Introduction to Quantum Information Science
Recitation 6
(10/6 and 10/10)

1. Practice with density matrices. Express the following mixtures of pure states as density matrices:

a) $|0\rangle$ with probability $\frac{1}{3}$ and $R_{\pi/3}|0\rangle$ with probability $\frac{1}{3}$ and $R_{2\pi/3}|0\rangle$ with probability $\frac{1}{3}$.
Recall

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Solution:

$$\begin{aligned} R_{\pi/3}|0\rangle &= \cos(\pi/3)|0\rangle + \sin(\pi/3)|1\rangle \\ R_{2\pi/3}|0\rangle &= \cos(2\pi/3)|0\rangle + \sin(2\pi/3)|1\rangle \\ &= -\cos(\pi/3)|0\rangle + \sin(\pi/3)|1\rangle \end{aligned}$$

Hence,

$$\begin{aligned} &\frac{1}{3}|0\rangle\langle 0| + \frac{1}{3}R_{\pi/3}|0\rangle\langle 0| R^\dagger_{\pi/3} + \frac{1}{3}R_{2\pi/3}|0\rangle\langle 0| R^\dagger_{2\pi/3} \\ =&\frac{1}{3}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} \cos(\frac{\pi}{3})^2 & \cos(\frac{\pi}{3})\sin(\frac{\pi}{3}) \\ \cos(\frac{\pi}{3})\sin(\frac{\pi}{3}) & \sin(\frac{\pi}{3})^2 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} \cos(\frac{\pi}{3})^2 & -\cos(\frac{\pi}{3})\sin(\frac{\pi}{3}) \\ -\cos(\frac{\pi}{3})\sin(\frac{\pi}{3}) & \sin(\frac{\pi}{3})^2 \end{bmatrix} \\ =&\frac{1}{3}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 2\cdot\left(\frac{1}{2}\right)^2 & 0 \\ 0 & 2\cdot\left(\frac{\sqrt{3}}{2}\right)^2 \end{bmatrix} \\ =&\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \end{aligned}$$

b) The states: $\{|0\rangle, |1\rangle, |+\rangle, |-\rangle, |i\rangle, |-i\rangle\}$ all with probability $\frac{1}{6}$

Solution: The maximally mixed state.
Recall that $I = \sum_i^n |\psi_i\rangle\langle\psi_i|$ for any orthonormal basis $\{|\psi_i\rangle\}$. So,

$$I = |0\rangle\langle 0| + |1\rangle\langle 1| = |+\rangle\langle +| + |-\rangle\langle -| = |i\rangle\langle i| + |-i\rangle\langle -i|,$$

and our density matrix is

$$\rho = 3\cdot\frac{1}{6}I = \frac{I}{2},$$

the maximally mixed state.

c) The Bell state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ with probability $1-2\epsilon$, $|00\rangle$ with probability ϵ, and $|11\rangle$ with probability ϵ.

Solution: The entries for the $|00\rangle$ and $|11\rangle$ states are clearly in the top left and bottom right corners respectively. For the Bell state, expand

$$|EPR\rangle\langle EPR| = \frac{|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|}{2},$$

which corresponds to the four corners. Combining these terms weighted by their respective probabilities yields

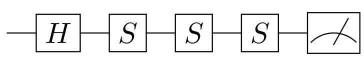
$$\begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1-2\epsilon}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1-2\epsilon}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

2. Unitary Evolution of Mixed States

Reminder:

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$$

a) Apply the following circuits to the mixed state $\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|+\rangle\langle +|$. What are the measurement probabilities if we measure in the $\{|i\rangle, |-i\rangle\}$ basis?



Solution: The first Hadamard doesn't do anything. For the next step, applying $S^3|0\rangle$ is easy. To find $S^3|+\rangle$, observe $S|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) = |i\rangle$, and the pattern continues such that the final state is

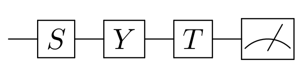
$$S^3 \rho S^{3\dagger} = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|-i\rangle\langle -i|.$$

Therefore, noting both $|i\rangle$ and $|-i\rangle$ have overlap 1/2 with $|1\rangle$, we have

$$\Pr[|i\rangle] = \frac{1}{4},$$

$$\Pr[|-i\rangle] = \frac{3}{4}.$$

b) Same as part (a).



Solution: First,

$$\begin{aligned} S\rho S^\dagger &= \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|i\rangle\langle i| \\ &= \frac{1}{2}|0\rangle\langle 0| + \frac{1}{4}(|0\rangle\langle 0| + i|1\rangle\langle 1|(-i)). \end{aligned}$$

(Be careful of those constants! The $1/\sqrt{2}$ factor in $|i\rangle$ appears twice so becomes $1/2$, and the factor of i needs to be conjugated when its given with the bra.) Then applying Y yields

$$\frac{1}{2}i\,|1\rangle\langle 1|\,(-i) + \frac{1}{4}\left(i\,|1\rangle\langle 1|\,(-i) + (-i)\,|0\rangle\langle 0|\,(i)\right) = \frac{1}{2}\,|1\rangle\langle 1| + \frac{1}{2}\,|i\rangle\langle i|\,.$$

Finally, applying T gives

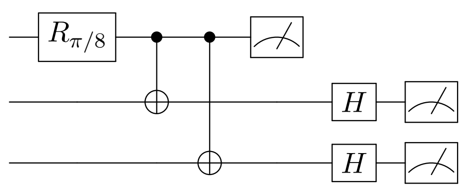
$$\begin{aligned}&\frac{1}{2}e^{i\pi/4}\,|1\rangle\,\langle 1|\,e^{-\pi/4} + \frac{1}{4}(|0\rangle + ie^{i\pi/4}\,|1\rangle)(\langle 0| - ie^{-i\pi/4}\,\langle 1|)\\ =\;& \frac{1}{2}\,|1\rangle\langle 1| + \frac{1}{4}(|0\rangle\langle 0| - ie^{-i\pi/4}\,|0\rangle\langle 1| + ie^{i\pi/4}\,|1\rangle\langle 0| + |1\rangle\langle 1|).\end{aligned}$$

Therefore,

$$\Pr[|i\rangle] = \frac{1}{4} + \frac{1}{4}\left(\frac{1}{2} + \frac{e^{-i\pi/4}}{2} + \frac{e^{i\pi/4}}{2} + \frac{1}{2}\right) = \frac{1}{2} + \frac{1}{4\sqrt{2}},$$

and the probability of $|-i\rangle$ is 1 minus that, which is $\frac{1}{2} - \frac{1}{4\sqrt{2}}$.

c) For the following circuit, when applied to $|000\rangle$, what is the mixed state of the system following the first measurement? What is the probability of measuring $|000\rangle$ at the end of the circuit?



Solution: The state of the system before measurement is

$$\cos(\pi/8)\,|000\rangle + \sin(\pi/8)\,|111\rangle\,.$$

After the first measurement, the first qubit collapses, and the other qubits collapse due to the entanglement. With probability $\cos^2(\pi/8)$ the system collapsed to $|000\rangle$ and with probability $\sin^2(\pi/8)$ the system collapsed to $|111\rangle$, so the state of the system is the probabilistic mixture

$$\rho = \cos^2(\pi/8)\,|000\rangle\langle 000| + \sin^2(\pi/8)\,|111\rangle\langle 111|\,.$$

Next, after the two Hadamards, the state is

$$I \otimes H^{\otimes 2}\rho\left(I \otimes H^{\otimes 2}\right)^{\dagger} = I \otimes H^{\otimes 2}\rho I \otimes H^{\otimes 2} = \cos^2(\pi/8)\,|0{+}{+}\rangle\langle 0{+}{+}| + \sin^2(\pi/8)\,|1{-}{-}\rangle\langle 1{-}{-}|\,.$$

Therefore, the probability of $|000\rangle$ is

$$\Pr[|000\rangle] = \langle 000|\,\rho\,|000\rangle = \cos^2(\pi/8)\cdot 1\cdot\frac{1}{2}\cdot\frac{1}{2} + \sin^2(\pi/8)\cdot 0 = \frac{\cos^2\pi/8}{4}.$$

3. Properties of density matrices.

a) Show that every probabilistic mixture of possibly many n-dimensional pure states can be written as a probabilistic mixture involving at most n pure states.

Solution: We write our mixture as an $n \times n$ density matrix ρ. The spectral theorem tell us that if A is Hermitian, there exists an orthonormal basis consisting of eigenvectors of A and each eigenvalue is real. All density matrices are Hermitian, so

$$\rho = \sum_{i=1}^{n} \lambda_i \left|\psi_i\right\rangle \left\langle\psi_i\right|$$

where $\{|\psi_i\rangle\}$ is an orthonormal basis of eigenvectors for ρ and $\{\lambda_i\}$ is the set of eigenvalues of ρ. This gives our desired probabilistic mixture:

$$|\psi_i\rangle \text{ with probability } \lambda_i.$$

b) Show that a density matrix ρ corresponds to a pure state if and only if $\rho^2 = \rho$. What's another term for a Hermitian matrix ρ that satisfies $\rho^2 = \rho$?

Solution: For the forward direction, suppose $\rho = |\psi\rangle \langle\psi|$ corresponds to a pure state. Then $\rho^2 = |\psi\rangle \langle\psi|\psi\rangle \langle\psi| = |\psi\rangle \langle\psi| = \rho$, as desired.

Next, for the reverse direction, suppose $\rho^2 = \rho$. Using the spectral decomposition of ρ, write

$$\rho = \sum_{i=1}^{n} \lambda_i \left|\psi_i\right\rangle \left\langle\psi_i\right|.$$

Squaring, we get

$$\begin{aligned}
\rho^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i\lambda_j \left|\psi_i\right\rangle \left\langle\psi_i|\psi_j\right\rangle \left\langle\psi_j\right| \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \delta_{ij}\lambda_i\lambda_j \left|\psi_i\right\rangle \left\langle\psi_j\right| \\
&= \sum_{i=1}^{n} \lambda_i^2 \left|\psi_i\right\rangle \left\langle\psi_i\right|.
\end{aligned}$$

By assumption, $\rho = \rho^2$, and matching corresponding eigenvectors and eigenvalues gives $\lambda_i^2 = \lambda_i$ for all i. Since the eigenvalues λ_i of a density matrix correspond to probabilities, they are between 0 and 1 and must sum to 1. The first property implies $\lambda_i \in \{0, 1\}$, and the second that there is exactly one value j such that $\lambda_j = 1$. This implies $\rho = |\psi_j\rangle \langle\psi_j|$, which is to say it is a pure state.

A Hermitian matrix A such that $A^2 = A$ is known as a *projector*. All of its eigenvalues must be 0 or 1.