DATA STRUCTURE FOR TRIANGULATIONS

We shall discuss the data structure to represent triangulations and facilitate the mesh adaptation procedure. There is a dilemma for the data structure in the implementation level. If more sophisticated data structure is used to easily traverse in the mesh, for example, to save the star of vertices or edges, it will simplify the implementation of most adaptive finite element subroutines. On the other hand, if the triangulation is changed, for example, a triangle is bisected, one has to update those data structure which in turn complicates the implementation.

Our solution is to maintain two basic data structure and construct auxiliary data structure inside each subroutine when it is necessary. It is not optimal in terms of the computational cost. But it will benefit the interface of accessing subroutines, simplify the coding and save the memory. Also as we shall see soon, the auxiliary data structure can be constructed by sparse matrixlization efficiently. This is an example we scarify a small factor of efficiency to gain the simplicity.

1. BASIC DATA STRUCTURE

The matrices `node(1:N,1:d)` and `elem(1:NT,1:d+1)` are used to represent a d-dimensional triangulation embedded in $\mathbb{R}^d$, where N is the number of vertices and NT is the number of elements. These two matrices represent two different structures of a triangulation: `elem` for the topology and `node` for the geometric embedding.

The matrix `elem` represents a set of abstract simplices. The index set $\{1, 2, \cdots, N\}$ is called the global index set of vertices. Here a vertex is thought as an abstract entity. By definition, `elem(t,1:d+1)` are the global indices of $d+1$ vertices which form the abstract d-simplex `t`. Note that any permutation of vertices of t will represent the same abstract simplex.

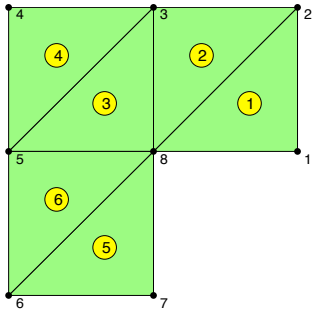
The matrix `node` gives the geometric realization of the simplicial complex. For example, for a 2-D triangulation, `node(k,1:2)` contain x- and y-coordinates of the k-th node. We shall always order the vertices of a simplex such that the signed volume is positive. That is in 2-D, three vertices of a triangle is ordered counter-clockwise and in 3-D, the ordering of vertices follows the right-hand rule. Note that an even permutation of vertices is still allowed to represent the same element.

As an example, `node` and `elem` matrices to represent the triangulation of the L-shape domain $\Omega = (-1,1)\times(-1,1)\backslash([0,1]\times[0,-1])$ are given in the Figure 1 (a) and (b).

2. AUXILIARY DATA STRUCTURE FOR 2-D TRIANGULATION

We discuss ways to extract the combinatorial structure of a triangulation by using `elem` array only. The combinatorial structure will benefit the implementation of finite element methods.

edge. We first complete the 2-D simplicial complex represented by `elem` by constructing 1-dimensional simplices, i.e., edges of the triangulation. We use `edge(1:NE,1:2)` to store indices of the starting and ending points of edges. The column is sorted in a way such that for the k-th edge, `edge(k,1)<edge(k,2)`. The following code will generate an `edge` matrix.



1	1	0
2	1	1
3	0	1
4	-1	1
5	-1	0
6	-1	-1
7	0	-1
8	0	0
	1	2

node

1	1	2	8
2	3	8	2
3	8	3	5
4	4	5	3
5	7	8	6
6	5	6	8
	1	2	3

elem

FIGURE 1. (Left) is a triangulation of the L-shape domain $(-1,1) \times (-1,1) \backslash ([0,1] \times [0,-1])$ and (Right) is its representation using `node` and `elem` matrices.

```
1  totalEdge = sort([elem(:,[2,3]); elem(:,[3,1]); elem(:,[1,2])],2);
2  [i,j,s] = find(sparse(totalEdge(:,2),totalEdge(:,1),1));
3  edge = [j,i];
4  bdEdge = [j(s==1),i(s==1)];
```

The first line collects all edges from the set of triangles and sorts the column such that `totalEdge(k,1) <totalEdge(k,2)`. The interior edges are repeated twice in `totalEdge`. We use the summation property of `sparse` command to merge the duplicated indices. The nonzero vector `s` takes value 1 (for boundary edges) or 2 (for interior edges). We then use `find` to return the nonzero indices which forms the `edge` set. We can also find the boundary edges using the subset of indices pair corresponding to the nonzero value 1. Note that we switch the order of `(i,j)` in line 3 to sort the edge set row-wise since the output of `find(sparse)` is sorted column-wise.

To construct `edge` matrix only, the above 3 line code can be further simplified to one line:

```
edge = unique(sort([elem(:,[2,3]); elem(:,[3,1]); elem(:,[1,2])],2),'rows');
```

The `unique` function provides more functionality which we shall explore more later. However, numerical tests show that the running time of `unique` is around 3 times of the combination `find(sparse)`.

Now we have three types of simplices for a 2-D simplicial complex:

0-simplex: `{1,2,...,N}`; 1-simplex: `edge`; 2-simplex: `elem`.

We shall discuss data structure to efficiently traverse in these simplices. These data structures use mainly the combinatorial property of a mesh, i.e., using the matrix `elem`. We do use some geometric properties of the 2-D planar triangulation. For example, we assume each edge is shared by at most two triangles, which may not hold for general abstract simplicial complex.

Following [**?**], we shall use the name convention `a2b` to represent the link form `a` to `b`. This link is usually the map from the local index set to the global index set. Throught out this paper, we denote the number of `node`, `elem`, and `edge` by

```
N = size(node,1); NT = size(elem,1); NE = size(edge,1);
```

node and elem. The `elem` matrix, by the definition, is a link from triangles to vertices, i.e., `elem` is `elem2node`. The link from vertices to triangles, namely given a vertex v, to find all triangles containing v, is stored in the sparse matrix:

```
t2v = sparse([1:NT,1:NT,1:NT], elem, 1, NT, N);
```

The `NT`$\times$ `N` matrix `t2v` is the incidence matrix between triangles and vertices. `t2v(t,i)=1` means the i-th node is a vertex of triangle t. If we look at `t2v` column-wise, the nonzero in the i-th column of `t2v(:,i)` will give all triangles containing the i-th node. Since sparse matrix is stored column-wise, the star of the i-th node can be efficiently found by

```
nodeStar = find(t2v(:,i));
```

1	1	1	0	0	0	0	0	1
2	0	1	1	0	0	0	0	1
3	0	0	1	0	1	0	0	1
4	0	0	1	1	1	0	0	0
5	0	0	0	0	0	1	1	1
6	0	0	0	0	1	1	0	1
	1	2	3	4	5	6	7	8

t2v

1	2	1	1
2	1	2	3
3	4	6	2
4	3	4	4
5	6	5	5
6	5	3	6
	1	2	3

neighbor

TABLE 1. `t2v` and `neighbor` matrices for the L-shape domain in Figure 1.

The cardinality of the node star, called *valence*, can be computed by the `accumarray` command. The following one line code

```
valence = accumarray(elem(:),ones(3*NT,1),[N 1]);
```

is equivalent to the double loop:

```
for t=1:NT
    for i=1:3
        valence(elem(t,i)) = valence(elem(t,i))+1;
    end
end
```

When `NT` is big, the `for t=1:NT` loop is not efficient in MATLAB. As we mentioned early, `sparse` and `accumarray` are two most commonly used commands to replace the `for` loop.

node and edge. The `edge` matrix, by the definition, is a link from edges to vertices. Sometimes we know only vertices of an edge, say v_i, v_j, and want to find the edge using these two nodes. Namely an index map from $(v_i, v_j) \to k$ such that `edge(k,:)=[vi vj]` or `[vj vi]`. We shall construct such mapping by the sparse matrix

```
node2edge = sparse(edge(:,[1,2]),edge(:,[2,1]),[1:NE,1:NE],N,N);
```

Here we repeat the `edge` matrix with the reverse order in the indices set to allow `i<j` or `j<i` such that if `[i,j]=edge(k,:)` then `node2edge(i,j)=node2edge(j,i)=k`. Thus `node2edge` is a symmetric matrix.

There is another way to construct the link `node` $\to$ `edge` using the product of sparse matrices. Let us introduce the incidence matrix between edges and vertices as

```
e2v = sparse([1:NE,1:NE],[edge(:,1);edge(:,2)],1,NE,N);
```

The sparse matrix `e2v` is of dimension `NE`$\times$`N` such that `e2v(e,v)=1` if `v` is a vertex of `e`. Then `e2v(:,i)` or `e2v(:,j)` contains all edges using the vertex `i` or `j`, respectively. The

intersection of `e2v(:,i)`∩`e2v(:,j)` is the edge using `i,j` as two nodes, which can be found by

```
find(e2v(:,i).*e2v(:,j));
```

`edge` and `elem`. The edge matrix is constructed using the element matrix. But there is no direct link between edges and triangles. One indirect link is through the path `elem` → `node` → `edge`. For example, `node2edge(elem(t,2),elem(t,3))` will give the index of the edge composed by the second and third vertices of the triangle `t`.

Since the access of a sparse matrix is not efficient especially in a large loop, we shall form a direct link `elem` → `edge`. We label three edges of a triangle such that the i-th edge is opposite to the i-th vertex. We define the matrix `elem2edge` as the map of local index of edges in each triangle to its global index. The following 3 line code will construct `elem2edge` using more output from `unique` function.

```
1 totalEdge = sort([elem(:,[2,3]); elem(:,[3,1]); elem(:,[1,2])],2);
2 [edge, i2, j] = unique(totalEdge,'rows');
3 elem2edge = reshape(j,NT,3);
```

Line 1 collects all edges element-wise. The size of `totalEdge` is thus `3NT×2`. By the construction, there is a natural index mapping from `totalEdge` to `elem`. In line 2, we apply `unique` function to obtain the edge matrix. The output index vectors `i2` and `j` contain the index mapping between `edge` and `totalEdge`. Here `i2` is a `NE×1` vector to index the last (2-nd in our case) occurrence of each unique value in `totalEdge` such that `edge = totalEdge(i2,:)`, while `j` is a `3NT×1` vector such that `totalEdge = edge(j,:)`. (Try `help unique` in MATLAB to learn more examples.) Then using the natural index mapping from `totalEdge` to `elem`, we reshape the `3NT×1` vector `j` to a `NT×3` matrix which is `elem2edge`.

We then define a `NE×4` matrix `edge2elem` such that `edge2elem(k,1)` and `edge2elem(k,2)` are two triangles sharing the k-th edge for an interior edge. If the k-th edge is on the boundary, then we set `edge2elem(k,1) = edge2elem(k,2)`. Furthermore, we shall record the local indices in `edge2elem(k,3:4)` such that `elem2edge(edge2elem(k,1),edge2elem(k,3))=k`. Similarly `edge2elem(k,4)` is the local index of k-th edge in `edge2elem(k,2)`.

To construct `edge2elem` matrix, we need to find out the index map from `edge` to `elem`. The following code is a continuation of the code constructing `elem2edge`.

```
1 i1(j(3*NT:-1:1)) = 3*NT:-1:1; i1=i1';
2 k1 = ceil(i1/NT); t1 = i1 - NT*(k1-1);
3 k2 = ceil(i2/NT); t2 = i2 - NT*(k2-1);
4 edge2elem = [t1,t2,k1,k2];
```

The code in line 1 uses `j` to find the first occurrence of each unique edge in the `totalEdge`. In MATLAB, when assign values using an index vector with duplication, the value at the repeated index will be the last one assigned to this location. Obvious `j` contains duplication of edge indices. For example, `j(1)=j(2)=4` which means `totalEdge(1,:)=totalEdge(2,:)=edge(4,:)`. We reverse the order of `j` such that `i1(4)=1` which is the first occurrence.

Using the natural index mapping from `totalEdge` to `elem`, for an index `i` between `1:N`, the formula `k=ceil(i/NT)` computes the local index of i-th edge, and `t=i-NT*(k-1)` is the global index of the triangle which `totalEdge(i,:)` belongs to. The `edge2elem` is just composed by `t1,t2,k1` and `k2`.

`elem` and `elem`. We use the matrix `neighbor(1:NT,1:3)` to record the neighboring triangles for each triangle. By definition, `neighbor(t,i)` is opposite to the i-th vertex of

the t-th triangle. If i is opposite to the boundary, then we set `neighbor(t,i)=t`. Using the index map between edge and elem, we can easily form the neighbor matrix by the following 2 lines of code.

```
1  ix = (i1 ~= i2);
2  neighbor = accumarray([[t1(ix),k1(ix)];[t2,k2]],[t2(ix);t1],[NT 3]);
```

In line 1, to avoid the duplication in the index array, we find the index set of interior edges by noting that if `e` is a boundary edge, then `i1(e)=i2(e)`. Since `t1` and `t2` share the same edge, we form the `neighbor` matrix by using `t1,k1` and `t2,k2` as indices set and `t2,t1` as the value in line 2.

We summarize the construction of these auxiliary data structures in the subroutine `auxstructure.m`.

```
1  function [neighbor,elem2edge,edge2elem,edge,bdEdge]=auxstructure(elem)
2  totalEdge = sort([elem(:,[2,3]); elem(:,[3,1]); elem(:,[1,2])],2);
3  [edge, i2, j] = unique(totalEdge,'rows');
4  NT = size(elem,1);
5  elem2edge = reshape(j,NT,3);
6  i1(j(3*NT:-1:1)) = 3*NT:-1:1; i1=i1';
7  k1 = ceil(i1/NT); t1 = i1 - NT*(k1-1);
8  k2 = ceil(i2/NT); t2 = i2 - NT*(k2-1);
9  ix = (i1 ~= i2);
10 neighbor = accumarray([[t1(ix),k1(ix)];[t2,k2]],[t2(ix);t1],[NT 3]);
11 bdEdge = edge((i1 == i2),:);
12 edge2elem = [t1,t2,k1,k2];
```

3. Auxiliary data structure for 3-D triangulation

Most codes discussed for 2-D triangulations can be generalized to 3-D triangulations in a straightforward way. Due to the page limit, we pick up the following important data structures to explain in detail.

`elem` and `face`. The `face` matrix, which represents the 2-D simplex, can be generated by the `unique` function of all element-wise faces. The link `elem2face`, `faceStar`, and `neighbor` can be constructed similarly using the index map. We list `auxstructure3.m` below and skip the explanation.

```
1  function [neighbor,elem2face,face2elem,face,bdFace] = auxstructure3(elem)
2  totalFace = [elem(:,[2 4 3]);elem(:,[1 3 4]);elem(:, [1 4 2]);elem(:, [1 2 3])];
3  [face, i2, j] = unique(sort(totalFace,2),'rows');
4  NT = size(elem,1);
5  elem2face = reshape(j,NT,4);
6  i1(j(4*NT:-1:1)) = 4*NT:-1:1; i1 = i1';
7  k1 = ceil(i1/NT); t1 = i1 - NT*(k1-1);
8  k2 = ceil(i2/NT); t2 = i2 - NT*(k2-1);
9  ix = (i1 ~= i2);
10 neighbor = accumarray([[t1(ix),k1(ix)];[t2,k2]],[t2(ix);t1],[NT 4]);
11 bdFace = face((i1 == i2),:);
12 face2elem = [t1,t2,k1,k2];
```

`elem` and `edge`. The `edge` matrix and `elem2edge` can be generated using `unique` commands as in the 2-D case.

```
1 totalEdge = sort([elem(t,[1 2]); elem(t,[1 3]); elem(t,[1 4]); ...
2                   elem(t,[2 3]); elem(t,[2 4]); elem(t,[3 4])],2);
3 [edge, i2, j] = unique(totalEdge,'rows');
4 elem2edge = reshape(j,NT,6);
```

We now discuss the construction of `edgeStar`. This link from `edge` to `elem` is important since the 3-D local mesh refinement is always cutting edges. Unlike the 2-D case, we cannot use a `NE`$\times$`2` dense matrix for `edgeStar` since the number of elements sharing one edge varies a lot. Again we shall resort to the sparse matrix.

```
1 t2v = sparse([1:NT,1:NT,1:NT,1:NT], elem(1:NT,:), 1, NT, N);
2 nodeStar1 = t2v(1:NT,edge(:,1));
3 nodeStar2 = t2v(1:NT,edge(:,2));
4 edgeStar = nodeStar1.*nodeStar2;
```

The elements containing an edge are characterized as the intersection of two stars of the ending nodes of this edge. The first line generates the incidence matrix `t2v`. Line 2 and 3 extract columns from `t2v`. The intersection is found by the Hadamard product of two sparse matrix `nodeStar1` and `nodeStar2`. The resulting sparse matrix `edgeStar` is a `NT`$\times$`NE` sparse matrix and `find(edgeStar(:,i))` will give the element indices containing the i-th edge.

In the construction of `elem2edge` and `edgeStar`, we use Hadamard product of sparse matrices to find the quantity associated with two index sets. This technique is crucial in 3-D refinement.