C S 358H: Intro to Quantum Information Science

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1 Kinda-dense coding

We said in class that superdense quantum coding requires 1 ebit of entanglement between Alice and Bob, in addition to 1 qubit of communication. In this problem, however, we'll see how to do a "poor man's" dense quantum coding with no entanglement, just 1 qubit of communication from Alice to Bob.

Suppose Alice knows two bits, x and y. She'd like to let Bob learn either bit of his choice, x or y, though not necessarily both of them (and she doesn't know which Bob is interested in).

Question 1.1

Question. *Suppose Alice is able to send 1 qubit to Bob, and nothing else. Describe a protocol that lets Bob learn the bit of his choice with probability* $\cos^2(\pi/8) \approx 85\%$. *You may assume Alice and Bob can perform any quantum gate/apply any unitary to qubits in their possession that you want. Give an analysis or proof that it achieves this success probability.*
Hint: You might find the following states useful:

$$\cos(\pi/8)\,|0\rangle + \sin(\pi/8)\,|1\rangle\,, \quad \sin(\pi/8)\,|0\rangle + \cos(\pi/8)\,|1\rangle$$
$$\cos(\pi/8)\,|0\rangle - \sin(\pi/8)\,|1\rangle\,, \quad \sin(\pi/8)\,|0\rangle - \cos(\pi/8)\,|1\rangle$$

Solution. A protocol that lets Bob learn the bit of his choice with probability $\cos^2(\pi/8)$ is

Protocol for Kinda-dense coding

1. **Alice:**
   (a) Prepare her qubit in the state $Z^y X^x R_y(\pi/8)\,|0\rangle$.
   (b) Send the qubit to Bob.
2. **Bob:**
   (a) If Bob wants to learn x, measure the qubit in the standard basis and return the output.
   (b) If Bob wants to learn y, measure the qubit in the Hadamard basis and return the output.

Figure 1: Protocol used by Alice and Bob

The success probability of Bob learning either bit for every possibility of x, y is $\cos^2(\pi/8)$ and therefore, the overall success probability is $\cos^2(\pi/8)$. It is easy to see why this is the case. We enumerate the state Alice sends to Bob for every possibility of x, y:

1. $x = 0, y = 0$: Alice sends $R_y(\pi/8)\,|0\rangle = \cos\pi/8\,|0\rangle + \sin\pi/8\,|1\rangle$. If Bob measures in the standard basis, he will get 0 with probability $\cos^2(\pi/8)$. If Bob measures in the Hadamard basis, he will get 0 with probability $\cos^2(\pi/8)$.

2. $x = 1, y = 0$: Alice sends $XR_y(\pi/8)\,|0\rangle = \sin\pi/8\,|0\rangle + \cos\pi/8\,|1\rangle$. If Bob measures in the standard basis, he will get 1 with probability $\cos^2(\pi/8)$. If Bob measures in the Hadamard basis, he will get 0 with probability $\cos^2(\pi/8)$.

3. $x = 0, y = 1$: Alice sends $ZR_y(\pi/8)\,|0\rangle = \cos\pi/8\,|0\rangle - \sin\pi/8\,|1\rangle$. If Bob measures in

the standard basis, he will get 0 with probability $\cos^2(\pi/8)$. If Bob measures in the Hadamard basis, he will get 1 with probability $\cos^2(\pi/8)$.

4. $x = 1, y = 1$: Alice sends $ZXR_y(\pi/8)\,|0\rangle = \sin\pi/8\,|0\rangle - \cos\pi/8\,|1\rangle$. If Bob measures in the standard basis, he will get 1 with probability $\cos^2(\pi/8)$. If Bob measures in the Hadamard basis, he will get 1 with probability $\cos^2(\pi/8)$.

Hence, proved. □

Question 1.2

Question. *Now suppose Alice can no longer send a qubit and she is limited to 1 bit of classical communication only. And suppose that now the bits x and y are uniformly random and independent of each other. Describe a protocol where Alice sends one classical bit to Bob and Bob learns the bit of his choice with 75% chance of success.*
In other words, describe a protocol where Alice sends a classical bit to Bob, then Bob decides whether he would like to learn x or y (Bob's choice is his own: you cannot assume it's uniformly random), he performs some series of steps which results in some designated output bit, and there is a 75% chance that this bit matches the value of the bit he wanted.

Solution. In the classical case, Alice simply sends $x \times y$ and Bob decodes the input bit as the bit he wants to learn. This protocol succeeds with 75% probability irrespective of Bob's choice because the inputs x and y are uniformly distributed. Without loss of generality, assume Bob wants to learn x. The probability of Bob learning x is 1 if $x = 0$ and 0.5 if $x = 1$ (since y will be 1 only half the time). Therefore, the net probability of Bob learning x is 0.75. The same argument holds for the case when Bob wants to learn y. □

2 Non-local operations

Question 2.1

Question. *Suppose Alice and Bob hold one qubit each of an arbitrary two-qubit state $|\psi\rangle$ that is possibly entangled. They can apply local operations (i.e. apply gates to any qubits they possess) and are allowed to classically communicate with each other. Their goal is to apply the CNOT gate to their state $|\psi\rangle$. Describe a protocol they can use to achieve this given two ebits of entanglement.*
(Recall 1 ebit = 1 EPR/Bell pair, one half controlled by Alice, the other controlled by Bob. Thus, 2 ebits = 2 EPR/Bell pairs shared by Alice and Bob.)

Solution. The protocol that Alice and Bob can use is

Non-Local CNOT Protocol

Assumption: We are given a procedure $\mathsf{teleport}(|\psi\rangle, A, B, |e\rangle) = |\phi\rangle$ *that teleports a qubit* $|\psi\rangle$ *from Alice to Bob using the entangled state* $|e\rangle$ *and the teleported qubit is stored in* $|\phi\rangle$*, the bit of* $|e\rangle$ *located at* B*.*

1. Alice and Bob perform $\mathsf{teleport}(|\psi_A\rangle, \text{Alice}, \text{Bob}, |e_1\rangle) = |\psi'_A\rangle$
2. Bob performs $\mathsf{CNOT}(|\psi'_A\rangle, |\psi_B\rangle)$.
3. Bob and Alice perform $\mathsf{teleport}(|\psi'_A\rangle, \text{Bob}, \text{Alice}, |e_2\rangle) = |\psi''_A\rangle$

Therefore, we have $|\psi''_A\psi_B\rangle = \mathsf{CNOT}(|\psi_A\psi_B\rangle)$.

Figure 2: Protocol used by Alice and Bob to execute a non-local CNOT gate

□

3 The GHZ Game

In the "GHZ game", there are three players, Alice, Bob, and Charlie, who are given bits x, y, and z respectively. We're promised that $x + y + z = 0 \pmod 2$; otherwise the bits can be arbitrary. The players' goal is, without communicating with each other, to output bits a, b, c respectively such that $a + b + c = \mathrm{OR}(x, y, z) \pmod 2$. In other words, they should collectively output an odd number of 1-bits if and only if at least one of the input bits is 1.

Question 3.1

Question. *Show that, in a classical universe, there is no strategy that causes the players to succeed for all four possible allowed inputs (x, y, z) with certainty.*

Proof. We assume a deterministic strategy adopted by every player (which may or may not be the same). Define the bit output by each player Alice, Bob and Charlie on receiving bit i as a_i, b_i, c_i respectively. We assume that this strategy works for the case when two of x, y, z are 1 and the third is 0. Now we show that for any strategy that works for these three cases will definitely fail when all x, y, z are 0.

$$\begin{aligned} a_0 + b_0 + c_0 &= (a_0 + b_1 + c_1) + (a_1 + b_0 + c_1) + (a_1 + b_1 + c_0) \\ &= 1 + 1 + 1 \pmod 2 \text{ (since the strategy works when two of } x, y, z \text{ are 1)} \\ &= 1 \pmod 2 \end{aligned} \tag{1}$$

Therefore, a strategy that works for the cases when two of x, y, z are 1 will always fail when all of x, y, z are 0. This implies that there is no deterministic strategy that works for all four possible allowed inputs (x, y, z) with certainty. $\square$

Question 3.2

Question. *Now suppose that the players share the state:*

$$\frac{|000\rangle - |011\rangle - |101\rangle - |110\rangle}{2}$$

Suppose that each player measures their qubit in the $\{|0\rangle, |1\rangle\}$ basis if their input bit is 0, or in the $\{|+\rangle, |-\rangle\}$ basis if their input bit is 1, and that they output $0/1$ based on what they see (the $|+\rangle$ state means they should output 0). Show that this lets the players win the GHZ game for all four possible input triples with certainty.

Proof. Without loss of generality, we can only consider the cases when all three players get 0's, or when Alice gets a 0 and Bob and Charlie get a 1. We can do this because the state shared by the three players is symmetric in the three bits. Also, since the order of measurements does not matter (the partial trace is unchanged because of measurements), we first measure Alice's state and then consider the state that Bob and Charlie are left with to make our computation easier.

We first consider the case when Alice measures $|0\rangle$. The state that Bob and Charlie are left with is

$$
\begin{aligned}
|BC\rangle &= \frac{|00\rangle - |11\rangle}{\sqrt{2}} \\
&= \frac{|{+}{-}\rangle + |{-}{+}\rangle}{\sqrt{2}}
\end{aligned} \tag{2}
$$

Now, either both Bob and Charlie receive a 0 or both receive a 1. If both receive a 0, they both measure in the $|0\rangle / |1\rangle$ basis and therefore both measure either 0 or 1, which gives $a + b + c = 0$. If both receive a 1, they both measure in the $|+\rangle / |-\rangle$ basis and therefore exactly one of them receives a 1 and the other receives a 0, which gives $a + b + c = 1$. This is what we want in both cases.
Now, if Alice had measured $|1\rangle$, the state that Bob and Charlie are left with is

$$
\begin{aligned}
|BC\rangle &= \frac{|01\rangle + |10\rangle}{\sqrt{2}} \\
&= \frac{|{+}{+}\rangle - |{-}{-}\rangle}{\sqrt{2}}
\end{aligned} \tag{3}
$$

Now, again, either both Bob and Charlie receive a 0 or both receive a 1. If both receive a 0, they both output different bits, which gives $a + b + c = 0$. If both receive a 1, they both output either a 0 or a 1, which gives $a + b + c = 1$. This is again what we want in both cases. Hence, proved. $\square$

Question 3.3

Question. *Design a protocol where the players instead share the so-called GHZ state*

$$
\frac{|000\rangle + |111\rangle}{\sqrt{2}}
$$

and still win with certainty. No communication between players is allowed.

Solution. Again, without loss of generality, we can only consider the cases when all three players get 0's, or when Alice gets a 0 and Bob and Charlie get a 1. We can do this because the state shared by the three players is symmetric in the three bits. Also, since the order of measurements does not matter (the partial trace is unchanged because of measurements), we first measure Alice's state and then consider the state that Bob and Charlie are left with to make our computation easier. However, this requires us to propose the same protocol for all three players.
We propose the following protocol for all players

Protocol used to win the GHZ Game

1. Input bit is 0: Measure in the $|+\rangle / |-\rangle$ basis and output 0 if $|+\rangle$ is observed.
2. Input bit is 1: Apply the Phase gate and then measure in the $|+\rangle / |-\rangle$ basis and output 0 if $|+\rangle$ is observed.

Figure 3: Protocol used by each of Alice, Bob and Charlie that wins the GHZ Game with 100% Probability

We now prove that this strategy works every time. First we consider the case when Alice measures a 0. The state that Bob and Charlie are left with is

$$\begin{aligned}
|ABC\rangle &= \frac{|000\rangle + |111\rangle}{\sqrt{2}} \\
&= |+\rangle \frac{|00\rangle + |11\rangle}{\sqrt{2}} + |-\rangle \frac{|00\rangle - |11\rangle}{\sqrt{2}} \\
\implies |BC\rangle &= \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{|{++}\rangle + |{--}\rangle}{\sqrt{2}}, \text{ if Alice measured a } |+\rangle \\
\implies (P \otimes P)\,|BC\rangle &= \frac{|00\rangle - |11\rangle}{\sqrt{2}} = \frac{|{+-}\rangle + |{-+}\rangle}{\sqrt{2}}
\end{aligned} \tag{4}$$

Now when both Bob and Charlie receive a 0, they both measure a 0 or a 1. which gives $a + b + c = 0$. If both receive a 1, they both apply a P gate before measuring and therefore, exactly one of them receives a 1, which gives $a + b + c = 1$. This is what we want in both cases.
Now we consider the states left with Bob and Charlie when Alice measured a 1,

$$\begin{aligned}
|BC\rangle &= \frac{|00\rangle - |11\rangle}{\sqrt{2}} = \frac{|{+-}\rangle + |{-+}\rangle}{\sqrt{2}} \text{ if Alice measured a } |-\rangle \\
\implies (P \otimes P)\,|BC\rangle &= \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{|{++}\rangle + |{--}\rangle}{\sqrt{2}}
\end{aligned} \tag{5}$$

In this case, the outputs are flipped, i.e., if both Bob and Charlie receives a 0, they output different bits, which gives $a + b + c = 0$, and if both receive a 1, they output the same bit, which gives $a + b + c = 1$, again giving us what we want in both cases. □

4 Noisy CHSH

Suppose Alice and Bob share a Bell pair $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Imagine that unbeknownst to Alice and Bob, their qubits are not completely isolated from the outside world: with probability ϵ, one of the qubits is measured in the $\{|0\rangle, |1\rangle\}$ basis by the "environment" and the state of their pair collapses to either the state $|00\rangle$ or $|11\rangle$ (with probability $1-\epsilon$, the qubits remain in the Bell state).

Recall: In the CHSH game, Alice and Bob receive independent random bits x and y respectively. Their goal is to output bits a and b respectively such that $a + b = xy \pmod 2$. No communication is allowed. In the "usual strategy", Alice does nothing to her qubit if $x = 0$ and she applies a $\frac{\pi}{4}$ counterclockwise rotation towards $|1\rangle$ if $x = 1$. Bob applies a $\frac{\pi}{8}$ counterclockwise rotation if $y = 0$ and he applies a $\frac{\pi}{8}$ *clockwise* rotation, toward $-|1\rangle$ if $y = 1$. Alice and Bob both measure their qubits in the $\{|0\rangle, |1\rangle\}$ basis and output whatever they see. This strategy wins $\cos^2\left(\frac{\pi}{8}\right) \approx 85\%$ of the time, while any classical strategy can win with probability at most 3/4.

Question 4.1

Question. *Work out an expression for the probability with which ALice and Bob win the CHSH game using this noisy Bell pair asusming they follow the usual strategy (the one reviewed above). Show your work.*

Solution. In the case when there is no noise, with probability $1-\epsilon$, Alice and Bob will still win with probability $\cos^2 \pi/8$. We now compute the probability of winning when the qubit gets collapsed to either $|00\rangle$ or $|11\rangle$.
Since Alice either applies a $\pi/4$ rotation or does nothing before measuring, and her qubit was either in the $|0\rangle$ or the $|1\rangle$ state to begin with, she has a 50% probability of measuring either a $|0\rangle$ or $|1\rangle$ in all cases.
On the other hand, Bob applies a $(-1)^b\pi/8$ rotation on receiving $y = b$. However, Bob starts with either $|0\rangle$ or $|1\rangle$ with equal probability. The net probability of measuring $|0\rangle$ is $1/2 \times \cos^2 \pi/8 + 1/2 \times \cos^2 \pi/8 = 1/2$. Therefore, Bob measures either a $|0\rangle$ or a $|1\rangle$ with equal probability.
Therefore, Alice and Bob send uniformly random bits in the case the noise destroys their entanglement. This leads to a success rate of 1/2 when at least one of x, y is 0 since they only win when $a = b$ in this case. They also have a 50% success rate when $x = y = 1$ since they win when $a \neq b$ in this case which happens with probability 1/2. Thus the net probability of winning is 50%.
Thus, the total probability of winning the noisy CHSH game is

$$\Pr[\text{win}] = (1-\epsilon)\cos^2\frac{\pi}{8} + \frac{\epsilon}{2} = \cos^2\frac{\pi}{8} - \epsilon\left(\cos^2\frac{\pi}{8} - \frac{1}{2}\right) \tag{6}$$

□

Question 4.2

Question. *How large does ϵ need to be before Alice and Bob's success probability with this strategy is no better than is possible with a classical strategy (with no Bell pair at all)?*

Solution. We can solve this by equating the probability computed in Equation 6 to 3/4, we get

$$\begin{aligned} \frac{3}{4} &= \cos^2\frac{\pi}{8} - \epsilon\left(\cos^2\frac{\pi}{8} - \frac{1}{2}\right) \\ \implies \epsilon &= \frac{4\cos^2\frac{\pi}{8} - 3}{4\cos^2\frac{\pi}{8} - 2} \approx 0.29 \end{aligned} \tag{7}$$

□