Introduction to Quantum Information Science
Homework 1

Due Wednesday, September 8th at 11:59 PM

1. Stochastic and Unitary Matrices.

a) [8 Points] Of the following matrices, which ones are stochastic? Which ones are unitary?

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ C = \begin{bmatrix} 1 & \frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix},\ D = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix},$$
$$E = \begin{bmatrix} 2 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix},\ F = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},\ G = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix},\ H = \begin{bmatrix} \frac{3i}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3i}{5} \end{bmatrix}$$

Solution:

A is **not stochastic** because column 2 does not sum to 1. A is **not unitary** since it has determinant 0, so it is not invertible.

B is **stochastic** because both columns sum to 1. B is **unitary** because $B = B^\dagger$ and $B^2 = I$.

C is **stochastic** because both columns sum to 1. C is **not unitary** because:
$$CC^\dagger = \begin{bmatrix} 1 & \frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 13 & 2 \\ 2 & 1 \end{bmatrix} \neq I.$$

D is **not stochastic** because it is not real. D is **unitary** because $DD^\dagger = \text{diag}(1, i)\text{diag}(1, -i) = \text{diag}(1, i(-i)) = \text{diag}(1, 1) = I$.

E is **not stochastic** because it is not positive. E is **not unitary** because:
$$EE^\dagger = \begin{bmatrix} 2 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 2 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 17 & -7 \\ -7 & 5 \end{bmatrix} \neq I.$$

F is **not stochastic** because it is not positive. F is **unitary** because:
$$FF^\dagger = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = I.$$

G is **not stochastic** because it is not positive. G is **unitary** because:
$$GG^\dagger = \frac{1}{25}\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix} = I.$$

H is **not stochastic** because it is not real. H is **not unitary** because:
$$HH^\dagger = \frac{1}{25}\begin{bmatrix} 3i & 4 \\ 4 & -3i \end{bmatrix}\begin{bmatrix} -3i & 4 \\ 4 & 3i \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 25 & 24i \\ -24i & 25 \end{bmatrix} \neq I.$$

b) [3 Points] Show that any stochastic matrix that's also unitary must be a permutation matrix.

Solution: In a stochastic matrix every column has 1-norm of 1 with non-negative real entries. In a unitary matrix every column and every row has a 2-norm of 1. Since each column has 1-norm of 1, all elements must be ≤ 1. But since if any element of the matrix a is strictly less than 1, then $a^2 < a$, so the 2-norm must be strictly smaller than the 1-norm. This is a contradiction because both norms are equal to 1.

So the matrix must be an 0-1 matrix. Since every column and every row has 2-norm of 1, each column and row can only contain a single 1. Thus the matrix is a permutation matrix.

c) [1 Point] Stochastic matrices preserve the 1-norms of nonnegative vectors, while unitary matrices preserve 2-norms. Give an example of a 2×2 matrix, other than the identity matrix, that the preserves 4-norm of real vectors $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$: that is, $a^4 + b^4$.

Solution: $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ changes the norm to $b^4 + a^4$ which is the same thing.

d) [Extra credit, 4 Points] Give a characterization of all real matrices that preserve the 4-*norms* of real vectors. Hopefully, your characterization will help explain why preserving the 2-norm, as quantum mechanics does, leads to a much richer set of transformations than preserving the 4-norm does.

Solution: There are multiple ways to solve this problem. We present two solutions here, the first based on an algebraic expansion, and the second based on geometric intuition that works with complex matrices.

(First solution) For a matrix A_{ij} and a vector ψ we demand that $\sum_i \psi_i^4 = \sum_r \left(\sum_c A_{rc}\psi_c\right)^4$ (we can drop the roots and absolute values because everything is real). Regard this as a polynomial in ψ_i and equate the coefficients. In particular, the coefficient of ψ_i^4 is 1 on the left hand side for all i, so it must also be 1 on the right hand side, while all other coefficients are zero.

We will find it useful to consider terms that look like $\left(\sum_r A_{ri}^2 A_{rj}^2\right)\psi_i^2\psi_j^2$. When $i \neq j$, then the coefficient $\sum_r A_{ri}^2 A_{rj}^2 = 0$. Since we are dealing with real numbers, $A_{ri}^2 > 0$ and $A_{rj}^2 > 0$, so the only way for $\sum_r A_{ri}^2 A_{rj}^2 = 0$ to be true is if for all r, $A_{ri}^2 A_{rj}^2 = 0$ and so $A_{ri}A_{rj} = 0$ when $i \neq j$. It is not hard to show that for a row r, if two or more values of i have $A_{ri} \neq 0$ then there must exist i and j such that $A_{ri}A_{rj} \neq 0$. The contrapositive states that no more than one entry in a row of the matrix can be nonzero. A further consequence of this is that there are at most n nonzero entries, since there is at most one for every row.

Since we must preserve the norm of all real vectors, our matrix must be full rank. Otherwise, vectors in the null space would be sent to the zero vector by definition. More importantly, every row and every column must have at least one non-zero element. From above, we only have at most n nonzero entries to go around, which means every row and every column must have exactly one nonzero entry now.

Finally, to preserve the 4-norm of the basis vectors, each of these nonzero entries must have absolute value 1, i.e. 1 or -1. The end result is that A must be a signed permutation matrix.

With the 2-norm this trick is not possible because the expansion of the right hand side does not give squares and thus non-negative values in the summation. Since we are no longer forcing a sum of non-negative terms to be equal to zero, we do not get constraints on each row like before.

(Second solution) Let U be a matrix that preserves the 4-norm. We claim the following:

1. For any vector x with a single nonzero entry, we must have $||Ux||_2 \geq ||x||_2$ because among all vectors of unit 4-norm, the standard basis vectors minimize the 2-norm.

2. For any vector y in which all of the entries are $\pm c$ for some $c \in \mathbb{R}$, we must have $||Uy||_2 \leq ||y||_2$ because among all vectors of unit 4-norm, the vectors with entries equal in magniture maximize the 2-norm.

Both of these follow from standard Lagrange multiplier arguments.

Fix $c = 1$ so that y is a vector in which the entries are all ± 1. In ket notation, we write:

$$y = \sum_{i=1}^{n} y_i \left|i\right\rangle$$

where $\left|i\right\rangle$ denotes the standard basis vector with 1 in the ith position. Consider what happens when we average $||Uy||_2^2 = \left\langle y\right| U^\dagger U \left|y\right\rangle$ over all vectors y that have this property:

$$\mathop{\mathbb{E}}_{y\in\{-1,1\}^n} \left[||Uy||_2^2\right] = \frac{1}{2^n} \sum_{y\in\{-1,1\}^n} \sum_{i=1}^{n}\sum_{j=1}^{n} y_i y_j \left\langle i\right| U^\dagger U \left|j\right\rangle.$$

Observe that when $i \neq j$, the sum over all y's means that the $\left\langle i\right| U^\dagger U \left|j\right\rangle$ terms are in 1-to-1 correspondence with the $-\left\langle i\right| U^\dagger U \left|j\right\rangle$ terms, and so they cancel. We are left with the $i = j$ terms, and because $y_i^2 = 1$ for all i this simplifies to:

$$\mathop{\mathbb{E}}_{y\in\{-1,1\}^n} \left[||Uy||_2^2\right] = \sum_{i=1}^{n} \left\langle i\right| U^T U \left|i\right\rangle = \sum_{i=1}^{n} ||U \left|i\right\rangle||_2^2 \geq \sum_{i=1}^{n} ||\left|i\right\rangle||_2^2 = n.$$

On the other hand, we know $\mathbb{E}_{y\in\{-1,1\}^n}\left[||Uy||_2^2\right] \leq n$ because $||Uy||_2^2 \leq ||y||_2^2 = n$. So, we conclude that U preserves the 2-norm of every $y \in \{-1,1\}^n$. This in turn implies that $\sum_{i=1}^{n} ||U\left|i\right\rangle||_2^2 = n$ and therefore that U preserves the 2-norm of every standard basis vector. In particular, the columns of U most all be unit vectors in the 2-norm.

Because U is assumed to be 4-norm preserving, the columns of U must *also* be unit vectors in the 4-norm. However, the only unit vectors in both the 2-norm and 4-norm are $e^{i\theta}$ times a standard basis vector. On the other hand, U must be invertible (otherwise it has $||Uv||_4 = ||0||_4 = 0$ for some nonzero v), and so the rows of U must be linearly independent. This immediately implies that every row and column must have exactly one nonzero entry, which is to say that U must be a generalized permutation matrix.

2. Tensor Products

a) [1 Point] Calculate the tensor product

$$\begin{bmatrix}\frac{2}{3}\\ \frac{1}{3}\end{bmatrix} \otimes \begin{bmatrix}\frac{1}{5}\\ \frac{4}{5}\end{bmatrix}.$$

Solution:

$$\begin{bmatrix}\frac{2}{3}\\ \frac{1}{3}\end{bmatrix} \otimes \begin{bmatrix}\frac{1}{5}\\ \frac{4}{5}\end{bmatrix} = \begin{bmatrix}\frac{2}{3}\begin{bmatrix}\frac{1}{5}\\ \frac{4}{5}\end{bmatrix}\\ \frac{1}{3}\begin{bmatrix}\frac{1}{5}\\ \frac{4}{5}\end{bmatrix}\end{bmatrix} = \begin{bmatrix}\frac{2}{15}\\ \frac{8}{15}\\ \frac{1}{15}\\ \frac{4}{15}\end{bmatrix}$$

b) [5 Points] Of the following length-4 vectors, decide which ones are factorizable as a tensor product of two 2×1 vectors, and factorize them. (Here the vector entries should be thought of as labeled by 00, 01, 10, and 11 respectively.)

$$A = \begin{bmatrix} \frac{2}{9} \\ \frac{1}{9} \\ \frac{4}{9} \\ \frac{2}{9} \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ C = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix},\ D = \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix},\ E = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}.$$

Solution:

A $= \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} \otimes \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$.

B $= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

C $= \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \otimes \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$.

D We have $ac = 0, ad = 1/2, bc = 1/2, cd = 0$. By the first and last equations at least two of a, b, c, d are 0, so at least one of the middle two equations must fail. D is not factorizable.

E $= \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

c) [3 Points] Prove that there's no 2×2 *real* matrix A such that

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

This observation perhaps helps to explain why the complex numbers play such a central role in quantum mechanics.

Solution: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, so $A^2 = \begin{bmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{bmatrix}$. We demand:

$$a^2 + bc = 1, \quad b(a+d) = 0, \quad c(a+d) = 0, \quad d^2 + bc = -1$$

From the middle two we know that at least one of b or $(a+d)$ is 0, and that at least one of c or $(a+d)$ is 0.

If $(a+d) = 0$ then $a = -d$ so $a^2 = d^2$. Thus $-1 = d^2 + bc = a^2 + bc = 1$. Contradiction.

If $(a+d) \neq 0$ then $b = c = 0$, so $bc = 0$. Thus $d^2 = -1$, which has no real solutions.

(Alternative solution) Recall that the determinant is multiplicative: $\det(AB) = \det(A)\det(B)$, so in particular $\det(A^2) = \det(A)^2 = -1$. This implies that $\det(A) = \pm i$, but the determinant of a real matrix is always a real number.

3. Dirac Notation

a) [2 Points] Let $|\psi\rangle = \frac{|0\rangle + 2|1\rangle}{\sqrt{5}}$ and $|\phi\rangle = \frac{2i|0\rangle + 3|1\rangle}{\sqrt{13}}$. What's $\langle\psi|\phi\rangle$?

Solution:

$$\langle\psi|\phi\rangle = \frac{\langle 0| + 2\langle 1|}{\sqrt{5}} \frac{2i|0\rangle + 3|1\rangle}{\sqrt{13}} = \frac{2i\langle 0|0\rangle + 4i\langle 1|0\rangle + 3\langle 0|1\rangle + 2\cdot 3\langle 1|1\rangle}{\sqrt{65}} = \frac{2i+6}{\sqrt{65}}.$$

b) [1 Point] Usually quantum states are normalized: $\langle\psi|\psi\rangle = 1$. The state $|\phi\rangle = 2i|0\rangle - 3i|1\rangle$ is not normalized. What constant A makes $|\psi\rangle = \frac{|\phi\rangle}{A}$ a normalized state?

Solution:

$$\langle\phi|\phi\rangle = (2i)^*(2i) + (-3i)^*(-3i) = 4 + 9 = 13,$$
$$1 = \langle\psi|\psi\rangle = \frac{\langle\phi|\phi\rangle}{A^2} = \frac{13}{A^2} \implies A^2 = 13.$$

Therefore, $A = \sqrt{13}$, up to a phase. Observe that a state's inner product with itself is always real.

c) [2 Points] Define $|i\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$ and $|-i\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}$. Show (explicitly or implicitly) that the vectors $|i\rangle$ and $|-i\rangle$ form an orthonormal basis for $\mathbb{C}^2$.

Solution: For orthonormality, we need to show that $\langle i|-i\rangle = \langle -i|i\rangle = 0$ and $\langle i|i\rangle = \langle -i|-i\rangle = 1$. Remember to take the complex conjugate for the bra:

$$\begin{aligned}
\langle i|-i\rangle &= \left(\frac{\langle 0| - i\langle 1|}{\sqrt{2}}\right)\left(\frac{|0\rangle - i|1\rangle}{\sqrt{2}}\right) = \frac{\langle 0|0\rangle - i\langle 0|1\rangle - i\langle 1|0\rangle - \langle 1|1\rangle}{2} = 0,\\
\langle i|i\rangle &= \left(\frac{\langle 0| - i\langle 1|}{\sqrt{2}}\right)\left(\frac{|0\rangle + i|1\rangle}{\sqrt{2}}\right) = \frac{\langle 0|0\rangle + i\langle 0|1\rangle - i\langle 1|0\rangle + \langle 1|1\rangle}{2} = 1,\\
\langle -i|-i\rangle &= \left(\frac{\langle 0| + i\langle 1|}{\sqrt{2}}\right)\left(\frac{|0\rangle - i|1\rangle}{\sqrt{2}}\right) = \frac{\langle 0|0\rangle - i\langle 0|1\rangle + i\langle 1|0\rangle + \langle 1|1\rangle}{2} = 1.
\end{aligned}$$

It's sufficient here to look at only $\langle i|-i\rangle$ and not $\langle -i|i\rangle$ since $\langle -i|i\rangle = (\langle i|-i\rangle)^\dagger = (\langle i|-i\rangle)^* = 0$.

To show this is a basis, there are two methods. First, in any n-dimensional vector space, a set of n vectors is a basis if and only if it is linearly independent. $\mathbb{C}^2$ is 2-dimensional and this is a set of 2 vectors. Usually, proving linear independence is laborious, but this is a small set, and recall that any set of nonzero vectors which is *pairwise orthogonal* is linearly independent. We've already seen $\langle i|-i\rangle = 0$, so we are done.

Or, second, we can show that any vector $|v\rangle = \left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right] \in \mathbb{C}^2$ can be written as a linear combination of the vectors $|i\rangle$ and $|-i\rangle$:

$$\begin{aligned}
|v\rangle &= \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\\
&= v_1|0\rangle + v_2|1\rangle\\
&= v_1\left(\frac{|i\rangle + |-i\rangle}{\sqrt{2}}\right) + v_2\left(\frac{-i|i\rangle + i|-i\rangle}{\sqrt{2}}\right)\\
&= \left(\frac{v_1 - iv_2}{\sqrt{2}}\right)|i\rangle + \left(\frac{v_1 + iv_2}{\sqrt{2}}\right)|-i\rangle.
\end{aligned}$$

d) [2 Points] Write the normalized vector $|\psi\rangle$ from part (b) in the $\{|i\rangle, |-i\rangle\}$-basis.

Solution: We project $|\psi\rangle$ onto each basis state. Recall $\text{proj}_{|w\rangle}(|v\rangle) = (\langle |v\rangle, |w\rangle\rangle) \cdot |w\rangle = \langle w|v\rangle \cdot |w\rangle$.

$$\alpha = \langle i|\psi\rangle = \frac{2i \langle 0|0\rangle + 3i^2 \langle 1|1\rangle}{\sqrt{2 \cdot 13}} = \frac{2i - 3}{\sqrt{26}},$$

where note we immediately dropped the terms with $\langle 0|1\rangle = 0$ and $\langle 1|0\rangle = 0$ terms.

$$\beta = \langle -i|\psi\rangle = \frac{2i \langle 0|0\rangle - 3i^2 \langle 1|1\rangle}{\sqrt{2 \cdot 13}} = \frac{2i + 3}{\sqrt{26}},$$

$$|\psi\rangle = \frac{(2i - 3)|i\rangle + (2i + 3)|-i\rangle}{\sqrt{26}}.$$

Note that here, you will get the right answer whether you compute $\langle i|\psi\rangle$ or $\langle \psi|i\rangle$, but this is not true in general. Often in quantum computing, we take the magnitude of our inner products which means the order does not matter. But when taking a projection, we refer to the bare inner product, and in complex numbers you can get a different answer when the order is switched.