

# COL872

## Problem Set 1

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## 1 Question 1

Question 1

**Question.** *guess*

*Proof.*



## 2 Question 2

### 2.1 Question 2.1

Question 1
<b>Question.</b> <i>guess</i>
<i>Proof.</i> <input type="checkbox"/>

### 2.2 Question 2.2

Question 1
<b>Question.</b> <i>guess</i>
<i>Proof.</i> <input type="checkbox"/>

### 3 Question 3

#### 3.1 Question 3.1

Question 1
<b>Question.</b> <i>guess</i>
<i>Proof.</i> <input type="checkbox"/>

#### 3.2 Question 3.2

Question 1
<b>Question.</b> <i>guess</i>
<i>Proof.</i> <input type="checkbox"/>

## 4 Question 4

Question 1

**Question.** *guess*

*Proof.*



## 5 Question 5

### 5.1 Question 5.1

Question 1
<b>Question.</b> <i>guess</i>
<i>Proof.</i> <input type="checkbox"/>

### 5.2 Question 5.2

Question 1
<b>Question.</b> <i>guess</i>
<i>Proof.</i> <input type="checkbox"/>

### 5.3 Question 5.3

Question 1
<b>Question.</b> <i>guess</i>
<i>Proof.</i> <input type="checkbox"/>

### 5.4 Question 5.4

Question 1
<b>Question.</b> <i>guess</i>
<i>Proof.</i> <input type="checkbox"/>

## 6 Question 6

### 6.1 Question 6.1

#### Question 6.1

**Question.** Let  $\rho$  be the density matrix for a mixed state over  $n$  qubits. In class, we saw that there exists a pure state  $|\psi\rangle$  over  $2n$  qubits such that measuring the last  $n$  qubits results in the density matrix  $\rho$ . Using Schmidt decomposition, prove that if  $|\psi_1\rangle$  and  $|\psi_2\rangle$  are two purifications of  $\rho$ , then there exists a unitary matrix  $\mathbf{U}$  acting over  $n$  qubits such that  $|\psi_2\rangle = (\mathbf{I}_n \otimes \mathbf{U}) |\psi_1\rangle$ . Here  $\mathbf{I}_n$  is the identity operation over the first  $n$  qubits.

*Proof.* Let,

$$\begin{aligned} |\psi_1\rangle &= \sum_i \lambda_{1i} |u_{1i}\rangle |v_{1i}\rangle \\ |\psi_2\rangle &= \sum_i \lambda_{2i} |u_{2i}\rangle |v_{2i}\rangle \end{aligned} \tag{1}$$

On measuring the last  $n$  qubits, we are left with,

$$\begin{aligned} \rho &= \rho_1 = \sum_i \lambda_{1i}^2 |u_{1i}\rangle \langle u_{1i}| \\ &= \rho_2 = \sum_i \lambda_{2i}^2 |u_{2i}\rangle \langle u_{2i}| \end{aligned} \tag{2}$$

Since  $\{|u_{1i}\rangle\}_i$  and  $\{|u_{2i}\rangle\}_i$  are orthonormal vectors, the two multi-sets  $\{\lambda_{1i}\}_i$  and  $\{\lambda_{2i}\}_i$  should be the same and they form the eigenvalues of  $\rho$ . Therefore, we can assume the ordering of  $\{|u_{1i}\rangle\}_i$  and  $\{|u_{2i}\rangle\}_i$  such that  $\lambda_{1i} = \lambda_{2i} = \lambda_i$  (direct equality holds since  $\lambda_{bi}$  are guaranteed to be positive by Schmidt decomposition).

Now, we represent  $|\psi_2\rangle$  such that the first  $n$  qubits have the same orthonormal vectors as  $|\psi_1\rangle$ . It is guaranteed that  $|u_{1i}\rangle = |u_{2i}\rangle$  if the multiplicity of  $\lambda_i^2$  is 1. Consider  $\lambda_p^2$  such that it has a multiplicity  $k > 1$ . The two sets of eigenvectors corresponding to this eigenvalue are  $S_1 = \{|u_{1i}\rangle | \lambda_i = \lambda_p\}$  and  $S_2 = \{|u_{2i}\rangle | \lambda_i = \lambda_p\}$ . Now, these two sets span the same subspace of  $n$  qubits. Therefore, we can write  $\sum_{|u_{2i}\rangle \in S_2} |u_{2i}\rangle |v_{2i}\rangle$  as,

$$\begin{aligned} \sum_{|u_{2i}\rangle \in S_2} |u_{2i}\rangle |v_{2i}\rangle &= \sum_{|u_{2i}\rangle \in S_2} \left( \sum_{|u_{1j}\rangle \in S_1} \alpha_{ij} |u_{1j}\rangle \right) |v_{2i}\rangle \\ &= \sum_{|u_{2i}\rangle \in S_2} \left( \sum_{|u_{1j}\rangle \in S_1} |u_{1j}\rangle \alpha_{ij} |v_{2i}\rangle \right) \\ &= \sum_{|u_{1j}\rangle \in S_1} |u_{1j}\rangle \left( \sum_{|u_{2i}\rangle \in S_2} \alpha_{ij} |v_{2i}\rangle \right) \\ &= \sum_{|u_{1j}\rangle \in S_1} |u_{1j}\rangle |v'_{2j}\rangle \end{aligned} \tag{3}$$

Note that  $\{|v'_{2j}\rangle | \lambda_j = \lambda_p\}$  is an orthonormal set since  $S_2$  is also an orthonormal set. Therefore,  $|\psi_2\rangle$  can be written as,

$$|\psi_2\rangle = \sum_i \lambda_i |u_{1i}\rangle |v'_{2i}\rangle = \sum_i \lambda_i |u_i\rangle |v'_{2i}\rangle \quad (4)$$

Now, since  $\{|v_{1i}\rangle\}_i$  and  $\{|v'_{2i}\rangle\}_i$  are both orthonormal sets, there exists a change of basis matrix (assuming that both sets span the entire set of  $n$  qubits, else, we can extend them to span the entire set), say  $\mathbf{U}$ . Therefore, we can write  $|\psi_2\rangle$  in terms of  $|\psi_1\rangle$  as,

$$|\psi_2\rangle = (\mathbf{I}_n \otimes \mathbf{U}) |\psi_1\rangle \quad (5)$$

This completes the proof.  $\square$

## 6.2 Question 6.2

### Question 6.2

**Question.** Let  $\rho_1, \rho_2$  be two density matrices, corresponding to mixed states over  $n$  qubits. Show that the following two statements are equivalent:

- $\rho_1$  and  $\rho_2$  have the same set of eigenvalues (counting multiplicities).
- There exists a pure state  $|\psi\rangle$  over  $2n$  qubits such that when the first  $n$  qubits are measured, the state of the remaining qubits is described by density matrix  $\rho_2$ . Similarly, when the last  $n$  qubits are measured, the state of the first  $n$  qubits is  $\rho_1$ .

*Proof.* ( $\implies$ ) Let the eigenvalues of  $\rho_1$  and  $\rho_2$  be  $\{\lambda_i^2\}_i$  and the eigenvectors be  $\{|u_i\rangle\}_i$  and  $\{|v_i\rangle\}_i$  respectively. Now, consider the pure state,

$$|\psi\rangle = \sum_i \lambda_i |u_i\rangle |v_i\rangle \quad (6)$$

Therefore, on measuring the first  $n$  qubits, we get  $\rho_2 = \sum_i \lambda_i^2 |v_i\rangle \langle v_i|$  and on measuring the last  $n$  qubits, we get  $\rho_1 = \sum_i \lambda_i^2 |u_i\rangle \langle u_i|$ . Therefore, we have shown the existence of a pure state  $|\psi\rangle$  over  $2n$  qubits which yields  $\rho_1$  on measuring the last  $n$  qubits and  $\rho_2$  on measuring the first  $n$  qubits.

( $\impliedby$ ) Using Schmidt decomposition, we can represent  $|\psi\rangle$  as  $\sum_i \lambda_i |u_i\rangle |v_i\rangle$ . Now, on measuring the last  $n$  qubits, we get the density matrix  $\rho_1 = \sum_i \lambda_i^2 |u_i\rangle \langle u_i|$  and on measuring the first  $n$  qubits, we get the density matrix as  $\rho_2 = \sum_i \lambda_i^2 |v_i\rangle \langle v_i|$ . Now, since  $\{|u_i\rangle\}_i$  and  $\{|v_i\rangle\}_i$  are both orthonormal sets, the eigenvalues of  $\rho_1$  and  $\rho_2$  are both  $\{\lambda_i^2\}_i$  (multi-set). Thus,  $\rho_1$  and  $\rho_2$  have the same set of eigenvalues.  $\square$



## 7 Question 7

### 7.1 Question 7.1

#### Question 7.1

**Question.** Consider the partial measurement of the first qubit in an  $n$  qubit system. Express this partial measurement as projective measurement.

*Proof.* Consider the partial measurement,

$$\begin{aligned}\mathbf{P}_0 &= (|0\rangle\langle 0|) \otimes \mathbf{I}_{n-1} \\ \mathbf{P}_1 &= (|1\rangle\langle 1|) \otimes \mathbf{I}_{n-1}\end{aligned}\tag{7}$$

This  $\{\mathbf{P}_i\}_{i \in [2]}$  is the projective measurement that is equivalent to the partial measurement of the first qubit in an  $n$  qubit system. Clearly, it satisfies the idempotence property and  $\mathbf{P}_0 + \mathbf{P}_1 = \mathbf{I}_n$ . Consider any pure state  $|\psi\rangle = \alpha_0 |0\rangle |\phi_0\rangle + \alpha_1 |1\rangle |\phi_1\rangle$ . Now, applying  $\mathbf{P}_b$  on  $|\psi\rangle$  gives,

$$\begin{aligned}\Pr[\text{first qubit} = b] &= \langle \psi | \mathbf{P}_b | \psi \rangle \\ &= \alpha_0^2 \langle \phi_0 | \langle 0 | \mathbf{P}_b | 0 \rangle | \phi_0 \rangle + \alpha_1^2 \langle \phi_1 | \langle 1 | \mathbf{P}_b | 1 \rangle | \phi_1 \rangle \\ &= \alpha_b^2 \langle \phi_b | \phi_b \rangle \\ &= \alpha_b^2\end{aligned}\tag{8}$$

The collapsed states are,

$$\begin{aligned}|\psi'_b\rangle &= \frac{\mathbf{P}_b |\psi\rangle}{\sqrt{\langle \psi | \mathbf{P}_b | \psi \rangle}} \\ &= \frac{((|b\rangle\langle b|) \otimes \mathbf{I}_{n-1}) (\alpha_0 |0\rangle \otimes |\psi_0\rangle + \alpha_1 |1\rangle \otimes |\psi_1\rangle)}{\alpha_b} \\ &= \frac{1}{\alpha_b} \cdot ((\alpha_0 |b\rangle\langle b| |0\rangle) \otimes |\psi_0\rangle + (\alpha_1 |b\rangle\langle b| |1\rangle) \otimes |\psi_1\rangle) \\ &= \frac{1}{\alpha_b} \cdot |b\rangle |\phi_b\rangle \\ &= |b\rangle |\phi_b\rangle\end{aligned}\tag{9}$$

These are exactly the same as the partial measurements. Therefore, the proposed  $\{\mathbf{P}_i\}_{i \in [2]}$  is the projective measurement that corresponds to the partial measurement of the first qubit in an  $n$  qubit system.  $\square$

## 7.2 Question 7.2

Question 1
<b>Question.</b> <i>guess</i>
<i>Proof.</i> <input type="checkbox"/>

## 7.3 Question 7.3

Question 1
<b>Question.</b> <i>guess</i>
<i>Proof.</i> <input type="checkbox"/>