Introduction to Quantum Information Science
Recitation 5
(~~9/29~~ 9/30 and 10/3)

1. First: Any questions about the solutions to homework 3?

2. Outer Products

a) We have seen that products of a bra and ket of the form $\langle\psi|\phi\rangle$ correspond to inner products, which result in scalars as output. Show that if instead we multiply a ket and bra together like $|\psi\rangle\langle\phi|$, then the result is a matrix. This new type of product is called the *outer product.*
We will be using this *a lot* in the next couple weeks.

Solution: This follows directly from the rules of matrix multiplication. $|\psi\rangle$ is a column vector and $\langle\phi|$ is a row vector so their product is of the form:

$$\begin{bmatrix}\psi_0\\ \psi_1\\ \vdots\\ \psi_{n-1}\end{bmatrix}\begin{bmatrix}\phi_0 & \phi_1 & \cdots & \phi_{n-1}\end{bmatrix} = \begin{bmatrix}\psi_0\phi_0 & \psi_0\phi_1 & \psi_0\phi_2 & \cdots\\ \psi_1\phi_0 & \psi_1\phi_1 & \psi_1\phi_2 & \cdots\\ \psi_2\phi_0 & \psi_2\phi_1 & \psi_2\phi_2 & \cdots\\ \vdots & & & \ddots\end{bmatrix}$$

3. The Trace You may recall from your linear algebra course that the trace of a square $n \times n$ matrix A, denoted $\mathrm{Tr}(A)$, can be defined in several equivalent ways. Here are a few:

1. The sum of the diagonal elements: $\mathrm{Tr}(A) = \sum_i^n A_{i,i}$.
2. The sum of the eigenvalues: $\mathrm{Tr}(A) = \sum_i^n \lambda_i$.
3. For any orthonormal basis of n elements $|\psi_i\rangle$, $\mathrm{Tr}(A) = \sum_i^n \langle\psi_i| A |\psi_i\rangle$.

Some important properties of the trace include that it's *linear* and that it's *cyclic*. See e.g. Wikipedia for more.

a) Write the matrix

$$\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$$

in the form

$$\sum_{i,j} \alpha_{i,j} |i\rangle\langle j|$$

Solution:

$$1\left|0\right\rangle\left\langle 0\right| + 2\left|0\right\rangle\left\langle 1\right| + 3\left|1\right\rangle\left\langle 0\right| + 4\left|1\right\rangle\left\langle 1\right|$$

b) Write the matrix from part (a) using Hadamard basis instead (a.k.a. the $\left|+\right\rangle / \left|-\right\rangle$ basis).

Solution: We can start with the matrix we found in part (a) and then apply a change of basis to both parts of the outer product. We do this by applying a Hadamard from both the left and the right

$$\begin{aligned} H(1\left|0\right\rangle\left\langle 0\right| + 2\left|0\right\rangle\left\langle 1\right| + 3\left|1\right\rangle\left\langle 0\right| + 4\left|1\right\rangle\left\langle 1\right|)H &= 1\left|+\right\rangle\left\langle +\right| + 2\left|+\right\rangle\left\langle -\right| + 3\left|-\right\rangle\left\langle +\right| + 4\left|-\right\rangle\left\langle -\right| \\ &= \begin{bmatrix} 5 & -1 \\ -2 & 0 \end{bmatrix} \end{aligned}$$

c) Write the CNOT gate in the form

$$\sum_{i,j} \alpha_{i,j} \left|i\right\rangle\!\left\langle j\right|$$

Solution:

$$\left|00\right\rangle\left\langle 00\right| + \left|01\right\rangle\left\langle 01\right| + \left|11\right\rangle\left\langle 10\right| + \left|10\right\rangle\left\langle 11\right|$$

d) Using the definition of CNOT in terms of outer product notationyou found in part (c), apply CNOT to the state $\left|\psi\right\rangle = \frac{1}{2}(\left|00\right\rangle + e^{-i\pi/4}\left|01\right\rangle + e^{i\pi/4}\left|10\right\rangle - \left|11\right\rangle)$.

Solution:

$$\frac{1}{2}(\left|00\right\rangle + e^{-i\pi/4}\left|01\right\rangle + e^{i\pi/4}\left|11\right\rangle - \left|10\right\rangle)$$

e) Let $\left|\psi\right\rangle = \sum_{i=0}^{n-1} v_i \left|i\right\rangle$ be a unit-length vector and let $M = \left|\psi\right\rangle\left\langle\psi\right|$. Compute $\mathrm{Tr}(M)$ by summing the diagonal entries of M.

Solution: The ith diagonal entry of M is $v_i v_i^* = |v_i|^2$, so the sum of the diagonal entries is $\sum_i |v_i|^2 = 1$ because $\left|\psi\right\rangle$ is a unit length vector.

f) Show that $\mathrm{Tr}(A \cdot B) = \mathrm{Tr}(B \cdot A)$. Generalize this to show the *cyclic property* of the trace $\mathrm{Tr}(A_1 \cdot A_2 \cdot \ldots \cdot A_n) = \mathrm{Tr}(A_2 \cdot \ldots \cdot A_n \cdot A_1)$.
Hint: Begin by writing A and B using outer product notation.

Solution: Let $A = \sum_{ij} a_i j \left|i\right\rangle\!\left\langle j\right|$ and $B = \sum_{ij} b_{ij} \left|i\right\rangle\!\left\langle j\right|$.

$$\begin{aligned} \mathrm{Tr}(A \cdot B) &= \sum_i \left\langle i\right| (\sum_{jk} a_{jk} \left|j\right\rangle\!\left\langle k\right|)(\sum_{lm} b_{lm} \left|l\right\rangle\!\left\langle m\right|) \left|i\right\rangle \\ &= \sum_i \left\langle i\right| (\sum_{jkl} a_{jk} b_{kl} \left\langle j\right| \left|l\right\rangle) \left|i\right\rangle \\ &= \sum_{ij} a_{ij} b_{ji} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{ji} a_{ji} b_{ij} && \text{(Change variable names)} \\
&= \sum_{ij} b_{ij} a_{ji} && \text{(Addition and multiplication are commutative)} \\
&= \mathrm{Tr}(B \cdot A) && \text{(By symmetry)}
\end{aligned}
$$

For the cyclic property, let $B = A_2 \cdot ... \cdot A_n$.

$$\mathrm{Tr}(A_1 \cdot A_2 \cdot ... \cdot A_n) = \mathrm{Tr}(A_1 \cdot B) = \mathrm{Tr}(B \cdot A) = \mathrm{Tr}(A_2 \cdot ... \cdot A_n \cdot A_1)$$

g) We can write the trace as $\mathrm{Tr}(M) = \sum_i \langle i| M |i\rangle$. Would the same be true if we replaced $\{|i\rangle\}$ with any other orthonormal basis $\{|v_i\rangle\}$? Prove it.

Solution: Let U be the unitary such that $\forall i \colon U|i\rangle = |v_i\rangle$.

$$\sum_i \langle v_i| M |v_i\rangle = \sum_i \langle i| U^\dagger M U |i\rangle$$

So the trace with respect to the new basis is the same as the trace of $U^\dagger M U$ in the standard basis. But from the cyclic invariance of the trace:

$$\mathrm{Tr}(U^\dagger M U) = \mathrm{Tr}(U U^\dagger M) = \mathrm{Tr}(M)$$

We can do the trace with respect to any orthonormal basis we want.

h) Show that $\mathrm{Tr}(M)$ is equal to the sum of the eigenvalues for any Hermitian matrix M.
Hint: Use the eigendecomposition of M.

Solution: The spectral theorem (i.e. that Hermitian matrices are diagonalizable) that says that we can write our Hermitian matrix as $M = UDU^\dagger$ where D is a diagonal matrix whose entries are the eigenvalues and U is some unitary. Using the cyclic property of the trace:

$$\mathrm{Tr}(M) = \mathrm{Tr}(UDU^\dagger) = \mathrm{Tr}(U^\dagger U D) = \mathrm{Tr}(D) = \sum_i \lambda_i.$$

4. Finally: Questions about anything else?