COL872
Problem Set 1

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6 Question 6

6.1 Question 6.1

Question 6.1

Question. *Let ρ be the density matrix for a mixed state over n qubits. In class, we saw that there exists a pure state $|\psi\rangle$ over $2n$ qubits such that measuring the last n qubits results in the density matrix ρ. Using Schmidt decomposition, prove that if $|\psi_1\rangle$ and $|\psi_2\rangle$ are two purifications of ρ, then there exists a unitary matrix $\mathbf{U}$ acting over n qubits such that $|\psi_2\rangle = (\mathbf{I}_n \otimes \mathbf{U}) |\psi_1\rangle$. Here $\mathbf{I}_n$ is the identity operation over the first n qubits.*

Proof. Let,

$$\begin{aligned} |\psi_1\rangle &= \sum_i \lambda_{1i} |u_{1i}\rangle |v_{1i}\rangle \\ |\psi_2\rangle &= \sum_i \lambda_{2i} |u_{2i}\rangle |v_{2i}\rangle \end{aligned} \tag{1}$$

On measuring the last n qubits, we are left with,

$$\begin{aligned} \rho = \rho_1 &= \sum_i \lambda_{1i}^2 |u_{1i}\rangle \langle u_{1i}| \\ = \rho_2 &= \sum_i \lambda_{2i}^2 |u_{2i}\rangle \langle u_{2i}| \end{aligned} \tag{2}$$

Since $\{|u_{1i}\rangle\}_i$ and $\{|u_{2i}\rangle\}_i$ are orthonormal vectors, the two multi-sets $\{\lambda_{1i}\}_i$ and $\{\lambda_{2i}\}_i$ should be the same and they form the eigenvalues of ρ. Therefore, we can assume the ordering of $\{|u_{1i}\rangle\}_i$ and $\{|u_{2i}\rangle\}_i$ such that $\lambda_{1i} = \lambda_{2i} = \lambda_i$ (direct equality holds since λ_{bi} are guaranteed to be positive by Schmidt decomposition).
Now, we represent $|\psi_2\rangle$ such that the first n qubits have the same orthonormal vectors as $|\psi_1\rangle$. It is guaranteed that $|u_{1i}\rangle = |u_{2i}\rangle$ if the multiplicity of λ_i^2 is 1. Consider λ_p^2 such that it has a multiplicity $k > 1$. The two sets of eigenvectors corresponding to this eigenvalue are $S_1 = \{|u_{1i}\rangle \,|\lambda_i = \lambda_p\}$ and $S_2 = \{|u_{2i}\rangle \,|\lambda_i = \lambda_p\}$. Now, these two sets span the same subspace of n qubits. Therefore, we can write $\sum_{|u_{2i}\rangle \in S_2} |u_{2i}\rangle |v_{2i}\rangle$ as,

$$\begin{aligned} \sum_{|u_{2i}\rangle \in S_2} |u_{2i}\rangle |v_{2i}\rangle &= \sum_{|u_{2i}\rangle \in S_2} \left(\sum_{|u_{1j}\rangle \in S_1} \alpha_{ij} |u_{1j}\rangle \right) |v_{2i}\rangle \\ &= \sum_{|u_{2i}\rangle \in S_2} \left(\sum_{|u_{1j}\rangle \in S_1} |u_{1j}\rangle \alpha_{ij} |v_{2i}\rangle \right) \\ &= \sum_{|u_{1j}\rangle \in S_1} |u_{1j}\rangle \left(\sum_{|u_{2i}\rangle \in S_2} \alpha_{ij} |v_{2i}\rangle \right) \\ &= \sum_{|u_{1j}\rangle \in S_1} |u_{1j}\rangle |v'_{2j}\rangle \end{aligned} \tag{3}$$

Note that $\{|v'_{2j}\rangle \,|\lambda_j = \lambda_p\}$ is an orthonormal set since S_2 is also an orthonormal set. Therefore, $|\psi_2\rangle$ can be written as,

$$|\psi_2\rangle = \sum_i \lambda_i |u_{1i}\rangle |v'_{2i}\rangle = \sum_i \lambda_i |u_i\rangle |v'_{2i}\rangle \tag{4}$$

Now, since $\{|v_{1i}\rangle\}_i$ and $\{|v'_{2i}\rangle\}_i$ are both orthonormal sets, there exists a change of basis matrix (assuming that both sets span the entire set of n qubits, else, we can extend them to span the entire set), say $\mathbf{U}$. Therefore, we can write $|\psi_2\rangle$ in terms of $|\psi_1\rangle$ as,

$$|\psi_2\rangle = (\mathbf{I}_n \otimes \mathbf{U}) |\psi_1\rangle \tag{5}$$

This completes the proof. □

6.2 Question 6.2

Question 6.2

Question. *Let ρ_1, ρ_2 be two density matrices, corresponding to mixed states over n qubits. Show that the following two statements are equivalent:*

- *ρ_1 and ρ_2 have the same set of eigenvalues (counting multiplicities).*
- *There exists a pure state $|\psi\rangle$ over $2n$ qubits such that when the first n qubits are measured, the state of the remaining qubits is described by density matrix ρ_2. Similarly, when the last n qubits are measured, the state of the first n qubits is ρ_1.*

Proof. ($\implies$) Let the eigenvalues of ρ_1 and ρ_2 be $\{\lambda_i^2\}_i$ and the eigenvectors be $\{|u_i\rangle\}_i$ and $\{|v_i\rangle\}_i$ respectively. Now, consider the pure state,

$$|\psi\rangle = \sum_i \lambda_i |u_i\rangle |v_i\rangle \tag{6}$$

Therefore, on measuring the first n qubits, we get $\rho_2 = \sum_i \lambda_i^2 |v_i\rangle \langle v_i|$ and on measuring the last n qubits, we get $\rho_1 = \sum_i \lambda_i^2 |u_i\rangle \langle u_i|$. Therefore, we have shown the existence of a pure state $|\psi\rangle$ over $2n$ qubits which yields ρ_1 on measuring the last n qubits and ρ_2 on measuring the first n qubits.
($\impliedby$) Using Schmidt decomposition, we can represent $|\psi\rangle$ as $\sum_i \lambda_i |u_i\rangle \langle v_i|$. Now, on measuring the last n qubits, we get the density matrix $\rho_1 = \sum_i \lambda_i^2 |u_i\rangle \langle u_i|$ and on measuring the first n qubits, we get the density matrix as $\rho_2 = \sum_i \lambda_i |v_i\rangle \langle v_i|$. Now, since $\{|u_i\rangle\}_i$ and $\{|v_i\rangle\}_i$ are both orthonormal sets, the eigenvalues of ρ_1 and ρ_2 are both $\{\lambda_i\}_i$ (multi-set). Thus, ρ_1 and ρ_2 have the same set of eigenvalues. □

7 Question 7

7.1 Question 7.1

Question 7.1

Question. *Consider the partial measurement of the first qubit in an n qubit system. Express this partial measurement is as projective measurement.*

Proof. Consider the projective measurement,

$$\begin{aligned}\mathbf{P}_0 &= (|0\rangle\langle 0|) \otimes \mathbf{I}_{n-1} \\ \mathbf{P}_1 &= (|1\rangle\langle 1|) \otimes \mathbf{I}_{n-1}\end{aligned} \tag{7}$$

This $\{\mathbf{P}_i\}_{i\in[2]}$ is the projective measurement that is equivalent to the partial measurement of the first qubit in an n qubit system. Clearly, it satisfies the idempotence property and $\mathbf{P}_0 + \mathbf{P}_1 = \mathbf{I}_n$. Consider any pure state $|\psi\rangle = \alpha_0 |0\rangle |\phi_0\rangle + \alpha_1 |1\rangle |\phi_1\rangle$. Now, applying $\mathbf{P}_b$ on $|\psi\rangle$ gives,

$$\begin{aligned}\Pr[\text{first qubit} = b] &= \langle\psi|\mathbf{P}_b|\psi\rangle \\ &= \alpha_0^2 \langle\phi_0 \langle 0|\mathbf{P}_b|0\rangle \phi_0\rangle + \alpha_1^2 \langle\phi_1 \langle 1|\mathbf{P}_b|1\rangle \phi_1\rangle \\ &= \alpha_b^2 \langle\phi_b|\phi_b\rangle \\ &= \alpha_b^2\end{aligned} \tag{8}$$

The collapsed states are,

$$\begin{aligned}|\psi_b'\rangle &= \frac{\mathbf{P}_b |\psi\rangle}{\sqrt{\langle\psi|\mathbf{P}_b|\psi\rangle}} \\ &= \frac{((|b\rangle\langle b|) \otimes \mathbf{I}_{n-1})(\alpha_0 |0\rangle \otimes |\psi_0\rangle + \alpha_1 |1\rangle \otimes |\psi_1\rangle)}{\alpha_b} \\ &= \frac{1}{\alpha_b} \cdot ((\alpha_0 |b\rangle\langle b| \, |0\rangle) \otimes |\psi_0\rangle + (\alpha_1 |b\rangle\langle b| \, |1\rangle) \otimes |\psi_1\rangle) \\ &= \frac{1}{\alpha_b} \cdot |b\rangle |\phi_b\rangle \\ &= |b\rangle |\phi_b\rangle\end{aligned} \tag{9}$$

These are exactly the same as the partial measurements. Therefore, the proposed $\{\mathbf{P}_i\}_{i\in[2]}$ is the projective measurement that corresponds to the partial measurement of the first qubit in an n qubit system. $\square$

7.2 Question 7.2

Question 7.2

Question. *The measurements discussed in class have the following (collapsing) property: once the measurement is applied to an n-qubit system, the state collapses to one of $\{|x\rangle\}_{x\in\{0,1\}^n}$, and any further measurements produce the same measurement. Does this property hold true for projective measurements?*

Proof. From the idempotence property of $\mathcal{P}$, we get that any further measurements will produce the same measurement. However, it need not be the case that the state will collapse to one of $\{|x\rangle\}_{x\in\{0,1\}^n}$. For instance, consider the following projective measurement on 1 qubit system,

$$\mathbf{P}_0 = |+\rangle\langle+|\,, \mathbf{P}_1 = |-\rangle\langle-| \tag{10}$$

This satisfies the idempotence property and the sum of the two projections is equal to $\mathbf{I}_1$. However, consider the collapsed state on input $|0\rangle$ with $\mathbf{P}_0$,

$$\frac{\mathbf{P}_0\,|0\rangle}{\sqrt{\langle 0|\mathbf{P}_0|0\rangle}} = \frac{\frac{1}{\sqrt{2}}\cdot|+\rangle}{\frac{1}{\sqrt{2}}} = |+\rangle \tag{11}$$

This is neither $|0\rangle$ nor $|1\rangle$. Therefore, the state does not necessarily collapse to one of the possible bit-strings in case of a projective measurement. □

7.3 Question 7.3

Question 7.3

Question. *Let $\mathcal{M} = \{\mathbf{M}_i\}_{i\in[t]}$ be a POVM applied to an n qubit pure state $|\psi\rangle$. Show that there exists a projective measurement $\mathcal{P} = \{\mathbf{P}_i\}_{i\in[t]}$ on a larger system over $n + \log t$ qubits, and a pure state $|\psi'\rangle$ on $n + \log t$ qubits such that $\langle\psi|\mathbf{M}_i|\psi\rangle = \langle\psi'|\mathbf{P}_i|\psi'\rangle$ for all $i \in [t]$.*

Proof. We first show that each $\mathbf{M}_i$ can be represented as a matrix of the form $|\alpha_i|^2\mathbf{V}_i^\dagger\cdot\mathbf{V}_i$ for some matrix $\mathbf{V}_i$. Since all $\mathbf{M}_i$ are Hermitian and psd, we can represent them as $|\alpha_i|^2\sum_j \lambda_j\,|u_j\rangle\langle u_j|$ such that all $|u_j\rangle$ are orthonormal vectors and thus the previous representation follows (where $|\alpha_i|^2 = \mathsf{Tr}(\mathbf{M}_i)$). Now consider the following state $|\psi'\rangle$,

$$|\psi'\rangle = \sum_{i\in[t]} \alpha_i\mathbf{V}_i\,|\psi\rangle\,|i\rangle \tag{12}$$

□

Now, consider the following projective measurement $\mathcal{P}$,

$$\mathbf{P}_i = \mathbf{I}_n \otimes (|i\rangle\langle i|) \tag{13}$$

Consider the probability of getting output i,

$$\begin{aligned}
\langle\psi'|\mathbf{P}_i|\psi'\rangle &= \sum_{j\in[t]} \langle\psi|\mathbf{V}_j^\dagger \alpha_j^* |\mathbf{I}_n| \alpha_j \mathbf{V}_j|\psi\rangle \cdot \langle j|(|i\rangle\langle i|)|j\rangle \\
&= \langle\psi|\mathbf{V}_i^\dagger \alpha_i^* |\mathbf{I}_n| \alpha_i \mathbf{V}_i|\psi\rangle \\
&= \langle\psi||\alpha_i|^2 \mathbf{V}_i^\dagger \mathbf{V}_i|\psi\rangle \\
&= \langle\psi|\mathbf{M}_i|\psi\rangle
\end{aligned} \tag{14}$$

Therefore, the probability of getting output i via the projective and the POVM is the same. Additionally, we require $\log t$ registers to store the values of i. Therefore, a POVM is a projective measurement on a larger space.