COL352
Problem Sheet 2

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1 Question 1

Question 1

Question. *Prove that* $L_1 = \{bin(p) : p \text{ is a prime number}\}$ *is not a regular language.*

Solution. □

2 Question 2

Question 2

Question. *The n-th Fibonacci number is defined as* $F_1 = 1, F_2 = 1,$ *and for all* $n \geq 3, F_n = F_{n-1} + F_{n-2}$.
Consider the language over $\Sigma = \{a\}$

$$L_2 = \{a^m | m = F_n\} \tag{1}$$

Is L_2 *regular? Justify your answer.*

Solution. □

3 Question 3

Question 3

Question. *If A is any language, let $A_{\frac{1}{2}-}$ denote the set of all first halves of strings in A so that*

$$A_{\frac{1}{2}-} = \{x| \text{ for some } y,\ |x| = |y| \text{ and } xy \in A\} \tag{2}$$

Show that if A is regular, then so is $A_{\frac{1}{2}-}$.

Solution. Let the DFA of A be $D = (Q, \Sigma, \delta, q_0, F)$. We propose the following DFA $D_{\frac{1}{2}-} = (Q_{\frac{1}{2}-}, \Sigma, \delta_{\frac{1}{2}-}, q_{0\frac{1}{2}-}, F_{\frac{1}{2}-})$ as the DFA which accepts $A_{\frac{1}{2}-}$:

$$\begin{aligned} Q_{\frac{1}{2}-} &= Q \times 2^Q \\ q_{0\frac{1}{2}-} &= (q_0, F) \\ \delta_{\frac{1}{2}-}((q, R), a) &= (\delta(q, a), \{s \in Q | \exists \alpha \in Q : \delta(s, \alpha) \in R\}) \\ F &= \{(q, R) | q \in R, R \in 2^Q\} \end{aligned} \tag{3}$$

Intuitively, $A_{\frac{1}{2}-}$ is the DFA which begins with the start symbol and the set of final states. Each time it processes a symbol, it advances the state and moves backwards on the transition for the set of states. Finally, if we arrive at a state which is a subset of the reverse traversal states, we know that we have arrived at a state which has a path using $|w|$ transitions which arrives at a final state in D.
We now formally prove the correctness using induction.

Claim 3.1. *After processing i symbols, let the state be (q_i, R_i). Then from all states $s \in R_i$ there exists a walk to F in i transitions. Additionally, each such state is present in R_i.*

Proof. The base case is true ($i = 0$) trivially. Now assume that the claim is true for i, consider the claim for $i + 1$.
Let the sequence of states be $(q_0, R_0), (q_1, R_1), \ldots, (q_i, R_i), (q_{i+1}, R_{i+1})$. We know that there is a path of i transitions from all states in R_i. From the construction of R_{i+1} we know that there is a transition from each state of R_{i+1} to a state of R_i. Therefore, adding this transition to the walk of i transitions in R_i, we get that there exists a walk to a F in $i + 1$ transitions. Also, each state that has a walk of $i + 1$ transitions is present in R_{i+1} since otherwise, there would be a state corresponding to that walk which won't be present in R_i. This would lead to a contradiction to our inductive hypothesis. Therefore, each such state having a walk of $i + 1$ transitions to F is present in R_{i+1}. □

We now use the above claim to prove the following:

1. For every word $w \in A$, $w_{\frac{1}{2}-}$ is recognised by $D_{\frac{1}{2}-}$
2. For every word w which is recognised by $D_{\frac{1}{2}-}$, there exists a word $w_0 \in A$.

To prove the first statement, we use the result of the previous claim. Let $|w| = 2n$. Now, after $D_{\frac{1}{2}-}$ processes n symbols, let the state be (q_n, R_n). Now, since w is accepted by D, there exists a walk of length n from q_n to a state in F. Therefore, $q_n \in R_n$. Thus, (q_n, R_n)

is an accepting state and it accepts $w_{\frac{1}{2}-}$.
For the second statement, we know that there exists a path of n transitions from any accepting state (q_n, R_n). We can construct a word w_0 by appending the transition symbols which will be arrive at a state in F. Therefore, for each word which is accepted by $D_{\frac{1}{2}-}$, we have a corresponding word that is accepted by D. $\square$

4 Question 4

Question 4

Question. *If A is any language, let $A_{\frac{1}{3}-\frac{1}{3}}$ denote the set of strings in A with the middle third removed so that*

$$A_{\frac{1}{3}-\frac{1}{3}} = \{xz| \text{ for some } y,\ |x| = |y| = |z| \text{ and } xyz \in A\} \quad (4)$$

Show that if A is regular, then $A_{\frac{1}{3}-\frac{1}{3}}$ is not necessarily regular.

Solution. We will show this by contradiction. Consider the language $A = 0^+1^+2^+$. A is regular. Now assume that $A_{\frac{1}{3}-\frac{1}{3}}$ is regular.
We also know that $B = 0^+2^+$ is regular. Consider the intersection of $A_{\frac{1}{3}-\frac{1}{3}}$ and B. We know that this language, say C, will only have 0 and 2. We will now show that the number of 0 and number of 2 in each word of C will be equal.
Consider any word w in C. Consider the corresponding word in A from which we got w in $A_{\frac{1}{3}-\frac{1}{3}}$, call it $w_0 = 0^a1^b2^c$. For the case $a = b = c$, it is trivial that the number of 0 and 2 in w will be equal. Otherwise, we know that at least one of a, b, c will be less than $\frac{a+b+c}{3}$. We will now show that if $w \in C$, then $b < \frac{a+b+c}{3}$. Assume that it is not the case. Then, wlog assume that a is the smallest number. Then, in w, we will also have 1 at some positions. However since $w \in C$, such a situation is not possible. Therefore, $\#0 = \#2$ in w.
Thus, we have proved that $C \subseteq \{0^n2^n, n > 0\}$. We now show that any word w in $\{0^n2^n, n > 0\}$ will be in C. This is easy to see since $0^n1^n2^n$ is in A and thus 0^n2^n will be in C. Therefore, $C = \{0^n2^n, n > 0\}$
However, we know that C is non regular. However, from our assumption that $A_{\frac{1}{3}-\frac{1}{3}}$ is regular and B is regular, C must be regular since it is the intersection of $A_{\frac{1}{3}-\frac{1}{3}}$ and B. This is a contradiction. Therefore, $A_{\frac{1}{3}-\frac{1}{3}}$ is irregular. Hence, proved. □

5 Question 5

Question 5

Question.

Solution. $\square$

6 Question 6

Question 6

Question.

Solution. $\square$

7 Question 7

Question 7

Question. *For every string* $x \in \{0,1\}^+$ *consider the number*

$$0.x = x[1] \cdot \frac{1}{2} + x[2] \cdot \frac{1}{2^2} + \cdots + x[|x|] \cdot \frac{1}{2^{|x|}}$$

where $|x|$ *is the length of* x*. For a real number* $\theta \in [0,1]$ *let*

$$L_\theta = \{x : 0.x \leq \theta\}$$

Prove that L_θ *is regular if and only if* θ *is rational.*

Solution. ($\Longleftarrow$) We construct the following DFA for any rational number $0.\alpha_1\alpha_2 \ldots \alpha_k\overline{\beta_1\beta_2 \ldots \beta_m}$. We know that every rational number can be represented this way since its binary representation is finite or infinte but recurring.
We define $D = (Q, \{0,1\}, \delta, q_1, F)$ as:

$$Q = \{q_1, q_2, \ldots, q_k, q_{k+1}, \ldots, q_{k+m}, q_{rej}, q_{acc}\}$$
$$\delta(q_i, a) = \begin{cases} q_{i+1}, & \text{if } a = \alpha_i, i \leq k \\ q_{acc}, & \text{if } a < \alpha_i, i \leq k \\ q_{rej}, & \text{if } a > \alpha_i, i \leq k \\ q_{k+1+(i-k-1 \mod m)}, & \text{if } a = \beta_{(i-k-1 \mod m)} \\ q_{acc}, & \text{if } a < \beta_{(i-k-1 \mod m)} \\ q_{rej}, & \text{if } a > \beta_{(i-k-1 \mod m)} \\ q_{acc}, & \text{if } i = acc \\ q_{rej}, & \text{if } i = rej \end{cases} \tag{5}$$
$$F = Q \setminus \{q_{rej}, q_1\}$$

It is easy to see that any number $0.x$ such that $0.x \leq \theta$ is accepted by D. This is because we move to the accepted as soon as any comparison is smaller and we move to the reject state as soon as any comparison is greater. For equality, we simply move to the next state and all such states are accepting.
Therefore, for any rational θ, we have constructed a DFA for L_θ. Therefore, L_θ is regular if theta is rational.
($\Longrightarrow$) □