Introduction to Quantum Information Science
Recitation, week 2

Review prerequisites and Practice with kets, tensors, and measurement

1. First: a brief lecture about p-norms p-norms are mentioned in lecture and the homework without definition. In case you don't recognize them, here is some review.

For any real $p \geq 1$, the p-norm of a vector x is

$$\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}.$$

Note that often the p-norm may be written using single bars, $|x|$ — to determine if you're dealing with a norm or an absolute value, ask yourself whether x is a vector or scalar. The most common p-norm is the 2-norm

$$\|x\|_2 := \sqrt{|x_1|^2 + |x_2|^2 + |x_3|^2 + \ldots},$$

and often we will drop the subscript for the 2-norm and just write $|x|$. This is just the standard Euclidean distance. Be careful to always write the absolute value symbols: although they don't matter with real numbers, in this class we work with complex numbers.

There are two other special p-norms. The 1-norm just gives the sum of the absolute values of all the entries in x. The ∞-norm is defined as

$$\|x\|_\infty = \lim_{p\to\infty} \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} = \max_i |x_i|.$$

There are many useful inequalities and theorems, some for all norms and some just for p-norms. You can search for these online. In particular, know that all norms satisfy the triangle inequality, are nonnegative, and $\|cx\| = |c|\|x\|$ for scalars c. And for p-norms, Hölder's Inequality and the Cauchy-Schwarz inequality are very useful.

2. Complex Numbers and Amplitude Review

a) Express $|a + ib|$, without any norms or absolute values (i.e. get rid of the $|\cdot|$) (e.g. $|1 + i| = \sqrt{2}$).

Solution:

$$|a + ib| = \sqrt{(a + ib)^*(a + ib)} = \sqrt{(a - ib)(a + ib)} = \sqrt{a^2 - i^2b^2} = \sqrt{a^2 + b^2}$$

Here we assumed $a, b \in \mathbb{R}$. Otherwise, we would need to write $|a|^2$, etc.

b) Express $(1 + i\sqrt{3})(1 - i)$ in the form $re^{i\theta}$ for some $r \in \mathbb{R}_{>0}$ and $\theta \in [0, 2\pi)$.

Solution:

$$1 + i\sqrt{3} = 2e^{i\pi/3}, \quad \text{and} \quad 1 - i = \sqrt{2}e^{-i\pi/4}$$

Hence,

$$(1 + i\sqrt{3})(1 - i) = 2e^{i\pi/3}\sqrt{2}e^{-i\pi/4} = 2\sqrt{2}e^{i\pi/12}$$

c) Compute $\sqrt{i}$ and i^i, writing your answer in terms of e.

Solution:

$$\sqrt{i} = \left(e^{i\pi/2}\right)^{\frac{1}{2}} = e^{i\pi/4}, \quad i^i = \left(e^{i\pi/2}\right)^i = e^{-\pi/2}$$

A student in class pointed out that the above answer is technically incomplete. It only gives the principal square root, when there are in fact two solutions to any square root. Technically, we should write that for any integer k,

$$\sqrt{i} = \left(e^{i(\pi/2+2k\pi)}\right)^{\frac{1}{2}} = e^{i(\pi/4+k\pi)}.$$

Then, restricting ourselves to angles between $-\pi$ and π, we set $k = -1$ and get $e^{-i3\pi/4}$.

It's important that you understand the more complete solution, but in this class, we'll almost always only need to think about non-equivalent solutions.

d) Let

$$u = \begin{bmatrix} i\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} e^{i\frac{\pi}{4}}\frac{3}{5} \\ i\frac{4}{5} \\ 0 \end{bmatrix}.$$

Compute the magnitude of the inner product $|\langle u, v\rangle|$.

Solution: Don't forget to take the complex conjugate!

$$|u \cdot v| = \left| -i\frac{1}{\sqrt{2}} \cdot e^{i\frac{\pi}{4}}\frac{3}{5} + 0 \cdot i\frac{4}{5} + \frac{1}{\sqrt{2}} \cdot 0 \right| = \frac{3}{5\sqrt{2}}$$

3. Linear Algebra Review

a) Compute the eigenvectors and eigenvalues of these three matrices:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Solution: You can do this by hand or using a computer — mathematical software, including WolframAlpha, is totally okay in this course! After all, our goal is to apply math to quantum computing, not to practice our arithmetic.

$$X \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \qquad X \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = -\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$Y \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} \qquad Y \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{bmatrix} = -\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{bmatrix}$$

$$Z\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix} \qquad Z\begin{bmatrix}0\\1\end{bmatrix}=-\begin{bmatrix}0\\1\end{bmatrix}$$

b) Let $A \in \mathbb{C}^{n\times n}$ be a matrix such that $A^2 = A$. What are the possible eigenvalues of A?

Solution: Let u be an eigenvector of A with eigenvalue λ. We can see that $\lambda u = Au = A^2u = \lambda^2 u$. This means that $\lambda^2 = \lambda$, which is only possible for $\lambda = 0$ and $\lambda = 1$.

c) Let $A \in \mathbb{R}^{n\times n}$ be a matrix such that its rows and columns are orthonormal vectors. Show that $A^\dagger A = AA^\dagger = I$.

Solution: Let a_i be the column vectors of A. We can see that $\left(A^\dagger A\right)_{ij} = a_i^\dagger a_j = \delta_{ij}$ where δ_{ij} is the Kronecker delta (takes value 0 unless $i = j$, in which case it takes value 1), because the column vectors are orthonormal. This matches perfectly with the identity matrix, so $A^\dagger A = I$. To prove $AA^\dagger = I$ we do the same things with the row vectors of A.

4. Tensor Products

a) Compute

$$\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix} \otimes \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}.$$

Solution:

$$\frac{1}{2}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}$$

b) Let A_{ij} mean the element in the i-th row and the j-th column of A. What does $A \otimes B$ look like abstractly?

Solution:

$$\begin{bmatrix}A_{11}B & A_{12}B & A_{13}B & \cdots\\ A_{21}B & A_{22}B & A_{23}B & \cdots\\ A_{31}B & A_{32}B & A_{33}B & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{bmatrix}$$

c) Show that $(A \otimes B)(u \otimes v) = (Au) \otimes (Bv)$.

Solution: Let us first see that for a matrix A and vector v, $(Au)_i = \sum_j A_{ij}u_j$. This leads us to

$$(A \otimes B)(u \otimes v) = \begin{bmatrix}A_{11}B & A_{12}B & A_{13}B & \cdots\\ A_{21}B & A_{22}B & A_{23}B & \cdots\\ A_{31}B & A_{32}B & A_{33}B & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{bmatrix}\begin{bmatrix}u_1 v\\ u_2 v\\ \vdots\end{bmatrix}$$

$$= \begin{bmatrix} \left(\sum_j A_{1j} u_j\right) Bv \\ \left(\sum_j A_{2j} u_j\right) Bv \\ \vdots \end{bmatrix}$$
$$= (Au) \otimes (Bv).$$

5. Measurement

a) We're given a qubit in the state

$$|\psi\rangle = \frac{\sqrt{3}}{2} |0\rangle + \frac{1}{2} |1\rangle .$$

What is the probability of seeing the outcome $|0\rangle$ when measuring in the $\{|0\rangle, |1\rangle\}$-basis?

Solution: By the Born rule this is just the absolute value squared of the corresponding amplitude: $|\frac{\sqrt{3}}{2}|^2 = \frac{3}{4}$.

b) What does $|\psi\rangle$ look like when written in the $\{|+\rangle, |-\rangle\}$-basis? And what is the probability of observing $|+\rangle$ if we were to measure in this basis instead?

Solution: It's a basic fact of linear algebra that for any orthonormal basis $\{|b_i\rangle\}$ we have

$$I = \sum_i |b_i\rangle\langle b_i| .$$

Hence,

$$\begin{aligned} |\psi\rangle &= \big(|+\rangle\langle+| + |-\rangle\langle-|\big) |\psi\rangle \\ &= \langle+|\psi\rangle |+\rangle + \langle-|\psi\rangle |-\rangle \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} |+\rangle + \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} |-\rangle \\ &\approx 0.97 |+\rangle + 0.26 |-\rangle . \end{aligned}$$

For the probability of observing $|+\rangle$ we can once again use the Borne rule. If computed without intermediate rounding (as it should be), this yields 0.93. . .

Note that a shortcut to this answer would've been to only compute $|\langle+|\psi\rangle|^2$.

c) What is the probability of making both the previous observations when consecutively measuring the *same* qubit? (The probability of seeing $|0\rangle$ then $|+\rangle$)

Solution: We already saw in a) that there's a $\frac{3}{4}$ chance to initially observe a $|0\rangle$. But once that happens the qubit will "snap" to $|0\rangle$. So the probability to then observe $|+\rangle$ becomes $|\langle+|0\rangle|^2 = \frac{1}{2}$. Overall that gives us a $\frac{3}{8}$ chance for the entire sequence of events.