# Introduction to Quantum Information Science
# Recitation 1
# (9/1 and 9/5)

**1. First: a brief lecture about $p$-norms**

**2. Complex Numbers and Amplitude Review**

**a)** Express $|a + ib|$ without writing any norms (i.e. get rid of the $|\dot{|}$) (e.g. $|1 + i| = \sqrt{2}$).

**Solution:**

$$|a + ib| = \sqrt{(a + ib)^*(a + ib)} = \sqrt{(a - ib)(a + ib)} = \sqrt{a^2 - i^2b^2} = \sqrt{a^2 + b^2}$$

**b)** Express $(1 + i\sqrt{3})(1 - i)$ in the form $re^{i\theta}$ for some $r \in \mathbb{R}_{>0}$ and $\theta \in [0, 2\pi)$.

**Solution:**

$$1 + i\sqrt{3} = 2e^{i\pi/3}, \quad \text{and} \quad 1 - i = \sqrt{2}e^{-i\pi/4}$$

Hence,

$$(1 + i\sqrt{3})(1 - i) = 2e^{i\pi/3}\sqrt{2}e^{-i\pi/4} = 2\sqrt{2}e^{i\pi/12}$$

**c)** Compute $\sqrt{i}$ and $i^i$, writing your answer in terms of $e$.

**Solution:**

$$\sqrt{i} = \left(e^{i\pi/2}\right)^{\frac{1}{2}} = e^{i\pi/4}, \quad i^i = \left(e^{i\pi/2}\right)^i = e^{-\pi/2}$$

As a student in class points out, the first answer is technically incomplete. It only gives the principal square root, when there are in fact two solutions to any square root. Technically, we should write, for $k$ any integer,

$$\sqrt{i} = \left(e^{i(\pi/2+2k\pi)}\right)^{\frac{1}{2}} = e^{i(\pi/4+k\pi)}.$$

Then, restricting ourselves to angles between $-\pi$ and $\pi$, we set $k = -1$ and get $e^{-i3\pi/4}$.

**d)** Let

$$u = \begin{bmatrix} i\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} e^{i\frac{\pi}{4}}\frac{3}{5} \\ i\frac{4}{5} \\ 0 \end{bmatrix}.$$

Compute the magnitude of the inner product $|\langle u, v\rangle|$.

**Solution:** Don't forget to take the complex conjugate!

$$|u \cdot v| = \left| -i\frac{1}{\sqrt{2}} \cdot e^{i\frac{\pi}{4}}\frac{3}{5} + 0 \cdot i\frac{4}{5} + \frac{1}{\sqrt{2}} \cdot 0 \right| = \frac{3}{5\sqrt{2}}$$

## 3. Linear Algebra Review

**a)** Compute the eigenvectors and eigenvalues of these three matrices:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

**Solution:**

$$X \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \qquad X \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = - \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$Y \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} \qquad Y \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{bmatrix} = - \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{bmatrix}$$

$$Z \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad Z \begin{bmatrix} 0 \\ 1 \end{bmatrix} = - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

**b)** Let $A \in \mathbb{C}^{n \times n}$ be a matrix such that $A^2 = A$. What are the possible eigenvalues of $A$?

**Solution:** Let $u$ be an eigenvector of $A$ with eigenvalue $\lambda$. We can see that $\lambda u = Au = A^2 u = \lambda^2 u$. This means that $\lambda^2 = \lambda$, which is only possible for $\lambda = 0$ and $\lambda = 1$.

**c)** Let $A \in \mathbb{R}^{n \times n}$ be a matrix such that its rows and columns are orthonormal vectors. Show that $A^\dagger A = AA^\dagger = I$.

**Solution:** Let $a_i$ be the column vectors of $A$. We can see that $\left(A^\dagger A\right)_{ij} = a_i^\dagger a_j = \delta_{ij}$ where $\delta_{ij}$ is the Kronecker delta (takes value 0 unless $i = j$, in which case it takes value 1), because the column vectors are orthonormal. This matches perfectly with the identity matrix, so $A^\dagger A = I$. To prove $AA^\dagger = I$ we do the same things with the row vectors of $A$.

## 4. Tensor Products

**a)** Compute

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

**Solution:**

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

**b)** Let $A_{ij}$ mean the element in the $i$-th row and the $j$-th column of $A$. What does $A \otimes B$ look like abstractly?

**Solution:**

$$\begin{bmatrix} A_{11}B & A_{12}B & A_{13}B & \cdots \\ A_{21}B & A_{22}B & A_{23}B & \cdots \\ A_{31}B & A_{32}B & A_{33}B & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

**c)** Show that $(A \otimes B)(u \otimes v) = (Au) \otimes (Bv)$.

**Solution:** Let us first see that for a matrix $A$ and vector $v$, $(Au)_i = \sum_j A_{ij}u_j$. This leads us to

$$\begin{aligned} (A \otimes B)(u \otimes v) &= \begin{bmatrix} A_{11}B & A_{12}B & A_{13}B & \cdots \\ A_{21}B & A_{22}B & A_{23}B & \cdots \\ A_{31}B & A_{32}B & A_{33}B & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} u_1 v \\ u_2 v \\ \vdots \end{bmatrix} \\ &= \begin{bmatrix} \left(\sum_j A_{1j}u_j\right) Bv \\ \left(\sum_j A_{2j}u_j\right) Bv \\ \vdots \end{bmatrix} \\ &= (Au) \otimes (Bv). \end{aligned}$$

## 5. Measurement

**a)** We're given a qubit in the state

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle .$$

What is the probability of seeing the outcome $|0\rangle$ when measuring in the $\{|0\rangle, |1\rangle\}$-basis?

**Solution:** By the Born rule this is just the absolute value squared of the corresponding amplitude: $|\frac{\sqrt{3}}{2}|^2 = \frac{3}{4}$.

**b)** What does $|\psi\rangle$ look like when written in the $\{|+\rangle, |-\rangle\}$-basis? And what is the probability of observing $|+\rangle$ if we were to measure in this basis instead?

**Solution:** It's a basic fact of linear algebra that for any orthonormal basis $\{|b_i\rangle\}$ we have

$$I = \sum_i |b_i\rangle\langle b_i| .$$

Hence,

$$\begin{aligned}
|\psi\rangle &= \big(|+\rangle\langle+| + |-\rangle\langle-|\big)\,|\psi\rangle \\
&= \langle+|\psi\rangle\,|+\rangle + \langle-|\psi\rangle\,|-\rangle \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} |+\rangle + \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} |-\rangle \\
&\approx 0.97\,|+\rangle + 0.26\,|-\rangle\,.
\end{aligned}$$

For the probability of observing $|+\rangle$ we can once again use the Borne rule. If computed without intermediate rounding this yields 0.93... Note that a shortcut to this answer would've been to only compute $|\langle+|\psi\rangle|^2$.

**c)** What is the probability of making both the previous observations when consecutively measuring the *same* qubit? (The probability of seeing $|0\rangle$ then $|+\rangle$ when first measuring in the $\{|0\rangle\,, |1\rangle\}$ then in the $\{|+\rangle\,, |-\rangle\}$-basis)

**Solution:** We already saw in a) that there's a $\frac{3}{4}$ chance to initially observe a $|0\rangle$. But once that happens the qubit will "snap" to $|0\rangle$. So the probability to then observe $|+\rangle$ becomes $|\langle+|0\rangle|^2 = \frac{1}{2}$. Overall that gives us a $\frac{3}{8}$ chance for the entire sequence of events.