Introduction to Quantum Information Science
Homework 5

Due Wednesday, October 6 at 11:59 PM

1. Kinda-dense coding We said in class that superdense quantum coding requires 1 ebit of entanglement between Alice and Bob, in addition to 1 qubit of communication. In this problem, however, we'll see how to do a "poor man's" dense quantum coding with no entanglement, just 1 qubit of communication from Alice to Bob.

Suppose Alice knows two bits, x and y. She'd like to let Bob learn either bit of his choice, x or y, though not necessarily both of them (and she doesn't know which Bob is interested in).

a) [6 Points] Suppose Alice is able to send 1 qubit to Bob, and nothing else. Describe a protocol that lets Bob learn the bit of his choice with probability $\cos^2(\pi/8) \approx 85\%$. You may assume Alice and Bob can perform any quantum gate/apply any unitary to qubits in their possession that you want. Give an analysis or proof that it achieves this success probability.
Hint: You might find the following states useful:

$$\cos(\pi/8)\left|0\right\rangle + \sin(\pi/8)\left|1\right\rangle, \quad \sin(\pi/8)\left|0\right\rangle + \cos(\pi/8)\left|1\right\rangle$$
$$\cos(\pi/8)\left|0\right\rangle - \sin(\pi/8)\left|1\right\rangle, \quad \sin(\pi/8)\left|0\right\rangle - \cos(\pi/8)\left|1\right\rangle$$

Solution: Alice sends the following states:

$$\begin{aligned} xy = 00 &\quad \to \quad \cos(\pi/8)\left|0\right\rangle + \sin(\pi/8)\left|1\right\rangle \\ xy = 01 &\quad \to \quad \cos(\pi/8)\left|0\right\rangle - \sin(\pi/8)\left|1\right\rangle \\ xy = 10 &\quad \to \quad \sin(\pi/8)\left|0\right\rangle + \cos(\pi/8)\left|1\right\rangle \\ xy = 11 &\quad \to \quad \sin(\pi/8)\left|0\right\rangle - \cos(\pi/8)\left|1\right\rangle \end{aligned}$$

If Bob wants to know x, he measures in the $\{\left|0\right\rangle, \left|1\right\rangle\}$ basis. He gets the cosine term with probability $\cos^2(\pi/8) \approx 0.85$, which corresponds to x.

If Bob wants to know y he measures in the $\{\left|+\right\rangle, \left|-\right\rangle\}$ basis. Write a general state as:

$$xy \quad \to \quad \left|\psi\right\rangle = \cos(\pi/8)\left|x\right\rangle + (-1)^y \sin(\pi/8)\left|1-x\right\rangle$$
$$\left|\left\langle +|\psi\right\rangle\right|^2 = \left|\cos(\pi/8)\left\langle +|x\right\rangle + (-1)^y \sin(\pi/8)\left\langle +|1-x\right\rangle\right|^2 = \left|\frac{\cos(\pi/8) + (-1)^y \sin(\pi/8)}{\sqrt{2}}\right|^2$$

If $y = 0$ then this is ≈ 0.85. If $y = 1$ then this is ≈ 0.15, meaning the probability of measuring $\left|-\right\rangle$ is ≈ 0.85.

b) [3 Points] Now suppose Alice can no longer send a qubit and she is limited to 1 bit of classical communication only. And suppose that now the bits x and y are uniformly random and independent of each other. Describe a protocol where Alice sends one classical bit to Bob and Bob learns the bit of his choice with 75% chance of success.
In other words, describe a protocol where Alice sends a classical bit to Bob, then Bob decides whether he would like to learn x or y (Bob's choice is his own: you cannot assume it's uniformly random), he performs some series of steps which results in some designated output bit, and there is a 75% chance that this bit matches the value of the bit he wanted.

Solution: Alice tosses a coin. If she gets heads she sends x, and y otherwise. Bob then just assumes that the bit Alice sent was the one that he wanted. With a 50% chance, it actually is the one he wants. Otherwise, it has a 50% chance of matching the bit he wanted by chance. Thus the success probability is $0.5 + 0.5 \cdot 0.5 = 3/4$.

2. Non-local operations [3 Points] Suppose Alice and Bob hold one qubit each of an arbitrary two-qubit state $|\psi\rangle$ that is possibly entangled. They can apply local operations (i.e. apply gates to any qubits they possess) and are allowed to classically communicate with each other. Their goal is to apply the CNOT gate to their state $|\psi\rangle$. Describe a protocol they can use to achieve this given two ebits of entanglement.
(Recall 1 ebit = 1 EPR/Bell pair, one half controlled by Alice, the other controlled by Bob. Thus, 2 ebits = 2 EPR/Bell pairs shared by Alice and Bob.)

Solution: Alice and Bob teleport Alice's bit to Bob using one ebit. Bob applies the CNOT. Then they use the second ebit to teleport it back.

3. The GHZ Game In the "GHZ game", there are three players, Alice, Bob, and Charlie, who are given bits x, y, and z respectively. We're promised that $x + y + z = 0 \pmod 2$; otherwise the bits can be arbitrary. The players' goal is, without communicating with each other, to output bits a, b, c respectively such that $a + b + c = \mathrm{OR}(x, y, z) \pmod 2$. In other words, they should collectively output an odd number of 1-bits if and only if at least one of the input bits is 1.

a) [2 Points] Show that, in a classical universe, there is no strategy that causes the players to succeed for all four possible allowed inputs (x, y, z) with certainty.

Solution: The following table gives the possible inputs and win conditions for the GHZ game:

x	y	z	$a \oplus b \oplus c$
0	0	0	0
0	1	1	1
1	0	1	1
1	1	0	1

This corresponds to the following system of equations:

$$
\begin{aligned}
a_0 \oplus b_0 \oplus c_0 &= 0 \\
a_0 \oplus b_1 \oplus c_1 &= 1 \\
a_1 \oplus b_0 \oplus c_1 &= 1 \\
a_1 \oplus b_1 \oplus c_0 &= 1
\end{aligned}
$$

Where the subscripts refer to the input bit assignments for x, y, z and so on. If you add all four of the above equations you'll get the contradiction $1 = 0$ and so there is no consistent strategy that always wins classically.

b) [6 Points] Now suppose that the players share the state:

$$\frac{|000\rangle - |011\rangle - |101\rangle - |110\rangle}{2}$$

Suppose that each player measures their qubit in the $\{|0\rangle, |1\rangle\}$ basis if their input bit is 0, or in the $\{|+\rangle, |-\rangle\}$ basis if their input bit is 1, and that they output 0/1 based on what they see (the $|+\rangle$ state means they should output 0). Show that this lets the players win the GHZ game for all four possible input triples with certainty.

Solution: Case # 1 $x = y = z = 0$: Win if $a = b = c = 0$ or $a = c = 1, b = 0$ or $a = b = 1, c = 0$ or $a = 0, b = c = 1$.

$$\begin{aligned}
\langle 000|\psi\rangle &= \frac{1}{2} \\
\langle 011|\psi\rangle &= -\frac{1}{2} \\
\langle 101|\psi\rangle &= -\frac{1}{2} \\
\langle 110|\psi\rangle &= -\frac{1}{2} \\
\text{Win Prob} &= 1
\end{aligned}$$

Case # 2 $x = y = 1, z = 0$: Win if $a = 1, b = c = 0$ or $a = 0, b = 1, c = 1$ or $a = b = 0, c = 1$ or $a = b = c = 1$.

$$\begin{aligned}
\langle 1++|\psi\rangle &= -\frac{1}{2} \\
\langle 0-+|\psi\rangle &= \frac{1}{2} \\
\langle 0+-|\psi\rangle &= \frac{1}{2} \\
\langle 1--|\psi\rangle &= \frac{1}{2} \\
\text{Win Prob} &= 1
\end{aligned}$$

Case #3 $x = 1, y = 0, z = 1$: See case # 2 for win conditions.

$$\begin{aligned}
\langle -0+|\psi\rangle &= \frac{1}{2} \\
\langle +1+|\psi\rangle &= -\frac{1}{2} \\
\langle +0-|\psi\rangle &= \frac{1}{2} \\
\langle -1-|\psi\rangle &= \frac{1}{2}
\end{aligned}$$

$$\text{Win Prob} = 1$$

Case # 4 $x = y = 1, z = 0$: Win conditions are the same as for case # 2.

$$\langle -+0|\psi\rangle = \frac{1}{2}$$
$$\langle +-0|\psi\rangle = \frac{1}{2}$$
$$\langle ++1|\psi\rangle = -\frac{1}{2}$$
$$\langle --1|\psi\rangle = \frac{1}{2}$$
$$\text{Win Prob} = 1$$

c) [Extra Credit, 2 Points] Design a protocol where the players instead share the so-called GHZ state

$$\frac{|000\rangle + |111\rangle}{\sqrt{2}}$$

and still win with certainty. No communication between players is allowed.

Solution: We can do this by first noting that $|\psi\rangle$ can be written in the $\{i, -i\}$ basis as:

$$|\psi\rangle = \frac{|iii\rangle + |-i-i-i\rangle}{\sqrt{2}}$$

We then just need to do a local transformation of the GHZ state to the starting state of our original protocol and proceed normally. The following operator applied to each qubit works:

$$V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$$

4. Noisy CHSH [10 Points] Suppose Alice and Bob share a Bell pair $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Imagine that unbeknownst to Alice and Bob, their qubits are not completely isolated from the outside world: with probability ϵ, one of the qubits is measured in the $\{|0\rangle, |1\rangle\}$ basis by the "environment" and the state of their pair collapses to either the state $|00\rangle$ or $|11\rangle$ (with probability $1-\epsilon$, the qubits remain in the Bell state).

Work out an expression for the probability with which Alice and Bob win the CHSH game using this noisy Bell pair assuming they follow the usual strategy (the one reviewed below). Show your work.

How large does ϵ need to be before Alice and Bob's success probability with this strategy is no better than is possible with a classical strategy (with no Bell pair at all)?

Make sure you answer both questions/parts!

Recall: In the CHSH game, Alice and Bob receive independent random bits x and y respectively. Their goal is to output bits a and b respectively such that $a + b = xy \pmod 2$. No communication is allowed. In the "usual strategy", Alice does nothing to her qubit if $x = 0$ and she applies a $\frac{\pi}{4}$ counterclockwise rotation towards $|1\rangle$ if $x = 1$. Bob applies a $\frac{\pi}{8}$ counterclockwise rotation if $y = 0$ and he applies a $\frac{\pi}{8}$ *clockwise* rotation, toward $-|1\rangle$ if $y = 1$. Alice and Bob both measure their qubits in the $\{|0\rangle, |1\rangle\}$ basis and output whatever they see. This strategy wins $\cos^2\left(\frac{\pi}{8}\right) \approx 85\%$ of the time, while any classical strategy can win with probability at most 3/4.

Solution: Suppose the state collapses to $|00\rangle$. There's 4 cases here given by the 4 possible input pairs Alice and Bob can receive:

1. If $(a,b)=(0,0)$, Alice measures in $\{|0\rangle,|1\rangle\}$ and Bob measures in $\{\left|\frac{\pi}{8}\right\rangle,\left|\frac{5\pi}{8}\right\rangle\}$. They win if the output bits are $(0,0)$ or $(1,1)$, which happens if the measurement outcomes are $|0\rangle\left|\frac{\pi}{8}\right\rangle$ or $|1\rangle\left|\frac{5\pi}{8}\right\rangle$. This occurs with probability $\cos^2(\frac{\pi}{8})+0=\cos^2(\frac{\pi}{8})$.

2. If $(a,b)=(0,1)$, Alice measures in $\{|0\rangle,|1\rangle\}$ and Bob measures in $\{\left|\frac{-\pi}{8}\right\rangle,\left|\frac{3\pi}{8}\right\rangle\}$. They win if the output bits are $(0,0)$ or $(1,1)$, which happens if the measurement outcomes are $|0\rangle\left|-\frac{\pi}{8}\right\rangle$ or $|1\rangle\left|\frac{3\pi}{8}\right\rangle$. This occurs with probability $\cos^2(\frac{\pi}{8})+0=\cos^2(\frac{\pi}{8})$.

3. If $(a,b)=(1,0)$, Alice measures in $\{|+\rangle,|-\rangle\}$ and Bob measures in $\{\left|\frac{\pi}{8}\right\rangle,\left|\frac{5\pi}{8}\right\rangle\}$. They win if the output bits are $(0,0)$ or $(1,1)$, which happens if the measurement outcomes are $|+\rangle\left|\frac{\pi}{8}\right\rangle$ or $|-\rangle\left|\frac{5\pi}{8}\right\rangle$. This occurs with probability $\frac{1}{2}\cos^2(\frac{\pi}{8})+\frac{1}{2}\cos^2(\frac{3\pi}{8})$.

4. If $(a,b)=(1,1)$, Alice measures in $\{|+\rangle,|-\rangle\}$ and Bob measures in $\{\left|-\frac{\pi}{8}\right\rangle,\left|\frac{3\pi}{8}\right\rangle\}$. They win if the output bits are $(0,1)$ or $(1,0)$, which happens if the measurement outcomes are $|+\rangle\left|\frac{3\pi}{8}\right\rangle$ or $|-\rangle\left|-\frac{\pi}{8}\right\rangle$. This occurs with probability $\frac{1}{2}\cos^2(\frac{3\pi}{8})+\frac{1}{2}\cos^2(\frac{\pi}{8})$.

Since the bits (a,b) are uniformly random, the total probability that Alice and Bob win the game in this case is:

$$\frac{3\cos^2(\frac{\pi}{8})+\cos^2(\frac{3\pi}{8})}{4}$$

Suppose instead that the state collapses to $|11\rangle$. We can go through the same cases:

1. If $(a,b)=(0,0)$, Alice measures in $\{|0\rangle,|1\rangle\}$ and Bob measures in $\{\left|\frac{\pi}{8}\right\rangle,\left|\frac{5\pi}{8}\right\rangle\}$. They win if the output bits are $(0,0)$ or $(1,1)$, which happens if the measurement outcomes are $|0\rangle\left|\frac{\pi}{8}\right\rangle$ or $|1\rangle\left|\frac{5\pi}{8}\right\rangle$. This occurs with probability $0+\cos^2(\frac{\pi}{8})=\cos^2(\frac{\pi}{8})$.

2. If $(a,b)=(0,1)$, Alice measures in $\{|0\rangle,|1\rangle\}$ and Bob measures in $\{\left|\frac{-\pi}{8}\right\rangle,\left|\frac{3\pi}{8}\right\rangle\}$. They win if the output bits are $(0,0)$ or $(1,1)$, which happens if the measurement outcomes are $|0\rangle\left|-\frac{\pi}{8}\right\rangle$ or $|1\rangle\left|\frac{3\pi}{8}\right\rangle$. This occurs with probability $0+\cos^2(\frac{\pi}{8})=\cos^2(\frac{\pi}{8})$.

3. If $(a,b)=(1,0)$, Alice measures in $\{|+\rangle,|-\rangle\}$ and Bob measures in $\{\left|\frac{\pi}{8}\right\rangle,\left|\frac{5\pi}{8}\right\rangle\}$. They win if the output bits are $(0,0)$ or $(1,1)$, which happens if the measurement outcomes are $|+\rangle\left|\frac{\pi}{8}\right\rangle$ or $|-\rangle\left|\frac{5\pi}{8}\right\rangle$. This occurs with probability $\frac{1}{2}\cos^2(\frac{3\pi}{8})+\frac{1}{2}\cos^2(\frac{\pi}{8})$.

4. If $(a,b)=(1,1)$, Alice measures in $\{|+\rangle,|-\rangle\}$ and Bob measures in $\{\left|-\frac{\pi}{8}\right\rangle,\left|\frac{3\pi}{8}\right\rangle\}$. They win if the output bits are $(0,1)$ or $(1,0)$, which happens if the measurement outcomes are $|+\rangle\left|\frac{3\pi}{8}\right\rangle$ or $|-\rangle\left|-\frac{\pi}{8}\right\rangle$. This occurs with probability $\frac{1}{2}\cos^2(\frac{\pi}{8})+\frac{1}{2}\cos^2(\frac{3\pi}{8})$.

Since the bits (a,b) are uniformly random, the total probability that Alice and Bob win the game in this case is:

$$\frac{3\cos^2(\frac{\pi}{8})+\cos^2(\frac{3\pi}{8})}{4}$$

, the same as the earlier case.

If the state does not collapse, then we saw in class that they win with probability $\cos^2(\frac{\pi}{8})$. So, overall, the win probability is:

$$\epsilon\cdot\frac{3\cos^2(\frac{\pi}{8})+\cos^2(\frac{3\pi}{8})}{4}+(1-\epsilon)\cdot\cos^2\left(\frac{\pi}{8}\right)$$

$$= \left(1 - \frac{\epsilon}{4}\right) \cos^2\left(\frac{\pi}{8}\right) + \frac{\epsilon}{4} \cos^2\left(\frac{3\pi}{8}\right)$$

Finally, to find how large an ϵ we need before we no longer do any better than the classical case, set the above expression equal to .75 and find that $\epsilon \approx .5858$.