COL352
Problem Sheet 2

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Contents

1 Question 1

Question 1

Question. *Show that every infinite Turing-recognizable language has an infinite decidable subset.*

Proof. Let the Turing-recognizable language be L. Consider an enumerator for this language E. Consider the following algorithm:

Algorithm 1 Return the first word of length n enumerated such that all words enumerated earlier are of shorter length else ϵ

```
procedure EnumerateFirst(E, n)
    w ← ϵ
    curr_w ← ϵ
    while |curr_w| < n do
        curr_w ← next(E)
        if |curr_w| > |w| then
            w ← curr_w
        end if
    end while
    if |w| > n then
        w ← ϵ
    end if
    return w
end procedure
```

We now define the following Turing machine $T_{decidable}$:

Algorithm 2 Turing machine of a decidable subset

```
procedure DecidableSubset(L, w)
    E ← enumerator(L)
    w' ← EnumerateFirst(E, |w|)
    if w' = w then
        return accept
    else
        return reject
    end if
end procedure
```

We claim that $T_{decidable}$ is a Turing machine and decidable. For this, we first prove that Algorithm 1 is correct and terminates. For correctness, notice that the while loop exits only if $|curr_w| \geq n$ and thus the length of the string in w after the termination of the while loop is at least n. Therefore, the final returned value is the first string enumerated with length equal to n (such that all strings before are of smaller length) or ϵ.
For termination, notice that there are only a finite possibilities before E enumerates a string with length $\geq n$. Therefore, the while loop terminates after a finite number of iterations.

We have shown that Algorithm 1 is decidable since it halts. Therefore, the string w' in ALgorithm 2 is generated in finite number of steps. The comparison of w' and w also completes in finite steps. Therefore, Algorithm 2 accepts or rejects w in finite number of steps. Therefore, $T_{decidable}$ is a decidable Turing machine.
We also need to show that $T_{decidable}$ recognizes an infinite language. This is obvious since L is infinite and therefore the following set forms an infinite sequence:

$$\{i | \forall i \in \mathbb{N} : EnumerateFirst(E, i) \neq \epsilon\} \tag{1}$$

This set is a bijection to the words recognised by $T_{decidable}$. Therefore, $L(T_{decidable})$ is also an infinite set.
Thus, we have shown that for every infinite Turing-recognizable language, we have an infinite decidable subset. $\square$

2 Question 2

Question 2

Question. *Show that single-tape TMs that cannot write on the portion of the tape containing the input string recognize only regular languages.*

Solution. sol □

3 Question 3

Question 3

Question. *ques*

Solution. sol □

4 Question 4

Question 4

Question. *ques*

Solution. sol □

5 Question 5

Question 5

Question. *ques*

Solution. sol ☐

6 Question 6

Question 6

Question. *ques*

Solution. sol $\square$

7 Question 7

Question 7

Question. *ques*

Solution. sol $\square$

8 Question 8

Question 8

Question. *ques*

Solution. sol □