C S 358H: Intro to Quantum Information Science

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Contents

1 More fun with matrices

Part a

Question. *Give an example of a 2×2 unitary matrix where the diagonal entries are 0 but all of the off-diagonal entries are nonzero.*

Solution. $\mathbf{A} = \begin{pmatrix} 0 & e^{i\theta_1} \\ e^{i\theta_2} & 0 \end{pmatrix}$ for any $\theta_1, \theta_2 \in [-\pi, \pi]$ is a family of unitary matrices with zero diagonal entries but non-zero off-diagonal entries. □

Part b

Question. *Give an example of a 4×4 unitary matrix satisfying the same condition.*

Solution. $\mathbf{A} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$ is a unitary and has zero diagonal entries and non-zero off-diagonal entries. □

Part c

Question. *Is it possible to have a 3×3 unitary matrix with this condition? If so, give an example. If not, prove it!*

Proof. No, it is impossible to have a 3×3 unitary matrix with this condition. We will prove this by contradiction. Assume that there exists a matrix A such that A is unitary and has zero diagonal entries and non-zero off-diagonal entries. Let $A = \begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix}$ where $a_{ij} (i \neq j) \in \mathbb{C}$. Since A is unitary, $AA^\dagger = \mathbf{I}$. We have

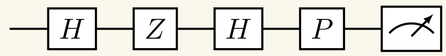
$$\begin{aligned}
AA^\dagger &= \begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix} \begin{pmatrix} 0 & a_{21}^* & a_{31}^* \\ a_{12}^* & 0 & a_{32}^* \\ a_{13}^* & a_{23}^* & 0 \end{pmatrix} \\
\implies \mathbf{I} &= \begin{pmatrix} |a_{12}|^2 + |a_{13}|^2 & a_{13}a_{23}^* & a_{12}a_{32}^* \\ a_{23}a_{13}^* & |a_{21}|^2 + |a_{23}|^2 & a_{21}a_{31}^* \\ a_{32}a_{12}^* & a_{31}a_{21}^* & |a_{31}|^2 + |a_{32}|^2 \end{pmatrix} \qquad (1) \\
\implies 0 &= a_{13}a_{23}^* = a_{12}a_{32}^* = a_{23}a_{13}^* = a_{21}a_{31}^* = a_{32}a_{12}^* = a_{31}a_{21}^* \\
\implies & \perp
\end{aligned}$$

This is a contradiction since we assumed that $a_{ij} \neq 0$ for $i \neq j$. Therefore, it is impossible to have a 3×3 unitary matrix with zero diagonal entries and non-zero off-diagonal entries. □

2 Single Qubit Quantum Circuits

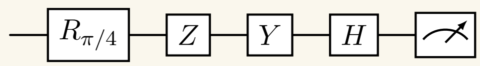
For the following circuits, calculate the output state before the measurement.

Part a

Question. *Measure in the* $\{|0\rangle, |1\rangle\}$*-basis:* $|0\rangle$ 

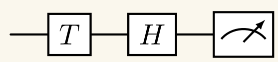
Solution. The state of the qubit right before the measurement will be, $|\psi\rangle = i|1\rangle$ which on measurement in the standard basis will give the output 1. □

Part b

Question. *Measure in the* $\{|+\rangle, |-\rangle\}$*-basis:* $|0\rangle$ 

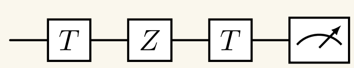
Solution. The state of the qubit right before the measurement will be, $|\psi\rangle = i|0\rangle$ which on measurement in the Hadamard basis will give either $|+\rangle$ or $|-\rangle$ with equal probability. □

Part c

Question. *Measure in the* $\{|+\rangle, |-\rangle\}$*-basis:* $|+\rangle$ 

Solution. Since we have to measure in the $|+\rangle, |-\rangle$ basis, we can add a Hadamard gate before the measurement and measure in the standard basis. This will cancel out the H gate and the resultant circuit is just executing a T gate on the $|+\rangle$ state. The resultant state after applying the gate is $\frac{1}{\sqrt{2}}(|0\rangle + e^{i\pi/4}|1\rangle)$. Therefore, the probabilites of getting $|+\rangle$ and $|-\rangle$ is 50% each.
Since the question also asks for the state before measurement, the state before measurement will be $\frac{1}{\sqrt{2}}(|+\rangle + e^{i\pi/4}|-\rangle)$. □

Part d

Question. *Measure in the* $\{|i\rangle, |-i\rangle\}$*-basis:* $|+\rangle$ 

Solution. The T gate is a rotation along the Z axis by $\pi/8$ radians and the Z gate is a rotation along the Z axis by $\pi/2$ radians. Therefore, the entire circuit can be seen as a rotation along the Z axis by $3\pi/4$ radians. Therefore, the resultant state is $\frac{1}{\sqrt{2}}(|0\rangle + e^{i3\pi/2}|1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) = |-i\rangle$. Therefore the probability of obtaining $|-i\rangle$ is 1. □

3 Miscellaneous

Part a

Question. *Normalize the state* $|0\rangle + |+\rangle$.

Solution. The state is equal to $(1 + \frac{1}{\sqrt{2}})|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$. Therefore, the normalization factor will be $\sqrt{2+\sqrt{2}}$ and the normalized state will be $\frac{1}{\sqrt{2\sqrt{2}}}\left(\sqrt{\sqrt{2}+1}\,|0\rangle + \sqrt{\sqrt{2}-1}\,|1\rangle\right)$. □

Part b

Question. *We say a quantum state vector* $|\psi\rangle$ *is an eigenvector or eigenstate of a matrix* Λ *if the following equation holds for some number* λ*:*

$$\Lambda|\psi\rangle = \lambda|\psi\rangle$$

λ *is called the eigenvalue of* $|\psi\rangle$*. Show that the normalized form of the state from part a) is an eigenstate of the* H *gate. What is the eigenvalue?*

Proof. If we represent the state in part a as $|\psi\rangle = a(|0\rangle + |+\rangle)$, where $a = \frac{1}{\sqrt{2+\sqrt{2}}}$, then we get,

$$\begin{aligned} H|\psi\rangle &= H\left(a(|0\rangle + |+\rangle)\right) \\ &= a(H(|0\rangle + |+\rangle)) \\ &= a(|+\rangle + |0\rangle) = 1 \cdot |\psi\rangle \end{aligned} \tag{2}$$

Therefore, $|\psi\rangle$ is an eigenstate of H with an eigenvalue of 1. □

Part c

Question. *What single-qubit states are reachable from* $|0\rangle$ *using only* H *and* P*, i.e. via any sequence of* H *and* P *gates? Are there finitely or infinitely many? Characterize all of the reachable states, up to global phase.*
Hint: Start by just playing with applying different sequences of matrices to the $|0\rangle$ *state and look for a pattern.*

Proof. There are a finitely many states that can be reached from $|0\rangle$ using only H and P. We first show the following,

$$PH|0\rangle = P|+\rangle = \frac{1}{\sqrt{2}}H(|0\rangle + i|1\rangle) = |i\rangle \tag{3}$$

Similarly, we have,

$$P^2H|0\rangle = P^2|+\rangle = |-\rangle \tag{4}$$

Also,

$$P^3H|0\rangle = P^3|+\rangle = |-i\rangle \tag{5}$$

We also state the following from Equations 3, 4, 5,

$$\begin{aligned} HPH\,|0\rangle &= |-i\rangle \\ HP^2H\,|0\rangle &= |1\rangle \\ HP^3H\,|0\rangle &= |i\rangle \end{aligned} \tag{6}$$

Also note that $P^4 = I$ and $H^2 = I$. Therefore, a non-trivial application of a unitary made up of a sequence of H and P gates will be of the form $H^h \cdot P^{p_n} \cdot H \cdot P^{p_{n-1}} \cdot H \cdot P^{p_{n-2}} \cdots H \cdot P^{p_1} \cdot H$ where $h \in \{0,1\} \;\wedge\; \forall i:\; p_i \in \{1,2,3\}$. In words, we start with a Hadamard gate and we alternate between phase gates and Hadamards in between and may or may not end with a Hadamard gate. Note that starting with a phase gate has no effect on the $|0\rangle$ state and hence we can ignore its effect. Therefore, the only set of states that are reachable by such a unitary belong to the set $\mathbb{S} = \{|0\rangle, |1\rangle, |+\rangle, |-\rangle, |i\rangle, |-i\rangle\}$. We can easily prove this by induction on i. $\square$

4 Distinguishability of states

Say you are given a state $|\psi\rangle$ that is either $|0\rangle$ or $|1\rangle$ but you don't know which. You can distinguish the two via a measurement in the $\{|0\rangle, |1\rangle\}$-basis.

Part a

Question. *But what if $|\psi\rangle$ is either $|0\rangle$ or $|+\rangle$ (with equal probability)? Give the protocol that distinguishes the two states with with a failure probability of* $\sin^2(\frac{\pi}{8}) \approx .146$. *Show explicitly that your protocol achieves this failure probability.*
Note that when we ask for a protocol, we mean some step-by-step algorithm that ends by outputting "I think this was $|0\rangle$" or "I think this was $|+\rangle$".
Hint: Read Section 5.2 of the textbook.

Proof. We propose the following protocol,

Protocol to distinguish between $|0\rangle$ and $|+\rangle$

1. Apply the $R_X(\pi/8)$ gate to the input state $|\psi\rangle$ (i.e., rotate the state by $\pi/8$ along the X axis).
2. Measure the state in the standard basis.
3. If the measurement result is $|0\rangle$, output $|0\rangle$, else output $|+\rangle$.

Figure 1: Distinguishing protocol

We now prove that the failure probability of this protocol is $\sin^2(\pi/8)$. If we had the $|0\rangle$ state, then the state of the qubit after applying the rotation gate is $\cos\frac{\pi}{8}|0\rangle + \sin\frac{\pi}{8}|1\rangle$. Alternatively, if we started with the $|+\rangle$ state, the state would be $\cos\frac{3\pi}{8}|0\rangle + \sin\frac{3\pi}{8}|1\rangle$. Now, the failure probability can be computed as,

$$\begin{aligned}\Pr[\text{failure}] &= \frac{1}{2}\cdot\Pr[\text{failure} \mid |\psi\rangle = |0\rangle] + \frac{1}{2}\cdot\Pr[\text{failure} \mid |\psi\rangle = |+\rangle] \\ &= \frac{1}{2}\cdot\sin^2\frac{\pi}{8} + \frac{1}{2}\cdot\cos^2\frac{3\pi}{8} \\ &= \frac{1}{2}\cdot\sin^2\frac{\pi}{8} + \frac{1}{2}\cdot\sin^2\frac{\pi}{8} = \sin^2\frac{\pi}{8}\end{aligned} \tag{7}$$

Hence, we have shown that the failure probability of this protocol is $\sin^2(\pi/8)$. □

Part b

Question. *Prove that this is optimal.*

Proof. Any protocol that will be used to distinguish the two states can be represented as a unitary on some n qubits, followed by measurements in the standard basis at the end. Any intermediate measurements can be deferred to the end by adding more qubits in the circuit (for each intermediate measurement, add an extra qubit, perform a CNOT between the qubit to be measured and the new qubit and instead measure the new qubit; now this new qubit isn't involved in any further operations so its measurement can be done at any point, we do

it at the end along with all other measurements).
Therefore, any protocol has the effect of performing a measurement in a certain orthonormal basis. However, we know that no unitary can change the inner product between two states. This implies that the angle between the two states is still fixed at $\theta = \cos^{-1} |\langle \psi_1 | \psi_2 \rangle|$. Thus, the most optimal basis to measure the two states will have each orthonormal state at an angle of some α and $\alpha + \theta$ from the two states (assuming that θ is acute, the argument is similar for an obtuse θ). Therefore the least failure probability for any protocol is $2 \cdot \frac{1}{2} \cdot \cos^2 \alpha = \cos^2(\pi/2 - \theta/2) = \sin^2 \theta/2$.
For the states $|0\rangle$ and $|+\rangle$, the angle is $\pi/4$ and therefore the best failure probability is $\sin^2 \pi/8$. $\square$

Part c

Question. *What is the failure probability if you measure in the* $\{|0\rangle, |1\rangle\}$ *basis?*

Solution. If we measure in the $\{|0\rangle, |1\rangle\}$ basis and guess that the state was $|+\rangle$ only if the measurement result is $|1\rangle$, the failure probability can be computed as,

$$\begin{aligned} \Pr[\text{failure}] &= \frac{1}{2} \cdot \Pr[\text{failure} \mid |\psi\rangle = |0\rangle] + \frac{1}{2} \cdot \Pr[\text{failure} \mid |\psi\rangle = |+\rangle] \\ &= 0 + \frac{1}{2} \cdot \frac{1}{2}, \text{since there is a 50\% probability of getting } |1\rangle \qquad (8) \\ &= \frac{1}{4} \end{aligned}$$

$\square$