MIXED FORMULATION FOR POISSON EQUATION

ABSTRACT. We give bases of RT and BDM spaces, and also how to handle boundary conditions.

1. BASIS OF VARIOUS MIXED ELEMENTS

We will give bases of spaces RT_0, RT_1, BDM_1 on both triangle and tetrahedron, utilizing both the barycentric coordinates and Cartesian coordinates. In this section, we denote the element K either a triangle (n=2) or tetrahedron (n=3), $e_{i,j}$ the edge of element K, $F_{i,j,k}$ the face of the tetrahedron K, and $\{\lambda_i\}_{i=1}^{n+1}$ the corresponding barycentric coordinates. Note that $RT_0 \subset BDM_1 \subset RT_1$.

1.1. Bases and dual bases on barycentric coordinates.

(1) triangle

(a) RT_0

The basis is: on edge $e_{i,j}$.

$$\boldsymbol{\phi}_{i,j} = \lambda_i \boldsymbol{rot}\lambda_j - \lambda_j \boldsymbol{rot}\lambda_i \tag{1.1}$$

The associated dual basis is:

$$\Phi_{i,j}(\boldsymbol{u}) = \int_{e_{i,j}} \boldsymbol{u} \cdot \boldsymbol{n}_{i,j} \, ds \tag{1.2}$$

(b) BDM_1

The bases are: on edge $e_{i,j}$

$$\boldsymbol{\phi}_{i,j}^1 = \lambda_i \boldsymbol{rot}\lambda_j, \ \boldsymbol{\phi}_{i,j}^2 = \lambda_j \boldsymbol{rot}\lambda_i$$

We can get better-to-use bases by combining the above two bases.

$$\begin{aligned} \boldsymbol{\phi}_{i,j} &= \boldsymbol{\phi}_{i,j}^1 - \boldsymbol{\phi}_{i,j}^2 \\ \boldsymbol{\psi}_{i,j} &= \boldsymbol{\phi}_{i,j}^1 + \boldsymbol{\phi}_{i,j}^2 \end{aligned}$$

The associated dual bases are:

$$\Phi_{i,j}(\boldsymbol{u}) = \int_{e_{i,j}} \boldsymbol{u} \cdot \boldsymbol{n}_{i,j} \, ds \tag{1.3}$$

$$\Psi_{i,j}(\boldsymbol{u}) = 3 \int_{e_{i,j}} \boldsymbol{u} \cdot \boldsymbol{n}_{i,j} (\lambda_i - \lambda_j) \, ds \tag{1.4}$$

(c) RT_1
The bases are:

$$\boldsymbol{\phi}^1_{i,j} = \lambda_i \boldsymbol{\phi}_{i,j},\ \boldsymbol{\phi}^2_{i,j} = \lambda_j \boldsymbol{\phi}_{i,j} \quad \text{on edge } e_{i,j}$$
$$\boldsymbol{\phi}^1_{K} = \lambda_k \boldsymbol{\phi}_{i,j},\ \boldsymbol{\phi}^2_{K} = \lambda_j \boldsymbol{\phi}_{i,k} \quad \text{on triangle } K$$

with $\boldsymbol{\phi}_{i,j}$ defined in (1.1).

(2) tetrahedron

(a) RT_0
The basis is: on face $F_{i,j,k}$

$$\boldsymbol{\phi}_{i,j,k} = 2(\lambda_i \nabla\lambda_j \times \nabla\lambda_k + \lambda_j \nabla\lambda_k \times \nabla\lambda_i + \lambda_k \nabla\lambda_i \times \nabla\lambda_j) \tag{1.5}$$

The associated dual basis is:

$$\Phi_{i,j,k}(\boldsymbol{u}) = \int_{F_{i,j,k}} \boldsymbol{u} \cdot \boldsymbol{n}_{i,j,k}\, ds \tag{1.6}$$

(b) BDM_1
The bases are: on face $F_{i,j,k}$

$$\begin{aligned} \boldsymbol{\phi}^1_{i,j,k} &= \lambda_k \nabla\lambda_i \times \nabla\lambda_j, \\ \boldsymbol{\phi}^2_{i,j,k} &= \lambda_j \nabla\lambda_k \times \nabla\lambda_i, \\ \boldsymbol{\phi}^3_{i,j,k} &= \lambda_i \nabla\lambda_j \times \nabla\lambda_k \end{aligned}$$

(c) RT_1
The bases are:

$$\left.\begin{aligned} \boldsymbol{\phi}^1_{i,j,k} &= \lambda_i \boldsymbol{\phi}_{i,j,k}, \\ \boldsymbol{\phi}^2_{i,j,k} &= \lambda_j \boldsymbol{\phi}_{i,j,k} \\ \boldsymbol{\phi}^3_{i,j,k} &= \lambda_k \boldsymbol{\phi}_{i,j,k} \end{aligned}\right. \quad \text{on face } F_{i,j,k}$$

$$\boldsymbol{\phi}^1_{K} = \lambda_l \boldsymbol{\phi}_{i,j,k},\ \boldsymbol{\phi}^2_{K} = \lambda_k \boldsymbol{\phi}_{i,j,l} \quad \text{on tetrahedron } K$$

with $\boldsymbol{\phi}_{i,j,k}$ defined in (1.5).

1.2. **Bases based on Cartesian coordinates: (Only 2D case is considered).** This part has commented out.

2. Solve Poisson Equation using mixed method

Suppose we are solving Poisson Equation with mixed Dirichlet and Neumann boundary conditions, utilizing mixed FEM.

$$\begin{cases} -\operatorname{div}(\alpha(x)\operatorname{grad} u) = f \quad in\,\Omega \\ \qquad\qquad\qquad u = g_D \;\; on\,\Gamma_D \\ \qquad\qquad\quad \partial_{\boldsymbol{n}} u = g_N \;\; on\,\Gamma_N \end{cases} \tag{2.1}$$

where $g_D \in H^{\frac{1}{2}}(\Gamma_D)$ and $g_N \in H^{-\frac{1}{2}}(\Gamma_N)$.

We denote the duality pairing of $H^{\frac{1}{2}}$ with $H^{-\frac{1}{2}}$ on Γ_D by $<,>_{\Gamma_D}$. Let $\boldsymbol{\sigma} = \alpha \operatorname{grad} u$, thus $\alpha^{-1}\boldsymbol{\sigma} = \operatorname{grad} u$, the week formulation then can be written as: Find $(\boldsymbol{\sigma}, u) \in H_{g_N,\Gamma_N}(\mathrm{div}, \Omega) \times L^2(\Omega)$ such that

$$\begin{cases} (\alpha^{-1}\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mathrm{div}\ \boldsymbol{\tau}, u) =< \boldsymbol{\tau} \cdot \boldsymbol{n}, g_D >_{\Gamma_D} & \forall\, \boldsymbol{\tau} \in H_{0,\Gamma_N}(\mathrm{div}, \Omega) \\ (\mathrm{div}\ \boldsymbol{\sigma}, v) = -(f, v) & \forall\, v \in L^2(\Omega) \end{cases} \tag{2.2}$$

where

$$H_{g,\Gamma}(\mathrm{div}, \Omega) = \{\boldsymbol{\sigma} \in H(\mathrm{div}, \Omega); \boldsymbol{\sigma} \cdot \boldsymbol{n} = g \ on\ \Gamma \subset \partial\Omega\}$$

We now consider the finite element approximation of equation (2.2). In the following, whenever it may be convenient, we will denote by the symbol $M(K)$ either the lowest Raviart-Thomas element (RT_0) or BDM element (BDM_1), both of which are conforming in $H(\mathrm{div}, \Omega)$. Let $\mathcal{M}_h$ be a shape regular mesh over Ω. We define the finite element spaces

$$X_{g,\Gamma,h} := \{\boldsymbol{\tau} \in H_{g,\Gamma}(\mathrm{div}; \Omega) : \boldsymbol{\tau}|_K \in M(K),\ \forall K \in \mathcal{M}_h\}$$

$$V_h := \{v \in L^2(\Omega) : v|_K \in P_0(K),\ \forall K \in \mathcal{M}_h\}.$$

where $P_k(K)$ is the k-th polynomial on K. The discrete problem can be written as: Find $(\boldsymbol{\sigma}_h, u_h) \in X_{g,\Gamma_N,h} \times V_h$ such that

$$\begin{cases} (\alpha^{-1}\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + (\mathrm{div}\ \boldsymbol{\tau}_h, u_h) =< \boldsymbol{\tau}_h \cdot \boldsymbol{n}, g_D >_{\Gamma_D} & \forall\, \boldsymbol{\tau}_h \in X_{0,\Gamma_N,h} \\ (\mathrm{div}\ \boldsymbol{\sigma}_h, v_h) = -(f, v_h) & \forall\, v_h \in V_h \end{cases} \tag{2.3}$$

2.1. **Handle boundary conditions.** When the coefficent α exsits, we have to do some modification for the Neumann boundary. We will discuss how to handle the mixed boundary conditions. We only need to modify the right handle side for the dof of boundary edges for Dirichlet boundary condition. As for the Neumann boundary condition, we can solve it by dual space. We consider triangular first here.

(1) Case 1: RT_0
For the dof associated with a boundary edge $e_{i,j}$, we can evaluate it in the way:

$$\Phi_{i,j}(\boldsymbol{\sigma}_h) = \int_{e_{i,j}} \boldsymbol{\sigma}_h \cdot \boldsymbol{n}_{i,j}\, ds = \int_{e_{i,j}} g_N\, ds$$

(2) Case 2: BDM_1
For the two dofs associated with a boundary edges $e_{i,j}$, we can evaluate them in the way:

$$\begin{aligned}
\Phi_{i,j}(\boldsymbol{\sigma}_h) &= \int_{e_{i,j}} \boldsymbol{\sigma}_h \cdot \boldsymbol{n}_{i,j} \, ds = \int_{e_{i,j}} g_N \, ds \\
\Psi_{i,j}(\boldsymbol{\sigma}_h) &= 3 \int_{e_{i,j}} \boldsymbol{\sigma}_h \cdot \boldsymbol{n}_{i,j} (\lambda_j - \lambda_i) \, ds = 3 \int_{e_{i,j}} g_N (\lambda_j - \lambda_i) \, ds
\end{aligned}$$

REFERENCES