Introduction to Quantum Information Science
Homework 7

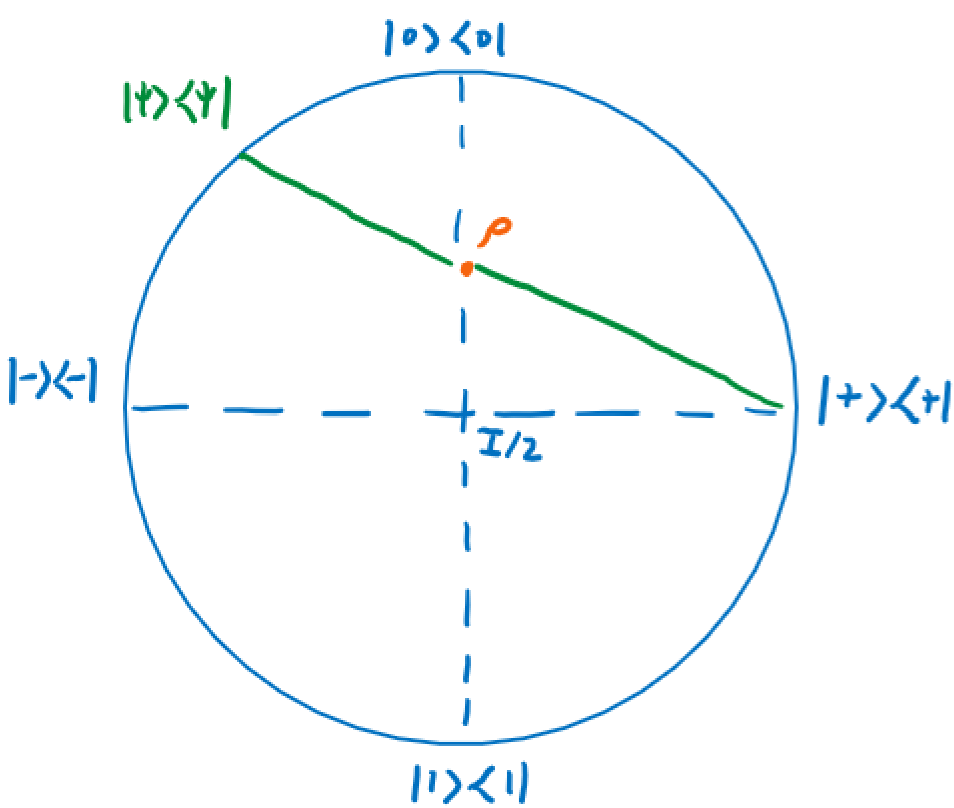
Due Wednesday, October 27 at 11:59 PM

Note: You should explain your reasoning, i.e. show your work, for all problems. You do not need to show us every step of each calculation, but every answer should include an explanation *written with words* of what you did.

1. Bloch Sphere [6 points] Give two different decompositions of the 1-qubit mixed state

$$\rho = \begin{bmatrix} \cos^2(\pi/8) & 0 \\ 0 & \sin^2(\pi/8) \end{bmatrix}$$

as a mixture of two pure states. Show your work. What do these decompositions correspond to physically? Draw a 2D-sketch of the Bloch sphere to aid your explanation.



Solution: First, there is the obvious decomposition: $\rho = \cos(\frac{\pi}{8})^2 |0\rangle\langle 0| + \sin(\frac{\pi}{8})^2 |1\rangle\langle 1|$. Geometrically, ρ is a mixture of the pure states $|0\rangle$ and $|1\rangle$ biased towards $|0\rangle$.

(First solution) To find a second decomposition, we need two pure states such that ρ is on the chord connecting them on the Bloch sphere. We can always pick the first one arbitrarily, so pick $|+\rangle\langle +|$. Now we must find $a, b, |\psi\rangle$ such that $\rho = a|+\rangle\langle +| + b|\psi\rangle\langle\psi|$. Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. We know α, β are real because our geometric diagram of ρ shows it exists in the real plane of the Bloch sphere. And we know $a, b \neq 0$ based on our picture and $\alpha, \beta \neq 0$ based on our picture (otherwise, the chord would intersect $|0\rangle\langle 0|$ or $|1\rangle\, 1$).

We then have the system

$$\rho = \begin{bmatrix} \cos(\pi/8)^2 & 0 \\ 0 & \sin(\pi/8)^2 \end{bmatrix} = a\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + b\begin{bmatrix} \alpha^2 & \alpha\beta \\ \alpha\beta & \beta^2 \end{bmatrix}.$$

which means

$$\frac{1}{2}a + b\alpha^2 = \cos^2\frac{\pi}{8} \tag{1}$$
$$\frac{1}{2}a + b\beta^2 = \sin^2\frac{\pi}{8} \tag{2}$$
$$\frac{1}{2}a + b\alpha\beta = 0. \tag{3}$$

Further, from our knowledge of states we know

$$a + b = 1 \tag{4}$$
$$\alpha^2 + \beta^2 = 1. \tag{5}$$

Eq (2) is actually equivalent to Eq (1) and (4), it doesn't provide any additional information. Plugging this system into a symbolic math program (e.g. WolframAlpha) gives

$$a = \frac{1}{4} \quad b = \frac{3}{4} \quad \alpha = \pm\frac{2+\sqrt{2}}{2\sqrt{3}} = \pm\frac{2\cos^2\frac{\pi}{8}}{\sqrt{3}} \quad \beta = \frac{2-\sqrt{2}}{2\sqrt{3}} = \frac{2\sin^2\frac{\pi}{8}}{\sqrt{3}}$$

such that $\rho = a|+\rangle\langle +| + b|\psi\rangle\langle\psi|$.

(Second/advanced solution) Next, begin with the identity and shift it as necessary to match ρ, writing writing

$$\rho = \frac{I + \left(2\cos(\frac{\pi}{8})^2 - 1\right)Z}{2} = \frac{1}{2}(I + \frac{1}{\sqrt{2}}Z).$$

we see this state has Bloch vector $\vec{r_\rho} = (0, 0, \frac{1}{\sqrt{2}})$. Then we can average together $\vec{r}_{|\psi\rangle\langle\psi|} = (a, 0, \frac{1}{\sqrt{2}})$ and $\vec{r}_{Z|\psi\rangle\langle\psi|Z} = (-a, 0, \frac{1}{\sqrt{2}})$ to obtain ρ. For these states to be pure we need $|\vec{r}| = 1$, so $a^2 + 1/2 = 1$ so $a = \sqrt{1/2}$. Since this lies in the XZ plane we know that α, β are real when $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

$$|\psi\rangle\langle\psi| = \frac{I}{2} + \frac{1}{\sqrt{2}}\frac{X}{2} + \frac{1}{\sqrt{2}}\frac{Z}{2} = \begin{bmatrix} \frac{2+\sqrt{2}}{4} & \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{2-\sqrt{2}}{4} \end{bmatrix} = \begin{bmatrix} \alpha^2 & \alpha\beta \\ \alpha\beta & \beta^2 \end{bmatrix}$$
$$\implies |\psi\rangle = \frac{\sqrt{2+\sqrt{2}}}{2}|0\rangle + \frac{\sqrt{2-\sqrt{2}}}{2}|1\rangle = \cos\frac{\pi}{8}|0\rangle + \sin\frac{\pi}{8}|1\rangle$$

Now we can write two more decompositions by rotating this state around using Z and S:

$$\rho = |\psi\rangle\langle\psi| + Z\,|\psi\rangle\langle\psi|\,Z^\dagger, \rho = S\,|\psi\rangle\langle\psi|\,S^\dagger + S^\dagger\,|\psi\rangle\langle\psi|\,S$$

Where

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}.$$

2. [6 points] Which of the following views about quantum mechanics necessarily lead to experimental predictions that are *different* from the predictions of conventional QM? Make two distinct lists: one list of the views that do, and one list of the views that don't. (Note: For this question, we're only interested in clear-cut predictions about the observed behaviors of "dumb physical systems" like masses or entangled particles, not about the experiences of conscious observers like Wigner's friend. Beyond that, though, we don't care how easy or hard the prediction is to test in practice.)

(A) The Many-Worlds Interpretation

(B) The Copenhagen Interpretation

(C) GRW (Ghirardi-Rimini-Weber) dynamical collapse

(D) Penrose's gravity-induced quantum state collapse

(E) Local hidden variables

(F) Nonlocal hidden variables (including Bohmian mechanics)

Solution: Does: C, D, E Does not: A, B, F

3. Quantum computation with real amplitudes 'Real quantum mechanics' is a hypothetical theory that's identical to standard quantum mechanics, except that the amplitudes always need to be real—and instead of unitary matrices, we're restricted to applying real orthogonal matrices.

a) [4 points] Prove that any standard quantum circuit acting on n qubits can be perfectly simulated by a real quantum circuit acting on $n+1$ qubits — and moreover, by a circuit containing exactly as many gates as the original circuit (although the gates might act on slightly more qubits than the gates of the original circuit). Your proof should give a mapping from complex-valued states to real-valued states and from unitary matrices to real orthogonal matrices.
Hint: Observe that, with $n+1$ qubits rather than n, you have 2^{n+1} amplitudes rather than just 2^n.

b) [2 points] To illustrate your construction, show how the phase gate gets converted into a purely real gate in your simulation.

c) [1 point] Conclude the proof that complex amplitudes are never actually needed for quantum computing speedups — positive and negative real amplitudes suffice — by explaining how measurements are performed.

Solution: We encode a complex amplitude α using two real amplitudes, $\Re(\alpha)$ and $\mathrm{Im}(\alpha)$:

$$|\psi\rangle = \sum_x \alpha_x\,|x\rangle \quad \to \quad |\psi\rangle' = \sum_x |x\rangle \otimes \big(\Re(\alpha_x)\,|0\rangle + \mathrm{Im}(\alpha_x)\,|1\rangle\big). \tag{6}$$

This uses $n+1$ real-valued qubits to encode n complex-valued qubits. Since $|\alpha_x|^2 = |\Re(\alpha_x)|^2 + |\mathrm{Im}(\alpha_x)|^2$, this remains normalized.

In order to map a general unitary to a real orthogonal matrix acting on states in our encoding, we use the fact that we can simulate the effect of multiplying complex numbers by using real-valued matrix-vector multiplication. Given a complex number $\alpha = a + ib$, we represent this as a two element column vector (this is equivalent to the encoding scheme we gave above for the amplitudes of the state):

$$a + ib \to \begin{bmatrix} a \\ b \end{bmatrix}$$

In order to multiply this by another complex number $\beta = c + id$ we multiply by the matrix:

$$\begin{bmatrix} c & -d \\ d & c \end{bmatrix}$$

In order to map a general unitary U in our simulation scheme we simply replace every number in our matrix with the corresponding 2×2 matrix above in a block-by-block fashion. For example:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \qquad \mapsto \qquad \begin{bmatrix} \Re(a) & -\mathrm{Im}(a) & \Re(b) & \mathrm{Im}(b) \\ \mathrm{Im}(a) & \Re(a) & \mathrm{Im}(b) & \Re(b) \\ \Re(c) & -\mathrm{Im}(c) & \Re(d) & \mathrm{Im}(d) \\ \mathrm{Im}(c) & \Re(c) & \mathrm{Im}(d) & \Re(d) \end{bmatrix} \begin{bmatrix} \Re(\alpha) \\ \mathrm{Im}(\alpha) \\ \Re(\beta) \\ \mathrm{Im}(\beta) \end{bmatrix}.$$

(b) The phase gate is originally defined as $P|0\rangle = |0\rangle$ and $P|1\rangle = i|1\rangle$. In our scheme, this is now a two-qubit gate:

$$P|00\rangle = |00\rangle, \quad P|01\rangle = |01\rangle, \quad P|10\rangle = |11\rangle, \quad P|11\rangle = -|10\rangle.$$

(c) To measure, we simply measure like usual, ignoring the extra qubit. Measuring $|\psi\rangle$ from Eq (6) gives $|x\rangle$ with probability $|\alpha_x|^2$. Measuring the first qubit of $|\psi\rangle'$ gives $|x\rangle$ with probability $\Pr[|x0\rangle] + \Pr[|x1\rangle] = |\Re(\alpha_x)|^2 + |\mathrm{Im}(\alpha_x)|^2 = |\alpha_x|^2$. The probabilities match, as desired.

4. Universal gate sets [6 points] Identify the following gate sets as either universal or not universal *in the sense — specifically — of able to approximate any target unitary to any desired precision.* If it is not, argue why (if it is, you do not have to give an argument).
Recall: $S = [\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}]$).

a) {All single qubit gates, CNOT}

Solution: This is universal.

b) {Toffoli, Hadamard}

Solution: Not universal: only real amplitudes.

c) {Toffoli, S}

Solution: Not universal: cannot create superposition.

d) {Toffoli, S, Hadamard}

Solution: This is universal, since it contains the stabilizer set (everything generated by $\{\text{CNOT}, S, H\}$), and Toffoli, which is not a stabilizer gate.

e) {Hadamard, S, Controlled Z}

Solution: Not universal: This is equivalent to {Hadamard, S, CNOT} which is the stabilizer set.

f) {Controlled Hadamard, Controlled S, NOT}

Solution: This is universal. Both controlled Hadamard and controlled S are not part of the stabilizer set. One way to see that this gate set is not contained in the stabilizer set is to construct the Toffoli gate: