COL352
Problem Sheet 2

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1 Question 1

Question 1

Question. *Prove that* $L_1 = \{bin(p) : p \text{ is a prime number}\}$ *is not a regular language.*

Solution. □

2 Question 2

Question 2

Question. *The n-th Fibonacci number is defined as $F_1 = 1, F_2 = 1,$ and for all $n \geq 3, F_n = F_{n-1} + F_{n-2}$.*
Consider the language over $\Sigma = \{a\}$

$$L_2 = \{a^m | m = F_n\} \tag{1}$$

Is L_2 regular? Justify your answer.

Solution. □

3 Question 3

Question 3

Question. *If A is any language, let $A_{\frac{1}{2}-}$ denote the set of all first halves of strings in A so that*

$$A_{\frac{1}{2}-} = \{x| \text{ for some } y,\ |x| = |y| \text{ and } xy \in A\} \tag{2}$$

Show that if A is regular, then so is $A_{\frac{1}{2}-}$.

Solution. Let the DFA of A be $D = (Q, \Sigma, \delta, q_0, F)$. We propose the following DFA $D_{\frac{1}{2}-} = (Q_{\frac{1}{2}-}, \Sigma, \delta_{\frac{1}{2}-}, q_{0\frac{1}{2}-}, F_{\frac{1}{2}-})$ as the DFA which accepts $A_{\frac{1}{2}-}$:

$$\begin{aligned} Q_{\frac{1}{2}-} &= Q \times 2^Q \\ q_{0\frac{1}{2}-} &= (q_0, F) \\ \delta_{\frac{1}{2}-}((q, R), a) &= (\delta(q, a), \{s \in Q | \exists \alpha \in Q : \delta(s, \alpha) \in R\}) \\ F &= \{(q, R) | q \in R, R \in 2^Q\} \end{aligned} \tag{3}$$

Intuitively, $A_{\frac{1}{2}-}$ is the DFA which begins with the start symbol and the set of final states. Each time it processes a symbol, it advances the state and moves backwards on the transition for the set of states. Finally, if we arrive at a state which is a subset of the reverse traversal states, we know that we have arrived at a state which has a path using $|w|$ transitions which arrives at a final state in D.
We now formally prove the correctness using induction.

Claim 3.1. *After processing i symbols, let the state be (q_i, R_i). Then from all states $s \in R_i$ there exists a walk to F in i transitions. Additionally, each such state is present in R_i.*

Proof. The base case is true ($i = 0$) trivially. Now assume that the claim is true for i, consider the claim for $i + 1$.
Let the sequence of states be $(q_0, R_0), (q_1, R_1), \ldots, (q_i, R_i), (q_{i+1}, R_{i+1})$. We know that there is a path of i transitions from all states in R_i. From the construction of R_{i+1} we know that there is a transition from each state of R_{i+1} to a state of R_i. Therefore, adding this transition to the walk of i transitions in R_i, we get that there exists a walk to a F in $i + 1$ transitions. Also, each state that has a walk of $i + 1$ transitions is present in R_{i+1} since otherwise, there would be a state corresponding to that walk which won't be present in R_i. This would lead to a contradiction to our inductive hypothesis. Therefore, each such state having a walk of $i + 1$ transitions to F is present in R_{i+1}. □

We now use the above claim to prove the following:

1. For every word $w \in A$, $w_{\frac{1}{2}-}$ is recognised by $D_{\frac{1}{2}-}$
2. For every word w which is recognised by $D_{\frac{1}{2}-}$, there exists a word $w_0 \in A$.

To prove the first statement, we use the result of the previous claim. Let $|w| = 2n$. Now, after $D_{\frac{1}{2}-}$ processes n symbols, let the state be (q_n, R_n). Now, since w is accepted by D, there exists a walk of length n from q_n to a state in F. Therefore, $q_n \in R_n$. Thus, (q_n, R_n)

is an accepting state and it accepts $w_{\frac{1}{2}-}$.

For the second statement, we know that there exists a path of n transitions from any accepting state (q_n, R_n). We can construct a word w_0 by appending the transition symbols which will be arrive at a state in F. Therefore, for each word which is accepted by $D_{\frac{1}{2}-}$, we have a corresponding word that is accepted by D. $\square$

4 Question 4

Question 4

Question. *If A is any language, let $A_{\frac{1}{3}-\frac{1}{3}}$ denote the set of strings in A with the middle third removed so that*

$$A_{\frac{1}{3}-\frac{1}{3}} = \{xz| \text{ for some } y,\ |x| = |y| = |z| \text{ and } xyz \in A\} \tag{4}$$

Show that if A is regular, then $A_{\frac{1}{3}-\frac{1}{3}}$ is not necessarily regular.

Solution. We will show this by contradiction. Consider the language $A = 0^+1^+2^+$. A is regular. Now assume that $A_{\frac{1}{3}-\frac{1}{3}}$ is regular.
We also know that $B = 0^+2^+$ is regular. Consider the intersection of $A_{\frac{1}{3}-\frac{1}{3}}$ and B. We know that this language, say C, will only have 0 and 2. We will now show that the number of 0 and number of 2 in each word of C will be equal.
Consider any word w in C. Consider the corresponding word in A from which we got w in $A_{\frac{1}{3}-\frac{1}{3}}$, call it $w_0 = 0^a1^b2^c$. For the case $a = b = c$, it is trivial that the number of 0 and 2 in w will be equal. Otherwise, we know that at least one of a, b, c will be less than $\frac{a+b+c}{3}$. We will now show that if $w \in C$, then $b < \frac{a+b+c}{3}$. Assume that it is not the case. Then, wlog assume that a is the smallest number. Then, in w, we will also have 1 at some positions. However since $w \in C$, such a situation is not possible. Therefore, $\#0 = \#2$ in w.
Thus, we have proved that $C \subseteq \{0^n2^n, n > 0\}$. We now show that any word w in $\{0^n2^n, n > 0\}$ will be in C. This is easy to see since $0^n1^n2^n$ is in A and thus 0^n2^n will be in C. Therefore, $C = \{0^n2^n, n > 0\}$
However, we know that C is non regular. However, from our assumption that $A_{\frac{1}{3}-\frac{1}{3}}$ is regular and B is regular, C must be regular since it is the intersection of $A_{\frac{1}{3}-\frac{1}{3}}$ and B. This is a contradiction. Therefore, $A_{\frac{1}{3}-\frac{1}{3}}$ is irregular. Hence, proved. $\square$

5 Question 5

Question 5

Question.

Solution. □

6 Question 6

Question 6

Question.

Solution. $\square$

7 Question 7

Question 7

Question.

Solution. □