Introduction to Quantum Information Science
Recitation 1

1. Complex Numbers and Amplitude Review

a) Compute $|a+ib|$ (e.g. $|1+i|=\sqrt{2}$).

Solution:

$$|a+ib| = \sqrt{(a+ib)^*(a+ib)} = \sqrt{(a-ib)(a+ib)} = \sqrt{a^2 - i^2b^2} = \sqrt{a^2+b^2}$$

b) Express $(1+i\sqrt{3})(1-i)$ in the form $re^{i\theta}$ for some $r \in \mathbb{R}_{>0}$ and $\theta \in [0, 2\pi)$.

Solution:

$$1+i\sqrt{3} = 2e^{i\pi/3}, \quad \text{and} \quad 1-i = \sqrt{2}e^{-i\pi/4}$$

Hence,

$$(1+i\sqrt{3})(1-i) = 2e^{i\pi/3}\sqrt{2}e^{-i\pi/4} = 2\sqrt{2}e^{i\pi/12}$$

c) Compute $\sqrt{i}$ and i^i.

Solution:

$$\sqrt{i} = \left(e^{i\pi/2}\right)^{\frac{1}{2}} = e^{i\pi/4}, \quad i^i = \left(e^{i\pi/2}\right)^i = e^{-\pi/2}$$

d) Let

$$u = \begin{bmatrix} i\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} e^{i\frac{\pi}{4}}\frac{3}{5} \\ i\frac{4}{5} \\ 0 \end{bmatrix}.$$

Compute $|u \cdot v|$.

Solution: Don't forget to take the complex conjugate!

$$|u \cdot v| = \left| -i\frac{1}{\sqrt{2}} \cdot e^{i\frac{\pi}{4}}\frac{3}{5} + 0 \cdot i\frac{4}{5} + \frac{1}{\sqrt{2}} \cdot 0 \right| = \frac{3}{5\sqrt{2}}$$

2. Linear Algebra Review

a) Compute the eigenvectors and eigenvalues of these three matrices:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Solution:

$$X \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \qquad X \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = - \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$Y \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} \qquad Y \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{bmatrix} = - \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{bmatrix}$$

$$Z \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad Z \begin{bmatrix} 0 \\ 1 \end{bmatrix} = - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

b) Let $A \in \mathbb{C}^{n \times n}$ be a matrix such that $A^2 = A$. What are the possible eigenvalues of A?

Solution: Let u be an eigenvector of A with eigenvalue λ. We can see that $\lambda u = Au = A^2 u = \lambda^2 u$. This means that $\lambda^2 = \lambda$, which is only possible for $\lambda = 0$ and $\lambda = 1$.

c) Let $A \in \mathbb{R}^{n \times n}$ be a matrix such that its rows and columns are orthonormal vectors. Show that $A^\dagger A = AA^\dagger = I$.

Solution: Let a_i be the column vectors of A. We can see that $\left(A^\dagger A\right)_{ij} = a_i^\dagger a_j = \delta_{ij}$ where δ_{ij} is the Kronecker delta (takes value 0 unless $i = j$, in which case it takes value 1), because the column vectors are orthonormal. This matches perfectly with the identity matrix, so $A^\dagger A = I$. To prove $AA^\dagger = I$ we do the same things with the row vectors of A.

3. Tensor Products

a) Compute

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Solution:

$$\frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

b) Let A_{ij} mean the element in the i-th row and the j-th column of A. What does $A \otimes B$ look like abstractly?

Solution:

$$\begin{bmatrix} A_{11}B & A_{12}B & A_{13}B & \cdots \\ A_{21}B & A_{22}B & A_{23}B & \cdots \\ A_{31}B & A_{32}B & A_{33}B & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

c) Show that $(A \otimes B)(u \otimes v) = (Au) \otimes (Bv)$.

Solution: Let us first see that for a matrix A and vector v, $(Au)_i = \sum_j A_{ij}u_j$. This leads us to

$$\begin{aligned} (A \otimes B)(u \otimes v) &= \begin{bmatrix} A_{11}B & A_{12}B & A_{13}B & \cdots \\ A_{21}B & A_{22}B & A_{23}B & \cdots \\ A_{31}B & A_{32}B & A_{33}B & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} u_1 v \\ u_2 v \\ \vdots \end{bmatrix} \\ &= \begin{bmatrix} \left(\sum_j A_{1j}u_j\right) Bv \\ \left(\sum_j A_{2j}u_j\right) Bv \\ \vdots \end{bmatrix} \\ &= (Au) \otimes (Bv). \end{aligned}$$

4. Measurement

a) We're given a qubit in the state

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle.$$

What is the probability of seeing the outcome $|0\rangle$ when measuring in the $\{|0\rangle, |1\rangle\}$-basis?

Solution: By the Born rule this is just the absolute value squared of the corresponding amplitude: $|\frac{\sqrt{3}}{2}|^2 = \frac{3}{4}$.

b) What does $|\psi\rangle$ look like when written in the $\{|+\rangle, |-\rangle\}$-basis? And what is the probability of observing $|+\rangle$ if we were to measure in this basis instead?

Solution: It's a basic fact of linear algebra that for any orthonormal basis $\{|b_i\rangle\}$ we have

$$I = \sum_i |b_i\rangle\langle b_i|.$$

Hence,

$$\begin{aligned} |\psi\rangle &= \big(|+\rangle\langle +| + |-\rangle\langle -|\big)|\psi\rangle \\ &= \langle +|\psi\rangle |+\rangle + \langle -|\psi\rangle |-\rangle \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} |+\rangle + \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} |-\rangle \\ &\approx 0.97|+\rangle + 0.26|-\rangle. \end{aligned}$$

For the probability of observing $|+\rangle$ we can once again use the Borne rule. If computed without intermediate rounding this yields $0.93\ldots$ Note that a shortcut to this answer would've been to only compute $|\langle +|\psi\rangle|^2$.

c) What is the probability of making both the previous observations when consecutively measuring the *same* qubit? (The probability of seeing $|0\rangle$ then $|+\rangle$ when first measuring in the $\{|0\rangle, |1\rangle\}$ then in the $\{|+\rangle, |-\rangle\}$-basis)

Solution: We already saw in a) that there's a $\frac{3}{4}$ chance to initially observe a $|0\rangle$. But once that happens the qubit will "snap" to $|0\rangle$. So the probability to then observe $|+\rangle$ becomes $|\langle +|0\rangle|^2 = \frac{1}{2}$. Overall that gives us a $\frac{3}{8}$ chance for the entire sequence of events.