Midterm Answer Key

Introduction to Quantum Information Science
Fall 2018

This is the midterm from fall 2018. We're releasing it to you now as practice for your midterm.

We will review this and possibly some other practice for the midterm in recitation the week before the exam (10/13 and 10/17). Please come if you'd like the practice.

In fall 2021, the exam will be written in Canvas. The exam will be open-book and open-note, but not open-friends: you may not discuss the exam with anyone besides the course staff until after the exam period is over.

We will create an optional "Midterm tech check" assignment so you can make sure you're comfortable with the interface, available the week before the exam.

This practice midterm may not cover all of the types of questions on the midterm. All of the content covered in the assignments up to homework 6, which are based on lectures up to 10/13, is fair-game. You should be comfortable with all of Chapters 1 to 15 except for Chapter 12 and Section 13.1 (interpretations of QM).

1. True or false? Write your answer on the provided line. [**20 Points, 2 Points per Part**]

__T__ **a)** The states $\frac{|0\rangle+i|1\rangle}{\sqrt{2}}$ and $\frac{|0\rangle-i|1\rangle}{\sqrt{2}}$ are orthogonal.

__T__ **b)** If the state ρ_{AB} is entangled between Alice and Bob, then Alice's reduced state is necessarily mixed (i.e., not pure).

__F__ **c)** Given a mixed state ρ_{AB}, if Alice's reduced state is mixed, then she necessarily shares entanglement with Bob.

__T__ **d)** Superdense coding requires Alice and Bob to share entanglement.

__F__ **e)** Quantum key distribution requires Alice and Bob to share entanglement.

__T__ **f)** Quantum teleportation requires Alice and Bob to share entanglement.

__F__ **g)** Although perfect cloning is impossible, given an unknown 1-qubit state $|\psi\rangle$, it's possible to produce a state *arbitrarily close* to $|\psi\rangle|\psi\rangle$.

__F__ **h)** While Alice and Bob can't use entanglement to send instantaneous signals, they *can* instantaneously check whether they've won the CHSH game.

__T__ **i)** Every k-qubit mixed state can be written as a mixture of at most 2^k pure states.

__T__ **j)** It's possible to "drag" a qubit from the $|+\rangle$ state to the $|-\rangle$ state, with arbitrarily high success probability, purely by making a sequence of two-outcome measurements.

2.

a) For each of the following states, say whether it's separable or entangled. If entangled, briefly justify your answer. If separable, give a factorization or a decomposition that proves separability.

b) [4 Points]

$$\frac{|00\rangle + i\,|01\rangle + i\,|10\rangle - |11\rangle}{2}$$

Solution. $\frac{1}{2}(|0\rangle + i\,|1\rangle) \otimes (|0\rangle + i\,|1\rangle) = |i\rangle\,|i\rangle$ *so unentangled.*

c) [4 Points]

$$\frac{|00\rangle + |01\rangle + |10\rangle - |11\rangle}{2}$$

Solution. $\frac{|0\rangle|+\rangle+|1\rangle|-\rangle}{\sqrt{2}}$. *This is a linear combination over orthogonal states, so it cannot be factored, so it is entangled.*

d) [4 Points]

$$\frac{1}{3}\begin{pmatrix}1 & 1 & 0 & 0\\ 1 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$$

Solution. $\frac{1}{3}\left|10\right\rangle\left\langle 10\right| + \frac{2}{3}\left|0+\right\rangle\left\langle 0+\right|$ *so unentangled.*

e) [Extra Credit, 5 Points]

$$\frac{1}{4}\begin{pmatrix}1 & 0 & 0 & -1\\ 0 & 1 & 1 & 0\\ 0 & 1 & 1 & 0\\ -1 & 0 & 0 & 1\end{pmatrix}$$

Solution.

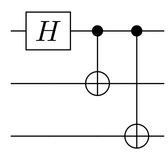
$$\left|i\right\rangle\left|i\right\rangle\left\langle i\right|\left\langle i\right| = \begin{pmatrix}1 & -i & -i & -1\\ i & 1 & 1 & -i\\ i & 1 & 1 & -i\\ -1 & i & i & 1\end{pmatrix}$$

$$\left|-i\right\rangle\left|-i\right\rangle\left\langle -i\right|\left\langle -i\right| = \begin{pmatrix}1 & i & i & -1\\ -i & 1 & 1 & i\\ -i & 1 & 1 & i\\ -1 & -i & -i & 1\end{pmatrix}$$

So the state we have is actually $\frac{\left|i\right\rangle\left|i\right\rangle\left\langle i\right|\left\langle i\right| + \left|-i\right\rangle\left|-i\right\rangle\left\langle -i\right|\left\langle -i\right|}{2}$ *so unentangled.*

f) Draw a quantum circuit, using standard 1- and 2-qubit gates, that maps the state $|000\rangle$ to the GHZ state $\frac{|000\rangle+|111\rangle}{\sqrt{2}}$, while also mapping $|100\rangle$ to $\frac{|000\rangle-|111\rangle}{\sqrt{2}}$. **[6 points]**

Solution.

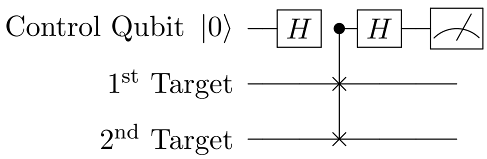


g) Draw a quantum circuit, again using standard 1- and 2-qubit gates, that tests whether the input state is $\frac{|000\rangle+|111\rangle}{\sqrt{2}}$ or $\frac{|000\rangle-|111\rangle}{\sqrt{2}}$, promised that one of those is the case. **[6 points]**

Solution. Run the circuit in reverse and measure the first qubit.

3. The Swap Test [26 Points]

The "swap test" is one of the most important subroutines in quantum algorithms and protocols, used to test whether two quantum states are close or far from one another. Given two one-qubit states (which we call "target qubits"), the swap test is given by the following quantum circuit:



where the measurement of the control qubit is in the $\{|0\rangle, |1\rangle\}$ basis. Here, the middle gate swaps the two target qubits if and only if the control qubit is $|1\rangle$. We say the test "accepts" if and only if the measurement outcome is $|0\rangle$.

a) Suppose the target qubits are both $|0\rangle$. With what probability does the swap test accept? **[3 Points]**

Solution.

The swap does nothing so the hadamards cancel and therefore we always accept.

b) More generally, suppose the target qubits are both $|v\rangle$. With what probability does the swap test accept? **[2 Points]**

Solution. The swap again does nothing so we always accept.

c) Suppose one target qubit is $|0\rangle$ and the other is $|1\rangle$. With what probability does the swap test accept? **[3 Points]**

Solution.

$$\begin{aligned}(H \otimes I \otimes I)CSWAP(H \otimes I \otimes I)\,|001\rangle &= (H \otimes I \otimes I)CSWAP\,|{+}01\rangle \\ &= (H \otimes I \otimes I)\frac{|001\rangle + |110\rangle}{\sqrt{2}} \\ &= \frac{|001\rangle + |101\rangle + |010\rangle - |110\rangle}{2}\end{aligned}$$

Probability of measuring $|0\rangle$ *on the first qubit is then* $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

d) More generally, suppose one target qubit is $|v\rangle$ and the other is $|w\rangle$, for any orthogonal $|v\rangle, |w\rangle$. With what probability does the swap test accept? **[2 Points]**

Solution. *Working through the same way, we end up with the state*

$$\frac{|0vw\rangle + |1vw\rangle + |0wv\rangle - |1wv\rangle}{2}$$

This also gives probability $\frac{1}{2}$ *of accepting.*

e) Suppose one target qubit is $|v\rangle$ and the other is $|w\rangle$, for some arbitrary $|v\rangle$ and $|w\rangle$. With what probability does the swap test accept, as a function of the inner product $\langle v|w\rangle$? **[Extra Credit, 5 Points]**

Solution.

Using what we have for 3d, we now need to figure out $\frac{\||vw\rangle+|wv\rangle\|^2}{4}$ *to get the probability of measuring* $|0\rangle$ *on the first qubit.*

$$\begin{aligned}\frac{\||vw\rangle + |wv\rangle\|^2}{4} &= \frac{1}{4}(\langle vw| + \langle wv|)(|vw\rangle + |wv\rangle) \\ &= \frac{2 + 2|\langle v|w\rangle|^2}{4} \\ &= \frac{1 + |\langle v|w\rangle|^2}{2}\end{aligned}$$

f) Here you'll investigate whether your answer to part (b) generalizes to the case where the two identical states being prepared are mixed rather than pure. Suppose the target qubits are both in the maximally mixed state $I/2$ (and not entangled with each other). With what probability does the swap test accept? *[Hint: Remember that the "1-qubit maximally mixed state" is just a fancy term for the uniform distribution over 0 and 1.]* **[5 Points]**

Solution.

With 50% probability they're the same computational basis state and accept always. With 50% probability they're the opposite basis state and accept with probability $\frac{1}{2}$.

$$\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 1 = \frac{3}{4}$$

g) Here you'll investigate why the swap test can't be used to counterfeit Wiesner's quantum money, or otherwise violate the No-Cloning Theorem, by simply swapping an unknown state against candidate states like $|0\rangle, |1\rangle, |+\rangle, |-\rangle$ until we find a match. Suppose we run the swap test with $|0\rangle$ and $|1\rangle$ as the target qubits, and suppose it accepts. What then is the post-measurement state of the target qubits? **[5 Points]**

Solution. *Based on our answer to 3c, we'd end up with* $\frac{|01\rangle+|10\rangle}{\sqrt{2}}$.

h) A physicist claims that "every photon in the universe is identical to every other one, apart from the known properties in which they differ like position, energy, and polarization." A philosopher replies, "logically, you can never *know* anything of that kind—for who's to say that photons don't have little RFID tags that differentiate them, which you simply haven't discovered yet?" Assuming quantum mechanics is true, how might something analogous to the swap test be used to address this dispute? **[5 Points]**

Solution. *We can run the SWAP test on two photons to test if their quantum states are different, and with enough trials we'd be able to tell if they were not the exact same state if we ever did not accept.*

4. Hardy's Paradox [25 Points]

Consider the state $|\psi\rangle = \frac{|00\rangle+|01\rangle+|10\rangle}{\sqrt{3}}$, shared by Alice and Bob.

a) Suppose Alice and Bob both measure their qubits in the $\{|0\rangle, |1\rangle\}$ basis. What is the probability that they both see $|1\rangle$? **[3 Points]**

Solution. *Since there is no amplitude on $|11\rangle$ there is a 0% probability.*

b) Suppose Alice and Bob both measure their qubits in the $\{|+\rangle, |-\rangle\}$ basis. What is the probability that they both see $|-\rangle$? **[5 Points]**

Solution.

$$|\langle --| \frac{|00\rangle + |01\rangle + |10\rangle}{\sqrt{3}}|^2 = \frac{1}{3}|\frac{1}{2} - \frac{1}{2} - \frac{1}{2}|^2 = \frac{1}{12}$$

c) Suppose one of the qubits is measured in the $\{|0\rangle, |1\rangle\}$ basis, and the outcome is $|0\rangle$. If the other qubit is now measured in the $\{|+\rangle, |-\rangle\}$ basis, what is the probability that the outcome will be $|-\rangle$? **[4 Points]**

Solution.

$$\frac{|00\rangle + |01\rangle + |10\rangle}{\sqrt{3}} = \frac{\sqrt{2}\,|0+\rangle + |10\rangle}{\sqrt{3}}$$

Thus if we measure the first qubit and get $|0\rangle$ then the second qubit will always be $|+\rangle$.

d) Consider the following "paradox." If Alice's qubit is $|0\rangle$, then Bob must see $|+\rangle$. Likewise, if Bob's qubit is $|0\rangle$, then Alice must see $|+\rangle$. Yet at least one of the two qubits must be $|0\rangle$, which means that either Alice or Bob *must* see $|+\rangle$. How can you reconcile this reasoning with your answer from part (b)? **[6 Points]**

Solution. *The converse does not hold true: If Bob sees $|+\rangle$ that does not mean Alice sees $|0\rangle$. Thus when they measure in the $\{|+\rangle, |-\rangle\}$ basis we get different results and can see $|--\rangle$.*

e) Calculate Alice's (or equivalently, Bob's) reduced density matrix. **[5 Points]**

Solution. *Based on 4c, we get*

$$\frac{2}{3}|+\rangle\langle+| + \frac{1}{3}|0\rangle\langle 0| = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

for Bob's qubit.

f) Calculate the entanglement entropy of $|\psi\rangle$, by extracting the eigenvalues from the relevant matrix. (You don't need to simplify your answer further or numerically estimate it.) **[Extra Credit, 5 Points]**

Solution. *The characteristic polynomial is*

$$(\frac{2}{3} - \lambda)(\frac{1}{3} - \lambda) - \frac{1}{9} = \lambda^2 - \lambda - \frac{1}{9} \Rightarrow \lambda = \frac{1 \pm \sqrt{1 - \frac{4}{9}}}{2}$$

From here we can use these eigenvalues to compute the entanglement entropy via

$$-\lambda_1 \log_2 \lambda_1 - \lambda_2 \log_2 \lambda_2$$