Introduction to Quantum Information Science
Homework 2

Due Wednesday, September 15 at 11:59 PM

1. More fun with matrices

a) [1 Point] Give an example of a 2×2 unitary matrix where the diagonal entries are 0 but all of the off-diagonal entries are nonzero.

Solution:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

b) [2 Points] Give an example of a 4×4 unitary matrix satisfying the same condition.

Solution:

$$\frac{1}{\sqrt{3}} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}$$

c) [2 Points] Is it possible to have a 3×3 unitary matrix with this condition? If so, give an example. If not, prove it!

Solution: No. Suppose there was such a matrix

$$U = \begin{bmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{bmatrix}.$$

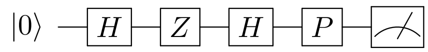
Because U is unitary, the columns are orthogonal to each other. Then, considering the first two columns, $0a + 0b^* + ef^* = ef^* = 0$. But this would imply that either $e = 0$ or $f = 0$: a contradiction.

2. Single Qubit Quantum Circuits For the following circuits, calculate the output state before the measurement. Then calculate the measurement probabilities in the specified basis. Here we use:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad R_{\pi/4} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$$

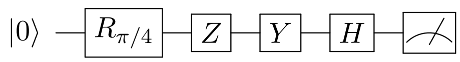
a) [2.5 Points] Measure in the $\{|0\rangle, |1\rangle\}$-basis:



Solution:

$$|0\rangle \to_H |+\rangle \to_Z |-\rangle \to_H |1\rangle \to_S i\,|1\rangle \to \text{Always measure } |1\rangle$$

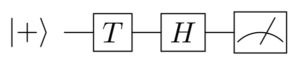
b) [2.5 Points] Measure in the $\{|+\rangle, |-\rangle\}$-basis:



Solution:

$$|0\rangle \to_{R_{\pi/4}} |+\rangle \to_Z |-\rangle \to_Y i\,|+\rangle \to_H i\,|0\rangle = i\frac{|+\rangle + |-\rangle}{\sqrt{2}} \to |+\rangle \text{ or } |-\rangle \text{ equally likely}$$

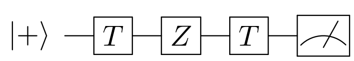
c) [2.5 Points] Measure in the $\{|+\rangle, |-\rangle\}$ basis:



Solution:

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \to_T \frac{|0\rangle + e^{i\pi/4}\,|1\rangle}{\sqrt{2}} \to_H \frac{|+\rangle + e^{i\pi/4}\,|-\rangle}{\sqrt{2}} \to |+\rangle \text{ or } |-\rangle \text{ equally likely}$$

d) [2.5 Points] Measure in the $\{|i\rangle, |-i\rangle\}$-basis:



Solution:

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \to_T \frac{|0\rangle + e^{i\pi/4}\,|1\rangle}{\sqrt{2}} \to_Z \frac{|0\rangle - e^{i\pi/4}\,|1\rangle}{\sqrt{2}} \to_T \frac{|0\rangle - i\,|1\rangle}{\sqrt{2}} = |-i\rangle \to \text{Always measure } |-i\rangle$$

3. Miscellaneous

a) [1 Point] Normalize the state $|0\rangle + |+\rangle$.

Solution:

$$\big(\langle 0| + \langle +|\big)\big(|0\rangle + |+\rangle\big) = \langle 0|0\rangle + \langle +|0\rangle + \langle 0|+\rangle + \langle +|+\rangle = 2 + \frac{2}{\sqrt{2}} = 2 + \sqrt{2}$$

$$\implies |\psi\rangle = \frac{|0\rangle + |+\rangle}{\sqrt{2+\sqrt{2}}} = \frac{|0\rangle + |+\rangle}{2\cos(\pi/8)}$$

b) [1 Point] We say a quantum state vector $|\psi\rangle$ is an eigenvector or eigenstate of a matrix Λ if the following equation holds for some number λ:

$$\Lambda |\psi\rangle = \lambda |\psi\rangle$$

λ is called the eigenvalue of $|\psi\rangle$. Show that the normalized form of the state from part a) is an eigenstate of the H gate. What is the eigenvalue?

Solution:

$$H\frac{|0\rangle + |+\rangle}{2\cos(\pi/8)} = \frac{|+\rangle + |0\rangle}{2\cos(\pi/8)} = \frac{|0\rangle + |+\rangle}{2\cos(\pi/8)} \quad \to \quad \text{eigenvalue} = 1$$

c) [6 Points] What single-qubit states are reachable from $|0\rangle$ using only H and P, i.e. via any sequence of H and P gates? Are there finitely or infinitely many? Characterize all of the reachable states, up to global phase.
Hint: Start by just playing with applying different sequences of matrices to the $|0\rangle$ state and look for a pattern.

Solution: As the following figure demonstrates, the states that can be generated are:

$$|0\rangle, |1\rangle, |+\rangle, |-\rangle, |i\rangle, |-i\rangle,$$

evidently a finite number.

$$\begin{tikzcd}[column sep=tiny]
 & & & & |0\rangle \arrow[dl, "H"'] \arrow[dr, "P"] & & & \\
 & & & |+\rangle \arrow[dl, "H"'] \arrow[dr, "P"] & & |0\rangle & & \\
 & & |0\rangle & & |i\rangle \arrow[dll, "H"'] \arrow[drr, "P"] & & & \\
 & & |-i\rangle \arrow[dl, "H"'] \arrow[dr, "P"] & & & & |-\rangle \arrow[dl, "H"'] \arrow[dr, "P"] & \\
 & |i\rangle & & |+\rangle & & |1\rangle \arrow[dl, "H"'] \arrow[dr, "P"] & & |-i\rangle \\
 & & & & |-\rangle & & |1\rangle &
\end{tikzcd}$$

The computation of $H|i\rangle = |-i\rangle$ is a little tricky. We have

$$H|i\rangle = \frac{\sqrt{+} + i\sqrt{-}}{\sqrt{2}} = \frac{(1+i)|0\rangle + (1-i)|1\rangle}{2} \cdot \frac{(1+i)}{(1-i)} = \frac{|0\rangle + -i|1\rangle}{\sqrt{2}} = |-i\rangle.$$

For the reverse, that $H|-i\rangle = |i\rangle$, note that H is Hermitian (i.e. $H = H^\dagger$), which combined with the fact it's unitary means that H is self-inverse. So since $|i\rangle \mapsto_H |-i\rangle$, then $|-i\rangle \mapsto_H |i\rangle$.

4. Distinguishability of states Say you are given a state $|\psi\rangle$ that is either $|0\rangle$ or $|1\rangle$ but you don't know which. You can distinguish the two via a measurement in the $\{|0\rangle, |1\rangle\}$-basis.

a) [6 Points] But what if $|\psi\rangle$ is either $|0\rangle$ or $|+\rangle$ (with equal probability)? Give the protocol that distinguishes the two states with with a failure probability of $\sin^2(\frac{\pi}{8}) \approx .146$. Show explicitly that your

protocol achieves this failure probability.
Note that when we ask for a protocol, we mean some step-by-step algorithm that ends by outputting "I think this was $|0\rangle$" or "I think this was $|+\rangle$".

Solution: (First solution) From the method given in the textbook, begin by computing the perpendicular bisector $|v\rangle$ of the target vectors. On a unit circle with $|0\rangle$ on the horizontal axis and $|1\rangle$ on the vertical, the two vectors fall at $\theta_1 = 0, \theta_2 = \frac{\pi}{4}$, so the bisector is at $\frac{\pi}{8}$. Next, we choose our measurement basis by rotating from the bisector by 45° c.c.w. and clockwise. This gives

$$|b_1\rangle = \cos(\frac{3\pi}{8})|0\rangle + \sin(\frac{3\pi}{8})|1\rangle$$

$$|b_2\rangle = \cos(\frac{-\pi}{8})|0\rangle + \sin(\frac{-\pi}{8})|1\rangle = \cos(\frac{\pi}{8})|0\rangle - \sin(\frac{\pi}{8})|1\rangle .$$

Our protocol is that if we measure and observe $|b_1\rangle$, then guess we had $|+\rangle$. If we observe $|b_2\rangle$, then guess we had $|0\rangle$. If the state was $|0\rangle$, the probability of observing $|b_1\rangle$ and making the wrong guess is $|\langle 0|b_1\rangle|^2 = \cos^2(\frac{3\pi}{8}) = \sin^2(\frac{\pi}{8})$. In the other case, the probability of failure is $1 - |\langle 0|b_2\rangle|^2 = 1 - \cos^2(\frac{\pi}{8}) = \sin^2(\frac{\pi}{8})$.

(Second solution) The best basis has $|0\rangle$ and $|+\rangle$ sandwiched in the middle, so one basis state must be $\propto |1\rangle + |+\rangle$. Normalize:

$$\big(\langle 1| + \langle +|\big)\big(|1\rangle + |+\rangle\big) = \langle 1|1\rangle + \langle +|1\rangle + \langle 1|+\rangle + \langle +|+\rangle = 2 + 2/\sqrt{2} = 2 + \sqrt{2}$$

So we measure the state $|\psi\rangle = \frac{|1\rangle + |+\rangle}{\sqrt{2+\sqrt{2}}}$. Probabilities are:

$$|\langle\psi|+\rangle|^2 = \left|\frac{\langle 1|+\rangle + \langle +|+\rangle}{\sqrt{2+\sqrt{2}}}\right|^2 = \left|\frac{\frac{1}{\sqrt{2}} + 1}{\sqrt{2+\sqrt{2}}}\right|^2 \approx 0.854$$

$$|\langle\psi|0\rangle|^2 = \left|\frac{\langle 1|0\rangle + \langle +|0\rangle}{\sqrt{2+\sqrt{2}}}\right|^2 = \left|0 + \frac{\frac{1}{\sqrt{2}}}{\sqrt{2+\sqrt{2}}}\right|^2 \approx 0.146$$

If we measure $|\psi\rangle$ then we say the input was $|+\rangle$, and it was $|0\rangle$ otherwise. The total failure probability is (1/2) measuring $|\psi\rangle$ from $|0\rangle$ plus (1/2) not measuring $|\psi\rangle$ from $|+\rangle$ = (0.146)/2 + (1-0.854)/2 = 0.146.

Note the bases from our two solutions are the same.

$$|b_1\rangle = \cos(\frac{3\pi}{8})|0\rangle + \sin(\frac{3\pi}{8})|1\rangle \cdot \frac{2\cos\frac{\pi}{8}}{2\cos\frac{\pi}{8}} = \frac{2\frac{1}{2}\left(\cos\frac{\pi}{2} + \cos\frac{\pi}{4}\right)|0\rangle + 2\frac{1}{2}\left(\sin\frac{\pi}{2} + \sin\frac{\pi}{4}\right)|1\rangle}{2\cos\frac{\pi}{8}} =$$

$$\frac{\cos\frac{\pi}{4}|0\rangle\left(1 + \cos\frac{\pi}{4}\right)|1\rangle}{2\cos\frac{\pi}{8}} = \frac{\frac{1}{\sqrt{2}}|0\rangle\left(1 + \frac{1}{\sqrt{2}}\right)|1\rangle}{2\cos\frac{\pi}{8}} = \frac{|1\rangle + |+\rangle}{\sqrt{2+\sqrt{2}}} = |\psi\rangle .$$

b) [Extra Credit, 5 Points] Prove that this is optimal.

Solution: Let's say the optimal state to measure is $|\psi\rangle = \sin\theta|0\rangle + \cos\theta e^{i\phi}|1\rangle$. This state should maximize $|\langle +|\psi\rangle|^2 = |\frac{\sin\theta + \cos\theta e^{i\phi}}{\sqrt{2}}|^2 = \frac{\sin^2\theta + (e^{i\phi}+e^{-i\phi})\sin\theta\cos\theta + \cos^2\theta}{2}$ and minimize $|\langle 0|\psi\rangle|^2 = \sin^2\theta$:

$$\text{Fidelity} = |\langle +|\psi\rangle|^2 - |\langle 0|\psi\rangle|^2 = \frac{\sin^2\theta + (e^{i\phi}+e^{-i\phi})\sin\theta\cos\theta + \cos^2\theta}{2} - \sin^2\theta$$

$$\frac{\partial \text{Fidelity}}{\partial\phi} = (1/2)(i\phi e^{i\phi} + i\phi e^{-i\phi})\sin\theta\cos\theta = 0 \quad \to \quad \phi = 0$$

$$\frac{\partial \text{Fidelity}}{\partial\theta} = \frac{1}{2}\left(\sin^2\theta + 2\sin\theta\cos\theta - \cos^2\theta\right) = \frac{\sin(2\theta) - \cos(2\theta)}{2} = 0$$

Since $\cos(2\theta) = -\sin(2\theta)$ happens when $2\theta = (4n+1)\pi/4$ we pick $\theta = \pi/8$ and get:

$$|\psi\rangle = \sin(\pi/8)|0\rangle + \cos(\pi/8)|1\rangle = \frac{1}{2}\sqrt{2-\sqrt{2}}\,|0\rangle + \frac{1}{2}\sqrt{2+\sqrt{2}}\,|1\rangle$$

$$= \frac{2|1\rangle + \sqrt{2}|0\rangle + \sqrt{2}|1\rangle}{2\sqrt{2+\sqrt{2}}} = \frac{|1\rangle + \frac{|0\rangle+|1\rangle}{\sqrt{2}}}{\sqrt{2+\sqrt{2}}} = \frac{|1\rangle + |+\rangle}{\sqrt{2+\sqrt{2}}}$$

c) [2 Points] What is the failure probability if you measure in the $\{|0\rangle, |1\rangle\}$ basis?

Solution: If we measure $|1\rangle$, we say the input state must have been $|+\rangle$. If we measure $|0\rangle$, we say the state was $|0\rangle$.

If we are given $|0\rangle$, then our identification is always correct. If we are given $|+\rangle$, the probability of measuring $|0\rangle$ is $|\langle +|0\rangle|^2 = |1/\sqrt{2}|^2 = 1/2$. Thus our average failure probability is $1/4$.