Introduction to Quantum Information Science
Recitation 3
(9/15 and 9/19)

1. Any questions about solutions to homework 1?

2. The Union Bound

a) Let A and B be random events. Prove the union bound: $\Pr[A \cup B] \leq \Pr[A] + \Pr[B]$. In words, the probability that A and B both occur is at most the sum of the probabilities of either one occurring.

Why are we reviewing this? Because the union bound is one of the most useful inequalities you can learn, and you'll use it when writing proofs in this course. Please consider getting it tattooed.

Solution: By the definition of probability we have

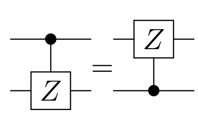
1. $\Pr[A] \geq 0$ for any event A.
2. $\Pr[\Omega] = 1$ where Ω is the entire probability space.
3. $\Pr[\cup_i A_i] = \sum_i \Pr[A_i]$ for any countable set of disjoint events $\{A_i\}$.

Therefore,

$$\begin{aligned} \Pr[A \cup B] &= \Pr[A \cup (B - A)] \\ &= \Pr[A] + \Pr[B - A] \\ &\leq \Pr[A] + \Pr[B - A] + \Pr[A \cap B] \\ &= \Pr[A] + \Pr[(B - A) \cup (A \cap B)] \\ &= \Pr[A] + \Pr[B]. \end{aligned}$$

3. Multi-Qubit Circuit Practice

a) Show that the following circuit identity holds:



(1)

Solution: Write the action of the controlled gate using outer products. The left hand side is

$$|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes Z = \begin{bmatrix} I & 0 \\ 0 & Z \end{bmatrix}$$

and the right hand side is

$$\begin{gathered}
I \otimes |0\rangle\langle 0| + Z \otimes |1\rangle\langle 1| = \\
(|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes |0\rangle\langle 0| + (|0\rangle\langle 0| - |1\rangle\langle 1|) \otimes |1\rangle\langle 1| = \\
|00\rangle\langle 00| + |10\rangle\langle 10| + |01\rangle\langle 01| - |11\rangle\langle 11| = \\
(|00\rangle\langle 00| + |01\rangle\langle 01|) + (|10\rangle\langle 10| - |11\rangle\langle 11|) = \\
|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes Z,
\end{gathered}$$

so the two circuits match.

b) Prepare a circuit that makes the following state starting from the all-zero state:

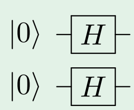
$$\frac{|00\rangle + \sqrt{2}\,|01\rangle + \sqrt{2}\,|10\rangle - |11\rangle}{\sqrt{6}}$$

We are given access to CNOT and any well defined single-qubit gate (feel free to define gates beyond those we've seen in class, just make sure you give an explicit description).

Solution:

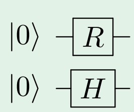
1. Support on all four components.
2. More $|01\rangle$ and $|10\rangle$ than other components.
3. Negative phase on $|11\rangle$.

We can get all four components by Hadamarding twice:



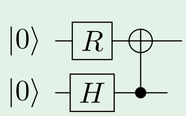
$$\rightarrow \quad \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}$$

Now we want the $\sqrt{2}$'s. Let's use a gate $R = \frac{1}{\sqrt{3}}\begin{bmatrix} \sqrt{2} & 1 \\ 1 & -\sqrt{2} \end{bmatrix}$. Check that it is unitary! This has $R\,|0\rangle = \frac{\sqrt{2}|0\rangle + |1\rangle}{\sqrt{3}}$. Let's replace a Hadamard gate with this:



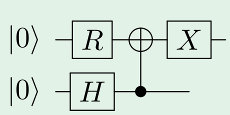
$$\rightarrow \quad \frac{\sqrt{2}\,|00\rangle + \sqrt{2}\,|01\rangle + |10\rangle + |11\rangle}{\sqrt{6}}$$

Now we need to move the $\sqrt{2}$ to the other components. Note that in the finite state the $\sqrt{2}$ makes the qubits more likely to be different, so they are correlated. So we will need a CNOT to correlate them:



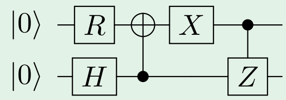
$$\rightarrow \quad \frac{\sqrt{2}\,|00\rangle + \sqrt{2}\,|11\rangle + |10\rangle + |01\rangle}{\sqrt{6}}$$

Now the bits are more likely to be the same, rather than different. No problem: just flip a bit:



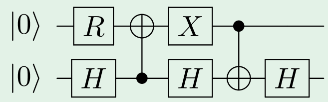
$$\rightarrow \quad \frac{|00\rangle + \sqrt{2}\,|10\rangle + \sqrt{2}\,|01\rangle + |11\rangle}{\sqrt{6}}$$

So close! Now we can flip the sign of $|11\rangle$ using CSIGN:



$$\rightarrow \quad \frac{|00\rangle + \sqrt{2}\,|10\rangle + \sqrt{2}\,|01\rangle - |11\rangle}{\sqrt{6}}$$

We have the desired state, but we were not allowed to use CSIGN. Remember the identity from the homework, though:



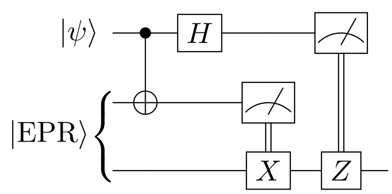
$$\rightarrow \quad \frac{|00\rangle + \sqrt{2}\,|10\rangle + \sqrt{2}\,|01\rangle - |11\rangle}{\sqrt{6}}$$

This might not be the simplest way to get that state, but it illustrates how you might think about how to build a quantum state using a circuit.

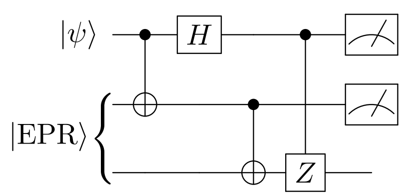
4. Deferred Measurements There is a general principle in quantum circuit design called the "Principle of Deferred Measurement." In a nutshell, it states that in any circuit where we perform intermediate measurements in the middle and condition future operations on the classical results of that measurement we can instead perform a conditional quantum operation and measure only at the end of the circuit.

To see this in action, show that the following two circuits produce the same final qubit at the end of the circuit.

Note: This is good practice for partial measurements.



and



where $|\text{EPR}\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$ and let $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$.

Solution: Let $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$. Then the first circuit evolves our state like so:

$$|\psi\rangle \otimes |\text{EPR}\rangle = \frac{\alpha\,|000\rangle + \alpha\,|011\rangle + \beta\,|100\rangle + \beta\,|111\rangle}{\sqrt{2}}$$

$$\text{CNOT} \downarrow$$

$$\frac{\alpha\,|000\rangle + \alpha\,|011\rangle + \beta\,|110\rangle + \beta\,|101\rangle}{\sqrt{2}}$$

$$H \downarrow$$

$$\frac{\alpha\,|000\rangle + \alpha\,|100\rangle + \alpha\,|011\rangle + \alpha\,|111\rangle + \beta\,|010\rangle - \beta\,|110\rangle + \beta\,|001\rangle - \beta\,|101\rangle}{2}$$

$$\text{measure} \downarrow$$

measure $|0\rangle$:

$$\frac{\alpha\,|00\rangle+\alpha\,|11\rangle+\beta\,|01\rangle-\beta\,|10\rangle}{\sqrt{2}}$$

classical CNOT $\downarrow$

$$\frac{\alpha\,|00\rangle+\alpha\,|10\rangle+\beta\,|01\rangle-\beta\,|11\rangle}{\sqrt{2}}$$

measure $\downarrow$

measure $|0\rangle$: $\alpha\,|0\rangle+\beta\,|1\rangle$ classical CSIGN $\downarrow$ $\alpha\,|0\rangle+\beta\,|1\rangle$

measure $|1\rangle$: $\alpha\,|0\rangle-\beta\,|1\rangle$ classical CSIGN $\downarrow$ $\alpha\,|0\rangle+\beta\,|1\rangle$

measure $|1\rangle$:

$$\frac{\alpha\,|00\rangle+\alpha\,|11\rangle+\beta\,|01\rangle-\beta\,|10\rangle}{\sqrt{2}}$$

classical CNOT $\downarrow$

$$\frac{\alpha\,|00\rangle+\alpha\,|10\rangle+\beta\,|01\rangle-\beta\,|11\rangle}{\sqrt{2}}$$

measure $\downarrow$

measure $|0\rangle$: $\alpha\,|0\rangle+\beta\,|1\rangle$ classical CSIGN $\downarrow$ $\alpha\,|0\rangle+\beta\,|1\rangle$

measure $|1\rangle$: $\alpha\,|0\rangle-\beta\,|1\rangle$ classical CSIGN $\downarrow$ $\alpha\,|0\rangle+\beta\,|1\rangle$

So no matter what we measure, the third qubit will ultimately end up in the state $|\psi\rangle$. On the other hand, for the second circuit we get:

$$|\psi\rangle\otimes|\mathrm{EPR}\rangle=\frac{\alpha\,|000\rangle+\alpha\,|011\rangle+\beta\,|100\rangle+\beta\,|111\rangle}{\sqrt{2}}$$

$$\text{CNOT}\downarrow$$

$$\frac{\alpha\,|000\rangle+\alpha\,|011\rangle+\beta\,|110\rangle+\beta\,|101\rangle}{\sqrt{2}}$$

$$H\downarrow$$

$$\frac{\alpha\,|000\rangle+\alpha\,|100\rangle+\alpha\,|011\rangle+\alpha\,|111\rangle+\beta\,|010\rangle-\beta\,|110\rangle+\beta\,|001\rangle-\beta\,|101\rangle}{2}$$

$$\text{CNOT}\downarrow$$

$$\frac{\alpha\,|000\rangle+\alpha\,|100\rangle+\alpha\,|010\rangle+\alpha\,|110\rangle+\beta\,|011\rangle-\beta\,|111\rangle+\beta\,|001\rangle-\beta\,|101\rangle}{2}$$

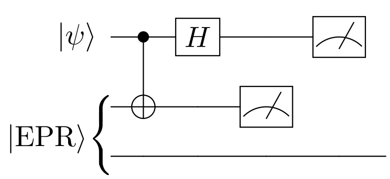
$$\text{CSIGN}\downarrow$$

$$\frac{\alpha\,|000\rangle+\alpha\,|100\rangle+\alpha\,|010\rangle+\alpha\,|110\rangle+\beta\,|011\rangle+\beta\,|111\rangle+\beta\,|001\rangle+\beta\,|101\rangle}{2}$$

$$=\frac{|01\rangle+|11\rangle+|00\rangle+|10\rangle}{2}\otimes|\psi\rangle$$

Once again, irregardless of the measurement result for the first two qubits, the third will be in the state $|\psi\rangle$. Also, it's easy to check that for both circuits all four measurement outcomes were equally likely.

5. Partial measurements in other bases Consider the following circuit:



Let $|\psi\rangle = |+\rangle$. Suppose that the measurements are not made in the standard $|0\rangle\,/\,|1\rangle$ basis, but in the $|+\rangle\,/\,|-\rangle$ basis.

a) What are the probabilities of observing $|+\rangle\,/\,|-\rangle$ on the middle qubit? And on the top qubit?

Note: If you find the math to solve this problem clunky, that's because it is. We actually are not teaching you the standard way to make partial measurements, which requires reduced density matrices and the partial trace. Instead, we rely on algebraic techniques and a small bit of hand-waving which relies on the intuition you should be building.

Solution: Borrowing from the previous problem's solution and plugging in the definition of $|\psi\rangle$, the state of the system before the measurements is

$$\frac{|000\rangle + |100\rangle + |011\rangle + |111\rangle + |010\rangle - |110\rangle + |001\rangle - |101\rangle}{\sqrt{2}}.$$

Because there are no further interactions between the top and middle qubits, the order in which we measure does not matter. So just by personal choice, begin by measuring the top qubit. Rewrite the state as

$$\begin{aligned} |\phi\rangle := & \frac{|0\rangle \otimes (|00\rangle + |11\rangle) + |1\rangle \otimes (|00\rangle + |11\rangle) + |0\rangle \otimes (|10\rangle + |01\rangle) - |1\rangle \otimes (|10\rangle + |01\rangle)}{\sqrt{2}} \\ & \frac{|+\rangle \otimes (|00\rangle + |11\rangle) + |-\rangle \otimes (|10\rangle + |01\rangle)}{\sqrt{2}} \\ & \frac{|+\rangle \otimes (|00\rangle + |11\rangle)}{\sqrt{2}} + \frac{|-\rangle \otimes (|10\rangle + |01\rangle)}{\sqrt{2}} \end{aligned}$$

From this, it is clear the probability of observing $|+\rangle$ on the first qubit is $1/2$, and similarly the probability of $|-\rangle$ is $1/2$.

Now to measure the middle qubit. Because we do not know with certainty what the result of the measurement on the first qubit is, we proceed by cases. If the first measurement observes $|+\rangle$, then the state of the system is the EPR state. As we've seen in class, the probability of observing $|+\rangle$ or $|-\rangle$ on either qubit of an EPR pair is 50%, and this could be verified by taking the inner product $|\langle +|\text{EPR}\rangle|^2$. If the first measurement observes $|-\rangle$, then the state of the system is $|\psi^+\rangle := \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$ (another one of the "Bell States"). If the answer is not yet clear, we can also rewrite the state as

$$|\psi^+\rangle = \frac{1}{2}(|+\rangle + |-\rangle) \otimes |0\rangle + \frac{1}{2}(|+\rangle - |-\rangle) \otimes |1\rangle .$$

From this, it is clear that the probability of observing $|+\rangle$ is again 50%. Since the middle qubit has a 50% chance of $|+\rangle$ in either case after measuring the first qubit, we conclude the middle qubit has a 50% chance overall of $|+\rangle$, and similarly 50% chance of $|-\rangle$.

b) If the top and middle qubits are observed to be in state $|+-\rangle$, then what state is the bottom qubit in?

Solution: In the previous solution, we already found that if the first qubit measured as $|+\rangle$, then the remaining two qubits are in the EPR state. Similarly to what we did in the previous solution, we rewrite the EPR state as

$$\frac{|{++}\rangle + |{--}\rangle}{\sqrt{2}},$$

which you can verify for yourself. From this, it is clear that if $|-\rangle$ is observed on the middle qubit, then the corresponding state of the third qubit is $|-\rangle$.

Note that whenever we write the remnant state, it should be normalized. For example, one might be tempted to have answered $\frac{|-\rangle}{\sqrt{2}}$ above, but that would not be a normalized state.