C S 358H: Intro to Quantum Information Science

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October 2024

Contents

1 Distinguishability of Mixed States

Question 1.1

Question. *Let ρ and σ be two different single qubit density matrices. Prove that ρ and σ are distinguishable — that is, there exists some measurement basis such that the probabilities of the outcomes is different in the two cases.*
Hint: Use the fact that $\sigma - \rho$ is necessarily a nonzero Hermitian matrix and the fact that any Hermitian matrix can be diagonalized.

Proof. Using the hint, we consider the fact that $\sigma - \rho$ is a nonzero Hermitian matrix and that it can be diagonalized. Therefore, there exists an eigenbasis for $\sigma - \rho$, say $\{|b_1\rangle, |b_2\rangle\}$, that has eigenvalues λ_1, λ_2. Now let us assume that measuring the two states in this eigenbasis gives probabilities p_σ, p_ρ respectively for measuring the first eigenbasis. Now consider the difference between these probabilties

$$\begin{aligned} p_\sigma - p_\rho &= \langle b_1|\sigma|b_1\rangle - \langle b_1|\rho|b_1\rangle \\ &= \langle b_1|\sigma - \rho|b_1\rangle = \lambda_1 \end{aligned} \tag{1}$$

Now, since $\{|b_1\rangle, |b_2\rangle\}$ is an eigenbasis for $\sigma - \rho$ and since the trace for $\sigma - \rho$ is 0, both eigenvalues have to be non-zero since $\sigma - \rho$ is non-zero to begin with. Therefore, we get that the probabilities of measuring $|b_1\rangle$ is different for both σ and ρ (and similarly the difference in probability of measuring $|b_2\rangle$ is also non-zero). Thus, the basis $\{|b_1\rangle, |b_2\rangle\}$ distinguishes between the two states. □

2 Reduced GHZ

Consider the n-qubit "Schrödinger cat state" (or "generalized GHZ state")

$$\frac{|0\cdots 0\rangle + |1\cdots 1\rangle}{\sqrt{2}}.$$

Question 2.1

Question. *What probability distribution over n-bit strings do we observe if we Hadamard the first $n-1$ qubits, then measure all n qubits in the $\{|0\rangle, |1\rangle\}$ basis? Show your work.*

Solution. We compute the density matrix of the cat state as

$$\rho = \frac{|0\cdots 0\rangle\langle 0\cdots 0| + |0\cdots 0\rangle\langle 1\cdots 1| + |1\cdots 1\rangle\langle 0\cdots 0| + |1\cdots 1\rangle\langle 1\cdots 1|}{2} \tag{2}$$

Now, we can write the Hadamard operator over the first $n-1$ qubits as

$$\mathbf{H}^{\otimes n-1} \otimes \mathbf{I} = (|+\rangle\langle 0| + |-\rangle\langle 1|)^{\otimes n-1} \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) \tag{3}$$

Now, if we apply the operator to ρ, we get

$$\begin{aligned}(\mathbf{H}^{\otimes n-1} \otimes \mathbf{I})\rho(\mathbf{H}^{\otimes n-1} \otimes \mathbf{I})^\dagger &= (\mathbf{H}^{\otimes n-1} \otimes \mathbf{I})\rho(\mathbf{H}^{\otimes n-1} \otimes \mathbf{I}) \\ \implies \rho' &= \frac{1}{2}((|+\rangle\langle +|)^{\otimes n-1} \otimes |0\rangle\langle 0| + (|+\rangle\langle -|)^{\otimes n-1} \otimes |0\rangle\langle 1| \\ &\quad + (|-\rangle\langle +|)^{\otimes n-1} \otimes |1\rangle\langle 0| + (|-\rangle\langle -|)^{\otimes n-1} \otimes |1\rangle\langle 1|)\end{aligned} \tag{4}$$

Now, we compute the probability of measuring a bitstring $|b\rangle = |b_1 \cdots b_n\rangle$ as

$$\begin{aligned}\Pr[\text{measuring } b] =&\ \langle b|\rho'|b\rangle = \langle b_1\cdots b_n|\rho'|b_1\cdots b_n\rangle \\ &= \frac{1}{2}(\frac{1}{2^{n-1}} \cdot |\langle b_n|0\rangle|^2 + \frac{(-1)^{\sum_{i=1}^{n-1} b_i}}{2^{n-1}} \cdot \langle b_n|0\rangle\langle 1|b_n\rangle \\ &\quad + \frac{\sum_{i=1}^{n-1} b_i}{2^{n-1}} \cdot \langle b_n|1\rangle\langle 0|b_n\rangle + \frac{1}{2^{n-1}} \cdot |\langle b_n|1\rangle|^2) \\ &= \frac{1}{2}\left(\frac{1}{2^{n-1}} \cdot |\langle b_n|0\rangle|^2 + \frac{1}{2^{n-1}} \cdot |\langle b_n|1\rangle|^2\right), \text{ the middle two terms vanish} \\ &= \frac{1}{2}\left(\frac{1}{2^{n-1}}\right) = \frac{1}{2^n}, \text{ since } b_n \text{ is either 0 or 1}\end{aligned} \tag{5}$$

See Question 2.3 for work that shows how we computed the middle two terms (that eventually vanish since either $\langle b_n|0\rangle$ or $\langle b_n|1\rangle$ is going to be 0). For the other two terms, we just have squares of the amplitudes since the two inner products we multiply are conjugates of each other. □

Question 2.2

Question. *Is this the same distribution or a different one, from what we'd have seen if we took the following state, applied Hadamards to the first* $n-1$ *qubits, and then measured all* n *qubits in the* $\{|0\rangle, |1\rangle\}$ *basis:*

$$\frac{|0\cdots 0\rangle\langle 0\cdots 0| + |1\cdots 1\rangle\langle 1\cdots 1|}{2}.$$

Show your work.

Solution. If we apply the apply the operator computed in Equation 3 to the above state (say its density matrix is σ), we get

$$\begin{aligned}(\mathbf{H}^{\otimes n-1}\otimes\mathbf{I})\sigma(\mathbf{H}^{\otimes n-1}\otimes\mathbf{I})^\dagger &= (\mathbf{H}^{\otimes n-1}\otimes\mathbf{I})\sigma(\mathbf{H}^{\otimes n-1}\otimes\mathbf{I}) \\ \implies \sigma' &= \frac{1}{2}(|+\rangle\langle +|^{\otimes n-1}\otimes|0\rangle\langle 0| + |-\rangle\langle -|^{\otimes n-1}\otimes|1\rangle\langle 1|)\end{aligned} \tag{6}$$

Now, we compute the probability of measuring a bitstring $|b\rangle = |b_1\cdots b_n\rangle$ as

$$\begin{aligned}\Pr[\text{measuring } b] &= \langle b|\rho'|b\rangle = \langle b_1\cdots b_n|\rho'|b_1\cdots b_n\rangle \\ &= \frac{1}{2}(\frac{1}{2^{n-1}}\cdot|\langle b_n|0\rangle|^2 + \frac{1}{2^{n-1}}\cdot|\langle b_n|1\rangle|^2) \\ &= \frac{1}{2}\left(\frac{1}{2^{n-1}}\right) = \frac{1}{2^n}, \text{ since } b_n \text{ is either 0 or 1}\end{aligned} \tag{7}$$

Therefore, the probability distributions are the same. □

Question 2.3

Question. *What probability distribution over* n*-bit strings do we observe if we Hadamard all* n *qubits, then measure all* n *qubits in the* $\{|0\rangle, |1\rangle\}$ *basis? Show your work.*

Solution. We can define the Hadamard operator over all n qubits as

$$\mathbf{H}^{\otimes n} = (|+\rangle\langle 0| + |-\rangle\langle 1|)^{\otimes n} \tag{8}$$

Now, if we apply the operator to ρ, we get

$$\begin{aligned}(\mathbf{H}^{\otimes n})\rho(\mathbf{H}^{\otimes n})^\dagger &= (\mathbf{H}^{\otimes n})\rho(\mathbf{H}^{\otimes n}) \\ \implies \rho'' &= \frac{1}{2}(|+\rangle\langle +|^{\otimes n} + |+\rangle\langle -|^{\otimes n} + |-\rangle\langle +|^{\otimes n} + |-\rangle\langle -|^{\otimes n})\end{aligned} \tag{9}$$

Now, we compute the probability of measuring a bitstring $|b\rangle = |b_1 \cdots b_n\rangle$ as

$$\begin{aligned}\Pr[\text{measuring } b] = \; & \langle b|\,\rho''\,|b\rangle = \langle b_1 \cdots b_n|\,\rho''\,|b_1 \cdots b_n\rangle \\ &= \frac{1}{2}\left(\frac{1}{2^n} + \frac{(-1)^{b\cdot 1^n}}{2^n} + \frac{(-1)^{b\cdot 1^n}}{2^n} + \frac{1}{2^n}\right),\ b\cdot 1^n \text{ is dot product of } b \text{ with } 1\cdots 1 \\ &= \begin{cases} \frac{1}{2^{n-1}} & \text{if } b\cdot 1^n = 0 \\ 0 & \text{otherwise} \end{cases}\end{aligned} \tag{10}$$

Note that we get the middle two terms as $\frac{(-1)^{b\cdot 1^n}}{2^n}$ since $\langle b_i|+\rangle \cdot \langle -|b_i\rangle = \langle b_i|-\rangle\,\langle +|b_i\rangle = -1$ only if bit $b_i = 1$. For the other two terms, we get the square of the product since the inner products are conjugates of each other.
Therefore, we have a probability of measuring bitstrings with even number of 1's as $\frac{1}{2^{n-1}}$ and we have 0 probability of measuring bitstrings with odd number of 1's. □

Question 2.4

Question. *Is this the same distribution or a different one, than if to the following state we apply Hadamards to all n qubits and then measure all n qubits in the $\{|0\rangle, |1\rangle\}$ basis:*

$$\frac{|0\cdots 0\rangle\,\langle 0\cdots 0| + |1\cdots 1\rangle\,\langle 1\cdots 1|}{2}.$$

Show your work.

Solution. If we apply the operator computed in Equation 8 to σ, we get

$$\begin{aligned}(\mathbf{H}^{\otimes n})\sigma(\mathbf{H}^{\otimes n})^\dagger &= (\mathbf{H}^{\otimes n})\sigma(\mathbf{H}^{\otimes n}) \\ \implies \sigma'' &= \frac{1}{2}(|+\rangle\,\langle +|^{\otimes n} + |-\rangle\,\langle -|^{\otimes n})\end{aligned} \tag{11}$$

Now, we compute the probability of measuring a bitstring $|b\rangle = |b_1 \cdots b_n\rangle$ as

$$\begin{aligned}\Pr[\text{measuring } b] = \; & \langle b|\,\sigma''\,|b\rangle = \langle b_1 \cdots b_n|\,\sigma''\,|b_1 \cdots b_n\rangle \\ &= \frac{1}{2}\left(\frac{1}{2^n} + \frac{1}{2^n}\right) = \frac{1}{2^n}\end{aligned} \tag{12}$$

Therefore, the probability distributions are different. □

3 Bloch Sphere

Question 3.1

Question. *Give two different decompositions of the 1-qubit mixed state*

$$\rho = \begin{bmatrix} \cos^2(\pi/8) & 0 \\ 0 & \sin^2(\pi/8) \end{bmatrix}$$

as a mixture of two pure states. Show your work. What do these decompositions correspond to physically? Draw a 2D-sketch of the Bloch sphere to aid your explanation.

Solution. One trivial decomposition for the given state ρ is $\cos^2 \pi/8$ probability of having the $|0\rangle$ state and $\sin^2 \pi/8$ probability of measuring the $|1\rangle$ state.
Now, for obtaining a second possible decomposition, we compute the coefficients of ρ in the generalised representation of any state in the Bloch sphere. We have

$$\begin{aligned} &\rho = \frac{1}{2}\left(\mathbf{I} + x\mathbf{X} + y\mathbf{Y} + z\mathbf{Z}\right) \\ &\implies \cos^{\pi}/8 = 1 + z, \sin^2 \pi/8 = 1 - z, x + iy = 0, x - iy = 0 \\ &\implies z = 2\cos^2 \pi/8 - 1 = \cos \pi/4 = \frac{1}{\sqrt{2}}, x = y = 0 \end{aligned} \tag{13}$$

Therefore, the state ρ is on the Z axis with the z coordinate of $\frac{1}{\sqrt{2}}$. Now, let us try to decompose the state with one state as the $|+\rangle$ state (this has coordinates $(1, 0, 0)$). Now, let us assume that the other state $|\psi\rangle$ will have coordinates (x, y, z). We can write ρ in terms of density matrices given by $|+\rangle$ and $|\psi\rangle$ as

$$\begin{aligned} &\rho = p\,|+\rangle\langle+| + (1-p)\,|\psi\rangle\langle\psi|\,, \text{ where } p \text{ is the probability of the } |+\rangle \text{ state} \\ &\implies \left(0, 0, \frac{1}{\sqrt{2}}\right) = p(1, 0, 0) + (1-p)(x, y, z) \\ &\implies 0 = p + (1-p)x, 0 = (1-p)y, \frac{1}{\sqrt{2}} = (1-p)z \\ &\implies x = \frac{p}{p-1}, y = 0, z = \frac{1}{\sqrt{2}(1-p)} \text{ } (y = 0 \text{ since } p = 1 \text{ gives a contradiction}) \end{aligned} \tag{14}$$

Now, we also know that $x^2 + y^2 + z^2 = 1$ since $|\psi\rangle$ is a pure state and therefore, we can solve for p and we get $p = \frac{1}{4}$. Now, we can compute x and z as

$$x = \frac{-1}{3}, z = \frac{2\sqrt{2}}{3} \tag{15}$$

Therefore, we can write $|\psi\rangle$ as

$$|\psi\rangle = \frac{2\sqrt{2}-1}{3}\,|0\rangle - \frac{1}{3}\,|1\rangle \tag{16}$$

Therefore, we have another decomposition for ρ as $|+\rangle$ with probability $\frac{1}{4}$ and $|\psi\rangle$ with probability $\frac{3}{4}$. We show the different decompositions in the Bloch sphere below

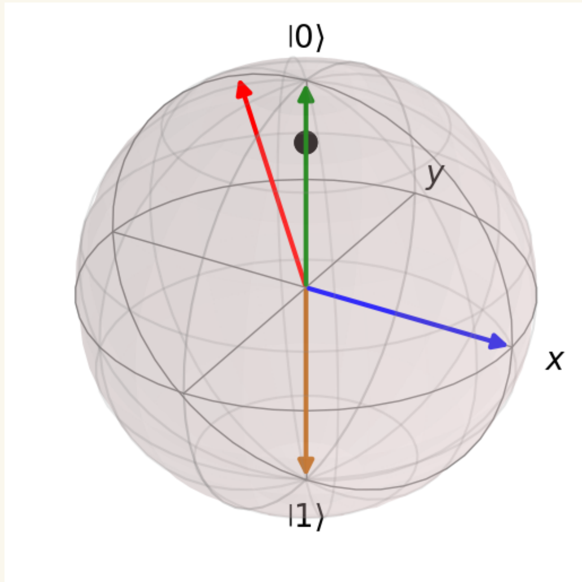

Figure 1: Bloch sphere indicating the different decompositions. The black circle indicates the position of ρ. The states in green and yellow are the states $|0\rangle$ and $|1\rangle$ respectively. The states in blue and red are the states $|+\rangle$ and $|\psi\rangle$ respectively. The lines connecting each of these pairs of states passes through ρ.

$\square$

4 Separable and Entangled States

$$|\psi_1\rangle = \frac{|00\rangle + i\,|01\rangle + i\,|10\rangle - |11\rangle}{2}$$
$$|\psi_2\rangle = \frac{3}{5}\,|01\rangle - \frac{4}{5}\,|10\rangle$$
$$|\psi_3\rangle = \frac{1}{\sqrt{3}}\,|00\rangle + \frac{1}{\sqrt{3}}\,|01\rangle + \frac{1}{\sqrt{6}}\,|10\rangle - \frac{1}{\sqrt{6}}\,|11\rangle$$

Question 4.1

Question. *Put the above states into Schmidt form:*

$$|\psi\rangle = \sum_i \lambda_i\,|\alpha_i\rangle\,|\beta_i\rangle$$

In other words, find orthonormal bases $\{|\alpha_0\rangle, |\alpha_1\rangle\}$ for the first qubit and $\{|\beta_0\rangle, |\beta_1\rangle\}$ for the second qubit, such that you can write the state without cross terms $|\alpha_0\rangle\,|\beta_1\rangle$ or $|\alpha_1\rangle\,|\beta_0\rangle$. Show your work.
Hint: you should not need to use the singular value decomposition to find the Schmidt form.

Solution. We can write $|\psi_1\rangle$ as

$$\begin{aligned} |\psi_1\rangle &= \frac{|00\rangle + i\,|01\rangle + i\,|10\rangle - |11\rangle}{2} \\ &= \frac{1}{\sqrt{2}}\,|0\rangle\,\frac{|0\rangle + i\,|1\rangle}{\sqrt{2}} + \frac{i}{\sqrt{2}}\,|1\rangle\,\frac{|0\rangle + i\,|1\rangle}{\sqrt{2}} \\ &= |ii\rangle \end{aligned} \tag{17}$$

Therefore, for $|\psi_1\rangle$ we have the orthonormal basis $\{|i\rangle, |-i\rangle\}$ for both qubits for the Schmidt form.

$|\psi_2\rangle$ is already in Schmidt form with the basis states $\{|0\rangle, |1\rangle\}$ for the first qubit and the basis states $\{|1\rangle, |0\rangle\}$ for the second qubit.

We can write $|\psi_3\rangle$ as

$$\begin{aligned} |\psi_3\rangle &= \frac{1}{\sqrt{3}}\,|00\rangle + \frac{1}{\sqrt{3}}\,|01\rangle + \frac{1}{\sqrt{6}}\,|10\rangle - \frac{1}{\sqrt{6}}\,|11\rangle \\ &= \sqrt{\frac{2}{3}}\,|0+\rangle + \sqrt{\frac{1}{3}}\,|1-\rangle \end{aligned} \tag{18}$$

Therefore, for $|\psi_3\rangle$ we have the orthonormal basis $\{|0\rangle, |1\rangle\}$ for the first qubit and $\{|+\rangle, |-\rangle\}$ for the second qubit for the Schmidt form. $\square$

Question 4.2

Question. *Calculate how many ebits of entanglement each of these states have. Please show your work/justify your reasoning for each of your answers. (Keep in mind, this answer need* *not be an integer.)*

Solution. $|\psi_1\rangle$ has an entanglement entropy of 0 since its Schmidt coefficients are 1 and all others are 0. Therefore, all terms would be 0 and hence the sum would also be 0.

We can compute the entanglement entropy of $|\psi_2\rangle$ as

$$E[|\psi_2\rangle] = -\left(\frac{3}{5}\right)^2 \log_2\left(\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2 \log_2\left(\frac{4}{5}\right)^2 \approx 0.942 \tag{19}$$

We can compute the entanglement entropy of $|\psi_3\rangle$ as

$$E[|\psi_3\rangle] = -\left(\frac{2}{3}\right) \log_2\left(\frac{2}{3}\right) - \left(\frac{1}{3}\right) \log_2\left(\frac{1}{3}\right) \approx 0.918 \tag{20}$$

□

Question 4.3

Question. *For each of the states you found with non-zero entanglement entropy in part (b), show explicitly that there exists no factorization of the states into a tensor product of two single qubit states.*

Proof. Note: After solving Question 5.1, an easy way to solve this question would be to just show that we have multiple non-zero Schmidt coefficients [the eigenvalues], thus implying that there exists no factorization of the states into a tensor product of two single qubit states since they are entangled. But since I had already solved this part before that, I am leaving the solution below unchanged.

Let us assume that we can decompose $|\psi_2\rangle$ into a tensor product of two single qubit states. Therefore, we can write

$$\begin{aligned} |\psi_2\rangle &= (a_1|0\rangle + b_1|1\rangle) \otimes (a_2|0\rangle + b_2|1\rangle) \\ &= a_1a_2|00\rangle + a_1b_2|01\rangle + b_1a_2|10\rangle + b_1b_2|11\rangle \\ \implies \frac{3}{5} &= a_1b_2, -\frac{4}{5} = b_1a_2, \quad a_1a_2 = b_1b_2 = 0 \end{aligned} \tag{21}$$

On solving these equations, we get a contradiction on applying the condition that the states are normalized. Therefore, $|\psi_2\rangle$ cannot be factorized into a tensor product of two single qubit states.

Similarly, let us assume that we can decompose $|\psi_3\rangle$ into a tensor product of two single qubit states. Therefore, we can write

$$\begin{aligned} |\psi_3\rangle &= (a_1|0\rangle + b_1|1\rangle) \otimes (a_2|0\rangle + b_2|1\rangle) \\ &= a_1a_2|00\rangle + a_1b_2|01\rangle + b_1a_2|10\rangle + b_1b_2|11\rangle \\ \implies \frac{1}{\sqrt{3}} &= a_1a_2, \frac{1}{\sqrt{3}} = a_1b_2, \frac{1}{\sqrt{6}} = b_1a_2, -\frac{1}{\sqrt{6}} = b_1b_2 \end{aligned} \tag{22}$$

Similarly, on solving these equations and enforcing the normality constraint, we arrive at a contradiction. Therefore, $|\psi_3\rangle$ cannot be factorized into a tensor product of two single qubit states. $\square$

5 Entanglement and Mixed States

Question 5.1

Question. *Suppose Alice has n qubits and Bob has n qubits of a shared $2n$-qubit pure state $|\psi\rangle$. Prove that the following are equivalent:*

i. Alice and Bob's systems are entangled.

ii. Alice's local density matrix is a mixed state.

iii. Both local density matrices are mixed states.

Recall that to prove several properties are equivalent, you must show they each have an if-and-only-if relationship with every other. One way to make this shorter is to write a chain of proofs, like A implies B, B implies C, and C implies A.
Hint: Consider the Schmidt decomposition.

Proof. Without loss of generality, we can write the state posessed by Alice and Bob using Schmidt decomposition as

$$|\psi_A\psi_B\rangle = \sum_{i=1}^{2^n} \lambda_i |\alpha_i\rangle |\beta_i\rangle \text{ , where } \{\alpha_i\}_1^{2^n} \text{ and } \{\beta_i\}_1^{2^n} \text{ are orthonormal bases vectors} \tag{23}$$

Now we prove **i** $\implies$ **ii**:
If Alice and Bob's systems are entangled, then there exists at least more than one $\lambda_i \neq 0$. Alice's local density matrix can be written as

$$\rho_A = \sum_{\lambda_i \neq 0} |\lambda_i|^2 |\alpha_i\rangle \langle\alpha_i| \tag{24}$$

Now, since we have more than one $\lambda_i \neq 0$, the density matrix ρ_A has more than one eigenvalues (since all $|\alpha_i\rangle$ are linearly independent) and hence it has a rank > 1. Therefore, Alice's local density matrix is a mixed state. Hence, $i \implies ii$.

Now we prove **ii** $\implies$ **iii**:
We can still write Alice's density matrix as written in Equation 24 (since we make no assumptions from i). Since Alice's density matrix is a mixed state, it has a rank > 1. Therefore, there exists more than one $\lambda_i \neq 0$. Bob's local density matrix can be written as

$$\rho_B = \sum_{\lambda_i \neq 0} |\lambda_i|^2 |\beta_i\rangle \langle\beta_i| \tag{25}$$

Since Bob's density matrix also has more than one non-zero eigenvalues (they share the same eigenvalues in the Schmidt decomposition), Bob's density matrix is also a mixed state. Hence, $ii \implies iii$.

Now we prove **iii** $\implies$ **i**:
Again we can still write Bob's density matrix as written in Equation 25 (since we make no assumptions from ii). Since Bob's density matrix is a mixed state, it has a rank > 1.

Therefore, there exists more than one $\lambda_i \neq 0$. Therefore, the original state $|\psi_A \psi_B\rangle$ has more than one λ_i that are non-zero. This implies that the original state is entangled. Hence, $iii \implies i$.

Thus, we have shown that $i \iff ii \iff iii$. Hence, proved. $\square$