COL872
Problem Set 1

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Contents

1 Question 1

Question 1

Question. *quess*

Proof. $\square$

2 Question 2

2.1 Question 2.1

Question 1

Question. *quess*

Proof. $\square$

2.2 Question 2.2

Question 1

Question. *quess*

Proof. $\square$

3 Question 3

3.1 Question 3.1

Question 1

Question. *quess*

Proof. $\square$

3.2 Question 3.2

Question 1

Question. *quess*

Proof. $\square$

4 Question 4

Question 1

Question. *quess*

Proof. $\square$

5 Question 5

5.1 Question 5.1

Question 1

Question. *quess*

Proof. □

5.2 Question 5.2

Question 1

Question. *quess*

Proof. □

5.3 Question 5.3

Question 1

Question. *quess*

Proof. □

5.4 Question 5.4

Question 1

Question. *quess*

Proof. □

6 Question 6

6.1 Question 6.1

Question 6.1

Question. *Let ρ be the density matrix for a mixed state over n qubits. In class, we saw that there exists a pure state $|\psi\rangle$ over $2n$ qubits such that measuring the last n qubits results in the density matrix ρ. Using Schmidt decomposition, prove that if $|\psi_1\rangle$ and $|\psi_2\rangle$ are two purifications of ρ, then there exists a unitary matrix $\mathbf{U}$ acting over n qubits such that $|\psi_2\rangle = (\mathbf{I}_n \otimes \mathbf{U}) |\psi_1\rangle$. Here $\mathbf{I}_n$ is the identity operation over the first n qubits.*

Proof. Let,

$$\begin{aligned} |\psi_1\rangle &= \sum_i \lambda_{1i} |u_{1i}\rangle |v_{1i}\rangle \\ |\psi_2\rangle &= \sum_i \lambda_{2i} |u_{2i}\rangle |v_{2i}\rangle \end{aligned} \tag{1}$$

On measuring the last n qubits, we are left with,

$$\begin{aligned} \rho = \rho_1 &= \sum_i \lambda_{1i}^2 |u_{1i}\rangle \langle u_{1i}| \\ = \rho_2 &= \sum_i \lambda_{2i}^2 |u_{2i}\rangle \langle u_{2i}| \end{aligned} \tag{2}$$

Since $\{|u_{1i}\rangle\}_i$ and $\{|u_{2i}\rangle\}_i$ are orthonormal vectors, the two multi-sets $\{\lambda_{1i}\}_i$ and $\{\lambda_{2i}\}_i$ should be the same and they form the eigenvalues of ρ. Therefore, we can assume the ordering of $\{|u_{1i}\rangle\}_i$ and $\{|u_{2i}\rangle\}_i$ such that $\lambda_{1i} = \lambda_{2i} = \lambda_i$ (direct equality holds since λ_{bi} are guaranteed to be positive by Schmidt decomposition).
Now, we represent $|\psi_2\rangle$ such that the first n qubits have the same orthonormal vectors as $|\psi_1\rangle$. It is guaranteed that $|u_{1i}\rangle = |u_{2i}\rangle$ if the multiplicity of λ_i^2 is 1. Consider λ_p^2 such that it has a multiplicity $k > 1$. The two sets of eigenvectors corresponding to this eigenvalue are $S_1 = \{|u_{1i}\rangle \, | \lambda_i = \lambda_p\}$ and $S_2 = \{|u_{2i}\rangle \, | \lambda_i = \lambda_p\}$. Now, these two sets span the same subspace of n qubits. Therefore, we can write $\sum_{|u_{2i}\rangle \in S_2} |u_{2i}\rangle |v_{2i}\rangle$ as,

$$\begin{aligned} \sum_{|u_{2i}\rangle \in S_2} |u_{2i}\rangle |v_{2i}\rangle &= \sum_{|u_{2i}\rangle \in S_2} \left(\sum_{|u_{1j}\rangle \in S_1} \alpha_{ij} |u_{1j}\rangle \right) |v_{2i}\rangle \\ &= \sum_{|u_{2i}\rangle \in S_2} \left(\sum_{|u_{1j}\rangle \in S_1} |u_{1j}\rangle \alpha_{ij} |v_{2i}\rangle \right) \\ &= \sum_{|u_{1j}\rangle \in S_1} |u_{1j}\rangle \left(\sum_{|u_{2i}\rangle \in S_2} \alpha_{ij} |v_{2i}\rangle \right) \\ &= \sum_{|u_{1j}\rangle \in S_1} |u_{1j}\rangle |v_{2j}'\rangle \end{aligned} \tag{3}$$

Note that $\{|v'_{2j}\rangle \, |\lambda_j = \lambda_p\}$ is an orthonormal set since S_2 is also an orthonormal set. Therefore, $|\psi_2\rangle$ can be written as,

$$|\psi_2\rangle = \sum_i \lambda_i |u_{1i}\rangle |v'_{2i}\rangle = \sum_i \lambda_i |u_i\rangle |v'_{2i}\rangle \tag{4}$$

Now, since $\{|v_{1i}\rangle\}_i$ and $\{|v'_{2i}\rangle\}_i$ are both orthonormal sets, there exists a change of basis matrix (assuming that both sets span the entire set of n qubits, else, we can extend them to span the entire set), say $\mathbf{U}$. Therefore, we can write $|\psi_2\rangle$ in terms of $|\psi_1\rangle$ as,

$$|\psi_2\rangle = (\mathbf{I}_n \otimes \mathbf{U}) |\psi_1\rangle \tag{5}$$

This completes the proof. □

6.2 Question 6.2

Question 6.2

Question. *Let ρ_1, ρ_2 be two density matrices, corresponding to mixed states over n qubits. Show that the following two statements are equivalent:*

- *ρ_1 and ρ_2 have the same set of eigenvalues (counting multiplicities).*
- *There exists a pure state $|\psi\rangle$ over $2n$ qubits such that when the first n qubits are measured, the state of the remaining qubits is described by density matrix ρ_2. Similarly, when the last n qubits are measured, the state of the first n qubits is ρ_1.*

Proof. ($\implies$) Let the eigenvalues of ρ_1 and ρ_2 be $\{\lambda_i^2\}_i$ and the eigenvectors be $\{|u_i\rangle\}_i$ and $\{|v_i\rangle\}_i$ respectively. Now, consider the pure state,

$$|\psi\rangle = \sum_i \lambda_i |u_i\rangle |v_i\rangle \tag{6}$$

Therefore, on measuring the first n qubits, we get $\rho_2 = \sum_i \lambda_i^2 |v_i\rangle \langle v_i|$ and on measuring the last n qubits, we get $\rho_1 = \sum_i \lambda_i^2 |u_i\rangle \langle u_i|$. Therefore, we have shown the existence of a pure state $|\psi\rangle$ over $2n$ qubits which yields ρ_1 on measuring the last n qubits and ρ_2 on measuring the first n qubits.
($\impliedby$) Using Schmidt decomposition, we can represent $|\psi\rangle$ as $\sum_i \lambda_i |u_i\rangle \langle v_i|$. Now, on measuring the last n qubits, we get the density matrix $\rho_1 = \sum_i \lambda_i^2 |u_i\rangle \langle u_i|$ and on measuring the first n qubits, we get the density matrix as $\rho_2 = \sum_i \lambda_i |v_i\rangle \langle v_i|$. Now, since $\{|u_i\rangle\}_i$ and $\{|v_i\rangle\}_i$ are both orthonormal sets, the eigenvalues of ρ_1 and ρ_2 are both $\{\lambda_i\}_i$ (multi-set). Thus, ρ_1 and ρ_2 have the same set of eigenvalues. □

7 Question 7

7.1 Question 7.1

Question 1

Question. *quess*

Proof. □

7.2 Question 7.2

Question 1

Question. *quess*

Proof. □

7.3 Question 7.3

Question 1

Question. *quess*

Proof. □