# Systems Theory Exercises - Realization Theory
# A.Y. 2025/26

**Exercise 1.** Consider the discrete time system $\Sigma$ described by the matrices

$$F = \begin{bmatrix} 1/2 & 0 & 1 & 0 \\ 0 & 1/2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad H = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}.$$

i) Verify that $\Sigma$ is not internally stable.

ii) Are there any state-feedback matrices $K$ such that the closed-loop system is internally stable?

Provide an adequate explanation and in case of affirmative answers, explicitly compute the feedback matrices.

iii) Verify that there is no state-feedback law from the second input that makes all the output unforced evolutions of the closed-loop system convergent to zero.

iv) Determine a state-feedback from both inputs that makes all the output unforced evolutions of the closed-loop system convergent to zero.

**Exercise 2.** Consider the following transfer function:

$$w(z) = \frac{z+a}{(z+1)^3}, \qquad a \in \mathbb{R}.$$

i) Construct, for any value of $a$, a discrete time state-space model that is a minimal realization for $w(z)$;

ii) corresponding to every realization determined at point i), compute a dead-beat controller;

iii) Does it exist for some value of $a$ a minimal realization for $w(z)$ whose matrix $F$ is the following:

$$F = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}?$$

**Exercise 3.** Given a strictly proper rational function $w(s) = p(s)/q(s)$, with $p$ and $q$ coprime polynomials, let $\Sigma = (F, g, H)$ be a minimal realization of it. Consider now the following transfer matrix of a system with two inputs and two outputs:

$$W^*(s) = \begin{bmatrix} w(s) & w(s) \\ w(s) & w(s) \end{bmatrix}.$$

It is required to:

i) build a minimal realization $\Sigma^*$ of $W^*(s)$, by exploiting the matrices $F, g$ and $H$;

ii) prove that the realization $\Sigma^*$ is reachable from only one input and observable from only one output.

**Exercise 4.** Determine, as the real parameters $a$ and $b$ vary, a minimal realization of the following proper rational transfer matrix:

$$W(s) = \begin{bmatrix} a/s^2 & b/s \\ 0 & 1/s \end{bmatrix}.$$

**Exercise 5.** Given the following strictly proper rational matrix:

$$W(z) = \begin{bmatrix} \dfrac{z-2}{z(z+2)} \\ \dfrac{z-2}{z(z+3)} \end{bmatrix},$$

i) build a minimal realization $\Sigma$ of $W(z)$;

ii) design, if possible, a state feedback for the system $\Sigma$ such that the closed-loop system turns out is non observable and all the output evolutions are null for $t \geq 2$.

**Exercise 6.** Consider the following rational function:

$$w_a(s) = \frac{s+a}{(s^2+3s+2)(s-a)},$$

with $a$ a real parameter. For every value of $a$, determine a continuous -ime linear dynamical system $\Sigma_a = (F_a, g_a, H_a)$ that is a minimal realization of $w_a(s)$.

**Exercise 7.** Consider a SISO discrete time linear dynamical system $\Sigma = (F, g, H)$. Knowing that corresponding to the input signal

$$u(t) = \begin{cases} t \cdot 2^t, & t \geq 0 \text{ and even;} \\ -2^t, & t \geq 0 \text{ and odd;} \end{cases}$$

the system responds, in pure forced evolution, with the output signal

$$y(t) = \begin{cases} 0, & t \geq 0 \text{ and even;} \\ -\dfrac{t-1}{4} \cdot 2^{t-1}, & t \geq 0 \text{ and odd;} \end{cases}$$

determine the system transfer function and its minimal realization.

**Exercise 8.** Consider the following rational matrix:

$$W(z) = \begin{bmatrix} \dfrac{1}{z+1} & \dfrac{1}{(z+1)(z+2)} \\ \dfrac{1}{z+1} & \dfrac{1}{(z+1)(z+2)} \end{bmatrix}.$$

(1) Build a discrete time system $\Sigma = (F, G, H)$ that is a minimal realization of $W(z)$.

(2) Design a dead-beat controller for $\Sigma$ that zeroes the state in as few steps as possible.

(3) Design a reduced order dead-beat observer for $\Sigma$.

(4) Write the equations of a regulator for $\Sigma$ that exploits the dead-beat controller and the observer designed at points (2) and (3).

**Exercise 9.** The impulse response of a discrete linear system with one input and one output is given for $t = 0, 1, 2, 3, \ldots$, by the following sequence:

$$0, 1, 1, a, \frac{1}{8}, \frac{1}{8}, \frac{a}{8}, \frac{1}{8^2}, \frac{1}{8^2}, \frac{a}{8^2}, \ldots,$$

where $a$ is a real number.

i) Determine the transfer function of the system.

ii) Build a realization of dimension 3 for such transfer function.

iii) Determine for what values of the parameter $a$ the realization of the previous point is not minimal and for such values build a minimal realization.

**Exercise 10.** Determine, as the parameters $a$ and $b$ vary in $\mathbb{R}$, a continuous time state space model $\Sigma$ that is a minimal realization of the following proper rational transfer matrix:

$$W(s) = \begin{bmatrix} 1/[(s+1)s] & b/s \\ 1/(s+1) & a/s \end{bmatrix}.$$

# SOLUTIONS OF SOME EXERCISES

**Exercise 1.**

i) The given system is not internally stable since it has two eigenvalues on the unitary circumference. The system is not even simply stable since it has a ramp type mode, that diverges.

ii) The non reachable subsystem can not be modified by a feedback action and since such subsystem is unstable, the overall closed-loop system can not be internally stable.

iii) Notice that

$$F + g_2 K = \begin{bmatrix} 1/2 & 0 & 1 & 0 \\ k_1 & 1/2 + k_2 & k_3 & 1 + k_4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and that the observability matrix of the closed-loop system does not depend on $K$

$$\mathcal{O} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1/2 & 0 & 2 & 1 \\ 1/4 & 0 & 5/2 & 3 \\ * & 0 & * & * \end{bmatrix}$$

and it has rank 3. Hence the observable subsystem remains of dimension 3 for any feedback from the second input and therefore it will have 1 as eigenvalue. We conclude that the non convergent mode associated to the eigenvalue 1 will always be observable from the output.

iv) Notice that, as already seen, by means of a feedback action from both inputs it is possible to make the first two rows of the matrix $F + GK$ arbitrary. Moreover it holds that

$$\begin{aligned} y_\ell(0) &= Hx_0 = [\,1 \quad 0 \quad 1 \quad 0\,]\, x_0 \\ y_\ell(1) &= H(F + GK)x_0 = [\,1/2 + k_{11} \quad k_{12} \quad 2 + k_{13} \quad 1 + k_{14}\,]\, x_0 \end{aligned}$$

By assuming $k_{11} = 1/2, k_{12} = 0, k_{13} = -2, k_{14} = -1$ one gets $H(F + GK) = 0$ and hence also $H(F + GK)^t = 0$ for any $t \geq 1$. From this we can conclude that $y_\ell(t) = 0$ for any $t \geq 1$.

**Exercise 2.** i) The given representation for the rational function:

$$w(z) = \frac{z + a}{(z + 1)^3}, \qquad a \in \mathbb{R},$$

is irreducible for any $a \neq 1$ and hence for any $a \neq 1$ its minimal realizations have dimension 3. For instance, the one in controllable canonical form:

$$F_a = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}, \quad g_a = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad H_a = [\,a \quad 1 \quad 0\,].$$

Instead for $a = 1$ an irreducible representation is

$$w(z) = \frac{1}{(z+1)^2},$$

and hence a minimal realization (in controllable canonical form) is:

$$F_c = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad g_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad H_c = [\,1 \quad 0\,].$$

ii) A dead-beat controller for $\Sigma_a = (F_a, g_a, H_a), a \neq 1$, is:

$$K_a = [\,1 \quad 3 \quad 3\,],$$

while for $\Sigma_1 = (F_1, g_1, H_1)$ is:

$$K_1 = [\,1 \quad 2\,].$$

iii) Since $w(z)$ is a scalar function and hence every realization of it has only one input, any minimal realization, being reachable, must have a cyclic state matrix, namely with one miniblock for each eigenvalue. The matrix

$$F = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

is not cyclic and hence for none of the values of $a$ there exists a minimal realization having $F$ as a state matrix.

**Exercise 4.** Let us first consider the case $a \neq 0$ and let us express the matrix $W(s)$ in the form

$$W(s) = \frac{1}{d(s)} N(s) = \frac{1}{s^2} \begin{bmatrix} a & bs \\ 0 & s \end{bmatrix} = \frac{1}{s^2} \left( \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + s \begin{bmatrix} 0 & b \\ 0 & 1 \end{bmatrix} \right).$$

A reachable realization of such matrix of dimension $\deg d(s) \cdot m = 2 \cdot 2 = 4$ is

$$F_c = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad G_c = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad H_c = \begin{bmatrix} a & 0 & 0 & b \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let us evaluate the observability of such realization as $a$ and $b$ vary. The observability matrix is:

$$\mathcal{O} = \begin{bmatrix} H_c \\ H_c F_c \\ H_c F_c^2 \\ H_c F_c^3 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 & b \\ 0 & 0 & 0 & 1 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Obviously this matrix has never rank 4. It has rank 3 for $a \neq 0$ and 1 for $a = 0$. The value of $b$, instead, turns out to be irrelevant. For $a \neq 0$ we express the system in standard observability form and by choosing as change of basis matrix $T$ the matrix that has as last column the generator $\mathbf{e}_2$ of the non-observable subspace and completing such column to a non-singular matrix, for instance:

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

one gets

$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

from which

$$T^{-1}F_cT = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \end{array}\right], \quad T^{-1}G_c = \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \end{array}\right], \quad H_cT = \left[\begin{array}{ccc|c} a & 0 & b & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

Therefore the minimal realization is:

$$F_m = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_m = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad H_m = \begin{bmatrix} a & 0 & b \\ 0 & 0 & 1 \end{bmatrix}.$$

For the case $a = 0$ one can still start from the previous realization by extracting the observable subspace of dimension 1. Alternatively it is possible to notice that for $a = 0$ the matrix becomes

$$W(s) = \begin{bmatrix} 0 & b/s \\ 0 & 1/s \end{bmatrix} = \begin{bmatrix} b \\ 1 \end{bmatrix} \frac{1}{s} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} b \\ 1 \end{bmatrix} (s-0)^{-1} \begin{bmatrix} 0 & 1 \end{bmatrix},$$

that immediately leads to the realization of dimension 1:

$$\Sigma = \left( [0], \begin{bmatrix} 0 & 1 \end{bmatrix}, \begin{bmatrix} b \\ 1 \end{bmatrix} \right).$$

**Exercise 5.** i) Let us rewrite the matrix $W(z)$ in the form

$$W(z) = \frac{1}{z(z+2)(z+3)} \begin{bmatrix} (z-2)(z+3) \\ (z-2)(z+2) \end{bmatrix} =: \frac{1}{d(z)} N(z)$$

and let us represent the polynomial matrix $N(z)$ in the form

$$N(z) = \begin{bmatrix} z^2 + z - 6 \\ z^2 - 4 \end{bmatrix} = \begin{bmatrix} -6 \\ -4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} z + \begin{bmatrix} 1 \\ 1 \end{bmatrix} z^2 =: B_0 + B_1 z + B_2 z^2.$$

Let us consider then a reachable realization of dimension $n \cdot m = 3 \cdot 1$, where $n = \deg d(z)$ and $m$ is the number of inputs:

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -6 & -5 \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad H = \begin{bmatrix} -6 & 1 & 1 \\ -4 & 0 & 1 \end{bmatrix}.$$

It is immediate to compute the observability matrix of the pair $(F, H)$ and verify that such realization is also observable and therefore, being it reachable and observable, it is minimal realization of $W(z)$. In fact, since $W(z)$ has three real distinct poles, it is not possible to find a realization of dimension smaller than 3.

ii) The system is reachable and hence for any choice of $K$ the closed-loop system $\Sigma_K = (F + gK, g, H)$ remains reachable. Requiring the closed-loop system $\Sigma_K = (F + gK, g, H)$ not to be observable amounts to require that it is not a minimal realization of its transfer function. We observe that $z - 2$ is a common factor to the numerators of the two elements of $W(z)$. Then if we attribute to the characteristic polynomial the zero $z = 2$ we certainly get $W_K(z) = H(zI_3 - F - gK)^{-1}g$ that - once the common terms have been canceled out - can be rewritten in the form

$$W_K(z) = \frac{1}{\tilde{d}(z)}\tilde{N}(z),$$

with $\tilde{d}(z)$ polynomial of degree 2. But then such transfer matrix admits a realization of dimension 2 and hence it is not minimal. In order to choose the remaining zeroes of the polynomial $\Delta_{F+gK}(z)$ one notices that if you want the output unforced evolution to be finite, then you have to guarantee that the observable modes are impulses, namely that the remaining eigenvalues of $F + gK$ are all in $z = 0$. Therefore our objective is to guarantee that $\Delta_{F+gK}(z) = (z - 2)z^2 = z^3 - 2z^2$. Being $(F, g)$ in controllable canonical form, it is immediate to check that the matrix $K$ that allows to reach such objective is:

$$K = \begin{bmatrix} 0 & 6 & 7 \end{bmatrix}.$$

**Exercise 6.** By exploiting, for instance, the controllable canonical form, we are able to provide a reachable realization of the transfer function, whose dimension is equal to the degree of the denominator $(s^2 + 3s + 2)(s - a) = s^3 + (3 - a)s^2 + (2 - 3a)s - 2a$ of $w_a(s)$ in the given representation, namely 3. One gets:

$$F_a = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2a & 3a - 2 & a - 3 \end{bmatrix} \qquad g_a = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad H_a = \begin{bmatrix} a & 1 & 0 \end{bmatrix}.$$

Let us check for which values of the parameter $a$ such realization is also observable: for such values it will be a minimal realization.

The observability matrix is

$$\mathcal{O}_a = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 2a & 3a - 2 & 2a - 3 \end{bmatrix}$$

and its determinant is $\det \mathcal{O}_a = 2a^3 - 6a^2 + 4a = 2a(a-1)(a-2)$. Therefore the provided realization in controllable canonical form of dimension 3 is minimal for any $a \in \mathbb{R}$ with $a \neq 0, 1, 2$. For such values, instead, it is easy to check that the non observable subspace has dimension 1 and hence the minimal realization has dimension 2. Rather than computing the standard observability form and eliminate the non observable part it is convenient to observe that for such values of the parameter $a$ (and just for them) a cancellation between numerator and denominator of $w_a(s)$ occurs. Therefore by rewriting $w_a(s)$ in simplified form and by constructing, starting from that writing, the new realizations in controllable canonical form of dimension 2, we are sure to obtain minimal realizations. Therefore one gets

$$w_0(s) = \frac{1}{s^2 + 3s + 2}, \qquad w_1(s) = \frac{1}{s^2 + s - 2}, \qquad w_2(s) = \frac{1}{s^2 - s - 2},$$

to which the following minimal realizations correspond

$$\begin{aligned}
\Sigma_0 &= \left( \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [1 \quad 0] \right) \\
\Sigma_1 &= \left( \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [1 \quad 0] \right) \\
\Sigma_2 &= \left( \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [1 \quad 0] \right).
\end{aligned}$$

**Exercise 7.** The $Z$-transform of the input signal is

$$\begin{aligned}
U(z) &= \sum_{t\ \geq 0} u(t) z^{-t} = \sum_{t\ \geq 0} (2t) 2^{2t} z^{-2t} + \sum_{t\ \geq 0} (-2^{2t+1}) z^{-2t-1} = \sum_{t\ \geq 0} (2t) 2^{2t} z^{-2t} - 2z^{-1} \sum_{t\ \geq 0} 4^t z^{-2t} \\
&= \sum_{t\ \geq 0} (2t) 2^{2t} z^{-2t} - 2z^{-1} \frac{1}{1 - 4z^{-2}} = \sum_{t\ \geq 0} (2t) 2^{2t} z^{-2t} - \frac{2z}{(z^2 - 4)}.
\end{aligned}$$

Observing that, posed

$$P(z) := \sum_{t \geq 0} 2^{2t} z^{-2t} = \sum_{t \geq 0} (4z^{-2})^t = \frac{1}{1 - 4z^{-2}} = \frac{z^2}{z^2 - 4},$$

one gets

$$\sum_{t \geq 0} (-2t) 2^{2t} z^{-2t-1} = \frac{dP(z)}{dz} = \frac{d}{dz} \left( \frac{z^2}{z^2 - 4} \right) = -\frac{8z}{(z^2 - 4)^2},$$

we obtain for $U(z)$ the following compact expression:

$$U(z) = \frac{8z^2}{(z^2 - 4)^2} - \frac{2z}{(z^2 - 4)} = \frac{8z^2 - 2z(z^2 - 4)}{(z^2 - 4)^2} = -\frac{2z^3 - 8z^2 - 8z}{(z^2 - 4)^2}.$$

The $Z$-transform of the output instead is

$$\begin{aligned}
Y(z) &= \sum_{t\ \geq 0} y(t) z^{-t} = \sum_{t\ \geq 0} -\frac{(2t+1) - 1}{4} 2^{(2t+1)-1} z^{-2t-1} = -\sum_{t\ \geq 0} \frac{t}{2} 2^{2t} z^{-2t-1} \\
&= \frac{1}{4} \frac{dP(z)}{dz} = \frac{1}{4} \left( -\frac{8z}{(z^2 - 4)^2} \right) = -\frac{2z}{(z^2 - 4)^2}.
\end{aligned}$$

Consequently, the transfer function of the system results to be

$$w(z) = \frac{Y(z)}{U(z)} = -\frac{2z}{(z^2-4)^2} \cdot \left( -\frac{(z^2-4)^2}{2z^3-8z^2-8z} \right) = \frac{2z}{2z^3-8z^2-8z} = \frac{1}{z^2-4z-4}.$$

Being such representation irreducible, a realization of dimension 2 is certainly minimal. Let us take, for instance, a realization in controllable canonical form:

$$F_c = \begin{bmatrix} 0 & 1 \\ 4 & 4 \end{bmatrix}, \quad g_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad H_c = [\,1 \quad 0\,].$$

**Exercise 9.** i) The transfer function is computed as

$$\begin{aligned}
W(z) &= \sum_{k=0}^{+\infty} w_k z^{-k} \\
&= 1z^{-1} + 1z^{-2} + az^{-3} + \frac{1}{8}\,z^{-4} + \frac{1}{8}\,z^{-5} + \frac{a}{8}\,z^{-6} + \frac{1}{8^2}\,z^{-7} + \frac{1}{8^2}\,z^{-8} + \frac{a}{8^2}\,z^{-9} + \dots \\
&= \left(1z^{-1} + \frac{1}{8}\,z^{-4} + \frac{1}{8^2}\,z^{-7} + \dots\right) + \left(1z^{-2} + \frac{1}{8}\,z^{-5} + \frac{1}{8^2}\,z^{-8} + \dots\right) + \\
&+ a\left(z^{-3} + \frac{1}{8}\,z^{-6} + \frac{1}{8^2}\,z^{-9} + \dots\right) \\
&= (z^{-1} + z^{-2} + az^{-3}) \left(1 + \frac{1}{8}\,z^{-3} + \frac{1}{8^2}\,z^{-6} + \dots\right) \\
&= (z^{-1} + z^{-2} + az^{-3}) \frac{1}{1 - \frac{z^{-3}}{8}} = \frac{a + z + z^2}{z^3 - \frac{1}{8}}
\end{aligned}$$

ii) A realization in controllable canonical form and of dimension 3 is given by

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/8 & 0 & 0 \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad H = [\,a \quad 1 \quad 1\,].$$

iii) The realizaton of dimension 3 is not minimal when the transfer function, posed in irreducible form, has a denominator of degree smaller than 3, namely when the polynomials

$$n(z) = z^2 + z + a, \quad d(z) = z^3 - 1/8 = (z - 1/2)(z^2 + z/2 + 1/4)$$

are not coprime.
Since $n(z)$ can not cancel out with $(z^2 + z/2 + 1/4)$, for any real value of the parameter $a$, the only chance is that $z = 1/2$ is a zero of $n(z)$, i.e.

$$a + \frac{1}{2} + (\frac{1}{2})^2 = 0 \qquad \text{from which} \qquad a = -\frac{3}{4}\,.$$

In such case the transfer function becomes

$$W(z) = \frac{(z-1/2)(z+3/2)}{(z-1/2)(z^2+z/2+1/4)} = \frac{z+3/2}{z^2+z/2+1/4}$$

and a minimal realization is

$$F = \begin{bmatrix} 0 & 1 \\ -1/4 & -1/2 \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad H = [\,3/2 \quad 1\,].$$