Systems Theory Exercises - Review

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1. **[Eigenvalues and their multiplicities]** Consider the following matrices. Identify the eigenvalues and their respective algebraic and geometric multiplicities. Consequently deduce the Jordan form of each matrix and its minimal polynomial:

   (a) $F = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

   (b) $F = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$;

   (c) $F = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$;

   (d) $F = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$;

   (e) $F = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{bmatrix}$.

2. **[Jordan form]** For each of the following matrices, denoted by the symbol F, determine the Jordan form J and a non singular matrix T such that $T^{-1}FT = J$:

   (a) $F = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

   (b) $F = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

   (c) $F = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

3. **[Unforced evolution of discrete time state space models]** Determine, for $t \in \mathbb{Z}_+$, the unforced state and output evolution of the discrete time state space model

$$\begin{aligned} \mathbf{x}(t+1) &= F\mathbf{x}(t) \\ \mathbf{y}(t) &= H\mathbf{x}(t) \end{aligned}$$

   corresponding to the following matrices F and H and the initial state $\mathbf{x}(0)$:

   (a) $F = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, H = \begin{bmatrix} 1 & 1 \end{bmatrix}, \mathbf{x}(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$;

(b) $F = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \mathbf{x}(0) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T;$

(c) $F = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \mathbf{x}(0) = \begin{bmatrix} 1 & 0 & 2 \end{bmatrix}^T.$

4. [**$\mathcal{Z}$-Transform**] Compute the $\mathcal{Z}$-transform of the following sequences:

   (a) $v(t) = 1 + \delta(t) + \frac{1}{2^{t-1}}$;

   (b) $v(t) = t \cdot \delta_{-1}(t) + 3^t \delta_{-1}(t+4) + 5j^t$;

   (c) $v(t) = \delta(t+1) + \delta(t-1) + \binom{t}{2}$;

   (d) $v(t) = \delta_{-1}(t) - \delta_{-1}(t-5) + \binom{t}{1} 5^t$;

   (e) $v(t) = \begin{cases} 1, & \text{for } t \in \mathbb{Z}_+, t \text{ even}; \\ 0, & \text{otherwise}; \end{cases}$

   (f) $v(t) = \begin{cases} 2^t, & \text{for } t \in \mathbb{Z}_+, t \text{ odd}; \\ 0, & \text{otherwise}; \end{cases}$

   (g) $v(t) = \begin{cases} 0, & \text{for } t = 0, 3; \\ 1, & \text{for } t = 1; \\ -1, & \text{for } t = 2; \\ 2, & \text{for } t \in \mathbb{Z}_+, t \geq 4; \end{cases}$

   (h) $v(t) = \begin{cases} 1, & \text{for } t \in \mathbb{Z}_+, t \text{ even}; \\ -2, & \text{otherwise}; \end{cases}$

   (i) $v(t) = \begin{cases} 1 + t, & \text{for } t \in \mathbb{Z}_+, t \text{ even}; \\ 0, & \text{otherwise}; \end{cases}$

   (l) $v(t) = \begin{cases} 2^t, & \text{for } t \in \mathbb{Z}_+, t \text{ multiple of } 3; \\ -1, & \text{for } t \in \mathbb{Z}_+, t \equiv 1 \text{ mod } 3 \; (i.e., t \text{ congruous to 1 modulus 3}); \\ 0, & \text{otherwise}. \end{cases}$

5. [**Inverse $\mathcal{Z}$-transform**] Compute the inverse $\mathcal{Z}$-transform of the following rational functions:

   (a) $V(z) = z^{-2} + \frac{1}{(z+4)(z+1)}$;

   (b) $V(z) = \frac{z+1}{z(z-1)(z-2)}$;

   (c) $V(z) = \frac{z^2}{z^2+10}$;

   (d) $V(z) = \frac{2z}{z^2 - \sqrt{3}z + 1}$;

   (e) $V(z) = \frac{z^2+1}{z(z+1)^2}$.

   (f) $V(z) = \frac{z}{z^2-1}$.

6. [**Study of the discrete time state space models via $\mathcal{Z}$-transform**] Determine, by operating in the $\mathcal{Z}$-transform domain, the forced state and output evolutions corresponding to the specific choices of the initial condition and of the input sequence:

(a)

$$\begin{aligned}
\mathbf{x}(t+1) &= \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \\
y(t) &= \begin{bmatrix} 1 & -1 \end{bmatrix} \mathbf{x}(t), \\
\mathbf{x}(0) &= \mathbf{0}, u(t) = 2^t \delta_{-1}(t-1);
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbf{x}(t+1) &= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u(t), \\
y(t) &= \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{x}(t), \\
\mathbf{x}(0) &= \begin{bmatrix} 1 & 0 \end{bmatrix}^T, u(t) = 0;
\end{aligned}$$

(c)

$$\begin{aligned}
\mathbf{x}(t+1) &= \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u(t), \\
y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}(t), \\
\mathbf{x}(0) &= \begin{bmatrix} 1 & 1 \end{bmatrix}^T, u(t) = \delta_{-1}(t).
\end{aligned}$$

7. [**Discrete time state space models stability**] Study the asymptotic and simple stability of the following state space models:

(a)

$$\begin{aligned}
\mathbf{x}(t+1) &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u(t), \\
y(t) &= \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{x}(t),
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbf{x}(t+1) &= \begin{bmatrix} 1/2 & 1 \\ 0 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \\
y(t) &= \begin{bmatrix} 1 & -1 \end{bmatrix} \mathbf{x}(t) - u(t),
\end{aligned}$$

(c)

$$\begin{aligned}
\mathbf{x}(t+1) &= \begin{bmatrix} 1/2 & a \\ 0 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(t), \qquad a \in \mathbb{R}, \\
y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t).
\end{aligned}$$

(d)

$$\begin{aligned}\mathbf{x}(t+1) &= \begin{bmatrix} a & 0 & 0 \\ 0 & 1/2 & 0 \\ 1 & 2 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(t), \qquad a \in \mathbb{R}, \\ y(t) &= \begin{bmatrix} 1 & -2 & 0 \end{bmatrix} \mathbf{x}(t).\end{aligned}$$

(e)

$$\begin{aligned}\mathbf{x}(t+1) &= \begin{bmatrix} -3a-3 & 1 \\ 0 & -1+a^2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t), \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t).\end{aligned}$$

8. **[Unforced evolution of the continuous time models]** Determine, for $t \in \mathbb{R}_+$, by operating in the time domain, the unforced state and output evolutions of the continuous time model

$$\begin{aligned}\dot{\mathbf{x}}(t) &= F\mathbf{x}(t) \\ \mathbf{y}(t) &= H\mathbf{x}(t)\end{aligned}$$

corresponding to the the following matrices F and H and the initial state $\mathbf{x}(0)$:

(a) $F = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, H = \begin{bmatrix} 1 & 1 \end{bmatrix}, \mathbf{x}(0) = \begin{bmatrix} 2 & -1 \end{bmatrix}^T;$

(b) $F = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 \end{bmatrix}, \mathbf{x}(0) = \begin{bmatrix} 1 & 1 \end{bmatrix}^T;$

(c) $F = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}, H = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}, \mathbf{x}(0) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T;$

(d) $F = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, H = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}, \mathbf{x}(0) = \begin{bmatrix} 2 & -1 & 2 \end{bmatrix}^T;$

9. **[Study of the continuous time models via Laplace transform]** Determine, by reasoning in the Laplace transform domain, the system state and output evolutions corresponding to the specific choices of the initial condition and input signal:

(a) $$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \\ y(t) = \begin{bmatrix} 1 & -1 \end{bmatrix} \mathbf{x}(t), \\ \mathbf{x}(0) = \mathbf{0}, u(t) = e^{2t}\delta_{-1}(t); \end{cases}$$

(b) $$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u(t), \\ y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{x}(t), \\ \mathbf{x}(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T, u(t) = 0; \end{cases}$$

$$
\text{(c)} \quad \begin{cases} \dot{\mathbf{x}}(t) &= \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \\ \\ \mathbf{y}(t) &= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \mathbf{x}(t), \\ \\ \mathbf{x}(0) &= \begin{bmatrix} 1 & 0 \end{bmatrix}^T, u(t) = e^{-t}\delta_{-1}(t). \end{cases}
$$

10. **[Continuous time models stability]** Determine the elementary modes and study the asymptotic and simple stability of the following state space models:

$$
\text{(a)} \quad \begin{cases} \dot{\mathbf{x}}(t) &= \begin{bmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} u(t), \\ \\ y(t) &= \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \mathbf{x}(t), \end{cases}
$$

$$
\text{(b)} \quad \begin{cases} \dot{\mathbf{x}}(t) &= \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1/2 & 0 \\ 0 & 1 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t), \\ \\ y(t) &= \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \mathbf{x}(t) - u(t), \end{cases}
$$

$$
\text{(c)} \quad \begin{cases} \dot{\mathbf{x}}(t) &= \begin{bmatrix} -1+a & a & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} u(t), \qquad a \in \mathbb{R}, \\ \\ y(t) &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}(t). \end{cases}
$$

SOLUTIONS

1. **[Eigenvalues and their multiplicities]** (In the following we denote by n_i the algebraic multiplicity of the eigenvalue λ_i, by s_i its geometric multiplicity and by m_i its multiplicity as a zero of the the minimal polynomial):

   (a) It is evident that $\Delta_F(z) = (z-1)^2(z-2)$ and therefore the eigenvalues of F are $\lambda_1 = 1$ and $\lambda_2 = 2$, the former with multiplicity two and the latter with multiplicity one in the characteristic polynomial. Therefore $n_1 = 2$ and $n_2 = 1$. Consequently $1 \leq s_1 \leq n_1 = 2$ while $s_2 = n_2 = 1$. From a direct evaluation of $\dim U_1 = \dim \ker(\lambda_1 I_3 - F) = 3 - \operatorname{rank}(I_3 - F)$ it follows that $s_1 = 2$. Then the Jordan form of F is

   $$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

   The multiplicities of λ_1 and λ_2 in the minimal polynomial coincide with the dimensions of the largest miniblocks associated with such eigenvalues in the Jordan form. Therefore $m_1 = 1$ and $m_2 = 1$, and the minimal polynomial is $\Psi_F(z) = (z-1)(z-2)$.

   (b) It is evident that $\Delta_F(z) = (z-1)^3$ and hence F has one only eigenvalue $\lambda_1 = 1$ with multiplicity $n_1 = 3$. Consequently $1 \leq s_1 \leq n_1 = 3$. From a direct evaluation of $\dim U_1 = \dim \ker(\lambda_1 I_3 - F) = 3 - \operatorname{rank}(I_3 - F)$ it follows that $s_1 = 2$. Then the Jordan form of F is

   $$J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   The multiplicity of λ_1 in the minimal polynomial coincides with the dimension of the largest miniblock associated with such eigenvalue in the Jordan form. Therefore $m_1 = 2$, and the minimal polynomial is $\Psi_F(z) = (z-1)^2$.

   (c) In this case $\Delta_F(z) = z^3$ and hence F has one only eigenvalue $\lambda_1 = 0$ with multiplicity $n_1 = 3$. Consequently $1 \leq s_1 \leq n_1 = 3$. From a direct evaluation of $\dim U_0 = \dim \ker(\lambda_1 I_3 - F) = 3 - \operatorname{rank} F$ it follows that $s_1 = 1$. Then the Jordan form of F is

   $$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

   The multiplicity of λ_1 in the minimal polynomial coincides with the dimension of the largest miniblock associated with such eigenvalue in the Jordan form. Therefore $m_1 = 3$ and the minimal polynomial is $\Psi_F(z) = z^3 = \Delta_F(z)$.

   (d) In this case $\Delta_F(z) = z^2(z-3)$ and hence the eigenvalues of F are $\lambda_1 = 0$ and $\lambda_2 = 3$, the former with multiplicity two and the latter with multiplicity one in the characteristic polynomial. Therefore $n_1 = 2$ and $n_2 = 1$. Consequently $1 \leq s_1 \leq n_1 = 2$ while $s_2 = n_2 = 1$. From a direct evaluation of $\dim U_0 = \dim \ker(\lambda_1 I_3 - F) = 3 - \operatorname{rank} F$ it follows that $s_1 = 1$. Then the Jordan form of F is

   $$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

The multiplicities of λ_1 and λ_2 in the minimal polynomial coincide with the dimensions of the largest miniblocks associated with such eigenvalues in the Jordan form. Therefore $m_1 = 2$ and $m_2 = 1$ and the minimal polynomial is $\Psi_F(z) = z^2(z-3) = \Delta_F(z)$.

(e) It is evident that $\Delta_F(z) = (z+1)^3$ and hence F has one only eigenvalue $\lambda_1 = -1$ with multiplicity $n_1 = 3$. Consequently $1 \le s_1 \le n_1 = 3$. From a direct evaluation of $\dim U_1 = \dim\ker(\lambda_1 I_3 - F) = 3 - \mathrm{rank}(-I_3 - F)$ it follows that $s_1 = 1$. Then the Jordan form of F is

$$J = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

The multiplicity of λ_1 in the minimal polynomial coincides with the dimension of the largest miniblock associated with such eigenvalue in the Jordan form. Therefore $m_1 = 3$ and the minimal polynomial is $\Psi_F(z) = (z+1)^3$.

2. [**Jordan Form**]:

(a) The matrix F is block diagonal therefore, by getting the Jordan form of the first block and the Jordan form of the second block and by suitably arranging them (if necessary), we get the Jordan form of the matrix F itself. Since the second block is a 1×1 matrix, we already have a Jordan form for it. As concerns the first block

$$F_1 = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$$

since it is a 2×2 matrix with two distinct eigenvalues, $\lambda_1 = 2$ and $\lambda_2 = 1$, both simple ($n_1 = n_2 = 1$), it turns out to be diagonalizable. The Jordan form of F_1 is

$$J_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

and a change of basis matrix $T_1 \in \mathbb{R}^{2\times 2}$ that brings F_1 in the Jordan form J_1 is of the type $T_1 = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$ with $\mathbf{v}_i$ (the vector with the coordinates of) an eigenvector of F_1 associated with λ_i. One gets, for example $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ that lead to

$$T_1 = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}.$$

Consequently

$$J = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) The matrix F is block diagonal therefore, by getting the Jordan form of the first block and the Jordan form of the second block and by suitably arranging them (if necessary), we get the Jordan form of the matrix F itself. Since the second block is a 1×1 matrix, we already have a Jordan form for it. As concerns the first block

$$F_1 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

since it is a 2×2 matrix with one only eigenvalue $\lambda_1 = 1$ with multiplicity $n_1 = 2$, is not diagonalizable. The Jordan form of F_1 is

$$J_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

and a change of basis matrix $T_1 \in \mathbb{R}^{2\times 2}$ that brings F_1 in the Jordan form J_1 is made up of two columns, namely $T_1 = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$. The first one is necessarily (the vector with the coordinates of) an eigenvector of F_1 associated with $\lambda_1 = 1$. The second one has to be chosen in such a way it satisfies

$$F_1 T_1 = T_1 J_1.$$

One gets, for example, $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. On the other hand $\mathbf{v}_2 = \begin{bmatrix} a \\ b \end{bmatrix}$ must satisfy the constraint $a = -1$. Then we chose, for instance, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and consequently we get

$$T_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Therefore

$$J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(c) The matrix F has an eigenvalue $\lambda_1 = 1$ with multiplicity $n_1 = 2$ and an eigenvalue $\lambda_2 = 2$ with multiplicity $n_2 = 1$. One needs to understand if its Jordan form is either

$$J_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

or

$$J_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

The first Jordan form corresponds to the case in which the geometric multiplicity s_1 of the eigenvalue λ_1 is unitary, the second one to the case in which $s_1 = 2$. It is immediate to verify that $\dim(\ker(\lambda_1 I_3 - F)) = 1$ and therefore the first case occurs. Therefore

$$J = J_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

The change of basis matrix $T \in \mathbb{R}^{3\times 3}$ is made up of three columns, namely $T = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$. The first one is necessarily (the vector with the coordinates of) an eigenvector of F associated with $\lambda_1 = 1$, while the third one is necessarily (the vector with the coordinates of) an eigenvector of F associated with $\lambda_2 = 2$. Finally, the second one, needs to be chosen in such a way that

$$FT = TJ.$$

One gets, for instance, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$. On the other hand, by assuming $\mathbf{v}_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and by imposing the condition $FT = TJ$ (in particular, by imposing the equality between the second column of the product matrix on the left side to the second column of the product matrix on the right side), one gets $2b = 1$ and $c = 0$. We chose then, for instance, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1/2 \\ 0 \end{bmatrix}$ and, consequently, we get

$$T = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1/2 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

3. **[Unforced evolution of discrete time state space models]**:

(a) It is immediate to verify that $F^t = \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix}$ from which it follows that

$$\begin{aligned} \mathbf{x}(t) &= F^t\mathbf{x}(0) = \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ y(t) &= HF^t\mathbf{x}(0) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1. \end{aligned}$$

(b) It is easy to verify that $F^t = \begin{bmatrix} 1 & 0 & 0 \\ -t & 1 & 0 \\ 0 & 0 & 2^t \end{bmatrix}$ from which it follows that

$$\begin{aligned} \mathbf{x}(t) &= F^t\mathbf{x}(0) = \begin{bmatrix} 1 & 0 & 0 \\ -t & 1 & 0 \\ 0 & 0 & 2^t \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1-t \\ 2^t \end{bmatrix} \\ y(t) &= HF^t\mathbf{x}(0) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1-t \\ 2^t \end{bmatrix} = 1 + 2^t. \end{aligned}$$

(c) It is immediate to verify that $F^t = \begin{bmatrix} 1 & 0 & 0 \\ 2^t-1 & 2^t & 0 \\ 0 & 0 & (-1)^t \end{bmatrix}$ from which it follows that

$$\begin{aligned}
\mathbf{x}(t) &= F^t\mathbf{x}(0) = \begin{bmatrix} 1 & 0 & 0 \\ 2^t-1 & 2^t & 0 \\ 0 & 0 & (-1)^t \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2^t-1 \\ 2(-1)^t \end{bmatrix} \\
y(t) &= HF^t\mathbf{x}(0) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2^t-1 \\ 2(-1)^t \end{bmatrix} = 1 + 2(-1)^t.
\end{aligned}$$

4. [**$\mathcal{Z}$-Transform**]:

(a) $V(z) = \frac{z}{z-1} + 1 + \frac{2z}{z-1/2}$.

(b) $V(z) = \frac{z}{(z-1)^2} + \frac{z}{z-3} + 5\frac{z}{z-j}$.

(c) $V(z) = z^{-1} + \frac{z}{(z-1)^3}$.

(d) $V(z) = \sum_{i=0}^{4} z^{-i} + 5 \cdot \frac{z}{(z-5)^2}$.

(e) $V(z) = \sum_{i=0}^{+\infty} 1z^{-2i} = \sum_{i=0}^{+\infty} (z^{-2})^i = \frac{1}{1-z^{-2}} = \frac{z^2}{z^2-1}$.

(f) $V(z) = \sum_{i=0}^{+\infty} 2^{2i+1}z^{-2i-1} = 2z^{-1}\sum_{i=0}^{+\infty}(4z^{-2})^i = 2z^{-1} \cdot \frac{1}{1-4z^{-2}} = \frac{2z}{z^2-4}$.

(g) $V(z) = 1 \cdot z^{-1} - 1 \cdot z^{-2} + \sum_{t=4}^{+\infty} 2z^{-t} = z^{-1} - z^{-2} + 2\left(\sum_{i=0}^{+\infty} z^{-t} - 1 - z^{-1} - z^{-2} - z^{-3}\right) = -2 - z^{-1} - 3z^{-2} - 2z^{-3} + 2\frac{z}{z-1}$.

(h) $V(z) = \sum_{i=0}^{+\infty} 1 \cdot z^{-2i} - \sum_{i=0}^{+\infty} 2 \cdot z^{-2i-1} = (1-2z^{-1}) \cdot \sum_{i=0}^{+\infty} z^{-2i} = (1-2z^{-1}) \cdot \frac{1}{1-z^{-2}} = \frac{z(z-2)}{z^2-1}$.

(i) $V(z) = \sum_{i=0}^{+\infty}(1+2i)z^{-2i} = -z^2\frac{d}{dz}\left(\sum_{i=0}^{+\infty} z^{-2i-1}\right) = -z^2\frac{d}{dz}\left(z^{-1} \cdot \frac{1}{1-z^{-2}}\right) = -z^2\frac{d}{dz}\left(\frac{z}{z^2-1}\right) = -z^2\frac{z^2-1-2z^2}{(z^2-1)^2} = \frac{z^2(z^2+1)}{(z^2-1)^2}$,
where it has been exploited the fact that

$$\frac{d}{dz}\left(\sum_{i=0}^{+\infty} z^{-2i-1}\right) = \sum_{i=0}^{+\infty}(-2i-1)z^{-2i-2} = -z^{-2} \cdot \sum_{i=0}^{+\infty}(2i+1)z^{-2i}.$$

(l) $V(z) = \sum_{i=0}^{+\infty} 2^{3i}z^{-3i} - \sum_{i=0}^{+\infty} 1 \cdot z^{-3i-1} = \sum_{i=0}^{+\infty}(8z^{-3})^i - z^{-1} \cdot \sum_{i=0}^{+\infty} z^{-3i} = \frac{1}{1-8z^{-3}} - z^{-1} \cdot \frac{1}{1-z^{-3}} = \frac{z^3}{z^3-8} - \frac{z^2}{z^3-1}$.

5. [**Inverse $\mathcal{Z}$-Transform**]:

(a) $v(t) = \delta(t-2) + \frac{1}{4}\delta(t) + \frac{1}{12}(-4)^t - \frac{1}{3}(-1)^t$.

(b) $v(t) = \frac{5}{4}\delta(t) + \frac{1}{2}\delta(t-1) - 2 + \frac{3}{4}2^t$.

(c) $v(t) = \frac{1}{2}(j\sqrt{10})^t + \frac{1}{2}(-j\sqrt{10})^t = \frac{(\sqrt{10})^t}{2}\left(e^{jt\pi/2} + e^{-jt\pi/2}\right) = (\sqrt{10})^t\cos\left(\frac{t\pi}{2}\right)$.

(d) $v(t) = -(2j)e^{j\pi t/6} + (2j)e^{-j\pi t/6} = 4\sin\left(t\frac{\pi}{6}\right)$.

(e) $v(t) = \delta(t-1) - 2\binom{t-1}{1}(-1)^{t-2}$ or, equivalently, $v(t) = -2\delta(t) + \delta(t-1) + 2(-1)^t\delta_{-1}(t) + 2\binom{t}{1}(-1)^{t-1}$.

(f) $v(t) = \frac{1}{2}\delta_{-1}(t-1) + \frac{1}{2}(-1)^{t-1}\delta_{-1}(t-1)$ or, equivalently,

$$v(t) = \begin{cases} 0, & \text{for } t \geq 0 \text{ and even;} \\ 1, & \text{for } t \geq 0 \text{ and odd.} \end{cases}$$

6. **[Study of the discrete time state space models via $\mathcal{Z}$-transform]**:

(a) The fact that the initial condition is zero ensures that the state and output evolutions that we compute are purely forced and therefore the following expressions hold:

$$\begin{aligned} X(z) &= (zI_n - F)^{-1}GU(z), & (1) \\ Y(z) &= [H(zI_n - F)^{-1}G + D]U(z). & (2) \end{aligned}$$

Since $u(t) = 2 \cdot (2^{t-1}\delta_{-1}(t-1))$, the $\mathcal{Z}$-transform of the given input is

$$U(z) = 2z^{-1} \cdot \frac{z}{z-2} = \frac{2}{z-2}.$$

Therefore we get:

$$\begin{aligned} X_f(z) &= \begin{bmatrix} z-1 & -3 \\ 0 & z-2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{2}{z-2} \\ &= \frac{1}{(z-1)(z-2)} \begin{bmatrix} 3 \\ z-1 \end{bmatrix} \frac{2}{z-2} = \begin{bmatrix} \frac{6}{(z-1)(z-2)^2} \\ \frac{2}{(z-2)^2} \end{bmatrix} \\ Y_f(z) &= \begin{bmatrix} 1 & -1 \end{bmatrix} X_f(z) = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{6}{(z-1)(z-2)^2} \\ \frac{2}{(z-2)^2} \end{bmatrix} \\ &= \frac{8-2z}{(z-1)(z-2)^2}. \end{aligned}$$

By applying the inverse $\mathcal{Z}$-transform to each term one gets

$$\begin{aligned} \mathbf{x}_f(t) &= \begin{bmatrix} 6\left(1 - 2^{t-1}\right)\delta_{-1}(t-1) + 6\binom{t-1}{1}2^{t-2} \\ \binom{t-1}{1}2^{t-1} \end{bmatrix} \\ y_f(t) &= \left(6 - 6\ 2^{t-1}\right)\delta_{-1}(t-1) + 4\binom{t-1}{1}2^{t-2}. \end{aligned}$$

(b) In this case the fact that the applied input is identically zero ensures that the state and output evolution that we compute are purely unforced and therefore the following expressions hold:

$$\begin{aligned} X(z) &= (zI_n - F)^{-1}z\mathbf{x}(0), & (3) \\ Y(z) &= H(zI_n - F)^{-1}z\mathbf{x}(0). & (4) \end{aligned}$$

Hence we get:

$$\begin{aligned} X_\ell(z) &= \begin{bmatrix} z-1 & 0 \\ 1 & z-1 \end{bmatrix}^{-1} z \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{z}{(z-1)^2} \begin{bmatrix} z-1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{z}{z-1} \\ -\frac{z}{(z-1)^2} \end{bmatrix} \\ Y_\ell(z) &= \begin{bmatrix} 1 & 1 \end{bmatrix} X_\ell(z) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{z}{z-1} \\ -\frac{z}{(z-1)^2} \end{bmatrix} \\ &= \frac{z}{z-1} - \frac{z}{(z-1)^2}. \end{aligned}$$

By applying the inverse $\mathcal{Z}$-transform to each term one gets, for $t \in \mathbb{Z}_+$,

$$\begin{aligned}
\mathbf{x}_\ell(t) &= \begin{bmatrix} 1 \\ -\binom{t}{1} \end{bmatrix} \\
y_\ell(t) &= 1 - \binom{t}{1}.
\end{aligned}$$

(c) In this case, being both the initial conditions and the input solicitation non null, we have to compute both the unforced and forced evolution components of state and output. As concerns the unforced evolution, by resorting to (3)-(4), we get:

$$\begin{aligned}
X_\ell(z) &= \begin{bmatrix} z+1 & -1 \\ 0 & z-1 \end{bmatrix}^{-1} z \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \frac{z}{(z-1)(z+1)} \begin{bmatrix} z \\ z+1 \end{bmatrix} = \begin{bmatrix} \frac{z^2}{z^2-1} \\ \frac{z}{z-1} \end{bmatrix} \\
Y_\ell(z) &= \begin{bmatrix} 0 & 1 \end{bmatrix} X_\ell(z) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{z^2}{z^2-1} \\ \frac{z}{z-1} \end{bmatrix} \\
&= \frac{z}{z-1}.
\end{aligned}$$

By applying the inverse $\mathcal{Z}$-transform to each term one gets, for $t \in \mathbb{Z}_+$,

$$\begin{aligned}
\mathbf{x}_\ell(t) &= \begin{bmatrix} \frac{1}{2}[(-1)^t + 1] \\ 1 \end{bmatrix} \\
y_\ell(t) &= 1.
\end{aligned}$$

As concerns the forced evolution, by resorting to (1)-(2), we get:

$$\begin{aligned}
X_f(z) &= \begin{bmatrix} z+1 & -1 \\ 0 & z-1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \frac{z}{z-1} \\
&= -\frac{z}{(z-1)^2(z+1)} \begin{bmatrix} 1 \\ z+1 \end{bmatrix} = -\begin{bmatrix} \frac{z}{(z-1)^2(z+1)} \\ \frac{z}{(z-1)^2} \end{bmatrix} \\
Y_f(z) &= \begin{bmatrix} 0 & 1 \end{bmatrix} X_f(z) = -\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{z}{(z-1)^2(z+1)} \\ \frac{z}{(z-1)^2} \end{bmatrix} \\
&= -\frac{z}{(z-1)^2}.
\end{aligned}$$

By applying the inverse $\mathcal{Z}$-transform to each term one gets,

$$\begin{aligned}
\mathbf{x}_f(t) &= -\begin{bmatrix} \frac{1}{4}[(-1)^t - 1]\delta_{-1}(t) + \frac{1}{2}\binom{t}{1} \\ \binom{t}{1} \end{bmatrix} \\
y_f(t) &= -\binom{t}{1}.
\end{aligned}$$

7. **[Discrete time state space models stability]**:

(a) It is immediate to notice that F is a Jordan miniblock of dimension 2 associated with the eigenvalue 1, therefore the system exhibits the modes 1 (bounded) and $t = \binom{t}{1}$ divergent. Then the system is unstable, since it is neither asymptotically stable nor simply stable.

(b) It is immediate to notice that F is diagonalizable with (simple) eigenvalues $1/2$ and -1. Since all the eigenvalues have moduli smaller than or equal to 1 and the ones with unitary modulus are simple, the system is simply stable but not asymptotically stable.

(c) It is immediate to verify that F is always diagonalizable with (simple) eigenvalues $1/2$ and 1. Since all the eigenvalues have moduli smaller than or equal to 1 and the ones with unitary modulus are simple, the system is simply stable, but not asymptotically stable.

(d) We distinguish the following cases: $a = 1$ and $a \neq 1$. For $a = 1$ we get $\Delta_F(z) = (z-1)^2(z-1/2)$. In this case we certainly do not have asymptotic stability. Then one needs to check if the system is simply stable. Since the dimension of the eigenspace associated with 1 is 1, it follows that in the Jordan form of F there is one only miniblock of dimension 2 associated with 1. Then the system is not simply stable. For $a \neq 1$ we have three eigenvalues (possibly not distinct if $a = 1/2$), one of which is 1. If $|a| > 1$ then we have instability, if $|a| \leq 1$ then we have simple stability (because, by excluding the case $a = 1$, the eigenvalues with unitary modulus are simple).

(e) We distinguish two cases depending on the fact that the two eigenvalues of the matrix F, $\lambda_1 = -3a-3$ and $\lambda_2 = -1+a^2$, are distinct or coincide. If $\lambda_1 = \lambda_2$, namely $-1+a^2 = -3a-3$, i.e. $a = -1, -2$, then the matrix F is already in Jordan form and exhibits only one miniblock of dimension 1 associated with the eigenvalue in exam. Specifically, for $a = -1$ one gets

$$F = J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

and for $a = -2$ one gets

$$F = J = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}.$$

If instead $\lambda_1 \neq \lambda_2$, namely $a \neq -1, -2$, then the matrix F is diagonalizable and its Jordan form is

$$J = \begin{bmatrix} -3a-3 & 0 \\ 0 & -1+a+^2 \end{bmatrix}.$$

Let us evaluate under what conditions both the eigenvalues have moduli smaller than 1. One gets that the two conditions

$$\begin{cases} |a^2-1| < 1 \\ |3a+3| < 1 \end{cases}$$

are fulfilled if and only if $-4/3 < a < -2/3$. For $a = -4/3$ the matrix F exhibits an eigenvalue with modulus smaller than 1 and another (simple) in 1, therefore one has simple stability. For $a = 2/3$ the matrix F exhibits an eigenvalue of modulus smaller than 1 and another (simple) in 1, therefore one has simple stability. For values outside of the interval $[-4/3, -2/3]$ the system is unstable.

8. **[Unforced evolution of the continuous time models]**:

(a) The generic power of the matrix F is

$$F^i = \begin{cases} \begin{bmatrix} (-1)^i & 0 \\ 0 & 1 \end{bmatrix}, & \text{if } i \text{ is even,} \\ \begin{bmatrix} (-1)^i & 0 \\ 1 & 1 \end{bmatrix}, & \text{if } i \text{ is odd} \end{cases}$$

from which it follows

$$F^i = \begin{bmatrix} (-1)^i & 0 \\ \frac{1-(-1)^i}{2} & 1 \end{bmatrix}, \qquad i \in \mathbb{Z}_+,$$

$$e^{Ft} = \sum_{i=0}^{+\infty} \begin{bmatrix} (-1)^i & 0 \\ \frac{1-(-1)^i}{2} & 1 \end{bmatrix} \frac{t^i}{i!} = \begin{bmatrix} e^{-t} & 0 \\ \frac{e^t-e^{-t}}{2} & e^t \end{bmatrix}.$$

Therefore one gets

$$\begin{aligned} \mathbf{x}_\ell(t) &= e^{Ft}\mathbf{x}(0) = \begin{bmatrix} e^{-t} & 0 \\ \frac{e^t-e^{-t}}{2} & e^t \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2e^{-t} \\ -e^{-t} \end{bmatrix} \\ y_\ell(t) &= H\mathbf{x}_\ell(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2e^{-t} \\ -e^{-t} \end{bmatrix} = e^{-t}. \end{aligned}$$

(b) The generic power of the matrix F is

$$F^i = 0, \qquad \forall\, i > 1,$$

from which it follows

$$e^{Ft} = I_2 + Ft = \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix}.$$

Therefore one gets

$$\begin{aligned} \mathbf{x}_\ell(t) &= e^{Ft}\mathbf{x}(0) = \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -t+1 \end{bmatrix} \\ y_\ell(t) &= H\mathbf{x}_\ell(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -t+1 \end{bmatrix} = 1. \end{aligned}$$

(c) The generic power of the matrix F is

$$F^i = \begin{cases} F = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}, & \text{if } i \text{ is odd;} \\ F^2 = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, & \text{id } i \text{ is even and greater than } 0. \end{cases}$$

One can resort to the unique expression

$$F^i = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ \frac{1+(-1)^i}{2} & \frac{-1+(-1)^i}{2} & (-1)^i \end{bmatrix},$$

valid for $i \geq 1$. From which it follows

$$
\begin{aligned}
e^{Ft} &= I_3 + \sum_{i=1}^{+\infty} \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ \frac{1+(-1)^i}{2} & \frac{-1+(-1)^i}{2} & (-1)^i \end{bmatrix} \frac{t^i}{i!} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ -\sum_{i=1}^{+\infty} \frac{t^i}{i!} & \sum_{i=0}^{+\infty} \frac{t^i}{i!} & 0 \\ \frac{1}{2}\sum_{i=1}^{+\infty} \frac{t^i}{i!} + \frac{1}{2}\sum_{i=1}^{+\infty}(-1)^i \frac{t^i}{i!} & -\frac{1}{2}\sum_{i=1}^{+\infty} \frac{t^i}{i!} + \frac{1}{2}\sum_{i=1}^{+\infty}(-1)^i \frac{t^i}{i!} & \sum_{i=0}^{+\infty}(-1)^i \frac{t^i}{i!} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ -(e^t - 1) & e^t & 0 \\ \frac{1}{2}\left(e^t - 1\right) + \frac{1}{2}\left(e^{-t} - 1\right) & -\frac{1}{2}\left(e^t - 1\right) + \frac{1}{2}\left(e^{-t} - 1\right) & e^{-t} \end{bmatrix}.
\end{aligned}
$$

Therefore one gets

$$
\begin{aligned}
\mathbf{x}_\ell(t) &= e^{Ft}\mathbf{x}(0) = \begin{bmatrix} 1 & 0 & 0 \\ -(e^t - 1) & e^t & 0 \\ \frac{1}{2}\left(e^t - 1\right) + \frac{1}{2}\left(e^{-t} - 1\right) & -\frac{1}{2}\left(e^t - 1\right) + \frac{1}{2}\left(e^{-t} - 1\right) & e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ -e^t + 1 \\ \frac{1}{2}e^t + \frac{3}{2}e^{-t} - 1 \end{bmatrix} \\
y_\ell(t) &= H\mathbf{x}_\ell(t) = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -e^t + 1 \\ \frac{1}{2}e^t + \frac{3}{2}e^{-t} - 1 \end{bmatrix} = -\frac{3}{2}e^t - \frac{3}{2}e^{-t} + 3.
\end{aligned}
$$

(d) The generic power of the matrix F is

$$
F^i = \begin{bmatrix} (-1)^i & 0 & 0 \\ (-1)^{i+1}2i & (-1)^i & 0 \\ 0 & 0 & (-1)^i \end{bmatrix}, \quad i \in \mathbb{Z}_+;
$$

From which it follows

$$
e^{Ft} = \sum_{i=0}^{+\infty} \begin{bmatrix} (-1)^i & 0 & 0 \\ (-1)^{i+1}2i & (-1)^i & 0 \\ 0 & 0 & (-1)^i \end{bmatrix} \frac{t^i}{i!} = \begin{bmatrix} e^{-t} & 0 & 0 \\ 2te^{-t} & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}.
$$

Therefore one gets

$$
\begin{aligned}
\mathbf{x}_\ell(t) &= e^{Ft}\mathbf{x}(0) = \begin{bmatrix} e^{-t} & 0 & 0 \\ 2te^{-t} & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2e^{-t} \\ (4t-1)e^{-t} \\ 2e^{-t} \end{bmatrix} \\
y_\ell(t) &= H\mathbf{x}_\ell(t) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2e^{-t} \\ (4t-1)e^{-t} \\ 2e^{-t} \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2e^{-t} \\ (4t-1)e^{-t} \end{bmatrix} = (4t+1)e^{-t}.
\end{aligned}
$$

9. **[Study of the continuous time models via Laplace transform]**:

(a) Since the initial condition is zero then the state and output evolutions are purely forced and the following expressions hold:

$$
\begin{aligned}
X(s) &= (sI_n - F)^{-1}GU(s), &(5)\\
Y(s) &= [H(sI_n - F)^{-1}G + D]U(s). &(6)
\end{aligned}
$$

Since $u(t) = e^{2t}\delta_{-1}(t)$, the Laplace transform of the given input is

$$U(s) = \frac{1}{s-2}.$$

Thus we get:

$$\begin{aligned}
X_f(s) &= \begin{bmatrix} s-1 & -3 \\ 0 & s-2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{s-2} \\
&= \frac{1}{(s-1)(s-2)} \begin{bmatrix} 3 \\ s-1 \end{bmatrix} \frac{1}{s-2} = \begin{bmatrix} \frac{3}{(s-1)(s-2)^2} \\ \frac{1}{(s-2)^2} \end{bmatrix} \\
Y_f(s) &= \begin{bmatrix} 1 & -1 \end{bmatrix} X_f(s) = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{3}{(s-1)(s-2)^2} \\ \frac{1}{(s-2)^2} \end{bmatrix} \\
&= \frac{4-s}{(s-1)(s-2)^2}.
\end{aligned}$$

By applying the inverse transform to each term one gets

$$\begin{aligned}
\mathbf{x}_f(t) &= \begin{bmatrix} 3\left(e^t - e^{2t} + t\ e^{2t}\right)\delta_{-1}(t) \\ t\ e^{2t}\delta_{-1}(t) \end{bmatrix} \\
y_f(t) &= \left(3e^t - 3e^{2t} + 2t\ e^{2t}\right)\delta_{-1}(t).
\end{aligned}$$

(b) In this case since the input applied is identically zero then the state and output evolutions are purely unforced and the following expressions hold:

$$\begin{aligned}
X_\ell(s) &= (sI_n - F)^{-1}\mathbf{x}(0), && (7) \\
Y_\ell(s) &= H(sI_n - F)^{-1}\mathbf{x}(0). && (8)
\end{aligned}$$

Thus we get:

$$\begin{aligned}
X_\ell(s) &= \begin{bmatrix} s & 0 \\ 1 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{s} \\ -\frac{1}{s(s-1)} \end{bmatrix} \\
Y_\ell(s) &= \begin{bmatrix} 1 & 1 \end{bmatrix} X_\ell(s) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s} \\ -\frac{1}{s(s-1)} \end{bmatrix} = \frac{s-2}{s(s-1)}.
\end{aligned}$$

By applying the inverse transform to each term one gets

$$\begin{aligned}
\mathbf{x}_\ell(t) &= \begin{bmatrix} 1 \\ 1-e^t \end{bmatrix} \\
y_\ell(t) &= 2 - e^t.
\end{aligned}$$

(c) In this case, being both the initial conditions and the input solicitation non null, we have to compute both the unforced (formulas (7)-(8)) and forced (formulas (5)-(6)) evolution components of state and output. As regards the unforced components one gets:

$$\begin{aligned}
X_\ell(s) &= \begin{bmatrix} s+1 & -1 \\ 0 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} \\ 0 \end{bmatrix} \\
Y_\ell(s) &= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} X_\ell(s) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} \\ -\frac{1}{s+1} \end{bmatrix}.
\end{aligned}$$

By applying the inverse transform to each term one gets

$$
\begin{aligned}
\mathbf{x}_\ell(t) &= \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix} \\
y_\ell(t) &= \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}.
\end{aligned}
$$

For the forced components, taking into account that

$$
U(s) = \frac{1}{s+1},
$$

one gets:

$$
\begin{aligned}
X_f(s) &= \begin{bmatrix} s+1 & -1 \\ 0 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{1}{s+1} = \begin{bmatrix} \frac{1}{(s+1)^2} \\ 0 \end{bmatrix} \\
Y_f(s) &= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} X_f(s) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{(s+1)^2} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{(s+1)^2} \\ -\frac{1}{(s+1)^2} \end{bmatrix}.
\end{aligned}
$$

By applying the inverse transform to each term one gets

$$
\begin{aligned}
\mathbf{x}_f(t) &= \begin{bmatrix} te^{-t}\delta_{-1}(t) \\ 0 \end{bmatrix} \\
y_f(t) &= \begin{bmatrix} te^{-t}\delta_{-1}(t) \\ -te^{-t}\delta_{-1}(t) \end{bmatrix}.
\end{aligned}
$$

10. [**Continuous time models stability**]:

(a) It is immediate to notice that F has two distinct eigenvalues, one in 0 with algebraic multiplicity 2 and one in -1 with algebraic multiplicity one. By evaluating the geometric multiplicity of the eigenvalue 0 we notice that it is one. Therefore the Jordan form of F exhibits only one miniblock associated with the eigenvalue 0. Consequently, the Jordan form of F is

$$
J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.
$$

Therefore the system exhibits the modes 1 (bounded), t (divergent) and e^{-t} (convergent). Then the system is unstable.

(b) It is immediate to notice that F has two distinct eigenvalues, one in -1 with algebraic multiplicity 2 and one in $1/2$ with algebraic multiplicity one. By evaluating the geometric multiplicity of the eigenvalue -1 we immediately notice that it is 2. Therefore the Jordan form of F exhibits two miniblocks associated with the eigenvalue -1. Consequently, the Jordan form of F is

$$
J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}.
$$

Therefore the system exhibits the modes $e^{t/2}$ (divergent) and e^{-t} (convergent). Then the system is unstable.

(c) It is immediate to notice that F has two eigenvalues $-1+a$ and 0. Such eigenvalues are distinct for any value of $a \neq 1$.

For $a \neq 1$ the system matrix has two distinct eigenvalues $\lambda_1 = -1+a$ and $\lambda_2 = 0$. The former with unitary algebraic multiplicity and the latter with algebraic multiplicity 2. For any $a \neq 1$, $\lambda_2 = 0$ has geometric multiplicity 1 and hence the Jordan form of F exhibits a miniblock associated with the eigenvalue 0. Consequently the Jordan form of F is

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1+a \end{bmatrix}.$$

Therefore the system exhibits the modes 1 (bounded), t (divergent) and $e^{(-1+a)t}$ (convergent if $a < 1$, divergent if $a > 1$). Then the system is always unstable.

For $a = 1$ the system matrix is a Jordan miniblock of dimension 3 associated with 0 thus the system exhibits the modes 1 (bounded), t and $t^2/2$ (both divergent) and it is not stable.