Introduction to Quantum Information Science
Recitation 2
(9/8 and 9/12)

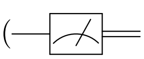
1. More on Unitaries

a) Show that the product of two unitary matrices is also a unitary matrix.

Solution: Let A and B be unitary matrices. In order to have that the product AB is unitary we want to show that $AB(AB)^\dagger = I$. $(AB)^\dagger = B^\dagger A^\dagger$ and so $AB(AB)^\dagger = ABB^\dagger A^\dagger = I$.

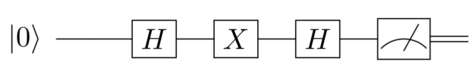
b) Generalize this to the product of an arbitrary number of unitary matrices.

Solution: Let $A_1, \cdots, A_n$ be unitary matrices. Similar to above, we have $(A_1 \cdots A_n)(A_1 \cdots A_n)^\dagger = (A_1 \cdots A_n)(A_n^\dagger \cdots A_1^\dagger) = I$.

2. Circuits And Measurement Find the probability of measuring $|0\rangle$ and $|1\rangle$ for the following circuits ( means measurement):

a)

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



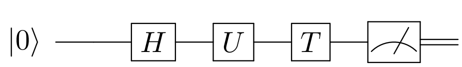
Solution:

$$|0\rangle \mapsto |+\rangle \mapsto |+\rangle \mapsto |0\rangle$$

We measure $|0\rangle$ with probability 1.

b)

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}, \quad U = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{-3}{5} \end{bmatrix}$$



Solution:

$$|0\rangle \mapsto |+\rangle$$

The effect of U on the standard basis is the following

$$|0\rangle \mapsto \frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle \quad |1\rangle \mapsto \frac{4}{5}|0\rangle - \frac{3}{5}|1\rangle$$

So U applied to the $|+\rangle$ state gives:

$$\begin{aligned} U|+\rangle &= U\frac{|0\rangle + |1\rangle}{\sqrt{2}} \\ &= \frac{\left(\frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle\right) + \left(\frac{4}{5}|0\rangle - \frac{3}{5}|1\rangle\right)}{\sqrt{2}} \\ &= \frac{\frac{7}{5}|0\rangle + \frac{1}{5}|1\rangle}{\sqrt{2}} \end{aligned}$$

Since we'll be measuring immediately afterwards in the standard basis, we can ignore the T gate; the relative phase it adds will drop out when we measure (note this is not true in general).

$$\begin{aligned} \Pr[|0\rangle] &= \left|\frac{7}{5\sqrt{2}}\right|^2 = \frac{49}{50} \\ \Pr[|1\rangle] &= \left|\frac{1}{5\sqrt{2}}\right|^2 = \frac{1}{50} \end{aligned}$$

3. Give the probability for measuring $|i\rangle$ and $|-i\rangle$ for the following circuits:

a)

$$\sqrt{\text{NOT}} = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Solution:

$$\begin{aligned} |i\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}} &\mapsto \frac{i|1\rangle + i(-i)|0\rangle}{\sqrt{2}} = |i\rangle \\ &\mapsto \frac{\big((1+i)|0\rangle + (1-i)|1\rangle\big) + i\big((1-i)|0\rangle + (1+i)|1\rangle\big)}{2\sqrt{2}} = \frac{1+i}{\sqrt{2}}|0\rangle \\ &\mapsto \frac{-1+i}{\sqrt{2}}|1\rangle \equiv |1\rangle\,. \end{aligned}$$

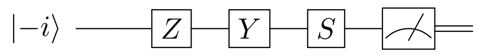
To get the measurement probabilities we want to take the magnitude squared of the inner products $\langle i|1\rangle$ and $\langle -i|1\rangle$

$$\Pr[|i\rangle] = |\langle i|1\rangle|^2 = \frac{1}{2}$$

$$\Pr[|-i\rangle] = |\langle -i|1\rangle|^2 = \frac{1}{2}$$

b)

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$



Solution:

$$|-i\rangle = \frac{|0\rangle - i\,|1\rangle}{\sqrt{2}} \mapsto \frac{|0\rangle + i\,|1\rangle}{\sqrt{2}} = |i\rangle\,.$$

From the previous problem we saw that $|i\rangle$ is an eigenstate of Y, so we can ignore this gate. Finally for S we have:

$$|i\rangle \mapsto |-\rangle$$

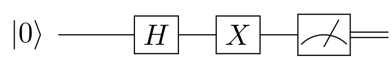
Now again to get our measurement probabilities we take the inner products of the measurement basis elements with our state.

$$\Pr[|i\rangle] = |\langle i|-\rangle|^2 = \left|\frac{(\langle 0| - i\,\langle 1|)(|0\rangle - |1\rangle)}{2}\right|^2 = \frac{1}{4}|(1+i)|^2 = \frac{1}{2}$$

$$\Pr[|-i\rangle] = \frac{1}{2}$$

4. Give the probability of measuring $|+\rangle$ and $|-\rangle$ for the following circuits:

a)

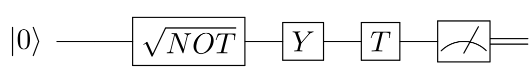


Solution:

$$|0\rangle \mapsto |+\rangle \mapsto |+\rangle$$

$$\Pr[|+\rangle] = 1$$

b)



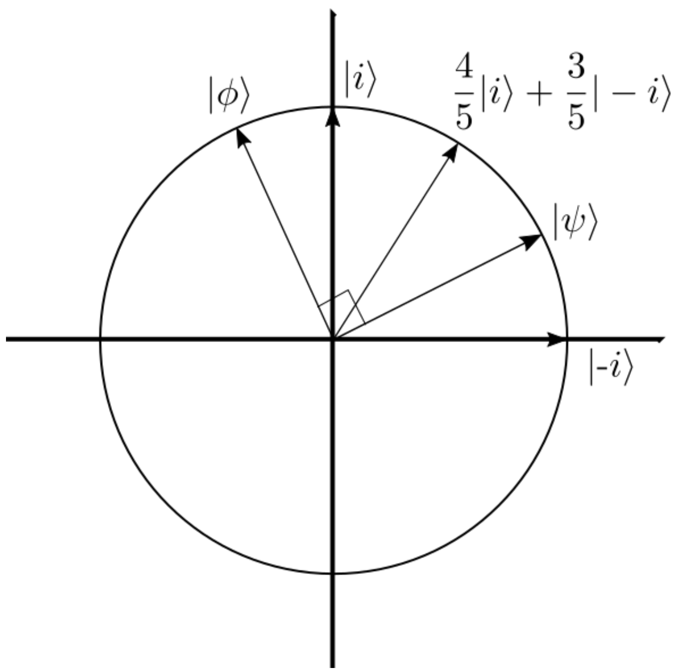
Solution:

$$\begin{aligned}
|0\rangle &\mapsto \frac{(1+i)\,|0\rangle + (1-i)\,|1\rangle}{2} \\
&\mapsto \frac{i(1+i)\,|1\rangle - i(1-i)\,|0\rangle}{2} = \frac{-(1+i)\,|0\rangle - (1-i)\,|1\rangle}{2} \\
&\mapsto \frac{-(1+i)\,|0\rangle - e^{i\pi/4}(1-i)\,|1\rangle}{2}.
\end{aligned}$$

Now take inner products with $\{|+\rangle, |-\rangle\}$ to get measurement probabilities.

$$\begin{aligned}
\Pr[|+\rangle] &= |\langle +|\psi\rangle|^2 \\
&= \left| \frac{\big(\langle 0| + \langle 1|\big)\big(-(1+i)\,|0\rangle - e^{i\pi/4}(1-i)\,|1\rangle\big)}{2\sqrt{2}} \right|^2 \\
&= \frac{\big|(i+1) + e^{i\pi/4}(i-1)\big|^2}{8} \\
&= \frac{2+\sqrt{2}}{4} \\
&= \cos(\pi/8)^2.
\end{aligned}$$

5. Distinguishability of Quantum States What is the best measurement basis to distinguish the state $|i\rangle$ and $\frac{4}{5}|i\rangle + \frac{3}{5}|-i\rangle$? Using this basis, give a protocol to distinguish the two states. With what probability does this protocol succeed, supposing that we are equally likely to be given either state?



Solution: See section 5.2 of the textbook for more details.

The best basis is $|\phi\rangle, |\psi\rangle$ as shown. Note that this is not the usual Bloch circle with $|0\rangle$ and $|1\rangle$ as the axes.

By requiring $|\phi\rangle$ and $|\psi\rangle$ to be orthogonal, the angle between $|\phi\rangle$ and $|i\rangle$ must be equal to the angle between $|\psi\rangle$ and $|-i\rangle$. For the optimal measurement, we want the angle between $|\phi\rangle$ and $|i\rangle$ to be the same as the angle between $|\psi\rangle$ and $\frac{4}{5}|i\rangle + \frac{3}{5}|-i\rangle$. Combining the previous two statements means that $|\psi\rangle$ must be an equal combination of (halfway between) $\frac{4}{5}|i\rangle + \frac{3}{5}|-i\rangle$ and $|-i\rangle$:

$$|\psi\rangle \propto |\alpha\rangle = (\frac{4}{5}|i\rangle + \frac{3}{5}|-i\rangle) + |-i\rangle \quad \rightarrow \quad \langle\alpha|\alpha\rangle = \left(\frac{4}{5}\langle i| + \frac{8}{5}\langle -i|\right)\left(\frac{4}{5}|i\rangle + \frac{8}{5}|-i\rangle\right)$$

$$= \frac{16}{25}\langle i|i\rangle + \frac{64}{25}\langle -i|-i\rangle = \frac{80}{25}$$

Therefore $|\psi\rangle = \frac{4}{\sqrt{80/25}}(\frac{1}{5}|i\rangle + \frac{2}{5}|-i\rangle)$.

Call $|\lambda\rangle = \frac{4}{5}|i\rangle + \frac{3}{5}|-i\rangle$.

To find $|\phi\rangle$ we flip the amplitudes and change signs: $|\phi\rangle = \frac{4}{\sqrt{80/25}}(\frac{2}{5}|i\rangle - \frac{1}{5}|-i\rangle)$.

But, we don't actually require $|\phi\rangle$; we could have left it implicit since we know its overlap with any state of interest is the complement of the overlap with $|\psi\rangle$. Calculate probabilities of measuring the two states of interest and find

$$|\langle\lambda|\psi\rangle|^2 = \left|(\frac{4}{5}\langle i| + \frac{3}{5}\langle -i|)(\frac{1}{\sqrt{80/25}}(\frac{4}{5}|i\rangle + \frac{8}{5}|-i\rangle))\right|^2 = \left|\frac{1}{\sqrt{80/25}}(\frac{16}{25} + \frac{24}{25})\right|^2 = .8,$$

$$|\langle i|\psi\rangle|^2 = \left|\langle i|(\frac{1}{\sqrt{80/25}}(\frac{4}{5}|i\rangle + \frac{8}{5}|-i\rangle))\right|^2 = \frac{1}{\sqrt{80/25}}\frac{4}{5} = .2,$$

where the probabilities of observing $|\phi\rangle$ would be 0.2 and 0.8 respectively.

Protocol: Measure in the basis. Assume that observing $|\psi\rangle$ means we were given $\frac{4}{5}|i\rangle + \frac{3}{5}|-i\rangle$. Assume that observing $|\phi\rangle$ means we were given $|i\rangle$. Protocol success probability assuming 50-50 chance of $|-\rangle$ or $|i\rangle$ is:

[Chance of given $\frac{4}{5}|i\rangle + \frac{3}{5}|-i\rangle$] $\times$ [chance to get $|\psi\rangle$] $= 0.5 \times 0.8 = 0.4$
plus
[Chance of given $|i\rangle$] $\times$ [chance to get $|\phi\rangle$] $= 0.5 \times (1 - .2) = 0.4$
equals
Total success probability: 0.8.