Introduction to Quantum Information Science
Recitation, week 4

Multi-qubit circuits and the Principle of Deferred Measurement

1. The Union Bound

a) Let A and B be random events. Prove the union bound: $\Pr[A \cup B] \leq \Pr[A] + \Pr[B]$. In words, the probability that A and B both occur is at most the sum of the probabilities of either one occurring.

Why are we reviewing this? Because the union bound is one of the most useful inequalities you can learn, and you'll use it when writing proofs in this course. Please consider getting it as a tattoo.

Solution: By the definition of probability we have

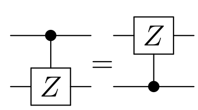
1. $\Pr[A] \geq 0$ for any event A.
2. $\Pr[\Omega] = 1$ where Ω is the entire probability space.
3. $\Pr[\cup_i A_i] = \sum_i \Pr[A_i]$ for any countable set of disjoint events $\{A_i\}$.

Therefore,

$$\begin{aligned} \Pr[A \cup B] &= \Pr[A \cup (B - A)] \\ &= \Pr[A] + \Pr[B - A] \\ &\leq \Pr[A] + \Pr[B - A] + \Pr[A \cap B] \\ &= \Pr[A] + \Pr[(B - A) \cup (A \cap B)] \\ &= \Pr[A] + \Pr[B]. \end{aligned}$$

2. Multi-Qubit Circuit Practice

a) Show that the following circuit identity holds:



(1)

Solution: **Method 1.** The action of a unitary is completely defined by its action on a basis. So, if we can show that the circuit on the left acts on $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ all the same as the circuit on the right, then we're right.

The circuits both act like identity unless the state is $|11\rangle$. If there is any 0, then either the control is 0, or the Z gate does nothing. Both circuits map $|11\rangle$ to $-|11\rangle$.

Method 2. If we're comfortable with outer product notation, we can write the action of the controlled gate using outer products. The left hand side is

$$|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes Z = \begin{bmatrix} I & 0 \\ 0 & Z \end{bmatrix}$$

and the right hand side is

$$\begin{gathered} I \otimes |0\rangle\langle 0| + Z \otimes |1\rangle\langle 1| = \\ (|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes |0\rangle\langle 0| + (|0\rangle\langle 0| - |1\rangle\langle 1|) \otimes |1\rangle\langle 1| = \\ |00\rangle\langle 00| + |10\rangle\langle 10| + |01\rangle\langle 01| - |11\rangle\langle 11| = \\ (|00\rangle\langle 00| + |01\rangle\langle 01|) + (|10\rangle\langle 10| - |11\rangle\langle 11|) = \\ |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes Z, \end{gathered}$$

so the two circuits match.

b) Prepare a circuit that makes the following state starting from the all-zero state:

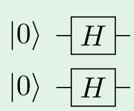
$$\frac{|00\rangle + \sqrt{2}\,|01\rangle + \sqrt{2}\,|10\rangle - |11\rangle}{\sqrt{6}}$$

We are given access to CNOT, any single-qubit gate we've seen in class, and the gate $R = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} & 1 \\ 1 & -\sqrt{2} \end{bmatrix}$.

Solution:

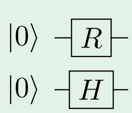
1. Support on all four components.
2. More $|01\rangle$ and $|10\rangle$ than other components.
3. Negative phase on $|11\rangle$.

We can get all four components by Hadamarding twice:



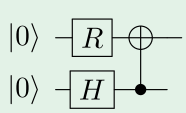
$$\rightarrow \quad \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}$$

Now we want the $\sqrt{2}$'s.. Let's use R. Before proceeding, you should check that it is unitary! This has $R\,|0\rangle = \frac{\sqrt{2}|0\rangle + |1\rangle}{\sqrt{3}}$. Let's replace a Hadamard gate with this:



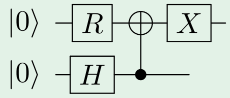
$$\rightarrow \quad \frac{\sqrt{2}\,|00\rangle + \sqrt{2}\,|01\rangle + |10\rangle + |11\rangle}{\sqrt{6}}$$

Now we need to move the $\sqrt{2}$ to the other components. Note that in the desired state the $\sqrt{2}$ makes the qubits more likely to be different, so they are correlated. So we will need a CNOT to correlate them:



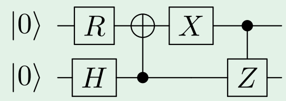
$$\rightarrow \quad \frac{\sqrt{2}\,|00\rangle + \sqrt{2}\,|11\rangle + |10\rangle + |01\rangle}{\sqrt{6}}$$

Now the bits are more likely to be the same, rather than different. No problem: just flip a bit:



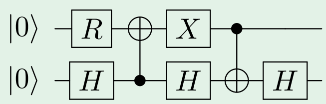
$$\rightarrow \quad \frac{|00\rangle + \sqrt{2}\,|10\rangle + \sqrt{2}\,|01\rangle + |11\rangle}{\sqrt{6}}$$

So close! Now we can flip the sign of $|11\rangle$ using CSIGN:



$$\rightarrow \quad \frac{|00\rangle + \sqrt{2}\,|10\rangle + \sqrt{2}\,|01\rangle - |11\rangle}{\sqrt{6}}$$

We have the desired state, but we were not allowed to use CSIGN. Remember the identity from the homework, though:



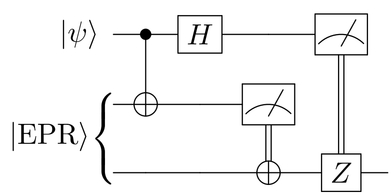
$$\rightarrow \quad \frac{|00\rangle + \sqrt{2}\,|10\rangle + \sqrt{2}\,|01\rangle - |11\rangle}{\sqrt{6}}$$

This might not be the simplest way to get that state, but it illustrates how you might think about how to build a quantum state using a circuit.

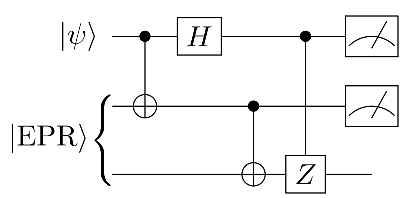
3. Deferred Measurements There is a general principle in quantum circuit design called the "Principle of Deferred Measurement." In a nutshell, it states that in any circuit where we perform intermediate measurements in the middle and condition future operations on the classical results of that measurement we can instead perform a conditional quantum operation and measure only at the end of the circuit.

To see this in action, show that the following two circuits produce the same final qubit at the end of the circuit.

Note: This is good practice for partial measurements.



and



where $|\mathrm{EPR}\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$ and let $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$.

Solution: Let $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$. Then the first circuit evolves our state like so:

$$|\psi\rangle \otimes |\mathrm{EPR}\rangle = \frac{\alpha\,|000\rangle + \alpha\,|011\rangle + \beta\,|100\rangle + \beta\,|111\rangle}{\sqrt{2}}$$

$$\text{CNOT} \downarrow$$

$$\frac{\alpha\,|000\rangle + \alpha\,|011\rangle + \beta\,|110\rangle + \beta\,|101\rangle}{\sqrt{2}}$$

$$H \downarrow$$

$$\frac{\alpha|000\rangle+\alpha|100\rangle+\alpha|011\rangle+\alpha|111\rangle+\beta|010\rangle-\beta|110\rangle+\beta|001\rangle-\beta|101\rangle}{2}$$

measure middle qubit $\downarrow$

measure $\|0\rangle$:	measure $\|1\rangle$:
$\frac{\alpha\|00\rangle+\alpha\|10\rangle+\beta\|01\rangle-\beta\|11\rangle}{\sqrt{2}}$	$\frac{\alpha\|01\rangle+\alpha\|11\rangle+\beta\|00\rangle-\beta\|10\rangle}{\sqrt{2}}$
classical CNOT $\downarrow$	classical CNOT $\downarrow$
$\frac{\alpha\|00\rangle+\alpha\|10\rangle+\beta\|01\rangle-\beta\|11\rangle}{\sqrt{2}}$	$\frac{\alpha\|00\rangle+\alpha\|10\rangle+\beta\|01\rangle-\beta\|11\rangle}{\sqrt{2}}$
measure first qubit $\downarrow$	measure first qubit $\downarrow$

measure $\|0\rangle$:	measure $\|1\rangle$:	measure $\|0\rangle$:	measure $\|1\rangle$:
$\alpha\|0\rangle+\beta\|1\rangle$	$\alpha\|0\rangle-\beta\|1\rangle$	$\alpha\|0\rangle+\beta\|1\rangle$	$\alpha\|0\rangle-\beta\|1\rangle$
classical CSIGN $\downarrow$	classical CSIGN $\downarrow$	classical CSIGN $\downarrow$	classical CSIGN $\downarrow$
$\alpha\|0\rangle+\beta\|1\rangle$	$\alpha\|0\rangle+\beta\|1\rangle$	$\alpha\|0\rangle+\beta\|1\rangle$	$\alpha\|0\rangle+\beta\|1\rangle$

So no matter what we measure, the third qubit will ultimately end up in the state $|\psi\rangle$. On the other hand, for the second circuit we get:

$$|\psi\rangle\otimes|\mathrm{EPR}\rangle=\frac{\alpha|000\rangle+\alpha|011\rangle+\beta|100\rangle+\beta|111\rangle}{\sqrt{2}}$$

$$\text{CNOT}\downarrow$$

$$\frac{\alpha|000\rangle+\alpha|011\rangle+\beta|110\rangle+\beta|101\rangle}{\sqrt{2}}$$

$$H\downarrow$$

$$\frac{\alpha|000\rangle+\alpha|100\rangle+\alpha|011\rangle+\alpha|111\rangle+\beta|010\rangle-\beta|110\rangle+\beta|001\rangle-\beta|101\rangle}{2}$$

$$\text{CNOT}\downarrow$$

$$\frac{\alpha|000\rangle+\alpha|100\rangle+\alpha|010\rangle+\alpha|110\rangle+\beta|011\rangle-\beta|111\rangle+\beta|001\rangle-\beta|101\rangle}{2}$$

$$\text{CSIGN}\downarrow$$

$$\frac{\alpha|000\rangle+\alpha|100\rangle+\alpha|010\rangle+\alpha|110\rangle+\beta|011\rangle+\beta|111\rangle+\beta|001\rangle+\beta|101\rangle}{2}$$

$$=\frac{|01\rangle+|11\rangle+|00\rangle+|10\rangle}{2}\otimes|\psi\rangle$$

Once again, regardless of the measurement result for the first two qubits, the third will be in the state $|\psi\rangle$. Also, it's easy to check that for both circuits all four measurement outcomes were equally likely.