Introduction to Quantum Information Science
Recitation, week 3

Properties of unitaries, Circuit notation, and Distinguishability of States

1. More on Unitaries

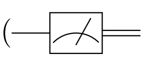
a) Show that the product of two unitary matrices is also a unitary matrix.

Solution: Let A and B be unitary matrices. In order to have that the product AB is unitary we want to show that $AB(AB)^\dagger = I$.

A common property of the complex conjugate is $(AB)^\dagger = B^\dagger A^\dagger$. And because the matrices are unitary, this is $II = I$.

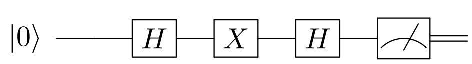
b) Generalize this to the product of an arbitrary number of unitary matrices.

Solution: Let $A_1, \cdots, A_n$ be unitary matrices. Similar to above, we have $(A_1 \cdots A_n)(A_1 \cdots A_n)^\dagger = (A_1 \cdots A_n)(A_n^\dagger \cdots A_1^\dagger) = I$.

2. Circuits And Measurement Find the probability of measuring $|0\rangle$ and $|1\rangle$ for the following circuits ( means measurement):

a)

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



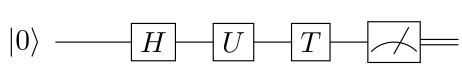
Solution:

$$|0\rangle \mapsto |+\rangle \mapsto |+\rangle \mapsto |0\rangle$$

We measure $|0\rangle$ with probability 1.

b)

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}, \quad U = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{-3}{5} \end{bmatrix}$$



Solution:

$$|0\rangle \mapsto |+\rangle$$

The effect of U on the standard basis is the following

$$|0\rangle \mapsto \frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle \quad |1\rangle \mapsto \frac{4}{5}|0\rangle - \frac{3}{5}|1\rangle$$

So U applied to the $|+\rangle$ state gives:

$$\begin{aligned} U|+\rangle &= U\frac{|0\rangle + |1\rangle}{\sqrt{2}} \\ &= \frac{\left(\frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle\right) + \left(\frac{4}{5}|0\rangle - \frac{3}{5}|1\rangle\right)}{\sqrt{2}} \\ &= \frac{\frac{7}{5}|0\rangle + \frac{1}{5}|1\rangle}{\sqrt{2}} \end{aligned}$$

Since we'll be measuring immediately afterwards in the standard basis, we can ignore the T gate; the relative phase it adds will drop out when we measure (note this is not true in general).

$$\begin{aligned} \Pr[|0\rangle] &= \left|\frac{7}{5\sqrt{2}}\right|^2 = \frac{49}{50} \\ \Pr[|1\rangle] &= \left|\frac{1}{5\sqrt{2}}\right|^2 = \frac{1}{50} \end{aligned}$$

3. Give the probability for measuring $|i\rangle$ and $|-i\rangle$ for the following circuits:

a)

$$\sqrt{\text{NOT}} = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Solution:

$$\begin{aligned} |i\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}} &\mapsto \frac{i|1\rangle + i(-i)|0\rangle}{\sqrt{2}} = |i\rangle \\ &\mapsto \frac{\big((1+i)|0\rangle + (1-i)|1\rangle\big) + i\big((1-i)|0\rangle + (1+i)|1\rangle\big)}{2\sqrt{2}} = \frac{1+i}{\sqrt{2}}|0\rangle \\ &\mapsto \frac{-1+i}{\sqrt{2}}|1\rangle \equiv |1\rangle\,. \end{aligned}$$

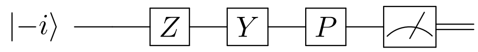
To get the measurement probabilities we want to take the squared magnitude of the inner products $\langle i|1\rangle$ and $\langle -i|1\rangle$

$$\Pr[|i\rangle] = |\langle i|1\rangle|^2 = \frac{1}{2}$$

$$\Pr[|-i\rangle] = |\langle -i|1\rangle|^2 = \frac{1}{2}$$

b)

$$P = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



Solution:

$$|-i\rangle = \frac{|0\rangle - i\,|1\rangle}{\sqrt{2}} \mapsto \frac{|0\rangle + i\,|1\rangle}{\sqrt{2}} = |i\rangle\,.$$

From the previous problem we saw that $|i\rangle$ is an eigenstate of Y, so we can ignore this gate. Finally for P we have:

$$|i\rangle \mapsto |-\rangle$$

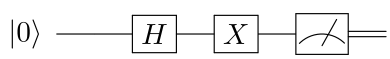
Now, again to get our measurement probabilities we take the inner products of the measurement basis elements with our state.

$$\Pr[|i\rangle] = |\langle i|-\rangle|^2 = \left|\frac{(\langle 0| - i\,\langle 1|)(|0\rangle - |1\rangle)}{2}\right|^2 = \frac{1}{4}|(1+i)|^2 = \frac{1}{2}$$

$$\Pr[|-i\rangle] = \frac{1}{2}$$

4. Give the probability of measuring $|+\rangle$ and $|-\rangle$ for the following circuits:

a)

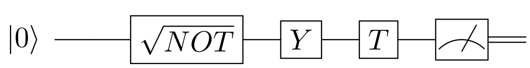


Solution:

$$|0\rangle \mapsto |+\rangle \mapsto |+\rangle$$

$$\Pr[|+\rangle] = 1$$

b)



Solution:

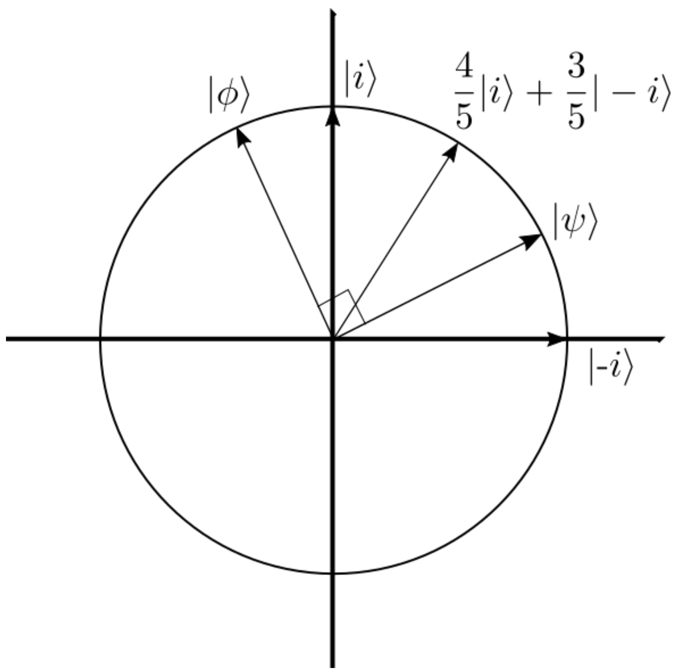
$$\begin{aligned} |0\rangle &\mapsto \frac{(1+i)\,|0\rangle + (1-i)\,|1\rangle}{2} \\ &\mapsto \frac{i(1+i)\,|1\rangle - i(1-i)\,|0\rangle}{2} = \frac{-(1+i)\,|0\rangle - (1-i)\,|1\rangle}{2} \\ &\mapsto \frac{-(1+i)\,|0\rangle - e^{i\pi/4}(1-i)\,|1\rangle}{2}. \end{aligned}$$

Now take inner products with $\{|+\rangle, |-\rangle\}$ to get measurement probabilities.

$$\begin{aligned}\Pr[|+\rangle] &= |\langle+|\psi\rangle|^2 \\ &= \left|\frac{\big(\langle 0| + \langle 1|\big)\big(-(1+i)|0\rangle - e^{i\pi/4}(1-i)|1\rangle\big)}{2\sqrt{2}}\right|^2 \\ &= \frac{\left|(i+1) + e^{i\pi/4}(1-i)\right|^2}{8} \\ &= \frac{2+\sqrt{2}}{4} \\ &= \cos^2(\pi/8),\end{aligned}$$

where a tool like Wolfram Alpha is very useful for finding the final two equalities.

5. Distinguishability of Quantum States What is the best measurement basis to distinguish the state $|i\rangle$ and $\frac{4}{5}|i\rangle + \frac{3}{5}|-i\rangle$? Using this basis, give a protocol to distinguish the two states. With what probability does this protocol succeed, supposing that we are equally likely to be given either state?



Solution: There are many, many possible ways to do this. Our solution below is very systematic, so it should always work. But, you might be able to find an answer must faster using a clever trick of geometry.

From reading Section 5.2 of the textbook, we know the optimal measurements basis, denoted $|\phi\rangle$ and $|\psi\rangle$, must be as illustrated in the figure above, with the two basis states equidistant from the midpoint between the two target states. Note that this is not the usual Bloch circle with $|0\rangle$ and $|1\rangle$ as the axes.

Another way of stating that that the basis states should be equidistant from the midpoint of the target states is that, for the optimal measurement, we want the angle between $|\phi\rangle$ and $|i\rangle$ to be the same as the angle between $|\psi\rangle$ and $\frac{4}{5}|i\rangle + \frac{3}{5}|-i\rangle$ — as explained in the textbook, any bias towards one or the other will only lower our success probability.

Since $|\phi\rangle$ and $|\psi\rangle$ to be orthogonal to be a basis, the angle between $|\phi\rangle$ and $|i\rangle$ must be equal to the angle between $|\psi\rangle$ and $|-i\rangle$.

Using some geometry and combining the previous two statements means that $|\psi\rangle$ must be an equal combination of (halfway between) $\frac{4}{5}|i\rangle + \frac{3}{5}|-i\rangle$ and $|-i\rangle$:

$$|\psi\rangle \propto |\alpha\rangle = (\frac{4}{5}|i\rangle + \frac{3}{5}|-i\rangle) + |-i\rangle .$$

We just have to normalize this state:

$$\begin{aligned} \langle\alpha|\alpha\rangle &= \left(\frac{4}{5}\langle i| + \frac{8}{5}\langle -i|\right)\left(\frac{4}{5}|i\rangle + \frac{8}{5}|-i\rangle\right) \\ &= \frac{16}{25}\langle i|i\rangle + \frac{64}{25}\langle -i|-i\rangle = \frac{80}{25}, \end{aligned}$$

so $|\psi\rangle = \frac{4}{\sqrt{80/25}}(\frac{1}{5}|i\rangle + \frac{2}{5}|-i\rangle)$.

To find $|\phi\rangle$, we flip the amplitudes and change signs: $|\phi\rangle = \frac{4}{\sqrt{80/25}}(\frac{2}{5}|i\rangle - \frac{1}{5}|-i\rangle)$. But, we don't actually require $|\phi\rangle$; we could have left it implicit since we know its overlap with any state of interest is just 1 minus the overlap with $|\psi\rangle$.

Protocol: Measure in the basis. Assume that observing $|\psi\rangle$ means we were given $\frac{4}{5}|i\rangle + \frac{3}{5}|-i\rangle$. Assume that observing $|\phi\rangle$ means we were given $|i\rangle$. Output the answer.

Probability of success Let $|\lambda\rangle = \frac{4}{5}|i\rangle + \frac{3}{5}|-i\rangle$, which was unnamed in the problem statement. First, calculate the probabilities of each measurements result for either of the two states of interest and find

$$|\langle\lambda|\psi\rangle|^2 = \left|(\frac{4}{5}\langle i| + \frac{3}{5}\langle -i|)(\frac{1}{\sqrt{80/25}}(\frac{4}{5}|i\rangle + \frac{8}{5}|-i\rangle))\right|^2 = \left|\frac{1}{\sqrt{80/25}}(\frac{16}{25} + \frac{24}{25})\right|^2 = .8,$$

$$|\langle i|\psi\rangle|^2 = \left|\langle i|\,(\frac{1}{\sqrt{80/25}}(\frac{4}{5}|i\rangle + \frac{8}{5}|-i\rangle))\right|^2 = \frac{1}{\sqrt{80/25}}\frac{4}{5} = .2,$$

where the probabilities of observing $|\phi\rangle$ would be 0.2 and 0.8 respectively. Therefore, the protocol success probability assuming 50-50 chance of $|-\rangle$ or $|i\rangle$ is:

[Chance of given $\frac{4}{5}|i\rangle + \frac{3}{5}|-i\rangle$] $\times$ [chance to get $|\psi\rangle$] $= 0.5 \times 0.8 = 0.4$
plus
[Chance of given $|i\rangle$] $\times$ [chance to get $|\phi\rangle$] $= 0.5 \times (1 - .2) = 0.4$
equals
Total success probability: 0.8.