Systems Theory Exercises - Observability
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Exercise 1. Given the discrete time system

$$x(t+1) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & a \end{bmatrix} x(t), \qquad \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & a \\ 0 & -2a & 2a \end{bmatrix} x(t),$$

where a is a real parameter, suppose that the initial state $x(0)$ is unknown and that the output values $y(t)$ are known for $t = 0, 1, 2, \dots$

i) Is it possible to determine $x(0)$ from the outputs $y(t)$ for $t = 0, 1, 2$?

ii) Is it possible to determine $x(3)$ from the outputs $y(t)$ for $t = 0, 1, 2$?

iii) Is it possible to determine $x(0)$ from the outputs $y(t)$ for $t = 0, 1$?

iv) Is it possible to determine $x(2)$ from the outputs $y(t)$ for $t = 0, 1$?

In each case, if possible, find the explicit expression of the desired state from the given output samples.

Exercise 2. Consider the continuous time system

$$\dot{x}(t) = Fx(t) + gu(t) \qquad y(t) = Hx(t),$$

with

$$F = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad H = [\,-1 \quad 1\,].$$

(1) Compute, if possible, the initial state $x(0)$, assuming that you are able to measure the functions $u(t) = 0$ and $y(t) = 6e^{-t} - 4$ in the time interval $[0, 1]$.

(2) Design a matrix K such that the closed-loop system, obtained by assuming $u(t) = Kx(t) + v(t)$, exhibits an unforced output evolution containing only the mode e^{-t} independently of $x(0)$.

(3) By referring to the closed-loop system designed at the previous point, compute all the initial states $x(0)$ compatible with the observation of the signals $v(t) = 0$ and $y(t) = e^{-t}$ in the time interval $[0, 1]$.

Exercise 3. Consider the following discrete time system

$$\begin{aligned} x(t+1) &= Fx(t) + gu(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t), \\ y(t) &= Hx(t) = [\,1/3 \quad 4/3 \quad 1\,]\, x(t) \qquad t \geq 0. \end{aligned}$$

i) Study the system observability and determine the set of non-observable states in $[0,1]$ (by this meaning at the time instants 0 and 1), the set of the non-observable states in $[1,2]$ and, finally, the set of the non-observable states in $[0,2]$.

ii) Design, if possible, a state feedback control input such that the resulting closed-loop system is asymptotically stable and its non-observable subsystem has only the eigenvalue $\lambda = -1/3$.

Exercise 4. Consider the following discrete time linear system:

$$\begin{aligned} x(t+1) &= Fx(t) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix} x(t) \\ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} &= \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} x(t) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} x(t). \end{aligned}$$

i) Compute the non-observable subspaces, corresponding to the various discrete-time intervals, from the first output only and from both outputs.

ii) Determine the set of initial states to which the unforced output evolutions satisfying the following constraints corresponds:

$$y_1(0) = 1, \quad y_2(0) = 0, \quad y_2(2) = 5.$$

Exercise 5. Given the discrete time system

$$\begin{aligned} x(t+1) &= Fx(t) + Gu(t) \\ y(t) &= Hx(t), \end{aligned}$$

with

$$F = \begin{bmatrix} 0 & 2 & 0 \\ 1 & -1 & 0 \\ 0 & 2 & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad H = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix},$$

suppose to observe $u(0), u(1), u(2), y(0), y(1)$ and $y(2)$, and assume that $x(0)$ is unknown.

i) For which values of $t \in \{0,1,2,3\}$ it is possible to compute the state $x(t)$?

ii) For such values of t explicitly compute $x(t)$.

Exercise 6. Let $\Sigma = (F,G,H)$ be a continuous time linear dynamical system of dimension 6, having as system matrix

$$F = \begin{bmatrix} \lambda_1 & 1 & & & & \\ 0 & \lambda_1 & & & & \\ & & \lambda_2 & 1 & & \\ & & 0 & \lambda_2 & & \\ & & & & \lambda_3 & \\ & & & & & \lambda_3 \end{bmatrix},$$

where λ_1, λ_2 and λ_3 are parameters taking real values.

i) Determine, as λ_1, λ_2 and λ_3 vary, the system modes.

ii) Determine, as λ_1, λ_2 and λ_3 vary, the minimal number of necessary outputs $p^* = p^*(\lambda_1, \lambda_2, \lambda_3)$, such that the system is observable.

Exercise 7. Consider the following discrete time system

$$\begin{aligned} x(t+1) &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a^2 & a^2 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) = Fx(t) + gu(t), \\ y(t) &= \begin{bmatrix} 0 & -3 & 1 \end{bmatrix} x(t) = Hx(t), \qquad t \geq 0, \end{aligned}$$

with a a real parameter. Study, as a in $\mathbb{R}$ varies, the system observability, and for those values of a for which the system is non-observable, determine the non-observable subspace $X^{no}(a)$.

Exercise 8. Given the discrete time system

$$x(t+1) = Fx(t) + gu(t) \qquad y(t) = Hx(t),$$

with

$$F = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}.$$

i) By supposing the input u to be known at every time instant and the output to be known at the time instants $0, 1$ and 2, determine for what instants $t \geq 0$, $x(t)$ can be uniquely determined.

ii) Design, if possible, a state feedback control $u(t) = Kx(t)$ that makes the system observable.

Exercise 9. Consider the following discrete time linear dynamical system:

$$\begin{aligned} x(t+1) &= Fx(t) + gu(t) = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t) \\ y(t) &= Hx(t) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} x(t), \qquad t \geq 0. \end{aligned}$$

Study the system observability and determine all the initial conditions $x(0)$ compatible with the observations $y(0) = 0$, $y(1) = 3$ and $y(2) = 7$ corresponding to the input samples $u(0) = 1$ and $u(1) = 0$.

Exercise 10. Consider the following continuous time linear dynamical system:

$$\begin{aligned} \dot{x}(t) &= Fx(t) + gu(t) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(t), \\ y(t) &= Hx(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \qquad t \geq 0. \\ x_0 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

i) Determine, if possible, a control input that brings the system state from $x(0) = x_0$ to $x(1) = 0$.

ii) Determine, if possible, a control input, with support in $[0, 1]$, such that for $t \geq 1$ the (unforced) output evolution is zero but the (unforced) state evolution is not.

Exercise 11. Consider the following discrete time linear dynamical system:

$$\begin{aligned} x(t+1) &= Fx(t) + gu(t) = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t) \\ y(t) &= Hx(t) = [\,1 \quad 0 \quad 0\,]\, x(t), \qquad\qquad t \geq 0. \end{aligned}$$

Study the system observability and determine all the initial conditions $x(0)$ that are compatible with the observations $y(0) = 3$, $y(1) = 3$ and $y(2) = 6$ corresponding to the input samples $u(0) = 1$ and $u(1) = 1$.

Exercise 12. Consider the discrete time linear system described by the following triple of matrices:

$$F = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad H = [\,2 \quad 1\,] \qquad G = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

i) Suppose to measure, for a fixed $k \geq 1$, the outputs $y(0), y(k), y(2k), \ldots$. Compute, if possible, $x(0)$.

ii) Suppose to measure, for a fixed $k \geq 1$, the outputs $y(1), y(k+1), y(2k+1), \ldots$. Compute, if possible, $x(0)$.

SOLUTIONS OF SOME EXERCISES

Exercise 2. (1) As a first step, we observe that the Jordan form of the matrix F is

$$J = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix},$$

and a matrix T such that

$$J = T^{-1} F T$$

is

$$T = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.$$

This implies that

$$e^{Ft} = Te^{Jt}T^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & e^{-t} \\ 0 & -e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1-e^{-t} \\ 0 & e^{-t} \end{bmatrix}.$$

There are two possible ways to solve this problem: either through the observability Gramian or by observing that

$$\begin{aligned} y_\ell(t) &= He^{Ft}x(0) = [\,-1 \quad 1\,] \begin{bmatrix} 1 & 1-e^{-t} \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} = [\,-1 \quad 2e^{-t}-1\,] \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \\ &= 2x_{20}e^{-t} - (x_{10} + x_{20}). \end{aligned}$$

And hence comparing the previous expression with $y(t) = 6e^{-t} - 4$ we get $x_{20} = 3, x_{10} = 1$, namely $x(0) = [\,1 \quad 3\,]^\top$.

(2) It is immediate to notice that the pair (F, g) is in controllable canonical form and hence it is reachable. Therefore by means of a state feedback one can attribute to $F + gK$ an arbitrary monic characteristic polynomial of degree 2. Obviously one of the eigenvalues of such a matrix must be in -1. The eigenvalue must be chosen in such a way that it (and hence the mode associated with it) is not observable for the pair $(F + gK, H)$. We notice that the transfer function of the system (F, g, K) is

$$w(s) = \frac{s-1}{s^2+s}$$

and we recall that the transfer function of the closed loop system is

$$w_K(s) = \frac{s-1}{\Delta_{F+gK}(s)}.$$

Clearly the system $(F+gK, g, K)$ is reachable. Therefore, if in the previous representation of the transfer functions, cancellations between numerator and denominator occur, and hence the realization is not minimal, it means that the realization is not observable. Therefore, by choosing $\Delta_{F+gK}(s) = (s+1)(s-1) = s^2 - 1$, then 1 is eigenvalue of the non-observable subsystem. Since the pair (F, g) is in controllable canonical form it is immediate to notice that the matrix K such that $\Delta_{F+gK}(s) = (s+1)(s-1) = s^2 - 1$ is given by

$$K = [\,1 \quad 1\,].$$

(3) The matrix $F + gK$ is

$$F + gK = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and it has the following eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

corresponding to $\lambda_1 = 1$ and $\lambda_2 = -1$, respectively. Every initial condition can be expressed as a function of $\mathbf{v}_1$ and $\mathbf{v}_2$ in the form

$$\mathbf{x}_0 = \alpha \mathbf{v}_1 + \beta \mathbf{v}_2$$

for appropriate real coefficients α and β, and the corresponding unforced output evolution is given by

$$y_\ell(t) = He^{Ft}\mathbf{x}_0 = H\left[\alpha e^t \mathbf{v}_1 + \beta e^{-t}\mathbf{v}_2\right] = \alpha e^t H\mathbf{v}_1 + \beta e^{-t} H\mathbf{v}_2 = 0 + 2\beta e^{-t},$$

where it has been exploited the fact that if $\mathbf{v}$ is eigenvector of F corresponding to the eigenvalue λ, then $e^{Ft}\mathbf{v} = e^{\lambda t}\mathbf{v}$. Consequently, the unforced output evolutions of the type e^{-t} for $t \in [0, 1]$ are all and only the ones corresponding to $\beta = 1/2$ and α arbitrary in $\mathbb{R}$, namely

$$\mathbf{x}_0 = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha - 1/2 \\ \alpha + 1/2 \end{bmatrix},$$

with α arbitrary in $\mathbb{R}$.

Exercise 3.

$$\begin{aligned}
X^{no}_{[0,1]} &= \ker \begin{bmatrix} H \\ HF \end{bmatrix} = \ker \begin{bmatrix} 1/3 & 4/3 & 1 \\ 0 & 4/3 & 4/3 \end{bmatrix} = \langle \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \rangle, \\
X^{no}_{[1,2]} &= \ker \begin{bmatrix} HF \\ HF^2 \end{bmatrix} = \ker \begin{bmatrix} 0 & 4/3 & 4/3 \\ 0 & 4/3 & 4/3 \end{bmatrix} = \langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \rangle, \\
X^{no}_{[0,2]} &= \ker \begin{bmatrix} H \\ HF \\ HF^2 \end{bmatrix} = \ker \begin{bmatrix} 1/3 & 4/3 & 1 \\ 0 & 4/3 & 4/3 \\ 0 & 4/3 & 4/3 \end{bmatrix} = \langle \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \rangle.
\end{aligned}$$

Clearly the system is not observable and $X^{no} = \langle \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \rangle$.

ii) The system is in controllable canonical form and hence reachable, so for every $K \in \mathbb{R}^{1\times 3}$ the system $(F + gK, g, H)$ is reachable and if cancellations take place between $H\text{adj}(zI - F - gK)g = H\text{adj}(zI - F)g$ and $\Delta_{F+gK}(z)$ they are due to unobservable eignevalues. Since

$$w(z) = H(zI - F)^{-1}g = \frac{(z+1)(z+1/3)}{z(z+1)(z-1)},$$

then

$$w_K(z) = H(zI - F - gK)^{-1}g = \frac{(z+1)(z+1/3)}{\Delta_{F+gK}(z)}.$$

So, it is sufficient to impose that $\Delta_{F+gK}(z)$ is a multiple of $z+1/3$ and all its remaining zeros have moduli smaller than 1. For instance, $\Delta_{F+gK}(z) = z^2(z+1/3)$. Since $F+gK$ is in companion form, it is easy to see that if we impose

$$F + gK = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1/3 \end{bmatrix},$$

namely we choose

$$K = [\,0 \quad -1 \quad -1/3\,],$$

we obtain the desired result.

Exercise 5. i) It is immediate to verify that the system is non-observable, since

$$\mathcal{O} = \mathcal{O}_{[0,2]} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ -1 & 3 & 0 \end{bmatrix}$$

is not full rank. Moreover rank $\mathcal{O}_{[0,2]} = 2$ and the non-observable subspace is

$$X^{no} = X^{no}_{[0,2]} = \langle \mathbf{e}_3 \rangle.$$

It is easy to notice that $\ker F^3 \supseteq X^{no}_{[0,2]}$ and hence the system is reconstructable (alternatively one can observe that $\sigma(F) = (1, -2, 0)$ and the only eigenvalue of the non-observable subsystem is 0). Therefore it is possible to determine $\mathbf{x}(3)$ starting from the given quantities. Finally, we observe that, generally, from the knowledge of $u(0), u(1), u(2), y(0), y(1)$ and $y(2)$ one can determine $X^{no}_{[0,2]}$ and for $t = 1, 2$ it holds that

$$\mathbf{x}(t) = F^t(\mathbf{x}_0 + X^{no}_{[0,2]}) + \mathbf{x}_f(t),$$

where $X^{no}_{[0,2]}$ and $\mathbf{x}_f(t)$ are given. Since $\ker F^2 \supseteq \ker F \supseteq X^{no}_{[0,2]}$, it follows that from the available data one can uniquely reconstruct $\mathbf{x}(1)$ and $\mathbf{x}(2)$ as well.

ii) In order to determine $\mathbf{x}(1), \mathbf{x}(2)$ and $\mathbf{x}(3)$ starting from the available data, we can follow the following procedure: we subtract the forced evolution component from $y(t), t = 0, 1, 2$, thus getting $y_\ell(0), y_\ell(1), y_\ell(2)$. Once we set

$$\begin{bmatrix} y_\ell(0) \\ y_\ell(1) \\ y_\ell(2) \end{bmatrix} = \begin{bmatrix} H \\ HF \\ HF^2 \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ -1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix},$$

we get

$$\mathbf{x}_0 + X^{no}_{[0,2]} = \begin{bmatrix} y_\ell(0) + y_\ell(1) \\ y_\ell(0) \\ 0 \end{bmatrix} + \langle \mathbf{e}_3 \rangle.$$

Indicating with $\bar{\mathbf{x}}_0$ the vector

$$\begin{bmatrix} y_\ell(0) + y_\ell(1) \\ y_\ell(0) \\ 0 \end{bmatrix},$$

one gets for $t = 1, 2, 3$

$$\mathbf{x}(t) = F^t \bar{\mathbf{x}}_0 + \sum_{k=0}^{t-1} F^{t-1-k} G u(k).$$

The numerical details of the computation are left as exercise to the student.

Exercise 7. The observability matrix of the system is given by

$$\mathcal{O} = \begin{bmatrix} H \\ HF \\ HF^2 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 1 \\ -a^2 & a^2 & -2 \\ 2a^2 & -3a^2 & a^2 - 2 \end{bmatrix}.$$

Such matrix has determinant $\det \mathcal{O} = -2a^2(a^2 - 9)$, that is zero if and only if either $a = 0$, or $a = \pm 3$. Therefore, the system is observable for every $a \notin \{0, 3, -3\}$. For $a = 0$ the matrix $\mathcal{O}$ becomes

$$\mathcal{O} = \begin{bmatrix} 0 & -3 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & -2 \end{bmatrix},$$

and hence its kernel, that represents the non-observable subspace of the system for $a = 0$, is given by

$$X^{no} = \langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rangle.$$

Similarly, for $a = 3$ and $a = -3$, one gets

$$X^{no} = \ker \begin{bmatrix} 0 & -3 & 1 \\ -9 & 9 & -2 \\ 18 & -27 & 7 \end{bmatrix} = \langle \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} \rangle.$$

Exercise 8. i) The system observability matrix is given by

$$\mathcal{O} = \begin{bmatrix} H \\ HF \\ HF^2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Such matrix is clearly singular, and hence the system is non-observable (in $[0, 2]$). From the knowledge of $u(0), u(1), y(0), y(1)$ and $y(2)$ we can estimate the initial state vector $x(0) = x_0$ up to a vector of

$$X^{no} = \ker \mathcal{O} = \langle \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rangle.$$

For $t > 0$, the state $x(t)$ is given by

$$x(t) = F^t x(0) + \sum_{i=0}^{t-1} F^{t-1-i} g u(i).$$

Obviously, being the input known at every time instant, the forced component of $x(t)$ is uniquely determined. As concerns the unforced component, instead, we observe that the initial state x_0 is known up to a vector $v \in X^{no}$. Since $X^{no} = \ker F = \ker F^t$ for any $t > 0$, it follows that $x(t)$ can be uniquely determined from the available data for any $t > 0$.

ii) If $K = [\,a \quad b \quad c\,]$, the closed-loop system matrix becomes

$$F + gK = \begin{bmatrix} 1 & 1 & 0 \\ a & 1+b & c \\ 0 & -1 & 0 \end{bmatrix}.$$

The observability closed-loop system matrix is given by

$$\mathcal{O}_K = \begin{bmatrix} H \\ H(F+gK) \\ H(F+gK)^2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 1-a & -b & -c \\ 1-a-ab & 1-a-b-b^2+c & -bc \end{bmatrix},$$

and its determinant coincides with $\det \mathcal{O}_K = c(2-2a-b)$. Therefore, by choosing, for example $c = 1$ and $a = b = 0$ one gets a controller that makes the system observable.

Exercise 9. The observability matrix of the system is

$$\mathcal{O} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 3 \\ 1 & 0 & 7 \end{bmatrix}.$$

Therefore it is immediate to notice that the system is not observable and the non-observable subspace is

$$X^{no} = \ker \mathcal{O} = \langle \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rangle.$$

By subtracting to the output evolution, its forced component, namely:

$$y_f(1) = Hgu(0) = 1, \qquad y_f(2) = HFgu(0) + Hgu(1) = 1,$$

one gets

$$y_l(0) = 0, \qquad y_l(1) = 3-1 = 2, \qquad y_l(2) = 7-1 = 6.$$

The system

$$\begin{bmatrix} y_l(0) \\ y_l(1) \\ y_l(2) \end{bmatrix} = \begin{bmatrix} H \\ HF \\ HF^2 \end{bmatrix} x(0),$$

namely

$$\begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 3 \\ 1 & 0 & 7 \end{bmatrix} x(0)$$

admits as particular solution $[-1\ 0\ 1]^T$, and hence the initial conditions compatible with the given input and output evolution are all and only the ones belonging to $[-1\ 0\ 1]^T + X^{no}$.

Exercise 10. i) Let us consider the reachability matrix of the system:

$$\mathcal{R} = [\, g \quad Fg\,] = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

It is evident that such matrix has rank 2 and hence the system is reachable, therefore a control input $u(\cdot)$ in $[0,1]$ that brings the system state to zero starting from $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, certainly exists. Indicating by

$$\begin{array}{rcll} R_1 & : & \mathcal{U}_{[0,1]} & \to \quad X \\ & : & u(\tau), \tau \in [0,1] & \mapsto \quad \int_0^1 e^{F(1-\tau)} g u(\tau) d\tau \end{array}$$

the map that associates to every input evolution in the time interval $[0,1]$, the corresponding state, at the time $t = 1$, determined as effect of the only forced evolution, then it holds

$$x(1) = e^{F1} x(0) + R_1 u(\cdot).$$

In order to determine the control input $u(\cdot)$ that solves the previous equation, it is sufficient to solve the equation

$$x(1) - e^{F1} x(0) = W_1 v, \tag{1}$$

where $v \in X = \mathbb{R}^2$ is unknown, and W_1 represents the reachability Gramian at the time $t = 1$, and then impose

$$u(\cdot) = R_1^* v, \tag{2}$$

where

$$\begin{array}{rcll} R_1^* & : & X & \to \quad \mathcal{U}_{[0,1]} \\ & : & x & \mapsto \quad g^T e^{F^T(1-\tau)} x, \tau \in [0,1] \end{array}$$

represents the adjoint transformation of R_1. Then one need to compute, as a first step, the expression of the exponential of the matrix F, and then the reachability Gramian.

Since the matrix F is a Jordan miniblock of dimension two, related to the eigenvalue 0, up to a row-column permutation, it follows that

$$e^{Ft} = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}.$$

Moreover

$$\begin{aligned} W_1 &= \int_0^1 e^{F(1-\tau)} g g^T e^{F^T(1-\tau)} d\tau = \int_0^1 \begin{bmatrix} 1 & 0 \\ 1-\tau & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} [2 \quad 0] \begin{bmatrix} 1 & 1-\tau \\ 0 & 1 \end{bmatrix} d\tau \\ &= 4 \int_0^1 \begin{bmatrix} 1 & 1-\tau \\ 1-\tau & (1-\tau)^2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 4/3 \end{bmatrix}. \end{aligned}$$

Therefore, since we assumed $x(0) = [1\ 1]^T$ and $x(1) = 0$, the equation (1) becomes

$$-\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 4/3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Such equation admits solution $v = [\,2 \quad -9/2\,]^T$, to which it corresponds, according to (2), the input signal $u(\tau) = -5 + 9\tau$, $\tau \in [0, 1]$.

ii) In order to make the output evolution equal to zero from $t = 1$, with a non null state evolution, it is necessary and sufficient that $x(t)$ is a (not identically zero) trajectory entirely contained in the non observable subspace for $t \geq 1$. We notice that

$$X^{no} = \ker\mathcal{O} = \ker \begin{bmatrix} H \\ HF \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rangle,$$

from which it follows that it is sufficient to impose that $x(1) \in X^{no}$ ($x(1) \neq 0$), and consequently $x(t) \in X^{no}$ for any $t \geq 1$, while the output will be identically zero. By exploiting the same formulas as before, by assuming as $x(1)$ the vector $[\,0 \quad 1\,]^T$, one gets $v = [\,1/2 \quad -3/2\,]^T$ and $u(\tau) = 3\tau - 2$, for $\tau \in [0, 1]$.

Exercise 11. The system observability matrix is

$$\mathcal{O} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

Therefore it is immediate to notice that the system is non observable and the non observable subspace is

$$X^{no} = \ker\mathcal{O} = \langle \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rangle.$$

By subtracting the forced component from the output evolution, namely:

$$y_f(0) = 0, \qquad y_f(1) = Hgu(0) = 1, \qquad y_f(2) = HFgu(0) + Hgu(1) = 2,$$

one gets

$$y_l(0) = 3, \qquad y_l(1) = 3 - 1 = 2, \qquad y_l(2) = 6 - 2 = 4.$$

The system

$$\begin{bmatrix} y_l(0) \\ y_l(1) \\ y_l(2) \end{bmatrix} = \begin{bmatrix} H \\ HF \\ HF^2 \end{bmatrix} x(0),$$

namely

$$\begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} x(0)$$

admits as particular solution $[3\ 0\ -1]^T$, and hence the initial conditions compatible with the given input and output evolution are all and only the ones belonging to $[3\ 0\ -1]^T + X^{no}$.

Exercise 12. i) It is immediate to notice that $F^k = F$ for any integer k greater than or equal to 1. Therefore, $y(k) = y(2k) = ... = HFx(0)$, for any $k \geq 1$. From this it follows that the only knowledge of $y(0)$ and $y(k)$ for a fixed $k \geq 1$ allows to set the equation

$$\begin{bmatrix} y(0) \\ y(k) \end{bmatrix} = \begin{bmatrix} H \\ HF \end{bmatrix} x(0) = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} x(0),$$

whose solution $x(0)$ is always uniquely determined. Hence

$$x(0) = \begin{bmatrix} 1/2 & -1/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y(0) \\ y(k) \end{bmatrix}.$$

ii) In this case since, for the same reasoning as before (i.e., $F^k = F$ for any $k \geq 1$), the knowledge of the sequence $y(1), y(k+1), y(2k+1), \ldots$, with k a positive and fixed integer, is equivalent to know the constant sequence $HFx(0), HFx(0), \ldots$, it is clear that from the given outputs sequence one can only determine the value of

$$HFx(0) = [\,0 \quad 1\,]\, x(0),$$

namely the second component of the initial state. Therefore it is not possible to exactly reconstruct the real value of $x(0)$.