COL872
Problem Set 1

Mallika Prabhakar (2019CS50440)
Sayam Sethi (2019CS10399)
Satwik Jain (2019CS10398)

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6 Question 6

6.1 Question 6.1

Question 1

Question. *Let ρ be the density matrix for a mixed state over n qubits. In class, we saw that there exists a pure state $|\psi\rangle$ over $2n$ qubits such that measuring the last n qubits results in the density matrix ρ. Using Schmidt decomposition, prove that if $|\psi_1\rangle$ and $|\psi_2\rangle$ are two purifications of ρ, then there exists a unitary matrix $\mathbf{U}$ acting over n qubits such that $|\psi_2\rangle = (\mathbf{I}_n \otimes \mathbf{U}) |\psi_1\rangle$. Here $\mathbf{I}_n$ is the identity operation over the first n qubits.*

Proof. Let,

$$\begin{aligned} |\psi_1\rangle &= \sum_i \lambda_{1i} |u_{1i}\rangle |v_{1i}\rangle \\ |\psi_2\rangle &= \sum_i \lambda_{2i} |u_{2i}\rangle |v_{2i}\rangle \end{aligned} \tag{1}$$

On measuring the last n qubits, we are left with,

$$\begin{aligned} \rho = \rho_1 &= \sum_i \lambda_{1i}^2 |u_{1i}\rangle \langle u_{1i}| \\ = \rho_2 &= \sum_i \lambda_{2i}^2 |u_{2i}\rangle \langle u_{2i}| \end{aligned} \tag{2}$$

Since $\{|u_{1i}\rangle\}_i$ and $\{|u_{2i}\rangle\}_i$ are orthonormal vectors, the two multi-sets $\{\lambda_{1i}\}_i$ and $\{\lambda_{2i}\}_i$ should be the same and they form the eigenvalues of ρ. Therefore, we can assume the ordering of $\{|u_{1i}\rangle\}_i$ and $\{|u_{2i}\rangle\}_i$ such that $\lambda_{1i} = \lambda_{2i} = \lambda_i$ (direct equality holds since λ_{bi} are guaranteed to be positive by Schmidt decomposition).
Now, we represent $|\psi_2\rangle$ such that the first n qubits have the same orthonormal vectors as $|\psi_1\rangle$. It is guaranteed that $|u_{1i}\rangle = |u_{2i}\rangle$ if the multiplicity of λ_i is 1. Consider λ_p such that it has a multiplicity $k > 1$. The two sets of eigenvectors corresponding to this eigenvalue are $S_1 = \{|u_{1i}\rangle \, | \lambda_i = \lambda_p\}$ and $S_2 = \{|u_{2i}\rangle \, | \lambda_i = \lambda_p\}$. Now, these two sets span the same subspace of n qubits. Therefore, we can write $\sum_{|u_{2i}\rangle \in S_2} |u_{2i}\rangle |v_{2i}\rangle$ as,

$$\begin{aligned} \sum_{|u_{2i}\rangle \in S_2} |u_{2i}\rangle |v_{2i}\rangle &= \sum_{|u_{2i}\rangle \in S_2} \left(\sum_{|u_{1j}\rangle \in S_1} \alpha_{ij} |u_{1j}\rangle \right) |v_{2i}\rangle \\ &= \sum_{|u_{2i}\rangle \in S_2} \left(\sum_{|u_{1j}\rangle \in S_1} |u_{1j}\rangle \alpha_{ij} |v_{2i}\rangle \right) \\ &= \sum_{|u_{1j}\rangle \in S_1} |u_{1j}\rangle \left(\sum_{|u_{2i}\rangle \in S_2} \alpha_{ij} |v_{2i}\rangle \right) \\ &= \sum_{|u_{1j}\rangle \in S_1} |u_{1j}\rangle |v'_{2j}\rangle \end{aligned} \tag{3}$$

Note that $\{v'_{2j} | \lambda_j = \lambda_p\}$ is an orthonormal set since S_2 is also an orthonormal set. Therefore, $|\psi_2\rangle$ can be written as,

$$|\psi_2\rangle = \sum_i \lambda_i |u_{1i}\rangle |v'_{2i}\rangle = \sum_i \lambda_i |u_i\rangle |v'_{2i}\rangle \tag{4}$$

Now, since $\{|v_{1i}\rangle\}_i$ and $\{|v'_{2i}\rangle\}_i$ are both orthonormal sets, there exists a change of basis matrix (assuming that both sets span the entire set of n qubits, else, we can extend them to span the entire set), say $\mathbf{U}$. Therefore, we can write $|\psi_2\rangle$ in terms of $|\psi_1\rangle$ as,

$$|\psi_2\rangle = (\mathbf{I}_n \otimes \mathbf{U}) |\psi_1\rangle \tag{5}$$

This completes the proof. □

6.2 Question 6.2

Question 1

Question. *quess*

Proof. □

7 Question 7

7.1 Question 7.1

Question 1

Question. *quess*

Proof. □

7.2 Question 7.2

Question 1

Question. *quess*

Proof. □

7.3 Question 7.3

Question 1

Question. *quess*

Proof. □