Introduction to Quantum Information Science
Homework 6

Due Wednesday, October 13 at 11:59 PM

Note: You should explain your reasoning, i.e. show your work, for all problems. You do not need to show us every step of each calculation, but every answer should include an explanation *written with words* of what you did.

1. Distinguishability of Mixed States [4 Points] Let ρ and σ be two different single qubit density matrices. Prove that ρ and σ are distinguishable — that is, there exists some measurement basis such that the probabilities of the outcomes is different in the two cases.
Hint: Use the fact that $\sigma - \rho$ is necessarily a nonzero Hermitian matrix and the fact that any Hermitian matrix can be diagonalized.

Solution: Our solution works for any number of qubits. Following the hint, we can write $\sigma - \rho = V\Lambda V^\dagger$ for some orthonormal matrix of eigenvectors V and diagonal matrix of eigenvalues Λ. Now measure in the basis given by V. Then for the ith eigenvector $|v_i\rangle$ the probability to observe it is $\langle v_i|\sigma|v_i\rangle$ and $\langle v_i|\rho|v_i\rangle$ respectively. But that means

$$\begin{aligned}\langle v_i|\sigma|v_i\rangle - \langle v_i|\rho|v_i\rangle &= \langle v_i|(\sigma-\rho)|v_i\rangle \\ &= \langle v_i|V\Lambda V^\dagger|v_i\rangle \\ &= \langle i|\Lambda|i\rangle \\ &= \lambda_i\end{aligned}$$

where λ_i is the ith eigenvalue of $\sigma - \rho$. And that we know to be non-zero for at least one i since otherwise $\sigma - \rho$ would've been the zero-matrix. Hence, there's at least one $|v_i\rangle$ that we'll observe with different probabilities.

2. Reduced GHZ Consider the n-qubit "Schrödinger cat state" (or "generalized GHZ state")

$$\frac{|0\cdots 0\rangle + |1\cdots 1\rangle}{\sqrt{2}}.$$

a) [3 Points] What probability distribution over n-bit strings do we observe if we Hadamard the first $n-1$ qubits, then measure all n qubits in the $\{|0\rangle, |1\rangle\}$ basis? Show your work.

Solution: After applying Hadamard gates to the first $n-1$ qubits our state will be

$$\frac{|+\cdots+0\rangle + |-\cdots-1\rangle}{\sqrt{2}}.$$

The probability of measuring $|i_1 \cdots i_n\rangle$ is thus

$$\left|\frac{\langle i_1 \cdots i_n|+\cdots+0\rangle + \langle i_1 \cdots i_n|-\cdots-1\rangle}{\sqrt{2}}\right|^2 = \frac{1}{2}\begin{cases} \left|\prod_{k=1}^{n-1} \langle i_k|+\rangle\right|^2 & \text{if } i_n = 0 \\ \left|\prod_{k=1}^{n-1} \langle i_k|-\rangle\right|^2 & \text{if } i_n = 1 \end{cases}$$
$$= \frac{1}{2^n}. \qquad \text{(regardless)}$$

In other words, we're dealing with the uniform distribution over all 2^n possible outcomes.

b) [3 Points] Is this the same distribution or a different one, from what we'd have seen if we took the following state, applied Hadamards to the first $n-1$ qubits, and then measured all n qubits in the $\{|0\rangle, |1\rangle\}$ basis:

$$\frac{|0\cdots 0\rangle\langle 0\cdots 0| + |1\cdots 1\rangle\langle 1\cdots 1|}{2}$$

. Show your work.

Solution: This state first turns into

$$\frac{|+\cdots+0\rangle\langle +\cdots+0| + |-\cdots-1\rangle\langle -\cdots-1|}{2}$$

which then for $|i_1 \cdots i_n\rangle$ gives the measurement probability

$$\frac{\langle i_1 \cdots i_n|+\cdots+0\rangle\langle +\cdots+0|i_1\cdots i_n\rangle + \langle i_1\cdots i_n|-\cdots-1\rangle\langle -\cdots-1|i_1\cdots i_n\rangle}{2}$$
$$= \frac{1}{2}\begin{cases} \prod_{k=1}^{n-1} \langle i_k|+\rangle\langle +|i_k\rangle & \text{if } i_n = 0 \\ \prod_{k=1}^{n-1} \langle i_k|-\rangle\langle -|i_k\rangle & \text{if } i_n = 1 \end{cases}$$
$$= \frac{1}{2^n} \qquad \text{(regardless)}$$

which again is the uniform distribution.

c) [2 Points] What probability distribution over n-bit strings do we observe if we Hadamard *all* n qubits, then measure all n qubits in the $\{|0\rangle, |1\rangle\}$ basis? Show your work.

Solution: For the state in a) we now get the probability

$$\left|\frac{\langle i_1 \cdots i_n|+\cdots+\rangle + \langle i_1 \cdots i_n|-\cdots-\rangle}{\sqrt{2}}\right|^2 = \frac{1}{2}\left|\prod_{k=1}^{n} \langle i_k|+\rangle + \prod_{k=1}^{n} \langle i_k|-\rangle\right|^2$$
$$= \frac{1}{2}\left|\left(\frac{1}{\sqrt{2}}\right)^n + \prod_{k=1}^{n} (-1)^{[i_k=1]}\frac{1}{\sqrt{2}}\right|^2$$
$$= \begin{cases} \frac{1}{2^{n-1}} & \text{if there are an even number of } i_k = 1 \\ 0 & \text{if there are an odd number of } i_k = 1 \end{cases}$$

which is the uniform distribution over basis vectors with an even number of ones in their description.

d) [2 Points] Is this the same distribution or a different one, than if to the following state we apply Hadamards to *all* n qubits and then measure all n qubits in the $\{|0\rangle, |1\rangle\}$ basis:

$$\frac{|0\cdots 0\rangle\langle 0\cdots 0| + |1\cdots 1\rangle\langle 1\cdots 1|}{2}.$$

Show your work.

Solution: No, in the state from b) we instead continue to get

$$\begin{aligned}
&\frac{\langle i_1\cdots i_n|+\cdots+\rangle\langle +\cdots+|i_1\cdots i_n\rangle + \langle i_1\cdots i_n|-\cdots-\rangle\langle -\cdots-|i_1\cdots i_n\rangle}{2} \\
=&\frac{1}{2}\left(\prod_{k=1}^{n}\langle i_k|+\rangle\langle +|i_k\rangle + \prod_{k=1}^{n}\langle i_k|-\rangle\langle -|i_k\rangle\right) \\
=&\frac{1}{2^n}, \qquad \text{(regardless)}
\end{aligned}$$

the uniform distribution over *all* possible observations.

3. Schmidt Form Recall that a bipartite state is in Schmidt form if it can be written as:

$$\sum_{i=1}^{n} \beta_i |u_i\rangle_A \otimes |v_i\rangle_B$$

where $\{u_i : 1 \le i \le n\}$ is an orthonormal basis and $\{v_i : 1 \le i \le n\}$ is an orthonormal basis (see Chapter 11 in the textbook). In this problem, we'll show that any bipartite state (state on two subsystems, possibly entangled)

$$|\psi\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{i,j} |i\rangle_A \otimes |j\rangle_B$$

can be put in Schmidt form.

a) [3 Points] Write the amplitudes of $|\psi\rangle$ in a matrix as follows:

$$M := \begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,n} \\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,n} \\ \vdots & & \ddots & \\ \alpha_{n,1} & & & \alpha_{n,n} \end{bmatrix}$$

Show that if Alice applies F to her half of the system and Bob applies G to his half of the system, then the matrix of amplitudes of the resulting state is FMG^T, where G^T is the transpose of G (not the conjugate transpose!).

Hint: Pay attention to indices in any summations you use; it may save you a lot of writing.

Solution: Taking the usual route, if Alice applies F to her qubit, then the resulting state will be

$$\begin{aligned}
\sum_{x,y} \alpha_{x,y}\big(F|x\rangle\big)\otimes|y\rangle &= \sum_{x,y}\alpha_{x,y}\left(\sum_z f_{z,x}|z\rangle\right)\otimes|y\rangle \\
&= \sum_x \sum_{y,z} f_{z,x}\alpha_{x,y}|z\rangle\otimes|y\rangle\ .
\end{aligned}$$

Therefore, the coefficient of the state $|i\rangle \otimes |j\rangle$ will be

$$\sum_x f_{i,x}\alpha_{x,j}$$

which is exactly the entry of FM at position i, j.

Similarly, if Bob applies G, then we get

$$\begin{aligned}\sum_{x,y} \alpha_{x,y} |x\rangle \otimes (G|y\rangle) &= \sum_{x,y} \alpha_{x,y} |x\rangle \otimes \left(\sum_z g_{z,y} |z\rangle\right) \\ &= \sum_y \sum_{x,z} \alpha_{x,y} g_{z,y} |x\rangle \otimes |z\rangle\end{aligned}$$

whose coefficient for $|i\rangle \otimes |j\rangle$ is

$$\sum_y \alpha_{i,y} g_{j,y}.$$

Again, since taking the transpose just switches indices, that's precisely the ij entry of MG^T.

b) [2 Points] The *singular value decomposition* of an $n \times n$ matrix M is a factorization:

$$M = U\Sigma V^\dagger$$

where U and V are $n \times n$ unitary matrices, and Σ is a diagonal matrix (here it *is* the conjugate-transpose, not the transpose). A fact from linear algebra, which you do not need to prove, is that every complex-valued square matrix has a singular value decomposition.

Using the singular value decomposition of M, show that Alice can apply a local unitary F and Bob can apply a local unitary G to make the matrix M of amplitudes diagonal. Conclude that the resulting state has the form:

$$(F_A \otimes G_B)|\psi\rangle = \sum_{i=1}^n \beta_i |i\rangle_A \otimes |i\rangle_B.$$

Solution: Let $F = U^\dagger$ and $G = V^T$. Then after applying these transformations Alice and Bob end up with the state described by the matrix

$$FMG^T = U^\dagger U\Sigma V^\dagger V = \Sigma.$$

Thus, if written in vector form it would look like

$$\sum_i \beta_i |ii\rangle.$$

c) [3 Points] Conclude that Alice and Bob's original state $|\psi\rangle$ can be written in Schmidt form by giving explicit bases $\{|u_i\rangle\}$ and $\{|v_i\rangle\}$ in terms of F and G. To be clear, you should give a formula for $\{|u_i\rangle\}$ and $\{|v_i\rangle\}$.

Solution: As a consequence, we can write the state in Schmidt form like so:

$$\sum_{i,j} \alpha_{i,j} |i\rangle \otimes |j\rangle = \sum_{i,j} \alpha_{i,j} \left(F^\dagger F |i\rangle\right) \otimes \left(G^\dagger G |j\rangle\right)$$

$$
\begin{aligned}
&= \left(F^\dagger \otimes G^\dagger\right) \sum_{i,j} \alpha_{i,j} F \left|i\right\rangle \otimes G \left|j\right\rangle \\
&= \left(F^\dagger \otimes G^\dagger\right) \sum_{i} \beta_i \left|ii\right\rangle \\
&= \sum_i \beta_i \left|f_i^\dagger\right\rangle \otimes \left|g_i^\dagger\right\rangle
\end{aligned}
$$

where $f_i^\dagger$ and $g_i^\dagger$ are the ith columns of $F^\dagger$ and $G^\dagger$ respectively.

4. Separable and Entangled States

$$
\begin{aligned}
\left|\psi_1\right\rangle &= \frac{\left|00\right\rangle + i\left|01\right\rangle + i\left|10\right\rangle - \left|11\right\rangle}{2} \\
\left|\psi_2\right\rangle &= \frac{3}{5}\left|01\right\rangle - \frac{4}{5}\left|10\right\rangle \\
\left|\psi_3\right\rangle &= \frac{1}{\sqrt{3}}\left|00\right\rangle + \frac{1}{\sqrt{3}}\left|01\right\rangle + \frac{1}{\sqrt{6}}\left|10\right\rangle - \frac{1}{\sqrt{6}}\left|11\right\rangle
\end{aligned}
$$

a) [3 Points] Put the above states into Schmidt form:

$$\left|\psi\right\rangle = \sum_i \lambda_i \left|\alpha_i\right\rangle \left|\beta_i\right\rangle$$

In other words, find orthonormal bases $\{\left|\alpha_0\right\rangle, \left|\alpha_1\right\rangle\}$ for the first qubit and $\{\left|\beta_0\right\rangle, \left|\beta_1\right\rangle\}$ for the second qubit, such that you can write the state without cross terms $\left|\alpha_0\right\rangle\left|\beta_1\right\rangle$ or $\left|\alpha_1\right\rangle\left|\beta_0\right\rangle$. Show your work. *Hint:* you should not need to use the singular value decomposition to find the Schmidt form.

Solution: For the first state use the base $\{\left|i\right\rangle, \left|-i\right\rangle\}$ for both qubits and write it as

$$\left|\psi_1\right\rangle = 1 \cdot \left|ii\right\rangle + 0 \cdot \left|-i-i\right\rangle.$$

The second state is in Schmidt form already if we use the computational basis for both qubits but flip the order of the basis for the second qubit.

Lastly, use $\{\left|0\right\rangle, \left|1\right\rangle\}$ and $\{\left|+\right\rangle, \left|-\right\rangle\}$ for the third state. Then

$$\left|\psi_3\right\rangle = \sqrt{\frac{2}{3}}\left|0+\right\rangle + \sqrt{\frac{1}{3}}\left|1-\right\rangle$$

is its Schmidt form.

b) [3 Points] Calculate how many ebits of entanglement each of these states have. Please show your work/justify your reasoning for each of your answers. (Keep in mind, this answer need not be an integer.)

Solution: The number of ebits is given by the entropy $\sum_i p_i(1/p_i)$ where the p_i are the nonzero probabilities of the different outcomes when looking at the state in Schmidt form. That means the number of ebits in $\left|\psi_1\right\rangle$ is

$$1\log_2(1/1) = 0.$$

For $\left|\psi_2\right\rangle$ it is

$$\frac{9}{25}\log_2(25/9) + \frac{16}{25}\log_2(25/16) \approx 0.94\,.$$

And in the case of $|\psi_3\rangle$ we get

$$\frac{2}{3}\log_2(3/2) + \frac{1}{3}\log_2(3) \approx 0.92\,.$$

c) [2 Points] For each of the states you found with non-zero entanglement entropy in part (b), show explicitly that there exists no factorization of the states into a tensor product of two single qubit states.

Solution: Recall that for any state of the form

$$a\,|00\rangle + b\,|01\rangle + c\,|10\rangle + d\,|11\rangle$$

if it can be factored into a tensor product, then $ad = bc$. For $|\psi_2\rangle$ and $|\psi_3\rangle$ however we get

$$0 \neq -\frac{12}{25}$$

and

$$-\frac{1}{\sqrt{18}} \neq \frac{1}{\sqrt{18}}$$

respectively.

5. Entanglement and Mixed States [6 Points] Suppose Alice has n qubits and Bob has n qubits of a shared $2n$-qubit pure state $|\psi\rangle$. Prove that the following are equivalent:

i. Alice and Bob's systems are entangled.

ii. Alice's local density matrix is a mixed state.

iii. Both local density matrices are mixed states.

Recall that to prove several properties are equivalent, you must show they each have an if-and-only-if relationship with every other. One way to make this shorter is to write a chain of proofs, like A implies B, B implies C, and C implies A.
Hint: Consider the Schmidt decomposition.

Solution: Consider the Schmidt form of $|\psi\rangle$

$$|\psi\rangle = \sum_i \beta_i\,|u_i\rangle\,|v_i\rangle\,.$$

This is entangled if and only if there are multiple nonzero β_i.

i. $\implies$ ii. If there are multiple nonzero β_i in the Schmidt form, then Alice's local density matrix $\sum_i |\beta_i|^2\,|u_i\rangle\langle u_i|$ has multiple eigenvalues, making it mixed.

ii. $\implies$ iii. If there are multiple nonzero $|\beta_i|^2$ for Alice, then the same applies to Bob's state $\sum_i |\beta_i|^2\,|v_i\rangle\langle v_i|$.

iii. $\implies$ i. If both have multiple nonzero $|\beta_i|^2$, then there are multiple nonzero β_i in the Schmidt form and the joint state is entangled.