COL872
Problem Set 4

Mallika Prabhakar (2019CS50440)
Sayam Sethi (2019CS10399)
Satwik Jain (2019CS10398)

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1 Question 1

1.1 Question 1.1

Question 1.1

Question. *Give an example of two projective measurements $\mathcal{P}, \mathcal{Q}$ that are non-commutative*

Proof. Let us consider $s, t = 2$ and the projective measurements be defined as follows:

$$\begin{aligned}\mathcal{P} &= \left\{\mathbf{P}_0 = |0\rangle\langle 0| = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \mathbf{P}_1 = |1\rangle\langle 1| = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right\}\\ \mathcal{Q} &= \left\{\mathbf{Q}_0 = |+\rangle\langle +| = \begin{pmatrix}1/2 & 1/2\\ 1/2 & 1/2\end{pmatrix}, \mathbf{Q}_1 = |-\rangle\langle -| = \begin{pmatrix}1/2 & -1/2\\ -1/2 & 1/2\end{pmatrix}\right\}\end{aligned} \tag{1}$$

Let us apply the projective methods in both orders: i.e. $\mathbf{Q}_0\mathbf{P}_0$ and $\mathbf{P}_0\mathbf{Q}_0$ to the qubit $|0\rangle$, which is the probability of getting 00 on $|0\rangle$

$\mathbf{Q}_0\mathbf{P}_0|0\rangle = |+\rangle$ with probability $= \frac{1}{2}$

$$\mathbf{Q}_0\mathbf{P}_0|0\rangle = \begin{pmatrix}1/2 & 1/2\\ 1/2 & 1/2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}1/2\\ 1/2\end{pmatrix} = \frac{1}{\sqrt{2}}|+\rangle \tag{2}$$

$\mathbf{P}_0\mathbf{Q}_0|0\rangle = |0\rangle$ with probability $= \frac{1}{4}$

$$\mathbf{P}_0\mathbf{Q}_0|0\rangle = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1/2 & 1/2\\ 1/2 & 1/2\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}1/2\\ 0\end{pmatrix} = \frac{1}{2}|0\rangle \tag{3}$$

Since both the residual states and respective probabilities of seeing 00 after both measurements are different, the two are NOT commutative

□

1.2 Question 1.2

Question 1.2

Question. *Show that the projective measurements $\mathcal{P} = \{\mathbf{P}_i\}_{i\in[s]}$ and $\mathcal{Q} = \{\mathbf{Q}_i\}_{i\in[t]}$ are commutative.*

Proof. Let $\{|\mathbf{v}_i\rangle\}_i$ be an orthonormal basis for $\mathbb{C}^N$. Let $\mathcal{S} = (S_1, \ldots, S_s)$ and $\mathcal{T} = (T_1, \ldots, T_t)$ be two partitions of $[N]$. Let $\mathbf{P}_i = \sum_{k\in S_i}|v_k\rangle\langle v_k|$ for each $i \in [s]$ and $\mathbf{Q}_j = \sum_{l\in T_j}|v_l\rangle\langle v_l|$ for each $j \in [t]$.

$$\begin{aligned}\mathbf{Q}_j\mathbf{P}_i &= \sum_{l\in T_j} |v_l\rangle\langle v_l| \sum_{k\in S_i} |v_k\rangle\langle v_k| \\ &= \sum_{l\in T_j}\sum_{k\in S_i} |v_l\rangle\langle v_l|\,|v_k\rangle\langle v_k| \\ &= \sum_{l\in T_j}\sum_{k\in S_i} |v_l\rangle\langle v_l|v_k\rangle\langle v_k| \\ &= \sum_{m\in T\cap S} |v_m\rangle\langle v_m|\end{aligned} \tag{4}$$

We considered the set $T \cap S$ because $\{\mathbf{v}_i\}_i$ is a set of orthonormal vectors and resultant matrix will only involve values where $k = l$ since the inner product would be zero in all other cases. Similarly, for $\mathbf{P}_i\mathbf{Q}_j$, we obtain the same matrix:

$$\mathbf{P}_i\mathbf{Q}_j = \sum_{k\in S_i} |v_k\rangle\langle v_k| \sum_{l\in T_j} |v_l\rangle\langle v_l| = \sum_{m\in T\cap S} |v_m\rangle\langle v_m| \tag{5}$$

Since the resultant matrix is same for both ways of application of measurement, the projective measurements $\mathcal{P}$ and $\mathcal{Q}$ are commutative. □

1.3 Question 1.3

Question 1.3

Question. *Is this the only case in which projective measurements are commutative? Prove or disprove.*

Proof. Using the same notation as in Question 1.21.2. Let us define another orthonormal basis for $\mathbb{C}^n$ as $\{|\mathbf{u}_i\rangle\}_i$ □

2 Question 2

2.1 Question 2.1

Question 2: Collapsing hash from claw-free functions

Question. *Show that* $\{H_k \equiv f_k\}_{k \in K}$ *is a collapsing hash function.*

Proof. We will Begin the proof by stating that the claw free function is a post-quantum CRHF. This can be said because For two bit-strings (x_0, x_1) in the domain of claw-free function, there can be two cases:

1. x_0 and x_1 have the same first bit. In this case no collision is possible between x_0 and x_1 as the claw-free function $f_k(b, \cdot)$ is injective $\forall k \in \mathcal{K}, b \in \{0, 1\}$
2. x_0 and x_1 have different first bit. In this case as well there is no collision possible due to the security definition of claw free function.

So In both the cases we have no q.p.t that can give a collision and thus claw free function is a collision resistant hash function.

□

2.2 Question 2.2

Question 2: Collapsing hash from claw-free functions

Question. *Construct a hash function family* $\mathcal{H}' = \{\mathcal{H}'_k : \{0,1\}^{n+2} \to \{0,1\}^n\}$ *using* $\mathcal{F}$*, and prove that* $\mathcal{H}'$ *is collapsing, assuming* $\mathcal{F}$ *is a claw-free function family.*

Proof. proof □

3 Question 3

3.1 Question 3.1

Question 3: Quantum Proof of Knowledge for Blum's protocol

Question. *Show that* $p'_{Ext} \geq poly(\varepsilon)$.

Proof. proof □

3.2 Question 3.2

Question 3.2

Question. *Let* $|\psi\rangle$ *be a pure state, and* $\mathbf{P} = (\mathbf{P}, \mathbf{I} - \mathbf{P})$ *any projective measurement such that* $\mathrm{Tr}\left(\mathbf{P} \cdot |\psi\rangle \langle\psi|\right) = 1 - \epsilon$. *Let* $\rho = |\psi\rangle \langle\psi|$ *and* ρ' *the post-measurement state, conditioned on measurement output being* 0. *Show a bound on* $||\rho - \rho'||_{tr}$ *in terms of* ϵ.

Proof. The post-measurement state and ρ' can be computed as follows:

$$\begin{aligned} |\psi'\rangle &= \frac{\mathbf{P}|\psi\rangle}{\sqrt{\mathrm{Tr}\left(\mathbf{P} \cdot |\psi\rangle \langle\psi|\right)}} = \frac{\mathbf{P}|\psi\rangle}{\sqrt{1-\epsilon}} \\ \rho' &= |\psi'\rangle \langle\psi'| = \frac{\mathbf{P}|\psi\rangle \langle\psi| \mathbf{P}^\dagger}{\mathrm{Tr}\left(\mathbf{P} \cdot |\psi\rangle \langle\psi|\right)} = \frac{\mathbf{P}\rho\mathbf{P}}{1-\epsilon} \end{aligned} \tag{6}$$

Note that the post-measurement state ρ' is also a pure state. We now state and prove the following claim,

Claim 3.1. *For any two pure states,* $|\psi\rangle$ *and* $|\phi\rangle$, *we have*

$$|| \, |\psi\rangle \langle\psi| - |\phi\rangle \langle\phi| \, ||_{tr} = \sqrt{1 - |\langle\psi|\phi\rangle|^2} \tag{7}$$

Proof. Since $|\psi\rangle$ and $|\phi\rangle$ are pure states, we can represent $|\phi\rangle$ as a rotation of $|\psi\rangle$ by some angle θ. Therefore, we can write $\rho_\psi - \rho_\phi$ as,

$$\begin{aligned} |\psi\rangle \langle\psi| - |\phi\rangle \langle\phi| &= |\psi\rangle \langle\psi| - \Big((\cos\theta |\psi\rangle + \sin\theta |\psi^\perp\rangle)(\cos\theta \langle\psi| + \sin\theta \langle\psi^\perp|)\Big) \\ &= (1 - \cos^2\theta) |\psi\rangle \langle\psi| - \sin\theta\cos\theta |\psi\rangle \langle\psi^\perp| - \sin\theta\cos\theta |\psi^\perp\rangle \langle\psi| \\ &\quad - \sin^2\theta |\psi^\perp\rangle \langle\psi^\perp| \end{aligned} \tag{8}$$

Now, if we represent this matrix in the $|\psi\rangle, |\psi^\perp\rangle$ basis, we get the eigenvalues as $\sin\theta$ and $-\sin\theta$. Therefore, the trace norm will be,

$$\begin{aligned} ||\rho_\psi - \rho_\phi||_{tr} &= \sum_i |\lambda_i|, \text{ (trace norm is sum of absolute values of eigenvalues)} \\ &= 2|\sin\theta| = 2\sqrt{1 - \cos^2\theta} \\ &= 2\sqrt{1 - |\langle\phi|\psi\rangle|^2} \end{aligned} \tag{9}$$

Hence, we have proven the result of the claim. □

Now, since we started with a pure state $|\psi\rangle$, the post-measurement state will also be a pure state (conditioned on the output). Therefore, we get the trace distance as,

$$\begin{aligned} ||\rho - \rho'||_{tr} &= 2\sqrt{1 - |\langle\psi|\psi'\rangle|^2} \\ &= 2\sqrt{1 - \left|\mathrm{Tr}\left(\langle\psi| \frac{\mathbf{P}|\psi\rangle}{\sqrt{1-\epsilon}}\right)\right|^2} \\ &= 2\sqrt{1 - \left|\mathrm{Tr}\left(\frac{\mathbf{P}|\psi\rangle\langle\psi|}{\sqrt{1-\epsilon}}\right)\right|^2} \\ &= 2\sqrt{1 - \left(\sqrt{1-\epsilon}\right)^2} \\ &= 2\sqrt{\epsilon} \end{aligned} \tag{10}$$

□

3.3 Question 3.3

Question 3.3

Question. *Show that if two states ρ, ρ' satisfy $||\rho - \rho'||_{tr} \leq \epsilon$, then for any unitary matrix $\mathbf{U}$, $||\mathbf{U}\cdot\rho\cdot\mathbf{U}^\dagger - \mathbf{U}\cdot\rho'\cdot\mathbf{U}^\dagger||_{tr} \leq \epsilon$.*

Proof. Since any unitary matrix is just a change of basis matrix, it does not change the eigenvalues of any matrix. Hence, the trace distance does not change since it is the sum of absolute values of the eigen values. Also, $\mathbf{U}\cdot\rho\cdot\mathbf{U}^\dagger$ is the density matrix corresponding to the the density matrix in the changed basis wrt $\mathbf{U}$. Therefore, the given inequality in the question holds. □

3.4 Question 3.4

Question x: description

Question. *question*

Proof. proof □

4 Question 4

4.1 Question 4

Question x: description

Question. *question*

Proof. proof □

5 Question 5

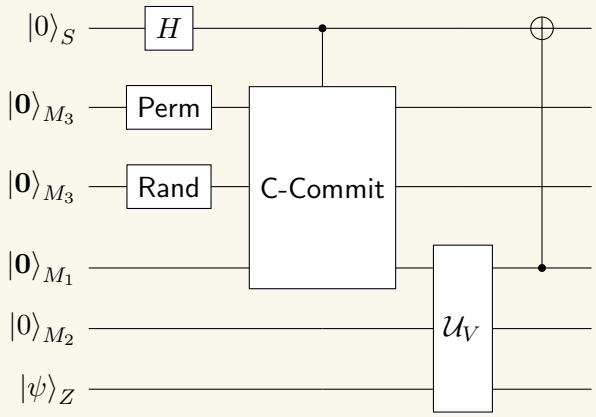
5.1 Question 5.1

Question 5.1

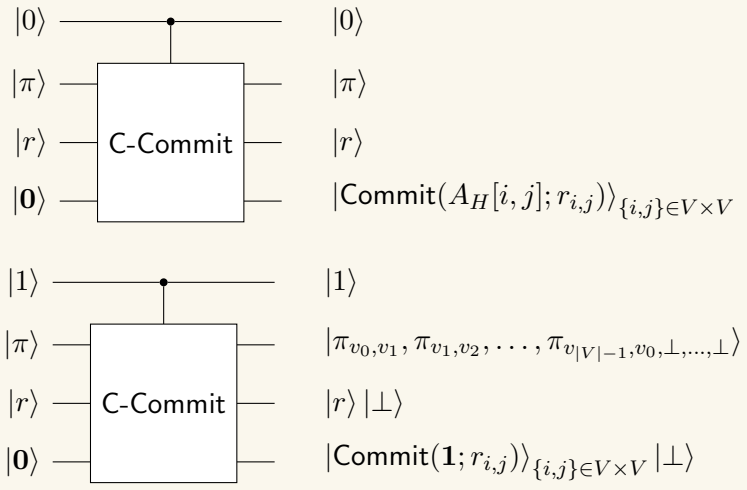
Question. *Similar to the graph-isomorphism protocol, construct a unitary* $\mathbf{Q}$ *for Blum's protocol such that,*

$$\mathbf{Q}\,|\mathbf{0}\rangle\,|\psi\rangle = \sqrt{p_\psi}\,|0\rangle\,|\psi_0\rangle + \sqrt{1-p_\psi}\,|1\rangle\,|\psi_1\rangle \tag{11}$$

Proof. Since the simulator (and prover) need to handle M_3 of different sizes, we define a register state (the register is of n qubits) as $\perp_n$ which corresponds to undefined. This is equivalent to having a register of $n+1$ qubits and the value in the last n registers is valid iff the first qubit is 1, else all the last n qubits have an invalid value. Now, we define the unitary $\mathbf{Q}$ as,



Here, H is the Hadamard gate, Perm creates a uniform superposition of all permutations on graph G with n vertices, Rand creates a uniform superposition for the randomness used by the commitment scheme (is effectively a Hadamard gate of appropriate size). The operator C-Commit takes in a control bit and two inputs (the message and the randomness) and stores the output in the third register. Depending on the control bit, it works as follows,



Note that the verifier cannot perform any computation on those registers that contain a $\perp$. This is simply possible in the case of a classical circuit by sending inputs of different sizes, however, since in the case of quantum circuits the input and output sizes are fixed. Since the circuit given has no measurement, it is a unitary of the form:

$$\mathbf{Q}\,|\mathbf{0}\rangle\,|\psi\rangle = \sqrt{p_\psi}\,|0\rangle_S\,|\psi_0\rangle + \sqrt{1-p_\psi}\,|1\rangle_S\,|\psi_1\rangle \tag{12}$$

□

5.2 Question 5.2

Question x: description

Question. *question*

Proof. proof □

5.3 Question 5.3

Question x: description

Question. *question*

Proof. proof □