Midterm

Introduction to Quantum Information Science
Tuesday, October 22nd

Your Name and EID: ______________________________

1.	/20
2.	/40
a)	/10
b)	/5
c)	/18
d)	/7
3.	/25
a)	/5
b)(Extra Credit)	/5
c)	/10
d)	/5
e)(Extra Credit)	/5
f)	/5
4.	/15
a)	/10
b)(Extra Credit)	/7
c)	/5
d)(Extra Credit)	/5
Total	/100

1. True or false? Write your answer on the provided line. **[20 Points, 2 Points per Part]**

_____ **a)** The BB84 QKD scheme requires both a quantum channel and an authenticated classical channel.

_____ **b)** Superdense coding, quantum teleportation, and Wiesner's quantum money are all examples of quantum protocols that require the use of entangled states.

_____ **c)** The states $\frac{|0\rangle+i|1\rangle}{\sqrt{2}}$ and $\frac{|1\rangle+i|0\rangle}{\sqrt{2}}$ can be perfectly distinguished by a measurement.

_____ **d)** The states $\frac{|0\rangle+i|1\rangle}{\sqrt{2}}$ and $\frac{|1\rangle-i|0\rangle}{\sqrt{2}}$ can be perfectly distinguished by a measurement.

_____ **e)** A superposition of pure states is called a mixed state.

_____ **f)** Since it only affects phases, the unitary transformation $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ has no physical effect.

_____ **g)** By measuring her qubit, Alice can instantaneously gain new information about the state of Bob's qubit, with no speed-of-light limitation.

_____ **h)** It's possible for Alice to learn about a bit of Bob's by measuring a bit of hers, even in some situations where the bits are not entangled.

_____ **i)** In the Elitzur-Vaidman bomb scenario, the bomb tester no longer works if the query qubit gets measured even when no bomb is present.

_____ **j)** Some matrices are unitary but not invertible.

2. Short Answer Exercises [40 Points Total]

a) For each of the following states, either factor them or write that they are entangled. **[2 Points per State]**

(i)

$$\frac{|00\rangle + i\,|01\rangle + i\,|10\rangle - |11\rangle}{2}$$

(ii)

$$\frac{|00\rangle + i\,|01\rangle + i\,|10\rangle + |11\rangle}{2}$$

(iii)

$$\frac{|00\rangle - |01\rangle - |10\rangle + |11\rangle + |20\rangle - |21\rangle}{\sqrt{6}}$$

(iv)

$$\sqrt{\frac{2}{3}}\,|+\rangle\,|+\rangle - \sqrt{\frac{1}{3}}\,|-\rangle\,|-\rangle$$

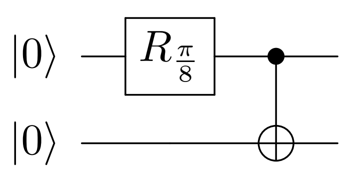
(v)

$$\frac{|000\rangle - |001\rangle - |010\rangle + |011\rangle - |100\rangle + |101\rangle + |110\rangle - |111\rangle}{\sqrt{8}}$$

b) [5 Points] Calculate how many ebits of entanglement are in the following 2-qubit state (You don't need to simplify your answer):

$$\frac{3}{5}|+\rangle|i\rangle - \frac{4}{5}|-\rangle|-i\rangle$$

c) [18 Points Total] Suppose we apply the following 2-qubit quantum circuit:



(i) What is the final state? **[3 Points]**

(ii) Now suppose the top qubit is measured in the $\{|0\rangle, |1\rangle\}$ basis. What are the probabilities of the two outcomes, and what are the states of the bottom qubit conditioned on those outcomes? [**3 Points**]

(iii) Suppose instead the top qubit is measured in the $\{|+\rangle, |-\rangle\}$ basis. What are the probabilities of the two outcomes, and what are the states of the bottom qubit conditioned on those outcomes? [**5 Points**]

(iv) Calculate the final mixed state of the bottom qubit. Does the mixed state depend on which basis we measured the top qubit in? [**7 Points**]

d) Draw a quantum circuit to prepare the state $\frac{|000\rangle+|011\rangle+|101\rangle+|110\rangle}{2}$. You can use any 1- or 2-qubit gates discussed in class (and these are all that's needed).

3. QUANTUM FINITE AUTOMATA [**25 points**]

A stream of bits arrive one by one. Unfortunately, you have only a single bit of memory with which to store information. Whenever a new bit arrives, you can act on your memory bit however you like, but you have no other way to make records—not even of how many bits there have been. Once the stream is finished—you don't know when in advance—you'll need to answer some particular yes-or-no question about the bit stream. (For those who know the term, yes, we're talking about a 2-state finite automaton.)

a) [**5 Points**] Suppose the question you'll need to answer is whether the total number of '1' bits is even. Describe a strategy that lets you answer this question with certainty.

b) [5 Points, Extra Credit] Suppose instead that the question is whether the total number of '1' bits is divisible by 3 (*not* the sum modulo 3, just that yes-or-no question). Again using one memory bit, is there a strategy to answer this question with certainty for all input streams? Why or why not?

c) [10 Points] Now suppose that, instead of 1 bit, your memory contains 1 qubit. Whenever a new bit arrives from the stream, you can act on your qubit by any unitary transformation you want, but can make no other records. Once the stream is finished, you can then measure your qubit to answer a question. Describe a quantum strategy to guess whether the number of '1' bits in the stream is divisible by 3, with success probability greater than $\frac{1}{2}$. Explicitly give the unitary transformations and the measurement.

d) [5 Points] With what probability does your strategy from (c) succeed? Does the answer depend on whether the number of '1' bits is or isn't divisible by 3?

e) [5 Points, Extra Credit] Is there a strategy that uses 1 qubit, and that determines whether the number of '1' bits is divisible by 3 with *certainty*? Why or why not?

f) [5 points] Suppose that, instead of just a final answer at the end, we'd like our algorithm to output *after each step* whether the number of '1' bits that it's seen *so far* is divisible by 3. Is there some special difficulty that arises here if one's memory is a qubit rather than a classical bit? If so, what?

4. The KS Game [15 Points]

Suppose Alice and Bob play the following game, called the "KS Game." A referee sends Alice a classical description of an orthonormal basis $B_1 = \{|u\rangle, |v\rangle, |w\rangle\}$ for $\mathbb{R}^3$, and sends Bob a classical description of some other orthonormal basis B_2 for $\mathbb{R}^3$. Alice must choose one of the three vectors in B_1, and Bob must choose one of the three vectors in B_2. The rule is this: whenever one player chooses a certain vector, the other player must choose that same vector *if* it's also in their basis. Alice and Bob win the game whenever they obey that rule, and lose whenever they violate it. Like with the CHSH game, they can agree on a strategy in advance but can't communicate once the game starts.

a) [10 Points] Suppose Alice and Bob share the 2-qutrit entangled state $\frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$. Describe a strategy that lets them win the KS Game with certainty, and explain why it works.

b) [7 Points, Extra Credit] Here's an interesting fact from geometry, called the Kochen-Specker Theorem. There is no way to take the real unit sphere, $x^2 + y^2 + z^2 = 1$, and color each of its points either blue or red in such a way that for every orthonormal basis $\{|u\rangle, |v\rangle, |w\rangle\}$, exactly one vector in the basis points to a red point. Assuming that theorem: in a classical world, is there a strategy that lets Alice and Bob win the KS Game with certainty, regardless of what challenges the referee sends? Prove your answer.

c) [5 points] Now suppose Alice and Bob are playing exactly the same game, except in 2 dimensions instead of 3 (i.e., the referee sends them orthonormal bases for $\mathbb{R}^2$ rather than $\mathbb{R}^3$). Explain how they can win the game with certainty, regardless of what the referee sends them, even with no use of entanglement.

d) [5 Points, Extra Credit] Suppose the bases were complex rather than real. Would your strategy from part (a) have to be modified? If so, how?