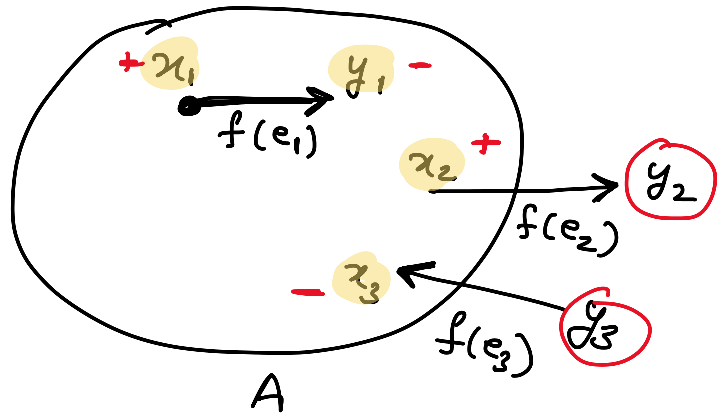
$$f_{in}(A) = 30 + 40$$

Lemma: $$\sum_{x \in A} \left(f_{out}(x) - f_{in}(x) \right) = f_{out}(A) - f_{in}(A), \quad A \subseteq V$$

Proof: LHS.

$$= \underbrace{\sum_{e \in out\,from(A)} f(e)}_{f_{out}(A)} - \underbrace{\sum_{e \in entry(A)} f(e)}_{f_{in}(A)}$$

$$e_i = (x_i, y_i)$$



$$= f_{out}(A) - f_{in}(A).$$

<u>Corollary</u>

$$f_{out}(A) - f_{in}(A) = \begin{cases} f_{out}(s) = val(f) & \text{if } s \in A,\ t \in \bar{A} \\ -f_{in}(t) = -val(f) & \text{if } t \in A,\ s \in \bar{A} \\ 0 & \text{o/w.} \end{cases}$$

Next class

$f \leftarrow$ flow that is returned by F.F. ALGO.

<u>CLAIM</u> : If "A := set of vertices reachable from s in G_f"

$s \in A$

$t \notin \bar{A}$

$f_{out}(A) - f_{in}(A) =$ max. flow possible.