C S 358H: Intro to Quantum Information Science

Sayam Sethi

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Contents

1 Stochastic and Unitary Matrices

Part a

Question. *Of the following matrices, which ones are stochastic? Which ones are unitary?*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ C = \begin{bmatrix} 1 & \frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix},\ D = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix},$$
$$E = \begin{bmatrix} 2 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix},\ F = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},\ G = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix},\ H = \begin{bmatrix} \frac{3i}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3i}{5} \end{bmatrix}$$

Solution. A matrix $\mathbf{A} = (a_{ij})$ is stochastic iff $\sum_i a_{ij} = 1 \wedge a_{ij} \geq 0$. Therefore, the stochastic matrices are B, C.
A matrix $\mathbf{A}$ is unitary iff $\mathbf{A}^\dagger \mathbf{A} = \mathbf{I}$. Therefore, the unitary matrices are B, D, F, G, H.
Note that matrix A is neither stochastic nor unitary. □

Part b

Question. *Show that any stochastic matrix that is also unitary must be a permutation matrix.*

Proof. Let $\mathbf{A}$ be a matrix that is stochastic and unitary. This implies,

$$\begin{gathered} \mathbf{A}^\dagger \mathbf{A} = \mathbf{I} \\ \sum_i a_{ij} = 1 \wedge a_{ij} \geq 0 \end{gathered} \tag{1}$$

Representing the above properties in terms of the matrix elements $\mathbf{A} = (a_{ij})$, we get the following,

$$\forall\ i : \sum_i a_{ij} \cdot a_{ji} = 1 \tag{2}$$

$$\forall\ i \neq k : \sum_i a_{ij} \cdot a_{ki} = 0 \tag{3}$$

$$\forall\ i : \exists\ p_i : a_{ip_i} \neq 0 \tag{4}$$

Now, from (3) and (4), we get that $a_{ij} = 0\ \forall\ i \neq p_j$, which implies that $a_{ip_i} = 1$ from (2). Therefore, $\mathbf{A}$ is a permutation matrix with $\Pi = \{p_i\}$. □

Part c

Question. *Stochastic matrices preserve the* 1*-norms of nonnegative vectors, while unitary matrices preserve* 2*-norms. Give an example of a* 2×2 *matrix, other than the identity matrix, that preserves the* 4*-norm of real vectors* $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$*: that is,* $a^4 + b^4$.

Solution. $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ preserves the 4-norm of the vector $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$. □

Part d

Question. *Give a characterization of all real matrices that preserve the* 4*-norms of real vectors. Hopefully, your characterization will help explain why preserving the* 2*-norm, as quantum mechanics does, leads to a much richer set of transformations than preserving the* 4*-norm does.*

Proof. ∎

2 Tensor Products

Part a

Question. *Calculate the tensor product*

$$\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{5} \\ \frac{4}{5} \end{bmatrix}.$$

Solution.

$$\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{5} \\ \frac{4}{5} \end{pmatrix} = \begin{pmatrix} \frac{2}{3}\begin{pmatrix} \frac{1}{5} \\ \frac{4}{5} \end{pmatrix} \\ \frac{1}{3}\begin{pmatrix} \frac{1}{5} \\ \frac{4}{5} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \frac{2}{15} \\ \frac{8}{15} \\ \frac{1}{15} \\ \frac{4}{15} \end{pmatrix} \tag{5}$$

□

Part b

Question. *Of the following length-4 vectors, decide which ones are factorizable as a tensor product of two 2×1 vectors, and factorize them. (Here the vector entries should be thought of as labeled by* 00, 01, 10, *and* 11 *respectively.)*

$$A = \begin{bmatrix} \frac{2}{9} \\ \frac{1}{9} \\ \frac{4}{9} \\ \frac{2}{9} \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, C = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}, D = \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}, E = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}.$$

Solution. The following matrices can be factorized as a tensor product of two 2×1 vectors:

$$A = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} \otimes \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} \tag{6}$$

$$B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{7}$$

$$C = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \tag{8}$$

$$E = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{9}$$

Matrix D cannot be factorized as a tensor product of two 2×1 vectors. □

Part c

Question. *Prove that there's no* 2×2 real *matrix* A *such that*

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

This observation perhaps helps to explain why the complex numbers play such a central role in quantum mechanics.

Proof. We will prove this by contradiction. Let there be a real matrix $\mathbf{A}$ such that $\mathbf{A}^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Notice that $det(\mathbf{A}^2) = -1$. However, we know that $det(\mathbf{A}^2) = det(\mathbf{A})^2$. Therefore, $det(\mathbf{A})^2 = -1 \implies det(\mathbf{A}) = \pm i$, which is not possible since $\mathbf{A}$ is a real matrix and hence its determinant will be real. This is a contradiction and hence there is no real matrix satisfying the condition. $\square$

3 Dirac Notation

Part a

Question. *Let* $|\psi\rangle = \frac{|0\rangle+2|1\rangle}{\sqrt{5}}$ *and* $|\phi\rangle = \frac{2i|0\rangle+3|1\rangle}{\sqrt{13}}$. *What's* $\langle\psi|\phi\rangle$*?*

Solution.

$$\langle\psi|\phi\rangle = \frac{1}{\sqrt{5}}\frac{2i}{\sqrt{13}}\langle 0|0\rangle + \frac{2}{\sqrt{5}}\frac{3}{\sqrt{13}}\langle 1|1\rangle = \frac{6+2i}{\sqrt{65}} \tag{10}$$

□

Part b

Question. *Usually quantum states are normalized:* $\langle\psi|\psi\rangle = 1$. *The state* $|\phi\rangle = 2i\,|0\rangle - 3i\,|1\rangle$ *is not normalized. What constant* A *makes* $|\psi\rangle = \frac{|\phi\rangle}{A}$ *a normalized state?*

Solution.

$$\langle\phi|\phi\rangle = -2i\cdot 2i\,\langle 0|0\rangle + 3i\cdot -3i\,\langle 1|1\rangle = 13 = A^2\,\langle\psi|\psi\rangle = A^2 \tag{11}$$

Therefore, $A = \pm\sqrt{13}$. □

Part c

Question. *Define* $|i\rangle = \frac{|0\rangle+i|1\rangle}{\sqrt{2}}$ *and* $|-i\rangle = \frac{|0\rangle-i|1\rangle}{\sqrt{2}}$. *Show (explicitly or implicitly) that the vectors* $|i\rangle$ *and* $|-i\rangle$ *form an orthonormal basis for* $\mathbb{C}^2$. *(Hint: show that any vector in* $\mathbb{C}^2$ *can be decomposed as a linear combination of* $|i\rangle$ *and* $|-i\rangle$*.)*

Proof. Consider any state $|\psi\rangle = a\,|0\rangle + b\,|1\rangle$. We can write this as,

$$\begin{aligned}
|\psi\rangle &= \frac{a-ib+a+ib}{2}\,|0\rangle + \frac{a-ib-(a+ib)}{2}i\,|1\rangle \\
&= \frac{a-ib}{\sqrt{2}}\frac{|0\rangle+i\,|1\rangle}{\sqrt{2}} + \frac{a+ib}{\sqrt{2}}\frac{|0\rangle-i\,|1\rangle}{\sqrt{2}} \\
&= \frac{a-ib}{\sqrt{2}}\,|i\rangle + \frac{a+ib}{\sqrt{2}}\,|-i\rangle
\end{aligned} \tag{12}$$

Since we can represent any state $|\psi\rangle$ as a linear combination of $|i\rangle, |-i\rangle$, they form an orthonormal basis for $\mathbb{C}^2$. □

Part d

Question. *Write the normalized vector* $|\psi\rangle$ *from part (b) in the* $\{|i\rangle, |-i\rangle\}$*-basis.*

Solution. For $|\psi\rangle$, $a = 2i/\sqrt{13}$ and $b = -3i/\sqrt{13}$ (assuming that we take the positive normalization constant), we get,

$$|\psi\rangle = \frac{2i-3}{\sqrt{26}}\,|i\rangle + \frac{2i+3}{\sqrt{26}}\,|-i\rangle \tag{13}$$

$\square$