TEORIA ELETROMAGNÉTICA - LISTA DE EXERCÍCIOS I

SÉRGIO CORDEIRO

Parte A: Conceitos e Definições

Definir/conceituar de forma fisicamente consistente:

a) Grandeza física; grandeza física escalar e grandeza física vetorial;
b) Campo; campo escalar e campo vetorial;
c) Vetor unitário;
d) Produto escalar entre dois vetores;
e) Produto vetorial entre dois vetores;
f) Gradiente de um campo escalar;
g) Fluxo de um campo vetorial;
h) Divergente de um campo vetorial;
i) Teorema de Gauss (ou da Divergência);
j) Rotacional de um campo vetorial;
k) Teorema de Stokes;
l) Campo vetorial conservativo;
m) Campo vetorial não-conservativo;
n) Laplaciano de um campo escalar;
o) Laplaciano de um campo vetorial;
p) Função Delta de Dirac unidimensional.

a) **Grandeza física** é qualquer propriedade de um fenômeno físico. Ela é uma **grandeza física escalar** quando pode ser quantificada através de um valor numérico simples, acompanhado por uma unidade de medida; e uma **grandeza física vetorial** quando, para quantificá-la, se necessita especificar um vetor, ou seja, um conjunto de n valores, se considerado um espaço $\mathbb{R}_n$.
b) **Campo** é uma distribuição espacial de uma grandeza física. A distribuição de uma grandeza física escalar é um **campo escalar** e a distribuição de uma grandeza física vetorial é um **campo vetorial**.
c) **Vetor unitário** é um vetor cujo módulo é igual à unidade.
d) O **produto escalar entre dois vetores** é uma grandeza escalar que pode ser entendida geometricamente como o tamanho da projeção de um vetor sobre o outro.

e) O **produto vetorial entre dois vetores** é uma grandeza vetorial que pode ser entendida geometricamente como a medida da área delimitada pelos vetores. A direção do resultado é perpendicular aos vetores e o sentido é dado, por convenção, pela regra da mão direita.
f) O **gradiente de um campo escalar** f é um campo vetorial que corresponde à máxima variação de f em cada ponto do espaço.
g) O **fluxo de um campo vetorial** $\vec{v}$ através de uma superfície S é um valor escalar dado pela somatória, sobre todos os pontos de S, da componente de $\vec{v}$ perpendicular à superfície.
h) O **divergente de um campo vetorial** $\vec{v}$ é um campo escalar que mede o espalhamento de $\vec{v}$ (ou seja, a componente radial de $\vec{v}$) em cada ponto do espaço. É definido como a relação entre o fluxo de $\vec{v}$ através de um volume infinitesimal v centrado no ponto e o tamanho de v.
i) O **Teorema de Gauss (ou da Divergência)** estabelece que o fluxo de um campo vetorial $\vec{v}$ através de uma superfície **fechada** S é igual à somatória do divergente de $\vec{v}$ em todos os pontos do espaço delimitado por S.
j) O **rotacional de um campo vetorial** $\vec{v}$ e um campo vetorial que indica as componentes tangenciais de $\vec{v}$ em cada ponto do espaço. É definido como a razão entre a máxima circulação de $\vec{v}$ através de uma curva ao redor do ponto e a área delimitada por essa curva.
k) O **Teorema de Stokes** estabelece que a **circulação** de um vetor $\vec{v}$ sobre uma curva C é igual ao fluxo do rotacional de $\vec{v}$ através da superfície delimitada por C.
l) Um campo é **conservativo**, ou ***potencial***, quando o trabalho realizado na translação, em equilíbrio, de um ponto a outro do espaço independe do caminho escolhido. Quando o campo é vetorial, ele deve ser ***irrotacional*** nesse caso.
m) Um campo é **não-conservativo** quando o trabalho realizado na translação, em equilíbrio, de um ponto a outro do espaço depende do caminho escolhido. Quando o campo é vetorial, ele deve ter rotacional não nulo.
n) O **laplaciano de um campo escalar** f é a medida do espaçamento das superfícies $f =$ constante em cada ponto do espaço. Segundo [MAXWELL 1873 1]:

> “I propose therefore to call $\nabla^2 q$ the *concentration* of q at the point P, because it indicates the excess of the value of q at that point over its mean in the neighbourhood of the point.”

o) O **laplaciano de um campo vetorial** $\vec{v}$ não tem interpretação física definida. Matematicamente, ele é a diferença entre o gradiente do divergente de $\vec{v}$ e o rotacional do rotacional de $\vec{v}$:

$$\vec{\nabla}^2\vec{v} = \vec{\nabla}(\vec{\nabla}\cdot\vec{v}) - \vec{\nabla}\times(\vec{\nabla}\times\vec{v})$$

p) A **função Delta de Dirac unidimensional** $\delta(x - x_0)$ é uma *função generalizada* (ou *distribuição*) que possui valor nulo em todo o seu domínio, com exceção do ponto $x = x_0$, onde assume valor infinito.

Parte B: Aplicação de Conceitos e Definições

1) Escreva a expressão cartesiana dos seguintes operadores diferenciais: a) $f\vec{\nabla}$; b) $\vec{A}\cdot\vec{\nabla}$; c) $\vec{A}\times\vec{\nabla}$.

a)

$$f\vec{\nabla} = f\left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right)$$

$$= f\frac{\partial}{\partial x}\hat{a}_x + f\frac{\partial}{\partial y}\hat{a}_y + f\frac{\partial}{\partial z}\hat{a}_z$$

b)

$$\vec{A}\cdot\vec{\nabla} = \Big(A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z\Big)\cdot\left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right)$$

$$= A_x\frac{\partial}{\partial x}\hat{a}_x + A_y\frac{\partial}{\partial y}\hat{a}_y + A_z\frac{\partial}{\partial y}\hat{a}_y$$

c)

$$\vec{A}\times\vec{\nabla} = \Big(A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z\Big)\times\left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right)$$

$$= \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$$

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2) Determine: a) a expressão cartesiana do operador $d\vec{l} \cdot \vec{\nabla}$; b) a relação entre o operador do item a) e a diferencial "d" de uma função dependente apenas do ponto.

a)

$$\begin{aligned} d\vec{l} \cdot \vec{\nabla} &= \left(dx\ \hat{a}_x + dy\ \hat{a}_y + dz\ \hat{a}_z \right) \cdot \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z \right) \\ &= \frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy + \frac{\partial}{\partial z}dz \end{aligned}$$

b)

$$\begin{aligned} d &= \frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy + \frac{\partial}{\partial z}dz + \frac{\partial}{\partial t}dt \\ &= \frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy + \frac{\partial}{\partial z}dz + 0 \\ &= d\vec{l} \cdot \vec{\nabla} \end{aligned}$$

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3) Mostre que: a) $d\vec{C} = (d\vec{l} \cdot \vec{\nabla})\vec{C} + \frac{\partial \vec{C}}{\partial t} dt$; b) $\frac{d\vec{C}}{dt} = (\vec{V} \cdot \vec{\nabla})\vec{C} + \frac{\partial \vec{C}}{\partial t}$, onde $\vec{C} = \frac{d\vec{l}}{dt}$.

a)

$$
\begin{aligned}
d\vec{C} &= \frac{\partial \vec{C}}{\partial x} dx + \frac{\partial \vec{C}}{\partial y} dy + \frac{\partial \vec{C}}{\partial z} dz + \frac{\partial \vec{C}}{\partial t} dt \\
&= \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy + \frac{\partial}{\partial z} dz \right) \vec{C} + \frac{\partial \vec{C}}{\partial t} dt \\
&= \left[\left(dx\ \hat{a}_x + dy\ \hat{a}_y + dz\ \hat{a}_z \right) \cdot \left(\frac{\partial}{\partial x} \hat{a}_x + \frac{\partial}{\partial y} \hat{a}_y + \frac{\partial}{\partial z} \hat{a}_z \right) \right] \vec{C} + \frac{\partial \vec{C}}{\partial t} dt \\
&= (d\vec{l} \cdot \vec{\nabla})\vec{C} + \frac{\partial \vec{C}}{\partial t} dt
\end{aligned}
$$

b)

$$
\begin{aligned}
\frac{d\vec{C}}{dt} &= \frac{1}{dt}\ d\vec{C} \\
&= \frac{1}{dt} \left[\left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy + \frac{\partial}{\partial z} dz \right) \vec{C} + \frac{\partial \vec{C}}{\partial t} dt \right] \\
&= \left(\frac{\partial}{\partial x} \frac{dx}{dt} + \frac{\partial}{\partial y} \frac{dy}{dt} + \frac{\partial}{\partial z} \frac{dz}{dt} \right) \vec{C} + \frac{\partial \vec{C}}{\partial t} \\
&= \left[\left(\frac{dx}{dt} \hat{a}_x + \frac{dy}{dt} \hat{a}_y + \frac{dz}{dt} \hat{a}_z \right) \cdot \left(\frac{\partial}{\partial x} \hat{a}_x + \frac{\partial}{\partial y} \hat{a}_y + \frac{\partial}{\partial z} \hat{a}_z \right) \right] \vec{C} + \frac{\partial \vec{C}}{\partial t} \\
&= \left[\vec{V} \cdot \vec{\nabla} \right] \vec{C} + \frac{\partial \vec{C}}{\partial t}
\end{aligned}
$$

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4) Usando coordenadas e componentes cartesianos, demostre que: a) $(\vec{A}\cdot\vec{\nabla})\vec{r}=\vec{A}$; b) $(\vec{A}\times\vec{\nabla})\cdot\vec{r}=0$; c) $(\vec{A}\times\vec{\nabla})\times\vec{r}=-2\vec{A}$.

a)

$$
\begin{aligned}
(\vec{A}\cdot\vec{\nabla})\vec{r} &= \left[\Big(A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z\Big)\cdot\left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right)\right](x\hat{a}_x + ... \\
&\qquad + y\hat{a}_y + z\hat{a}_z) \\
&= \left[A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z}\right](x\hat{a}_x + y\hat{a}_y + z\hat{a}_z) \\
&= A_x\frac{\partial}{\partial x}(x\hat{a}_x + y\hat{a}_y + z\hat{a}_z) + A_y\frac{\partial}{\partial y}(x\hat{a}_x + y\hat{a}_y + z\hat{a}_z) + ... \\
&\qquad + A_z\frac{\partial}{\partial z}(x\hat{a}_x + y\hat{a}_y + z\hat{a}_z) \\
&= A_x\hat{a}_x + \vec{0} + \vec{0} + \vec{0} + A_y\hat{a}_y + \vec{0} + \vec{0} + \vec{0} + A_z\hat{a}_z \\
&= \vec{A}
\end{aligned}
$$

b)

$$
\begin{aligned}
(\vec{A} \times \vec{\nabla}) \cdot \vec{r} &= \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} \cdot (x\hat{a}_x + y\hat{a}_y + z\hat{a}_z) \\
&= \left[\left(A_y \frac{\partial}{\partial z} - A_z \frac{\partial}{\partial y} \right) \hat{a}_x + \left(A_z \frac{\partial}{\partial x} - A_x \frac{\partial}{\partial z} \right) \hat{a}_y + \ldots \right. \\
&\quad \left. + \left(A_x \frac{\partial}{\partial y} - A_y \frac{\partial}{\partial x} \right) \hat{a}_z \right] \cdot \cdot (x\hat{a}_x + y\hat{a}_y + z\hat{a}_z) \\
&= \left(A_y \frac{\partial}{\partial z} - A_z \frac{\partial}{\partial y} \right) x + \left(A_z \frac{\partial}{\partial x} - A_x \frac{\partial}{\partial z} \right) y + \ldots \\
&\quad + \left(A_x \frac{\partial}{\partial y} - A_y \frac{\partial}{\partial x} \right) z \\
&= 0 + 0 + 0 \\
&= 0
\end{aligned}
$$

c)

$$(\vec{A} \times \vec{\nabla}) \times \vec{r} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_y \frac{\partial}{\partial z} - A_z \frac{\partial}{\partial y} & A_z \frac{\partial}{\partial x} - A_x \frac{\partial}{\partial z} & A_x \frac{\partial}{\partial y} - A_y \frac{\partial}{\partial x} \\ x & y & z \end{vmatrix}$$

$$\begin{aligned} &= \left[\left(A_z \frac{\partial}{\partial x} - A_x \frac{\partial}{\partial z}\right) z - \left(A_x \frac{\partial}{\partial y} - A_y \frac{\partial}{\partial x}\right) y\right] \hat{a_x} + ... \\ &\quad + \left[\left(A_x \frac{\partial}{\partial y} - A_y \frac{\partial}{\partial x}\right) x - \left(A_y \frac{\partial}{\partial z} - A_z \frac{\partial}{\partial y}\right) z\right] \hat{a}_y + ... \\ &\quad + \left[\left(A_y \frac{\partial}{\partial z} - A_z \frac{\partial}{\partial y}\right) y - \left(A_z \frac{\partial}{\partial x} - A_x \frac{\partial}{\partial z}\right) x\right] \hat{a}_z \end{aligned}$$

$$\begin{aligned} &= [-A_x - A_x] \, \hat{a_x} + [-A_y - A_y] \, \hat{a}_y + [-A_z - A_z] \, \hat{a}_z \\ &= -2\vec{A} \end{aligned}$$

5) Usando coordenadas e componentes cartesianos, demonstre que $(\vec{A}\cdot\vec{\nabla})\frac{\vec{r}}{r^n} = \frac{\vec{A}-n(\vec{A}\cdot\hat{a}_r)\hat{a}_r}{r^n}$.

$$
\begin{aligned}
(\vec{A}\cdot\vec{\nabla})\frac{\vec{r}}{r^n} &= (\vec{A}\cdot\vec{\nabla})(\vec{r}\,r^{-n}) \\
&= \Big[\,(\vec{A}\cdot\vec{\nabla})\vec{r}\Big]\,r^{-n} + \Big[(\vec{A}\cdot\vec{\nabla})r^{-n}\Big]\,\vec{r} \\
&= \vec{A}\,r^{-n} + \left[\left(A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{\frac{-n}{2}}\right]\vec{r} \\
&= \frac{\vec{A}}{r^n} + \left[A_x\left(\frac{-n}{2}\right)(x^2+y^2+z^2)^{\frac{-n-2}{2}}\ (2x) + \ldots\right. \\
&\qquad A_y\left(\frac{-n}{2}\right)(x^2+y^2+z^2)^{\frac{-n-2}{2}}\ (2y) + \ldots \\
&\qquad \left.+\ A_z\left(\frac{-n}{2}\right)(x^2+y^2+z^2)^{\frac{-n-2}{2}}\ (2z)\right]\vec{r} \\
&= \frac{\vec{A}}{r^n} - n\ \frac{x\ A_x + y\ A_y + z\ A_z}{(x^2+y^2+z^2)^{\frac{n+2}{2}}}\ \vec{r} \\
&= \frac{\vec{A}}{r^n} - n\ \frac{\vec{A}\cdot\vec{r}}{r^{n+2}}\ \vec{r} \\
&= \frac{\vec{A}}{r^n} - n\ \frac{\vec{A}\cdot\hat{a}_r}{r^n}\ \hat{a}_r
\end{aligned}
$$

6) Na mecânica quântica, define-se o operador vetorial momento angular como $\vec{L} = -j\vec{r} \times \vec{\nabla}$, onde $j \equiv \sqrt{-1}$. Mostre que $\vec{L} \times \vec{L} = j\vec{L}$.

$$
\begin{aligned}
\vec{L} \times \vec{L} &= (-j\vec{r} \times \vec{\nabla}) \times (-j\vec{r} \times \vec{\nabla}) \\
&= -j \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ x & y & z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} \times -j \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ x & y & z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} \\
&= j^2 \left[\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right) \hat{a}_x + \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right) \hat{a}_y + \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) \hat{a}_z \right] \times ... \\
&\quad \times \left[\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right) \hat{a}_x + \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right) \hat{a}_y + \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) \hat{a}_z \right]
\end{aligned}
$$

$$
= - \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} & z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} & x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \\ y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} & z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} & x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \end{vmatrix}
$$

$$
\begin{aligned}
&= - \left\{ \left[\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right) \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) - \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right) \right] \hat{a}_x + ... \right. \\
&\quad + \left. \left[\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right) - \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right) \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) \right] \hat{a}_y \right\} + ... \\
&\quad + \left[\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right) \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right) - \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right) \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right) \right] \hat{a}_z \Big\} \\
&\quad \vdots
\end{aligned}
$$

$$\vdots$$

$$\begin{aligned}
\vec{L} \times \vec{L} = -\Bigg\{&\left[z\frac{\partial}{\partial y} - yz\frac{\partial^2}{\partial x^2} + yz\frac{\partial^2}{\partial x^2} - y\frac{\partial}{\partial z}\right]\hat{a}_x + ... \\
&+\left[x\frac{\partial}{\partial z} - xz\frac{\partial^2}{\partial y^2} + xz\frac{\partial^2}{\partial y^2} - z\frac{\partial}{\partial x}\right]\hat{a}_y + ... \\
&+\left[y\frac{\partial}{\partial x} - xy\frac{\partial^2}{\partial z^2} + xy\frac{\partial^2}{\partial z^2} - x\frac{\partial}{\partial y}\right]\hat{a}_z\Bigg\} \\
&= \left[y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right]\hat{a}_x + \left[z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right]\hat{a}_y + \left[x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right]\hat{a}_z \\
&= \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ x & y & z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} \\
&= -j^2 \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ x & y & z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} \\
&= j\left\{-j \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ x & y & z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}\right\} \\
&= j\vec{L}
\end{aligned}$$

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7) Usando coordenadas e componentes cartesianos, demonstre as seguintes equações: a) $\vec{\nabla}(f+g) = \vec{\nabla} f + \vec{\nabla} g$; b) $\vec{\nabla}(fg) = (\vec{\nabla} f)g + (\vec{\nabla} g)f$.

a)

$$\begin{aligned}
\vec{\nabla}(f+g) &= \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right)(f+g) \\
&= \frac{\partial}{\partial x}(f+g)\;\hat{a}_x + \frac{\partial}{\partial y}(f+g)\;\hat{a}_y + \frac{\partial}{\partial z}(f+g)\;\hat{a}_z \\
&= \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}\right)\hat{a}_x + \left(\frac{\partial f}{\partial y} + \frac{\partial g}{\partial y}\right)\hat{a}_y + \left(\frac{\partial f}{\partial z} + \frac{\partial g}{\partial z}\right)\hat{a}_z \\
&= \frac{\partial f}{\partial x}\hat{a}_x + \frac{\partial g}{\partial x}\hat{a}_x + \frac{\partial f}{\partial y}\hat{a}_y + \frac{\partial g}{\partial y}\hat{a}_y + \frac{\partial f}{\partial z}\hat{a}_z + \frac{\partial g}{\partial z}\hat{a}_z \\
&= \frac{\partial f}{\partial x}\hat{a}_x + \frac{\partial f}{\partial y}\hat{a}_y + \frac{\partial f}{\partial z}\hat{a}_z + \frac{\partial f}{\partial x}\hat{a}_x + \frac{\partial g}{\partial y}\hat{a}_y + \frac{\partial g}{\partial z}\hat{a}_z \\
&= \vec{\nabla} f + \vec{\nabla} g
\end{aligned}$$

b)

$$\begin{aligned}
\vec{\nabla}(fg) &= \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right)(fg) \\
&= \frac{\partial}{\partial x}(fg)\;\hat{a}_x + \frac{\partial}{\partial y}(fg)\;\hat{a}_y + \frac{\partial}{\partial z}(fg)\;\hat{a}_z \\
&= \left(g\frac{\partial f}{\partial x} + f\frac{\partial g}{\partial x}\right)\hat{a}_x + \left(g\frac{\partial f}{\partial y} + f\frac{\partial g}{\partial y}\right)\hat{a}_y + \left(g\frac{\partial f}{\partial z} + f\frac{\partial g}{\partial z}\right)\hat{a}_z \\
&= g\frac{\partial f}{\partial x}\hat{a}_x + g\frac{\partial f}{\partial y}\hat{a}_y + g\frac{\partial f}{\partial z}\hat{a}_z + f\frac{\partial g}{\partial x}\hat{a}_x + f\frac{\partial g}{\partial y}\hat{a}_y + f\frac{\partial g}{\partial z}\hat{a}_z \\
&= g\vec{\nabla} f + f\vec{\nabla} g
\end{aligned}$$

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8) Se $f = f\left(g(\vec{r})\right)$, demonstre que $\vec{\nabla}[f(g)] = \frac{df}{dg}\vec{\nabla}g$.

$$\begin{aligned}
\vec{\nabla}[f(g)] &= \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right) f(g(\vec{r})) \\
&= \frac{\partial}{\partial x} f(g(\vec{r}))\ \hat{a}_x + \frac{\partial}{\partial y} f(g(\vec{r}))\ \hat{a}_y + \frac{\partial}{\partial z} f(g(\vec{r}))\ \hat{a}_z \\
&= \frac{df}{dg}\frac{\partial g}{\partial x}\hat{a}_x + \frac{df}{dg}\frac{\partial g}{\partial y}\frac{\partial y}{\partial y}\hat{a}_y + \frac{df}{dg}\frac{\partial g}{\partial z}\frac{\partial z}{\partial z}\hat{a}_z \\
&= \frac{df}{dg}\left[\frac{\partial g}{\partial x}\hat{a}_x + \frac{\partial g}{\partial y}\hat{a}_y + \frac{\partial z}{\partial z}\hat{a}_z\right] \\
&= \frac{df}{dg}\vec{\nabla}g
\end{aligned}$$

---

9) Demonstre que: $\vec{\nabla}(\vec{U}\cdot\vec{r}) = \vec{U}$.

$$\begin{aligned}
\vec{\nabla}(\vec{U}\cdot\vec{r}) &= \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right)\Big[(U_x\hat{a}_x + U_y\hat{a}_y + U_z\hat{a}_z)\cdot \ldots \\
&\quad \cdot(x\hat{a}_x + y\hat{a}_y + z\hat{a}_z)\Big] \\
&= \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right)(xU_x + yU_y + zU_z) \\
&= U_x\hat{a}_x + U_y\hat{a}_y + U_z\hat{a}_z \\
&= \vec{U}
\end{aligned}$$

---

10) Usando coordenadas e componentes cartesianos, demonstre que $\vec{\nabla}\frac{\vec{U}\cdot\hat{a}_r}{r^2} = \frac{\vec{U}-3(\vec{U}\cdot\hat{a}_r)\hat{a}_r}{r^3}$.

$$\begin{aligned}
\vec{\nabla}\frac{\vec{U}\cdot\hat{a}_r}{r^2} &= \vec{\nabla}\frac{(U_x\hat{a}_x + U_y\hat{a}_y + U_z\hat{a}_z)\cdot\vec{r}}{r^3} \\
&= \vec{\nabla}\left\{\left[(U_x\hat{a}_x + U_y\hat{a}_y + U_z\hat{a}_z)\cdot(x\hat{a}_x + y\hat{a}_y + z\hat{a}_z\right] r^{-3}\right\} \\
&= \vec{\nabla}\left\{(xU_x + yU_x + zU_x)r^{-3}\right\} \\
&= r^{-3}\vec{\nabla}\left\{(xU_x + yU_y + zU_z)\right\} + (xU_x + yU_y + zU_z)\vec{\nabla}\left\{r^{-3}\right\} \\
&= r^{-3}\vec{U} + (xU_x + yU_y + zU_z)\left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right)\left\{r^{-3}\right\} \\
&= r^{-3}\vec{U} + (\vec{U}\cdot\vec{r})\left(\frac{\partial r^{-3}}{\partial x}\hat{a}_x + \frac{\partial r^{-3}}{\partial y}\hat{a}_y + \frac{\partial r^{-3}}{\partial z}\hat{a}_z\right) \\
&= r^{-3}\vec{U} + (\vec{U}\cdot\vec{r})\left(-3r^{-4}\frac{\partial r}{\partial x}\hat{a}_x - 3r^{-4}\frac{\partial r}{\partial y}\hat{a}_y - 3r^{-4}\frac{\partial r}{\partial z}\hat{a}_z\right) \\
&= r^{-3}\vec{U} - 3r^{-4}(\vec{U}\cdot\vec{r})\left(\frac{\partial(x^2+y^2+z^2)^{\frac{1}{2}}}{\partial x}\hat{a}_x + \frac{\partial(x^2+y^2+z^2)^{\frac{1}{2}}}{\partial y}\hat{a}_y + \ldots\right. \\
&\qquad \left. + \frac{\partial(x^2+y^2+z^2)^{\frac{1}{2}}}{\partial z}\hat{a}_z\right) \\
&= r^{-3}\vec{U} - 3r^{-3}(\vec{U}\cdot\hat{a}_r)\left(\frac{1}{2}(x^2+y^2+z^2)^{\frac{-1}{2}}(2x)\hat{a}_x + \ldots\right. \\
&\qquad \left. + \frac{1}{2}(x^2+y^2+z^2)^{\frac{-1}{2}}(2y)\hat{a}_y + \frac{1}{2}(x^2+y^2+z^2)^{\frac{-1}{2}}(2z)\hat{a}_z\right) \\
&= \frac{1}{r^3}\left(\vec{U} - 3(\vec{U}\cdot\hat{a}_r)\frac{x\hat{a}_x + y\hat{a}_y + z\hat{a}_z}{(x^2+y^2+z^2)^{\frac{1}{2}}}\right) \\
&= \frac{\vec{U} - 3(\vec{U}\cdot\hat{a}_r)\hat{a}_r}{r^3}
\end{aligned}$$

11) Mostre que $\vec{t} = \vec{\nabla} f \times \vec{\nabla} g$ é um vetor tangente à curva determinada pela interseção das superfícies $f(\vec{r}) = \mathbf{constante}$ e $g(\vec{r}) = \mathbf{constante}$.

De acordo com as propriedades do gradiente, $\vec{\nabla} f$ será um vetor ortogonal à superfície $f(\vec{r}) = \mathbf{constante}$ e $\vec{\nabla} g$ será um vetor ortogonal à superfície $g(\vec{r}) = \mathbf{constante}$. Todos os pontos da curva C, determinada pela interseção dessas superfícies, pertencem a ambas simultaneamente. Portanto, em cada um desses pontos os vetores $\vec{\nabla} f$ e $\vec{\nabla} g$ serão ortogonais a C. O produto vetorial de dois vetores é sempre ortogonal a ambos os vetores, portanto, $\vec{t}$ estará na mesma direção da curva C em cada um dos seus pontos.
Isso não é válido no caso especial de os vetores $\vec{\nabla} f$ e $\vec{\nabla} g$ estarem na mesma direção (por exemplo, duas superfícies cilíndricas tangentes na direção dos eixos de simetria), porque nesse caso $\vec{t}$ será nulo.

---

12) Forneça um vetor unitário normal ao plano de equação cartesiana $k_x x + k_y y + k_z z = C$, onde $\vec{k} \equiv k_x \hat{a}_x + k_y \hat{a}_y + k_z \hat{a}_z$ e C são uniformes.

$$
\begin{aligned}
\vec{\nabla} C &= \frac{\partial C}{\partial x}\hat{a}_x + \frac{\partial C}{\partial y}\hat{a}_y + \frac{\partial C}{\partial z}\hat{a}_z \\
&= k_x \hat{a}_x + k_y \hat{a}_y + k_z \hat{a}_z \\
\vec{a}_N &= \frac{\vec{\nabla} C}{|\vec{\nabla} C|} \\
&= \frac{k_x \hat{a}_x + k_y \hat{a}_y + k_z \hat{a}_z}{\sqrt{k_x^2 + k_y^2 + k_z^2}}
\end{aligned}
$$

---

13) Usando coordenadas e componentes esféricos, determine um vetor unitário normal à superfície esférica de equação $r = a$, onde a é uniforme.

$$\begin{aligned}
\vec{\nabla} a &= \frac{\partial a}{\partial r}\hat{a}_r + \frac{1}{r\sin\theta}\frac{\partial a}{\partial \phi}\hat{a}_\phi + \frac{1}{r}\frac{\partial a}{\partial \theta}\hat{a}_\theta \\
&= \hat{a}_r \\
\vec{a}_N &= \frac{\vec{\nabla} a}{|\vec{\nabla} a|} \\
&= \frac{\hat{a}_r}{1} \\
&= \hat{a}_r
\end{aligned}$$

---

14) Repita o problema 13) usando coordenadas e componentes cilíndricas circulares.

$$\begin{aligned}
a &= r \\
&= (\rho^2 + z^2)^{\frac{1}{2}} \\
\vec{\nabla} a &= \left(\frac{\partial}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial \phi}\hat{a}_\phi + \frac{\partial}{\partial z}\hat{a}_z\right)(\rho^2 + z^2)^{\frac{1}{2}} \\
&= \frac{1}{2}(2\rho)(\rho^2 + z^2)^{\frac{-1}{2}}\hat{a}_\rho + \vec{0} + \frac{1}{2}(2z)(\rho^2 + z^2)^{\frac{-1}{2}}\hat{a}_z \\
&= \frac{\rho\hat{a}_\rho + z\hat{a}_z}{\sqrt{\rho^2 + z^2}} \\
\vec{a}_N &= \frac{\vec{\nabla} a}{|\vec{\nabla} a|} \\
&= \frac{\rho\hat{a}_\rho + z\hat{a}_z}{1} \\
&= \rho\hat{a}_\rho + z\hat{a}_z
\end{aligned}$$

---

15) Usando coordenadas e componentes cartesianos, demonstre as seguintes equações: a) $\vec{\nabla}\cdot(\vec{A}+\vec{B})=\vec{\nabla}\cdot\vec{A}+\vec{\nabla}\cdot\vec{B}$; b) $\vec{\nabla}\cdot(f\vec{A})=\vec{\nabla}f\cdot\vec{A}+f\vec{\nabla}\cdot\vec{A}$.

a)

$$\begin{aligned}
\vec{\nabla}\cdot(\vec{A}+\vec{B}) &= \left(\frac{\partial}{\partial x}\hat{a}_x+\frac{\partial}{\partial y}\hat{a}_y+\frac{\partial}{\partial z}\hat{a}_z\right)(A_x\hat{a}_x+A_y\hat{a}_y+A_z\hat{a}_z+...\\
&\qquad + B_x\hat{a}_x+B_y\hat{a}_y+B_z\hat{a}_z)\\
&= \frac{\partial A_x}{\partial x}+\frac{\partial B_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial B_y}{\partial y}+\frac{\partial A_z}{\partial z}+\frac{\partial B_z}{\partial z}\\
&= \frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}+\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}\\
&= \vec{\nabla}\cdot\vec{A}+\vec{\nabla}\cdot\vec{B}
\end{aligned}$$

b)

$$\begin{aligned}
\vec{\nabla}\cdot(f\vec{A}) &= \left(\frac{\partial}{\partial x}\hat{a}_x+\frac{\partial}{\partial y}\hat{a}_y+\frac{\partial}{\partial z}\hat{a}_z\right)\cdot(fA_x\hat{a}_x+fA_y\hat{a}_y+fA_z\hat{a}_z)\\
&= A_x\frac{\partial f}{\partial x}+f\frac{\partial A_x}{\partial x}+A_y\frac{\partial f}{\partial y}+f\frac{\partial A_y}{\partial y}+A_z\frac{\partial f}{\partial z}+f\frac{\partial A_z}{\partial z}\\
&= A_x\frac{\partial f}{\partial x}+A_y\frac{\partial f}{\partial y}+A_z\frac{\partial f}{\partial z}+f\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right)\\
&= \Big(A_x\hat{a}_x+A_y\hat{a}_y+A_z\hat{a}_z\Big)\cdot\left(\frac{\partial f}{\partial x}\hat{a}_x+\frac{\partial f}{\partial y}\hat{a}_y+\frac{\partial f}{\partial z}\hat{a}_z\right)+...\\
&\qquad + f\left(\frac{\partial}{\partial x}\hat{a}_x+\frac{\partial}{\partial y}\hat{a}_y+\frac{\partial}{\partial z}\hat{a}_z\right)\cdot\Big(A_x\hat{a}_x+A_y\hat{a}_y+A_z\hat{a}_z\Big)\\
&= \vec{A}\cdot\vec{\nabla}f+f\vec{\nabla}\cdot\vec{A}
\end{aligned}$$

---

16) Demonstre as equações: a) $\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = (\vec{\nabla} \times \vec{A}) \cdot \vec{B} - (\vec{\nabla} \times \vec{B}) \cdot \vec{A}$; b) $\vec{\nabla} \cdot (\vec{U} \times \vec{r}) = 0$.

a)

$$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right) \cdot \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right) \cdot \Big[(A_yB_z - B_yA_z)\hat{a}_x + \ldots$$

$$+ (A_zB_x - B_zA_x)\hat{a}_y + (A_xB_y - B_xA_y)\hat{a}_z \Big]$$

$$= \frac{\partial}{\partial x}(A_yB_z - B_yA_z) + \frac{\partial}{\partial y}(A_zB_x - B_zA_x) + \frac{\partial}{\partial z}(A_xB_y - B_xA_y)$$

$$= \frac{\partial}{\partial x}(A_yB_z) + \frac{\partial}{\partial y}(A_zB_x) + \frac{\partial}{\partial z}(A_xB_y) - \frac{\partial}{\partial x}(B_yA_z) - \ldots$$

$$- \frac{\partial}{\partial y}(B_zA_x) - \frac{\partial}{\partial z}(B_xA_y)$$

$$= \left(\frac{\partial}{\partial x}A_y\hat{a}_z + \frac{\partial}{\partial y}A_z\hat{a}_x + \frac{\partial}{\partial z}A_x\hat{a}_y\right) \cdot (B_z\hat{a}_z + B_x\hat{a}_x + B_y\hat{a}_y) - \ldots$$

$$- \left(\frac{\partial}{\partial x}B_y\hat{a}_z + \frac{\partial}{\partial y}B_z\hat{a}_x + \frac{\partial}{\partial z}B_x\hat{a}_y\right) \cdot (A_z\hat{a}_z + A_x\hat{a}_x + A_y\hat{a}_y)$$

$$= \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \cdot \vec{B} - \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix} \cdot \vec{A}$$

$$= (\vec{\nabla} \times \vec{A}) \cdot \vec{B} - (\vec{\nabla} \times \vec{B}) \cdot \vec{A}$$

b)

$$\begin{aligned}
\vec{\nabla} \cdot (\vec{U} \times \vec{r}) &= (\vec{\nabla} \times \vec{U}) \cdot \vec{r} - (\vec{\nabla} \times \vec{r}) \cdot \vec{U} \\
&= \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ U_x & U_y & U_z \end{vmatrix} \cdot \vec{r} - \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \cdot \vec{U} \\
&= (\vec{0} + \vec{0} + \vec{0}) \cdot \vec{r} - (\vec{0} + \vec{0} + \vec{0}) \cdot \vec{U} \\
&= 0
\end{aligned}$$

17) Usando coordenadas e componentes esféricos, mostre que: a) $\vec{\nabla} \cdot \vec{r} = 3$; b) $\vec{\nabla} \cdot \left(f(r)\vec{r} \right) = 3f + r\frac{df}{dr}$; c) Determine a função de r cujo produto por $\vec{r}$ seja solenoidal.

a)

$$\begin{aligned}\vec{\nabla} \cdot \vec{r} &= \left(\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \ \ \hat{a}_r + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \ \ \hat{a}_\phi + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\sin\theta \ \ \hat{a}_\theta \right) \cdot (r\hat{a}_r) \\ &= \frac{1}{r^2}\frac{\partial r^3}{\partial r} \\ &= 3\end{aligned}$$

b)

$$\begin{aligned}\vec{\nabla} \cdot \left(f(r)\vec{r} \right) &= \vec{r} \cdot \vec{\nabla} f(r) + f(r)\vec{\nabla} \cdot \vec{r} \\ &= (r\hat{a}_r) \cdot ... \\ & \quad \left(\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \ \ \hat{a}_r + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \ \ \hat{a}_\phi + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\sin\theta \ \ \hat{a}_\theta \right) f(r) + ... \\ & \quad 3f(r) \\ &= (r\hat{a}_r) \cdot \left(\frac{\partial f(r)}{\partial r}\hat{a}_r \right) + 3f(r) \\ &= r\frac{df}{dr} + 3f\end{aligned}$$

c)

$$\begin{aligned}\vec{\nabla} \cdot \left(f(r)\vec{r} \right) = 0 \implies & r\frac{df}{dr} + 3f = 0 \\ & \frac{df}{f} = -3\frac{dr}{r} \\ & \ln(f) = -3\ln(f) + C_1 \qquad \forall r > 0 \\ & f = \frac{C}{r^3} \qquad\qquad\qquad \forall r > 0\end{aligned}$$

---

18) Calcule a divergência do vetor velocidade linear $\vec{v}$ de um corpo rígido que gira com velocidade angular $\vec{\omega}$ em torno de um eixo fixo.

$$\begin{aligned}
\vec{\nabla}\cdot\vec{v} &= \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\ \hat{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial\phi}\ \hat{a}_\phi + \frac{\partial}{\partial z}\hat{a}_z\right)\cdot(\omega\rho\hat{a}_\phi)\\
&= \frac{1}{\rho}\phi\omega\rho\\
&= 0
\end{aligned}$$

---

19) Usando coordenadas e componentes esféricos, determine a divergência do vetor $\vec{v}(\vec{r}) = \frac{1}{r^2}f(\theta,\phi)\hat{a}_r + \csc\theta g(\phi,r)\hat{a}_\theta + h(r,\theta)\hat{a}_\phi$.

$$\begin{aligned}
\vec{\nabla}\cdot\vec{v}(\vec{r}) &= \left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\ \ \hat{a}_r + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\ \ \hat{a}_\phi + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\ \ \hat{a}_\theta\right)\cdot ...\\
&\quad\cdot\left(\frac{1}{r^2}f(\theta,\phi)\hat{a}_r + \csc\theta\ g(\phi,r)\hat{a}_\theta +\ h(r,\theta)\hat{a}_\phi\right)\\
&= \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{1}{r^2}f(\theta,\phi) + \frac{1}{r\sin\theta}\phi h(r,\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\csc\theta\ g(\phi,r)\\
&= 0+0+0
\end{aligned}$$

---

20) Seja um sistema de coordenadas esféricas r, θ, ϕ. a) A partir da expressão do gradiente de f em coordenadas e componentes esféricos, atribua ao operador nabla uma expressão neste sistema; b) Aplique o operador determinado em a) escalarmente ao vetor $\vec{v} = v_r \hat{a}_r + v_\theta \hat{a}_\theta + v_\phi \hat{a}_\phi$; c) Compare o resultado do item b) com a expressão correta da divergência do vetor $\vec{v}$.

a)

$$\vec{\nabla}_e = \frac{\partial}{\partial r}\hat{a}_r + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\hat{a}_\phi + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{a}_\theta$$

b)

$$\vec{\nabla}_e \cdot \vec{v} = \left(\frac{\partial}{\partial r}\hat{a}_r + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\hat{a}_\phi + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{a}_\theta\right) \cdot (v_r\hat{a}_r + v_\phi\hat{a}_\phi + v_\theta\hat{a}_\theta)$$

$$= \frac{\partial v_r}{\partial r} + \frac{1}{r\sin\theta}\phi v_\phi + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta}$$

c)

$$\vec{\nabla} \cdot \vec{v} = \left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2 \;\; \hat{a}_r + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \;\; \hat{a}_\phi + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta \;\; \hat{a}_\theta\right) \cdot (v_r\hat{a}_r + \ldots$$

$$+ v_\phi\hat{a}_\phi + v_\theta\hat{a}_\theta)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}r^2 v_r + \frac{1}{r\sin\theta}\phi v_\phi + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta \; v_\theta$$

---

21) Usando coordenadas e componentes cartesianos, demonstre que: a) $\vec{\nabla}\times(\vec{A}+\vec{B})=\vec{\nabla}\times\vec{A}+\vec{\nabla}\times\vec{B}$; b) $\vec{\nabla}\times(f\vec{A})=\vec{\nabla}f\times\vec{A}+f\vec{\nabla}\times\vec{A}$.

a)

$$\begin{aligned}
\vec{\nabla}\times(\vec{A}+\vec{B}) &= \vec{\nabla}\times(A_x\hat{a}_x+A_y\hat{a}_y+A_z\hat{a}_z+B_x\hat{a}_x+B_y\hat{a}_y+B_z\hat{a}_z)\\
&= \vec{\nabla}\times[(A_x+B_x)\hat{a}_x+(A_y+B_y)\hat{a}_y+(A_z+B_z)\hat{a}_z]\\
&= \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x+B_x & A_y+B_y & A_z+B_z \end{vmatrix}\\
&= \left[\frac{\partial}{\partial y}(A_z+B_z)-\frac{\partial}{\partial z}(A_y+B_y)\right]\hat{a}_x+...\\
&\quad +\left[\frac{\partial}{\partial z}(A_x+B_x)-\frac{\partial}{\partial x}(A_z+B_z)\right]\hat{a}_y+...\\
&\quad +\left[\frac{\partial}{\partial x}(A_y+B_y)-\frac{\partial}{\partial y}(A_x+B_x)\right]\hat{a}_z\\
&= \frac{\partial}{\partial y}A_z\hat{a}_x-\frac{\partial}{\partial z}A_y\hat{a}_x+\frac{\partial}{\partial z}A_x\hat{a}_y-\frac{\partial}{\partial x}A_z\hat{a}_y+\frac{\partial}{\partial x}A_y\hat{a}_z-...\\
&\quad -\frac{\partial}{\partial y}A_x\hat{a}_z+\frac{\partial}{\partial y}B_z\hat{a}_x-\frac{\partial}{\partial z}B_y\hat{a}_x+\frac{\partial}{\partial z}B_x\hat{a}_y-...\\
&\quad -\frac{\partial}{\partial x}B_z\hat{a}_y+\frac{\partial}{\partial x}B_y\hat{a}_z-\frac{\partial}{\partial y}B_x\hat{a}_z\\
&= \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}+\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}\\
&= \vec{\nabla}\times\vec{A}+\vec{\nabla}\times\vec{B}
\end{aligned}$$

b)

$$\vec{\nabla} \times (f\vec{A}) = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fA_x & fA_y & fA_z \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y} fA_z - \frac{\partial}{\partial z} fA_y\right] \hat{a}_x + \left[\frac{\partial}{\partial z} fA_x - \frac{\partial}{\partial x} fA_z\right] \hat{a}_y + ...$$

$$+ \left[\frac{\partial}{\partial x} fA_y - \frac{\partial}{\partial y} fA_x\right] \hat{a}_z$$

$$= A_z \frac{\partial f}{\partial y} \hat{a}_x + f \frac{\partial A_z}{\partial y} \hat{a}_x - A_y \frac{\partial f}{\partial z} \hat{a}_x - f \frac{\partial A_y}{\partial z} \hat{a}_x + A_x \frac{\partial f}{\partial z} \hat{a}_y + f \frac{\partial A_x}{\partial z} - ...$$

$$- A_z \frac{\partial f}{\partial x} \hat{a}_y - f \frac{\partial A_z}{\partial x} \hat{a}_y + A_y \frac{\partial f}{\partial x} \hat{a}_z + f \frac{\partial A_y}{\partial x} \hat{a}_z - A_x \frac{\partial f}{\partial y} \hat{a}_z - f \frac{\partial A_x}{\partial y} \hat{a}_z$$

$$= \left[A_z \frac{\partial f}{\partial y} - A_y \frac{\partial f}{\partial z}\right] \hat{a}_x + \left[A_x \frac{\partial f}{\partial z} - A_z \frac{\partial f}{\partial x}\right] \hat{a}_y + \left[A_y \frac{\partial f}{\partial x} - A_x \frac{\partial f}{\partial y}\right] \hat{a}_z + ...$$

$$f \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right] \hat{a}_x + f \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right] \hat{a}_y + f \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right] \hat{a}_z$$

$$= \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} + f \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \vec{\nabla} f \times \vec{A} + f \vec{\nabla} \times \vec{A}$$

22) Usando coordenadas e componentes esféricos, mostre que $\vec{\nabla} \times (f(r)\vec{r}) = \vec{0}$.

$$
\begin{aligned}
\vec{\nabla} \times (f(r)\vec{r}) &= \vec{\nabla} f(r) \times \vec{r} + f(r)\vec{\nabla} \times \vec{r} \\
&= \left(\frac{\partial f(r)}{\partial r}\hat{a}_r + \frac{1}{r\sin\theta}\frac{\partial f(r)}{\partial \phi}\hat{a}_\phi + \frac{1}{r}\frac{\partial f(r)}{\partial \theta}\hat{a}_\theta \right) \times (r\hat{a}_r) + ... \\
&\qquad + f(r)\frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\theta\hat{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ r & 0 & 0 \end{vmatrix} \\
&= \left(\frac{\partial f(r)}{\partial r}\hat{a}_r \right) \times (r\hat{a}_r) + f(r)\frac{1}{r^2\sin\theta}\left[r\frac{\partial r}{\partial \phi}\hat{a}_\theta - r\sin\theta\frac{\partial r}{\partial \theta}\hat{a}_\phi \right] \\
&= \vec{0} + f(r)\frac{1}{r^2\sin\theta}(\vec{0} - \vec{0}) \\
&= \vec{0}
\end{aligned}
$$

---

23) Demonstre que $\vec{\nabla} \times (\vec{U} \times \vec{r}) = 2\vec{U}$.

$$\begin{aligned}
\vec{\nabla} \times (\vec{U} \times \vec{r}) &= \vec{\nabla} \times \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ U_x & U_y & U_z \\ x & y & z \end{vmatrix} \\
&= \vec{\nabla} \times [(U_y z - U_z y)\hat{a}_x + (U_z x - U_x z)\hat{a}_y + (U_x y - U_y x)\hat{a}_z] \\
&= \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ U_y z - U_z y & U_z x - U_x z & U_x y - U_y x \end{vmatrix} \\
&= \left[\frac{\partial}{\partial y}(U_x y - U_y x) - \frac{\partial}{\partial z}(U_z x - U_x z)\right]\hat{a}_x + ... \\
&\quad + \left[\frac{\partial}{\partial z}(U_y z - U_z y) - \frac{\partial}{\partial x}(U_x y - U_y x)\right]\hat{a}_y + ... \\
&\quad + \left[\frac{\partial}{\partial x}(U_z x - U_x z) - \frac{\partial}{\partial y}(U_y z - U_z y)\right]\hat{a}_z \\
&= [U_x + U_x]\hat{a}_x + [U_y + U_y]\hat{a}_y + [U_z + U_z]\hat{a}_z \\
&= 2\vec{U}
\end{aligned}$$

---

24) Utilizando coordenadas e componentes cartesianos, mostre que, se $\vec{A}$ obedece à equação vetorial $\vec{A}\cdot(\vec{\nabla}\times\vec{A}) = 0$, então, qualquer que seja a função f, $(f\vec{A})\cdot[\vec{\nabla}\times(f\vec{A})] = 0$.

$$
\begin{aligned}
(f\vec{A})\cdot[\vec{\nabla}\times(f\vec{A})] &= f(A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z)\cdot\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fA_x & fA_y & fA_z \end{vmatrix} \\
&= (fA_x\hat{a}_x + fA_y\hat{a}_y + fA_z\hat{a}_z)\cdot\left[\left(\frac{\partial}{\partial y}fA_z - \frac{\partial}{\partial z}fA_y\right)\hat{a}_x + \ldots\right. \\
&\quad \left. + \left(\frac{\partial}{\partial z}fA_x - \frac{\partial}{\partial x}fA_z\right)\hat{a}_y + \left(\frac{\partial}{\partial x}fA_y - \frac{\partial}{\partial y}fA_x\right)\hat{a}_z\right] \\
&= (fA_x\hat{a}_x + fA_y\hat{a}_y + fA_z\hat{a}_z)\cdot\ldots \\
&\quad \cdot\left[\left(A_z\frac{\partial f}{\partial y} - A_y\frac{\partial f}{\partial z} + f\frac{\partial A_z}{\partial y} - f\frac{\partial A_y}{\partial z}\right)\hat{a}_x + \ldots\right. \\
&\quad + \left(A_x\frac{\partial f}{\partial z} - A_z\frac{\partial f}{\partial x} + f\frac{\partial A_x}{\partial z} - f\frac{\partial A_z}{\partial x}\right)\hat{a}_y + \ldots \\
&\quad \left. + \left(A_y\frac{\partial f}{\partial x} - A_x\frac{\partial f}{\partial y} + f\frac{\partial A_y}{\partial x} - f\frac{\partial A_x}{\partial y}\right)\hat{a}_z\right] \\
&= fA_x\left(A_z\frac{\partial f}{\partial y} - A_y\frac{\partial f}{\partial z} + f\frac{\partial A_z}{\partial y} - f\frac{\partial A_y}{\partial z}\right) + \ldots \\
&\quad + fA_y\left(A_x\frac{\partial f}{\partial z} - A_z\frac{\partial f}{\partial x} + f\frac{\partial A_x}{\partial z} - f\frac{\partial A_z}{\partial x}\right) + \ldots \\
&\quad + fA_z\left(A_y\frac{\partial f}{\partial x} - A_x\frac{\partial f}{\partial y} + f\frac{\partial A_y}{\partial x} - f\frac{\partial A_x}{\partial y}\right) \\
&\quad \vdots
\end{aligned}
$$

$$
\begin{aligned}
&\vdots\\
(f\vec{A})\cdot[\vec{\nabla}\times(f\vec{A})] &= fA_xA_z\frac{\partial f}{\partial y} - fA_xA_y\frac{\partial f}{\partial z} + fA_yA_x\frac{\partial f}{\partial z} - fA_yA_z\frac{\partial f}{\partial x} + ...\\
&\quad + fA_zA_y\frac{\partial f}{\partial x} - fA_zA_x\frac{\partial f}{\partial y} + f^2A_x\left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right] + ...\\
&\quad + f^2A_y\left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right] + f^2A_z\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]\\
&= 0 + 0 + 0 + f^2(A_x\hat{a}_x + A_y\hat{a}_y + A_x\hat{a}_x)\cdot ...\\
&\quad \cdot\left(\left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]\hat{a}_x + \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]\hat{a}_y + ...\right.\\
&\quad \left. + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]\hat{a}_z\right)\\
&= f^2\vec{A}\cdot(\vec{\nabla}\vec{A})\\
&= f^2\cdot 0\\
&= 0
\end{aligned}
$$

---

25) Mostre que: a) em geral, o rotacional de um vetor não é perpendicular ao vetor; b) se um vetor só depender das coordenadas de um plano e só tiver componentes nesse plano, então seu rotacional lhe será perpendicular; c) se um vetor tiver direção uniforme, então, mesmo que seu módulo dependa das três coordenadas, seu rotacional lhe será perpendicular.

a)

$$\begin{aligned}
\vec{A}\cdot[\vec{\nabla}\times\vec{A}] &= (A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z)\cdot\begin{vmatrix}\hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z\end{vmatrix} \\
&= (A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z)\cdot\left[\left(\frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y\right)\hat{a}_x + \dots\right. \\
&\qquad \left. + \left(\frac{\partial}{\partial z}A_x - \frac{\partial}{\partial x}A_z\right)\hat{a}_y + \left(\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right)\hat{a}_z\right] \\
&= (A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z)\cdot\left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{a}_x + \dots\right. \\
&\qquad \left. + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{a}_z\right] \\
&= A_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + A_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + A_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \\
&\not\equiv 0
\end{aligned}$$

b)

$$\begin{aligned}
A_z = \frac{\partial A_x}{\partial z} = \frac{\partial A_y}{\partial z} = 0 \implies \vec{A}\cdot[\vec{\nabla}\times\vec{A}] &= A_x(0) + A_y(0) + 0\cdot\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \\
&= 0
\end{aligned}$$

c)

$$A_x = \textbf{constante} \wedge A_y = \textbf{constante} \wedge A_z = \textbf{constante} \implies \dots$$

$$\begin{aligned}
\vec{A}\cdot[\vec{\nabla}\times\vec{A}] &= A_x(0) + A_y(0) + A_z(0) \\
&= 0
\end{aligned}$$

---

26) Demonstre que, mesmo que o rotacional do vetor $\vec{v} = v(\rho)\hat{a}_z$ obedeça, em relação a $\vec{v}$, à regra da mão direita, o rotacional do vetor $\vec{v'} = v(\rho)\hat{a}_z - C\hat{a}_z$, onde C é um escalar uniforme e positivo, pode obedecer ou não.

$$\vec{v} \times (\vec{\nabla} \times \vec{v}) = [v(\rho)\hat{a}_z] \times \left(\frac{1}{\rho} \begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial\rho} & \phi & \frac{\partial}{\partial z} \\ 0 & 0 & v(\rho) \end{vmatrix} \right)$$

$$= [v(\rho)\hat{a}_z] \times \left(\frac{1}{\rho} \left[-\rho \, \frac{\partial v(\rho)}{\partial\rho} \hat{a}_\phi \right] \right)$$

$$= [v(\rho)\hat{a}_z] \times \left(-\frac{\partial v(\rho)}{\partial\rho} \hat{a}_\phi \right)$$

$$= \left(v(\rho) \, \frac{\partial v(\rho)}{\partial\rho} \right) \hat{a}_\rho$$

$$\hat{a}_{\vec{v}} = \mathbf{sgn}(v) \; \hat{a}_z$$

$$\hat{a}_{\vec{\nabla}\times\vec{v}} = -\,\mathbf{sgn}\left(\frac{\partial v}{\partial\rho} \right) \hat{a}_\phi$$

$$\hat{a}_{\vec{v}\times(\vec{\nabla}\times\vec{v})} = \mathbf{sgn}(v) \;\; \mathbf{sgn}\left(\frac{\partial v}{\partial\rho} \right) \hat{a}_\rho$$

v	$\frac{\partial v}{\partial\rho}$	$\hat{a}_{\vec{v}}$	$\hat{a}_{\vec{\nabla}\times\vec{v}}$	$\hat{a}_{\vec{v}\times(\vec{\nabla}\times\vec{v})}$	Sistema
> 0	> 0	$+\hat{a}_z$	$-\hat{a}_\phi$	$+\hat{a}_\rho$	dextrógiro
> 0	< 0	$+\hat{a}_z$	$+\hat{a}_\phi$	$-\hat{a}_\rho$	dextrógiro
< 0	> 0	$-\hat{a}_z$	$-\hat{a}_\phi$	$-\hat{a}_\rho$	dextrógiro
< 0	> 0	$-\hat{a}_z$	$+\hat{a}_\phi$	$+\hat{a}_\rho$	dextrógiro

O sistema formado pelas bases $\left\{ \hat{a}_{\vec{v}}, \hat{a}_{\vec{\nabla}\times\vec{v}}, \hat{a}_{\vec{v}\times(\vec{\nabla}\times\vec{v})} \right\}$ é dextrógiro em todas as situações.

$$
\begin{aligned}
\vec{v'} \times (\vec{\nabla} \times \vec{v'}) &= ([v(\rho) - C]\hat{a}_z) \times \left(\frac{1}{\rho} \begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial\rho} & \phi & \frac{\partial}{\partial z} \\ 0 & 0 & v(\rho) - C \end{vmatrix} \right) \\
&= [(v(\rho) - C)\hat{a}_z] \times \left(-\frac{\partial v(\rho)}{\partial \rho} \hat{a}_\phi \right) \\
&= \left(\left[v(\rho) - C \right] \frac{\partial v(\rho)}{\partial \rho} \right) \hat{a}_\rho
\end{aligned}
$$

$$
\begin{aligned}
\hat{a}_{\vec{v'}} &= \mathbf{sgn}(v - C)\ \hat{a}_z \\
\hat{a}_{\vec{\nabla} \times \vec{v'}} &= -\ \mathbf{sgn} \left(\frac{\partial v}{\partial \rho} \right) \hat{a}_\phi \\
\hat{a}_{\vec{v'} \times (\vec{\nabla} \times \vec{v'})} &= \mathbf{sgn}(v - C)\ \ \mathbf{sgn} \left(\frac{\partial v}{\partial \rho} \right) \hat{a}_\rho
\end{aligned}
$$

O sistema formado pelas bases $\left\{ \hat{a}_{\vec{v'}}, \hat{a}_{\vec{\nabla} \times \vec{v'}}, \hat{a}_{\vec{v'} \times (\vec{\nabla} \times \vec{v'})} \right\}$ também é dextrógiro em todas as situações. Isso era esperado, uma vez que $v(\rho) - C$ pode ser escrito $v'(\rho)$, enquadrando-se assim no primeiro caso estudado.

---

27) Mostre que $\vec{A} \times \vec{B}$ será solenoidal quando $\vec{A}$ e $\vec{A}$ forem irrotacionais.

$$
\begin{aligned}
\vec{\nabla} \cdot (\vec{A} \times \vec{B}) &= (\vec{\nabla} \times \vec{A}) \cdot \vec{B} - (\vec{\nabla} \times \vec{B}) \cdot \vec{A}) \\
&= \vec{0} \cdot \vec{A} - \vec{0} \cdot \vec{B} \\
&= 0
\end{aligned}
$$

---

28) Demonstre que: a) $\nabla^2(f+g)=\nabla^2 f+\nabla^2 g$; b) $\nabla^2(fg)=(\nabla^2 f)g+2\vec{\nabla}f\cdot\vec{\nabla}g+f(\nabla^2 g)$.

a)

$$\begin{aligned}
\nabla^2(f+g) &= \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)(f+g)\\
&= \frac{\partial^2(f+g)}{\partial x^2}+\frac{\partial^2(f+g)}{\partial y^2}+\frac{\partial^2(f+g)}{\partial z^2}\\
&= \frac{\partial f}{\partial x}+\frac{\partial g}{\partial x}+\frac{\partial f}{\partial y}+\frac{\partial g}{\partial y}+\frac{\partial f}{\partial z}+\frac{\partial g}{\partial z}\\
&= \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}+\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}+\frac{\partial^2 g}{\partial z^2}\\
&= \nabla^2 f+\nabla^2 g
\end{aligned}$$

b)

$$\begin{aligned}
\nabla^2(fg) &= \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)(fg)\\
&= \frac{\partial}{\partial x}\frac{\partial}{\partial x}(fg)+\frac{\partial}{\partial y}\frac{\partial}{\partial y}(fg)+\frac{\partial}{\partial z}\frac{\partial}{\partial z}(fg)\\
&= \frac{\partial}{\partial x}\left(g\frac{\partial f}{\partial x}+f\frac{\partial g}{\partial x}\right)+\frac{\partial}{\partial y}\left(g\frac{\partial f}{\partial y}+f\frac{\partial g}{\partial y}\right)+\frac{\partial}{\partial z}\left(g\frac{\partial f}{\partial z}+f\frac{\partial g}{\partial z}\right)\\
&= g\frac{\partial f}{\partial x}+\frac{\partial g}{\partial x}\frac{\partial f}{\partial x}+f\frac{\partial g}{\partial x}+\frac{\partial f}{\partial x}\frac{\partial g}{\partial x}+g\frac{\partial f}{\partial y}+\frac{\partial g}{\partial y}\frac{\partial f}{\partial y}+f\frac{\partial g}{\partial y}+\frac{\partial g}{\partial y}\frac{\partial f}{\partial y}+...\\
&\quad +g\frac{\partial f}{\partial z}+\frac{\partial g}{\partial z}\frac{\partial f}{\partial z}+f\frac{\partial g}{\partial z}+\frac{\partial g}{\partial z}\frac{\partial f}{\partial z}\\
&= g\nabla^2 f+f\nabla^2 g+2\left(\frac{\partial f}{\partial x}\hat{a}_x+\frac{\partial f}{\partial y}\hat{a}_y+\frac{\partial f}{\partial z}\hat{a}_z\right)\cdot\left(\frac{\partial g}{\partial x}\hat{a}_x+\frac{\partial g}{\partial y}\hat{a}_y+\frac{\partial g}{\partial z}\hat{a}_z\right)\\
&= g\nabla^2 f+f\nabla^2 g+2(\vec{\nabla}f)\cdot(\vec{\nabla}g)
\end{aligned}$$

---

30) Mostre que: a) $\vec{\nabla}\cdot(\vec{\nabla}f\times\vec{\nabla}g)=0$; b) $\vec{\nabla}\times(f\vec{\nabla}g)=\vec{\nabla}f\times\vec{\nabla}g$; c) $\vec{\nabla}\times(f\vec{\nabla}g+g\vec{\nabla}f)=\vec{0}$.

a)

$$\begin{aligned}\vec{\nabla}\cdot(\vec{\nabla}f\times\vec{\nabla}g)&=[\vec{\nabla}\times\vec{\nabla}f]\cdot\vec{\nabla}g-[\vec{\nabla}\times\vec{\nabla}g]\cdot\vec{\nabla}f\\&=\vec{0}\cdot\vec{\nabla}g-\vec{0}\cdot\vec{\nabla}f\\&=0\end{aligned}$$

b)

$$\begin{aligned}\vec{\nabla}\times(f\vec{\nabla}g)&=(\vec{\nabla}f)\times(\vec{\nabla}g)+f\vec{\nabla}\times(\vec{\nabla}g)\\&=(\vec{\nabla}f)\times(\vec{\nabla}g)+f\vec{0}\\&=(\vec{\nabla}f)\times(\vec{\nabla}g)\end{aligned}$$

c)

$$\begin{aligned}\vec{\nabla}\times(f\vec{\nabla}g+g\vec{\nabla}f)&=\vec{\nabla}(f\vec{\nabla}g)+\vec{\nabla}(g\vec{\nabla}f)\\&=(\vec{\nabla}f)\times(\vec{\nabla}g)+(\vec{\nabla}g)\times(\vec{\nabla}f)\\&=(\vec{\nabla}f)\times(\vec{\nabla}g)-(\vec{\nabla}f)\times(\vec{\nabla}g)\\&=\vec{0}\end{aligned}$$

---

31) Seja r a coordenada radial esférica. a) Calcule $\nabla^2 r^n (r \neq 0 \; se \; n < 2)$ usando coordenadas esféricas; b) Forneça os valores de n que fazem r^n harmônico; c) Integre a equação de Laplace para uma função f(r) somente da coordenada radial e determine sua solução geral; d) Compare as respostas dos itens b) e c).

a)

$$\nabla^2 r^n = \left[\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] r^n \quad = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial r^n}{\partial r}\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 n r^{n-1})$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(n r^{n+1})$$

$$= \frac{1}{r^2}\; n(n+1) r^n$$

$$= n(n+1) r^{n-2}$$

b)

$$\nabla^2 r^n = 0 \implies n(n+1) r^{n-2} = 0$$

$$n = 0 \vee n = -1$$

c)

$$\nabla^2 f(r) = 0 \implies \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} f(r)\right) = 0$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} f(r)\right) = 0$$

$$r^2 \frac{\partial}{\partial r} f(r) = C_1$$

$$\frac{\partial}{\partial r} f(r) = C_1 r^{-2}$$

$$f(r) = -C_1 r^{-1} + C_2$$

$$f(r) = A + \frac{B}{r}$$

---

32) Repita o problema 31) trocando a coordenada radial r pela coordenada radial cilíndrica ρ .

a)

$$\begin{aligned}
\nabla^2 \rho^n &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\rho^n\right)\\
&= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\, n\rho^{n-1}\right)\\
&= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(n\rho^n\right)\\
&= \frac{1}{\rho}\, n^2\rho^{n-1}\\
&= n^2\rho^{n-2}
\end{aligned}$$

b)

$$\nabla^2\rho^n = 0 \implies n^2\rho^{n-2} = 0$$

$$n = 0$$

c)

$$\nabla^2 f(\rho) = 0 \implies \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2} f(\rho) = 0$$

$$\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f(\rho)}{\partial\rho}\right) = 0$$

$$\rho\frac{\partial f(\rho)}{\partial\rho} = C_1$$

$$\frac{\partial f(\rho)}{\partial\rho} = \frac{C_1}{\rho}$$

$$f(\rho) = C_1\ln(\rho) + C_2$$

$$f(\rho) = A + B\ln(\rho)$$

33) Determine a validade da equação $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A}$ usando coordenadas e componentes cartesianos.

$$
\begin{aligned}
\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) &= \vec{\nabla} \times \left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{a}_z \right] \\
&= \left(\frac{\partial}{\partial y} \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right] - \frac{\partial}{\partial z} \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right] \right) \hat{a}_x + ... \\
&+ \left(\frac{\partial}{\partial z} \left[\frac{\partial A_z}{\partial x} - \frac{\partial A_y}{\partial z} \right] - \frac{\partial}{\partial x} \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right] \right) \hat{a}_y + ... \\
&+ \left(\frac{\partial}{\partial x} \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right] - \frac{\partial}{\partial y} \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right] \right) \hat{a}_z \\
&= \left(\frac{\partial^2 A_y}{\partial y \partial x} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} + \frac{\partial^2 A_z}{\partial z \partial x} \right) \hat{a}_x + ... \\
&+ \left(\frac{\partial^2 A_z}{\partial z \partial y} - \frac{\partial^2 A_y}{\partial z^2} - \frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_x}{\partial x \partial y} \right) \hat{a}_y + ... \\
&+ \left(\frac{\partial^2 A_x}{\partial x \partial z} - \frac{\partial^2 A_z}{\partial x^2} - \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_y}{\partial y \partial z} \right) \hat{a}_z \\
&= \left(\frac{\partial^2 A_y}{\partial y \partial x} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} + \frac{\partial^2 A_z}{\partial z \partial x} \right) \hat{a}_x + ... \\
&+ \left(\frac{\partial^2 A_z}{\partial z \partial y} - \frac{\partial^2 A_y}{\partial z^2} - \frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_x}{\partial x \partial y} \right) \hat{a}_y + ... \\
&+ \left(\frac{\partial^2 A_x}{\partial x \partial z} - \frac{\partial^2 A_z}{\partial z^2} - \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_y}{\partial y \partial z} \right) \hat{a}_z \\
&\vdots
\end{aligned}
$$

$$\vdots$$

$$\begin{aligned}
\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) &= \left(\frac{\partial}{\partial x} \left[\frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} + \frac{\partial A_x}{\partial x} \right] - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} - \frac{\partial^2 A_x}{\partial x^2} \right) \hat{a}_x + ... \\
&+ \left(\frac{\partial}{\partial y} \left[\frac{\partial A_z}{\partial z} + \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right] - \frac{\partial^2 A_y}{\partial z^2} - \frac{\partial^2 A_y}{\partial x^2} - \frac{\partial^2 A_y}{\partial y^2} \right) \hat{a}_y + ... \\
&+ \left(\frac{\partial}{\partial z} \left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right] - \frac{\partial^2 A_z}{\partial z^2} - \frac{\partial^2 A_z}{\partial y^2} - \frac{\partial^2 A_z}{\partial z^2} \right) \hat{a}_z \\
&= \left(\frac{\partial}{\partial x} [\vec{\nabla} \cdot \vec{A}] \right) \hat{a}_x - \nabla^2 A_x \hat{a}_x + ... \\
&+ \left(\frac{\partial}{\partial y} [\vec{\nabla} \cdot \vec{A}] \right) \hat{a}_y - \nabla^2 A_y \hat{a}_y + ... \\
&+ \left(\frac{\partial}{\partial z} [\vec{\nabla} \cdot \vec{A}] \right) \hat{a}_z - \nabla^2 A_z \hat{a}_z \\
&= \left(\frac{\partial}{\partial x} \hat{a}_x + \frac{\partial}{\partial y} \hat{a}_y + \frac{\partial}{\partial z} \hat{a}_z \right) [\vec{\nabla} \cdot \vec{A}] - \vec{\nabla}^2 \vec{A} \\
&= \vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A}
\end{aligned}$$

---

34) Com a velocidade $\vec{v}_n = C_n \rho^n \hat{a}_\phi,\ C_n > 0 \forall n$, e usando coordenadas e componentes cilíndricas, calcule: a) $\vec{\nabla} \times (\vec{\nabla} \times \vec{v}_n)$; b) $\vec{\nabla}(\vec{\nabla} \cdot \vec{v}_n)$; c $(\nabla^2 v_n)\hat{a}_\phi$; d) $\vec{\nabla}^2 v_n \hat{a}_\phi$. Mostre que: e) a equação $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A}$ não vale quando nela entende-se que $\vec{\nabla}^2 \vec{v}_n = (\nabla^2 v_n)\hat{a}_\phi$; f) a equação $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A}$ se verifica quando nela entende-se, corretamente, que $\vec{\nabla}^2 \vec{v}_n = \vec{\nabla}^2(v_n \hat{a}_\phi)$. g) Forneça os valores de n com os quais a velocidade é harmônica.

a)

$$\begin{aligned}
\vec{\nabla} \times (\vec{\nabla} \times \vec{v}_n) &= \vec{\nabla} \times \left[\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho\, C_n \rho^n)\right] \hat{a}_z \\
&= \vec{\nabla} \times \left[\frac{1}{\rho}\frac{\partial}{\partial \rho} C_n \rho^{n+1}\right] \hat{a}_z \\
&= \vec{\nabla} \times \left[\frac{1}{\rho}(n+1) C_n \rho^n\right] \hat{a}_z \\
&= \vec{\nabla} \times [(n+1) C_n \rho^{n-1}] \hat{a}_z \\
&= -\frac{\partial}{\partial \rho}[(n+1) C_n \rho^{n-1}] \hat{a}_\phi \\
&= -(n+1)(n-1) C_n \rho^{n-2} \hat{a}_\phi \\
&= \frac{1-n^2}{\rho^2}\, C_n \rho^n \hat{a}_\phi \\
&= \frac{1-n^2}{\rho^2}\, \vec{v}_n
\end{aligned}$$

b)

$$\begin{aligned}
\vec{\nabla}(\vec{\nabla} \cdot \vec{v}_n) &= \vec{\nabla} \frac{1}{\rho}\frac{\partial}{\partial \rho} 0 + \frac{1}{\rho}\frac{\partial}{\partial \phi}(C_n \rho^n) + \frac{\partial}{\partial z} 0 \\
&= \vec{\nabla}(0+0+0) \\
&= \vec{0}
\end{aligned}$$

c)

$$
\begin{aligned}
(\nabla^2 v_n)\hat{a}_\phi &= \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}(C_n\rho^n\right)\hat{a}_\phi \\
&= \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\ nC_n\rho^{n-1}\right)\hat{a}_\phi \\
&= \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}nC_n\rho^{n}\right)\hat{a}_\phi \\
&= \left(\frac{1}{\rho}n^2C_n\rho^{n-1}\right)\hat{a}_\phi \\
&= (n^2C_n\rho^{n-2})\hat{a}_\phi \\
&= \frac{n^2}{\rho^2}v_n
\end{aligned}
$$

d)

$$
\begin{aligned}
\vec{\nabla}^2(v_n\hat{a}_\phi) &= \left[\nabla^2(C_n\rho^n) + \frac{1}{\rho^2}(-C_n\rho^n)\right]\hat{a}_\phi \\
&= \left[n^2C_n\rho^{n-2} - C_n\rho^{n-2}\right]\hat{a}_\phi \\
&= \frac{n^2-1}{\rho^2}\vec{v}_n
\end{aligned}
$$

e)

$$
\vec{\nabla}\times(\vec{\nabla}\times\vec{v}_n) = \frac{1-n^2}{\rho^2}\ \vec{v}_n
$$

$$
\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) = 0
$$

$$
(\nabla^2 v_n)\hat{a}_\phi = \frac{n^2}{\rho^2}v_n
$$

$$
\begin{aligned}
\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - (\nabla^2 v_n)\hat{a}_\phi &= 0 - \frac{n^2}{\rho^2}v_n \\
&= -\frac{n^2}{\rho^2}v_n
\end{aligned}
$$

$$
\therefore \vec{\nabla}\times(\vec{\nabla}\times\vec{v}_n) \neq \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - (\nabla^2 v_n)\hat{a}_\phi
$$

f)

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{v}_n) = \frac{1-n^2}{\rho^2}\ \vec{v}_n$$

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) = 0$$

$$\vec{\nabla}^2(v_n \hat{a}_\phi) = \frac{n^2-1}{\rho^2} v_n$$

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2(v_n \hat{a}_\phi) = 0 - \frac{n^2-1}{\rho^2} v_n$$

$$= \frac{1-n^2}{\rho^2} v_n$$

$$\therefore \vec{\nabla} \times (\vec{\nabla} \times \vec{v}_n) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2(v_n \hat{a}_\phi)$$

g)

$$\nabla^2 v_n = 0 \implies \frac{n^2}{\rho^2} v_n = 0$$

$$n = 0$$

$$\vec{\nabla}^2 \vec{v}_n = \vec{0} \implies \frac{n^2-1}{\rho^2} \vec{v}_n = \vec{0}$$

$$n = 1 \vee n = -1$$

---

35) Forme alguns operadores que envolvam duas vezes o operador nabla e uma vez uma função: a) escalar e b) vetorial.

a)

$$\nabla^2_f g = \frac{1}{f} \vec{\nabla} \cdot \vec{\nabla}(fg)$$

$$\nabla^f g = \vec{\nabla} \cdot \vec{\nabla}(f(g))$$

b)

$$\vec{\nabla}^2_v \vec{A} = \frac{1}{|\vec{v}|} \vec{\nabla}[\vec{\nabla} \cdot (\vec{v} \times \vec{A})]$$

$$\vec{\nabla}^v \vec{A} = \vec{v} \times \vec{\nabla}(\vec{\nabla} \cdot \vec{A})$$

---

36) Forme alguns operadores de terceira ordem que envolvam exclusivamente o operador nabla.

$$\vec{\nabla}^3 \vec{A} = \vec{\nabla} \times (\vec{\nabla} \times [\vec{\nabla} \times \vec{A}])$$
$$\nabla^3 \vec{A} = \vec{\nabla} \cdot (\vec{\nabla}[\vec{\nabla} \cdot \vec{A}])$$

---

37) Mostre que, se a superfície S do Teorema de Stokes for fechada, será obtido um resultado compatível com o Teorema de Gauss.

$$\iint_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{S} = \oint_C \vec{v} \cdot d\vec{l} \implies \oiint_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{S} = \oint_0 \vec{v} \cdot d\vec{l}$$
$$= 0$$

$$\oiint_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{S} = 0 \implies \iiint_V \vec{\nabla} \cdot (\vec{\nabla} \times \vec{v}) dv = 0 \implies \ldots$$
$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{v}) = 0$$

---

38) Demonstre que $\iiint_v \vec{\nabla} \times \vec{A}\, dv = \oiint_S d\vec{S} \times \vec{A}$.

$$\iiint_v \vec{\nabla} \cdot \vec{A}\, dv = \oiint_S \vec{A} \cdot d\vec{S} \implies \ldots$$
$$\iiint_v \vec{\nabla} \cdot (\vec{A} \times \vec{U})\, dv = \oiint_S (\vec{A} \times \vec{U}) \cdot d\vec{S}$$
$$\iiint_v \left([\vec{\nabla} \times \vec{A}] \cdot \vec{U} - [\vec{\nabla} \times \vec{U}] \cdot \vec{A} \right) dv = \oiint_S (d\vec{S} \times \vec{A}) \cdot \vec{U}$$
$$\iiint_v ([\vec{\nabla} \times \vec{A}] \cdot \vec{U} - \vec{0} \cdot \vec{A})\, dv = \oiint_S (d\vec{S} \times \vec{A}) \cdot \vec{U} \qquad \forall \vec{U}$$
$$\iiint_v \vec{\nabla} \times \vec{A}\, dv = \oiint_S d\vec{S} \times \vec{A}$$

---

39) Demonstre: a) a Identidade de Green $\iiint\limits_v (f\nabla^2 g + \vec{\nabla} f \cdot \vec{\nabla} g)\; dv = \oiint\limits_S f\vec{\nabla} g \cdot d\vec{S}$; b) o Teorema de Green $\iiint\limits_v (f\nabla^2 g - g\nabla^2 f)\; dv = \oiint\limits_S (f\vec{\nabla} g - g\vec{\nabla} f) \cdot d\vec{S}$.

a)

$$\iiint\limits_v \vec{\nabla} \cdot \vec{A}\; dv = \oiint\limits_S \vec{A} \cdot d\vec{S} \implies \ldots$$

$$\iiint\limits_v \vec{\nabla} \cdot (f\vec{\nabla} g)\; dv = \oiint\limits_S (f\vec{\nabla} g) \cdot d\vec{S}$$

$$\iiint\limits_v [\vec{\nabla} f \cdot \vec{\nabla} g + f\vec{\nabla} \cdot \vec{\nabla} g]\; dv = \oiint\limits_S (f\vec{\nabla} g) \cdot d\vec{S}$$

$$\iiint\limits_v [\vec{\nabla} f \cdot \vec{\nabla} g + f\nabla^2 g]\; dv = \oiint\limits_S (f\vec{\nabla} g) \cdot d\vec{S}$$

b)

$$\iiint\limits_v (f\nabla^2 g + \vec{\nabla} f \cdot \vec{\nabla} g)\; dv = \oiint\limits_S f\vec{\nabla} g \cdot d\vec{S} \implies \ldots$$

$$\iiint\limits_v (f\nabla^2 g + \vec{\nabla} f \cdot \vec{\nabla} g)\; dv - \iiint\limits_v (g\nabla^2 f + \vec{\nabla} g \cdot \vec{\nabla} f)\; dv = \oiint\limits_S f\vec{\nabla} g \cdot d\vec{S} - \ldots$$

$$- \oiint\limits_S g\vec{\nabla} f \cdot d\vec{S}$$

$$\iiint\limits_v (f\nabla^2 g - g\nabla^2 f)\; dv = \oiint\limits_S (f\vec{\nabla} g - g\vec{\nabla} f) \cdot d\vec{S}$$

---

40) Demonstre as seguintes equações: a) $\iint\limits_S d\vec{S} \times \vec{\nabla} f = \oint\limits_C f d\vec{l}$; b) $\oiint\limits_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{S} = 0$; c) $\oiint\limits_S d\vec{S} = \vec{0}$.

a)

$$\iint\limits_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{S} = \oint\limits_C \vec{v} \cdot d\vec{l} \implies ...$$

$$\iint\limits_S (\vec{\nabla} \times f\, \vec{U}) \cdot d\vec{S} = \oint\limits_C f\, \vec{U} \cdot d\vec{l}$$

$$\iint\limits_S (\vec{\nabla} f \times \vec{U} + f \vec{\nabla} \times \vec{U}) \cdot d\vec{S} = \oint\limits_C (f d\vec{l}) \cdot \vec{U}$$

$$\iint\limits_S (\vec{\nabla} f \times \vec{U}) \cdot d\vec{S} = \oint\limits_C (f d\vec{l}) \cdot \vec{U}$$

$$\iint\limits_S (d\vec{S} \times \vec{\nabla} f) \cdot \vec{U} = \oint\limits_C (f d\vec{l}) \cdot \vec{U} \qquad \forall \vec{U}$$

$$\iint\limits_S d\vec{S} \times \vec{\nabla} f = \oint\limits_C f d\vec{l}$$

b)

$$\iint\limits_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{S} = \oint\limits_C \vec{A} \cdot d\vec{l} \implies ...$$

$$\oiint\limits_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{S} = \oint\limits_0 \vec{A} \cdot d\vec{l}$$

$$\oiint\limits_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{S} = 0$$

c)

$$\iiint_V \vec{\nabla} f \; dv = \oiint_S f d\vec{S} \implies \ldots$$

$$\iiint_V \vec{\nabla} 1 \; dv = \oiint_S d\vec{S}$$

$$\iiint_V 0 \; dv = \oiint_S d\vec{S}$$

$$\oiint_S d\vec{S} = 0$$

---

41) $\oiint_S \vec{r} \cdot d\vec{S} = 3v$, onde v é o volume delimitado pela superfície S fechada.

$$\begin{aligned}\oiint_S \vec{r} \cdot d\vec{S} &= \iiint_v \vec{\nabla} \cdot \vec{r} \; dv \\ &= \iiint_v \vec{\nabla} \cdot [x\hat{a}_x + y\hat{a}_y + z\hat{a}_z] \; dv \\ &= \iiint_v 3dv \\ &= 3v\end{aligned}$$

---

42) Demonstre a equação $\oint_C \vec{r} \cdot d\vec{r} = 0$: a) usando o Teorema de Stokes; b) identificando o integrando com uma diferencial exata.

a)

$$\oint_C \vec{r} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{r}) \cdot d\vec{S}$$

$$= \iint_S \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \cdot d\vec{S}$$

$$= \iint_S \vec{0} \cdot d\vec{S}$$

$$= 0$$

b)

$$\oint_C \vec{r} \cdot d\vec{r} = \oint_C (r\hat{a}_r) \cdot (dr\hat{a}_r + rd\theta\hat{a}_\theta + r\sin\theta d\phi\hat{a}_\phi)$$

$$= \oint_C rdr$$

$$= \int_{-\infty}^{\infty} rdr$$

$$= \left.\frac{r^2}{2}\right|_{-\infty}^{\infty}$$

$$= 0$$

---

43) Mostre que $\oint\limits_C f\vec{\nabla}g \cdot d\vec{l} = -\oint\limits_C g\vec{\nabla}f \cdot d\vec{l}$.

$$\oint\limits_C \vec{A} \cdot d\vec{l} = \iint\limits_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{S} \implies \ldots$$

$$\oint\limits_C \vec{\nabla}[fg] \cdot d\vec{l} = \iint\limits_S (\vec{\nabla} \times \vec{\nabla}[fg]) \cdot d\vec{S}$$

$$\oint\limits_C (f\vec{\nabla}g + g\vec{\nabla}f) \cdot d\vec{l} = \iint\limits_S \vec{0} \cdot d\vec{S}$$

$$\oint\limits_C f\vec{\nabla}g \cdot d\vec{l} + \oint\limits_C g\vec{\nabla}f \cdot d\vec{l} = 0$$

$$\oint\limits_C f\vec{\nabla}g \cdot d\vec{l} = -\oint\limits_C g\vec{\nabla}f \cdot d\vec{l}$$

---

44) Mostre que $\oint\limits_C (x\ dy - y\ dx) = 2A$, onde A é a área da superfície plana compreendida pela curva plana fechada C.

$$\oint\limits_C (x\ dy - y\ dx) = \oint\limits_C (-y\hat{a}_x + x\hat{a}_y) \cdot (dx\hat{a}_x + dy\hat{a}_y)$$

$$= \oint\limits_C (-y\hat{a}_x + x\hat{a}_y) \cdot d\vec{l}$$

$$= \iint\limits_S \vec{\nabla} \times (-y\hat{a}_x + x\hat{a}_y) \cdot d\vec{S}$$

$$= \iint\limits_S \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & 0 \end{vmatrix} \cdot d\vec{S}$$

$$= \iint\limits_S (\hat{a}_z + \hat{a}_z) \cdot (dy\ dz\ \hat{a}_x + dx\ dz\ \hat{a}_y + dx\ dy\ \hat{a}_z)$$

$$= \iint\limits_S 2\ dx\ dy$$

$$= 2\iint\limits_S dx\ dy$$

$$= 2A$$

---

45) Com o vetor $\vec{v}_n = C_n \rho^n \hat{a}_\phi,\ C_n > 0 \forall n$, e usando coordenadas cilíndricas circulares: a) calcule a circulação de $\vec{v}_n$ ao longo da circunferência C: $\rho = b, z = z_0$, onde $b > 0$ e z_0 são uniformes, orientando-os positivamente no sentido trigonométrico; b) determine o fluxo do rotacional de $\vec{v}_n$ através do círculo S limitado pela circunferência definida no item a); c) verifique se, com a curva dada no item a) e a superfície no item b), o vetor $\vec{v}_n$ obedece ao Teorema de Stokes; explique uma eventual resposta negativa; d) repita os itens anteriores quando, no plano z = z0, a curva C for o conjunto das circunferências $\rho = a$ e $\rho = b (0 < a < b)$ e S a coroa circular por elas definida.

a)

$$\begin{aligned}
\oint_C \vec{v}_n \cdot d\vec{l} &= \oint_C C_n \rho^n \hat{a}_\phi \cdot (d\rho \hat{a}_\rho + \rho d\phi \hat{a}_\phi + dz \hat{a}_z) \\
&= \oint_C C_n \rho^{n+1} d\phi \\
&= \int_0^{2\pi} C_n b^{n+1} d\phi \\
&= C_n b^{n+1} \ \phi \Big|_0^{2\pi} \\
&= 2\pi C_n b^{n+1}
\end{aligned}$$

b)

$$\iint_S \vec{\nabla} \times \vec{v}_n \cdot d\vec{S} = \iint_S \left(\frac{1}{\rho} \frac{\partial}{\partial \rho} [\rho \; C_n \rho^n] \hat{a}_z \right) \cdot d\vec{S}$$

$$= \iint_S \left(\frac{1}{\rho} \frac{\partial}{\partial \rho} [C_n \rho^{n+1}] \hat{a}_z \right) \cdot d\vec{S}$$

$$= \iint_S \left(\frac{1}{\rho} \; [C_n (n+1) \rho^n] \hat{a}_z \right) \cdot d\vec{S}$$

$$= \iint_S (C_n (n+1) \rho^{n-1}] \hat{a}_z) \cdot d\vec{S}$$

$$= \iint_S (C_n (n+1) \rho^{n-1}])(\rho d\rho d\phi)$$

$$= C_n (n+1) \int_0^{2\pi} \int_0^b \rho^n d\rho d\phi$$

$$= C_n (n+1) \int_0^{2\pi} \frac{1}{n+1} \rho^{n+1} \Big|_0^b d\phi$$

$$= C_n \int_0^{2\pi} b^{n+1} d\phi$$

$$= C_n b^{n+1} \;\; \phi \Big|_0^{2\pi}$$

$$= 2\pi C_n b^{n+1}$$

c)

$$\oint_C \vec{v}_n \cdot d\vec{l} = \iint_S \vec{\nabla} \times \vec{v}_n \cdot d\vec{S}$$

O teorema é válido em qualquer ponto do espaço, com exceção da origem, onde o valor de v_n é infinito.

d)

$$\oint_C \vec{v}_n \cdot d\vec{l} = \oint_C C_n \rho^{n+1} d\phi$$

$$= \int_0^{2\pi} C_n a^{n+1} d\phi + \int_{2\pi}^0 C_n b^{n+1} d\phi$$

$$= 2\pi C_n (a^{n+1} - b^{n+1})$$

$$\iint_S \vec{\nabla} \times \vec{v}_n \cdot d\vec{S} = \iint_S C_n (n+1) \rho^n d\rho d\phi)$$

$$= C_n(n+1) \left(\int_0^{2\pi} \int_0^a \rho^n d\rho d\phi + \int_{2\pi}^0 \int_0^b \rho^n d\rho d\phi \right)$$

$$= C_n \int_0^{2\pi} (a^{n+1} - bn + 1) d\phi$$

$$= 2\pi C_n (a^{n+1} - b^{n+1})$$

---

46) Partindo da equação $\delta(\vec{r}-\vec{r}_0) = \delta(x-x_0)\delta(y-y_0)\delta(z-z_0)$, deduza as seguintes equações: a) $\delta(\vec{r}-\vec{r}_0) = 0$, se $\vec{r} \neq \vec{r}_0$; b) $\iiint\limits_v \delta(\vec{r}-\vec{r}_0)dv = \begin{cases} 0, \ se \ \vec{r}_0 \notin v \\ 1, \ se \ \vec{r}_0 \in v \end{cases}$;

c) $\iiint\limits_v f(\vec{r})\delta(\vec{r}-\vec{r}_0)dv = \begin{cases} 0, \ se \ \vec{r}_0 \notin v \\ f(\vec{r}_0), \ se \ \vec{r}_0 \in v \end{cases}$.

a)

$$\vec{r} \neq \vec{r}_0 \implies x \neq x_0 \vee y \neq y_0 \vee z \neq z_0$$

$$\delta(x-x_0) = 0 \vee \delta(y-y_0) = 0 \vee \delta(z-z_0) = 0$$

$$\delta(x-x_0)\delta(y-y_0)\delta(z-z_0) = 0$$

$$\delta(\vec{r}-\vec{r}_0) = 0$$

b)

$$r_0 \notin v \implies \delta(\vec{r}-\vec{r}_0) = 0 \forall \vec{r} \in v$$

$$\begin{aligned} \iiint\limits_v \delta(\vec{r}-\vec{r}_0)dv &= \iiint\limits_v 0 \ dv \\ &= 0 \end{aligned}$$

$$r_0 \in v \implies \delta(\vec{r}-\vec{r}_0) = \begin{cases} 0, & \vec{r} \neq \vec{r}_0 \\ \delta(0), & \vec{r} = \vec{r}_0 \end{cases}$$

$$\begin{aligned} \iiint\limits_v \delta(\vec{r}-\vec{r}_0)dv &= \iiint\limits_v \delta(0) \ dv \\ &= 1 \end{aligned}$$

c)

$$r_0 \notin v \implies \delta(\vec{r} - \vec{r}_0) = 0 \forall \vec{r} \in v$$

$$\begin{aligned} \iiint_v f(\vec{r})\delta(\vec{r} - \vec{r}_0) dv &= \iiint_v 0 \ dv \\ &= 0 \end{aligned}$$

$$r_0 \in v \implies \delta(\vec{r} - \vec{r}_0) = \begin{cases} 0, & \vec{r} \neq \vec{r}_0 \\ \delta(0), & \vec{r} = \vec{r}_0 \end{cases}$$

$$\begin{aligned} \iiint_v f(\vec{r})\delta(\vec{r} - \vec{r}_0) dv &= \iiint_v f(\vec{r})\delta(0) \ dv \\ &= f(\vec{r}_0) \end{aligned}$$

---

47) Mostre que: $\delta(x) = \lim_{k\to\infty} \left(\frac{k}{\sqrt{\pi}} e^{-k^2x^2} \right)$.

$$\begin{aligned} \lim_{k\to\infty} \left(\frac{k}{\sqrt{\pi}} e^{-k^2x^2} \right) \Bigg|_{x=0} &= \lim_{k\to\infty} \left(\frac{k}{\sqrt{\pi}} e^{0} \right) \\ &= \infty \end{aligned}$$

$$\begin{aligned} \lim_{k\to\infty} \left(\frac{k}{\sqrt{\pi}} e^{-k^2x^2} \right) \Bigg|_{x\neq 0} &= \lim_{k\to\infty} \left(\frac{k}{\sqrt{\pi}} e^{-k2} \right) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \int_{-\infty}^{\infty} \lim_{k\to\infty} \left(\frac{k}{\sqrt{\pi}} e^{-k^2x^2} \right) dx &= \lim_{k\to\infty} \left(\int_{-\infty}^{\infty} \frac{k}{\sqrt{\pi}} e^{-k^2x^2} dx \right) \\ &= \lim_{\sigma\to 0} \left(\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx \right) \\ &= \lim_{\sigma\to 0} 1 \\ &= 1 \end{aligned}$$

---

48) Mostre que: $\delta(\vec{r}) = \lim\limits_{k\to\infty}\left(\frac{k}{\sqrt{\pi}}e^{-k^2r^2}\right)$.

$$\begin{aligned}
\lim_{k\to\infty}\left(\frac{k}{\sqrt{\pi}}e^{-k^2r^2}\right) &= \lim_{k\to\infty}\left(\frac{k}{\sqrt{\pi}}e^{-k^2(x^2+y^2+z^2)}\right) \\
&= \lim_{k\to\infty}\left(\frac{k}{\sqrt{\pi}}e^{-k^2x^2}\ e^{-k^2y^2}\ e^{-k^2z^2}\right) \\
&= \lim_{k\to\infty}\left(\frac{k}{\sqrt{\pi}}e^{-k^2x^2}\right)\left(\frac{k}{\sqrt{\pi}}e^{-k^2y^2}\right)\left(\frac{k}{\sqrt{\pi}}e^{-k^2z^2}\right) \\
&= \delta(x)\delta(y)\delta(v) \\
&= \delta(\vec{r})
\end{aligned}$$

---

49) Mostre que: $\int_{-\beta}^{\beta} e^{jkx}dk = \frac{2}{x}\sin\beta x$.

$$\begin{aligned}
\int_{-\beta}^{\beta} e^{jkx}dk &= \frac{1}{jx}\ e^{jkx}\Big|_{-\beta}^{\beta} \\
&= \frac{e^{j\beta x} - e^{-j\beta x}}{jx} \\
&= \frac{2}{x}\ \frac{e^{j\beta x} - e^{-j\beta x}}{j} \\
&= \frac{2}{x}\sin\beta x
\end{aligned}$$

---

50) Mostre que: $\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{jkx}dk$.

$$
\begin{aligned}
x = 0 \implies \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{jkx}dk &= \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} e^{0}dk\right) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \\
&= \infty \\
x \neq 0 \implies \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{jkx}dk &= \frac{1}{2\pi}\lim_{\beta\to\infty}\int_{-\beta}^{\beta} e^{jkx}dk \\
&= \frac{1}{2\pi}\lim_{\beta\to\infty}\left(\frac{2}{x}\sin\beta x\right) \\
&= \frac{1}{\pi x}\lim_{\beta\to\infty}(\sin\beta x) \\
&= \textbf{indefinido}
\end{aligned}
$$

$$
\begin{aligned}
\int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{jkx}dk\right)dx &= \lim_{\beta\to\infty}\left[\int_{-\infty}^{\infty}\left(\frac{1}{\pi x}\sin\beta x\right)dx\right] \\
&= \frac{1}{\pi}\lim_{\beta\to\infty}\left[\int_{-\infty}^{\infty}\frac{\sin\beta x}{\beta x}\beta dx\right] \\
&= \frac{1}{\pi}\lim_{\beta\to\infty}\left[\int_{-\infty}^{\infty}\textbf{sinc}(u)\,du\right] \\
&= \frac{1}{\pi}\lim_{\beta\to\infty}\pi \\
&= 1
\end{aligned}
$$

Apesar de o valor da função ser indefinido para $x \neq 0$, ela exibe a propriedade importante, em termos práticos, que é a da filtragem (*sifting*), portanto pode ser considerada como uma expressão da Delta de Dirac.

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