Inversion of a Tridiagonal Jacobi Matrix

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ABSTRACT

A formula for the inverse of a general tridiagonal matrix is given in terms of the principal minors.

Let $A_n = [a_{ij}]$ be an $n \times n$ tridiagonal Jacobi matrix such that $a_{ii} = b_i$, $a_{i,i+1} = c_i$, $a_{i,i-1} = a_i$, and $a_{ij} = 0$ for $|i - j| > 1$. We set $|A_n| = \theta_n$. The principal minors θ_i satisfy

$$\theta_i = b_i\theta_{i-1} - a_i c_{i-1}\theta_{i-2}, \quad \theta_{-1} = 0, \quad \theta_0 = 1, \qquad i = 1, 2, \ldots, n.$$

We also define the sequence $\{\phi_i\}$ by the recurrence formula

$$\begin{aligned} \phi_i &= b_i\phi_{i+1} - c_i a_{i+1}\phi_{i+2}, \quad \phi_{n+1} = 1, \quad \phi_{n+2} = 0, \\ i &= n, n-1, \ldots, 3, 2, 1. \end{aligned}$$

LEMMA. $\theta_j\phi_{j+1} - a_{j+1}c_j\theta_{j-1}\phi_{j+2} = \theta_n, \quad j = n, n-1, \ldots, 2, 1.$

Let $A_n^{-1} = [\alpha_{ij}]$.

THEOREM.

$$\alpha_{ij} = \begin{cases} (-1)^{i+j} c_i c_{i+1} \cdots c_{j-1}\theta_{i-1}\phi_{j+1}/\theta_n, & i < j, \\ \theta_{i-1}\phi_{i+1}/\theta_n, & i = j, \\ (-1)^{i+j} a_{j+1}a_{j+2} \cdots a_i\theta_{j-1}\phi_{i+1}/\theta_n, & i > j. \end{cases}$$

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Explicit inverses of many of the well-known tridiagonal matrices can now be easily deduced from the theorem. Observe that this is a discrete analog, for the not necessarily symmetric case, of the Green's function for a Sturm-Liouville problem.

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