Systems Theory Exercises - State Observers and Regulators A.Y. 2025/26

Exercise 1. Consider the following discrete time system

$$
\begin{aligned}
x(t+1) &= Fx(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -\frac{3}{4} & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix} x(t) \\
y(t) &= Hx(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(t).
\end{aligned}
$$

Design, if possible, an asymptotic state observer that satisfies the following two properties: (1) condition $e(0) \in \operatorname{span}\{e_1, e_4\}$ implies $e(t) = \frac{e(0)}{2^t}$ and (2) for every of $e(0)$, $e(t)$ goes to zero faster than $\left(\frac{2}{3}\right)^t$.

Exercise 2. Given the continuous time system

$$
\dot{x}(t) = Fx(t) + gu(t) \qquad y(t) = Hx(t),
$$

with

$$
F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad g = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 1 \end{bmatrix},
$$

(1) Design a state feedback controller K such that the closed-loop system $\Sigma_K = (F + gK, g, H)$ is non-observable and exhibits e^{-2t} among its modes.

(2) Design an asymptotic observer for Σ_K and explicitly write its equations.

(3) Design a stabilizing asymptotic regulator for Σ, that uses, if possible, the same controller used at point (1) (and an opportune asymptotic observer), and explicitly write its equations.

Exercise 3. Given the discrete time system:

$$
\begin{aligned}
x(t+1) &= Fx(t) + gu(t) = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u(t) \\
y(t) &= Hx(t) = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} x(t),
\end{aligned}
$$

(1) Design a dead-beat controller for Σ that zeroes the state in the least number of steps.

(2) Design a dead-beat observer for Σ that zeroes the estimation error in the least number of steps.

(3) Write the equations of a regulator for Σ that exploits the dead-beat controller and the dead-beat observer designed at points (1) and (2).

Exercise 4. Given the discrete time system

$$\begin{aligned} x(t+1) &= Fx(t) + Gu(t) \\ y(t) &= Hx(t), \end{aligned}$$

with

$$F = \begin{bmatrix} 0 & 2 & 0 \\ 1 & -1 & 0 \\ 0 & 2 & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad H = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix},$$

i) Design, if possible, an asymptotic observer whose estimation error $e(t)$ satisfies the equation

$$e(t+2) = \frac{1}{4} e(t), \qquad \forall t \geq 0,$$

for every initial condition $e(0)$.

ii) Design, if possible, an asymptotic observer whose estimation error $e(t)$ satisfies the equation

$$e(t+2) = \frac{1}{4} e(t), \qquad \forall t \geq 1,$$

for every initial condition $e(0)$.

Exercise 5. Given the continuous time system:

$$\begin{aligned} \dot{x}(t) &= Fx(t) + gu(t) = \begin{bmatrix} -1 & -2 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(t) \\ y(t) &= Hx(t) = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} x(t), \end{aligned}$$

i) Design, if possible, a state observer whose estimation error has always a periodic evolution.

ii) Design, if possible, an asymptotic state observer, whose estimation error evolves as a linear combination of the modes

$$e^{-t}, te^{-t}, e^{-2t}.$$

Exercise 6. Consider the following discrete time system

$$\begin{aligned} x(t+1) &= \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) = Fx(t) + gu(t), \\ y(t) &= \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} x(t) = Hx(t), \qquad t \geq 0. \end{aligned}$$

i) Demonstrate the existence of dead-beat observers for the system and provide a complete parametrization for them.

ii) Demonstrate that, generally, for the aforementioned system, a state x_0 is indistinguishable form the null state in $[t, T]$, with $T - t \geq 2$, $t \geq 0$, if and only if it is such in $[t, t+2]$. Is it true that x_0 is indistinguishable from the null state in $[t, T]$, with $T - t \geq 2$, $t \geq 0$, if and only if it is such in $[T - 2, T]$?

Exercise 7. Consider the discrete time system

$$\begin{aligned} x(t+1) &= Fx(t) = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix} x(t) \\ y(t) &= Hx(t) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(t). \end{aligned}$$

i) Design, if possible, a (full order) closed-loop dead-beat observer based on the first output only and whose estimation error goes to zero in the least possible number of steps.

ii) Design, if possible, a (full order) closed-loop observer such that the estimation error $e(t) := x(t) - \hat{x}(t)$ asymptotically behaves as $1/2^t$.

Exercise 8. Given the discrete time system

$$\begin{aligned} x(t+1) &= Fx(t) + gu(t) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(t) \\ y(t) &= Hx(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} x(t), \end{aligned}$$

i) Design, if possible, a dead-beat observer of reduced order for the system;

ii) Design, if possible, an asymptotic regulator for the system, such that the resulting system, obtained from the connection between system and regulator, has only -3 as eigenvalue.

Exercise 9. Given the discrete time model:

$$\begin{aligned} x(t+1) &= Fx(t) = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} x(t) \\ y(t) &= Hx(t) = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} x(t), \end{aligned}$$

Determine, if possible, a dead-beat observer of reduced order that zeroes the estimation error in as few steps as possible.

Exercise 10. Consider the following discrete time linear dynamical system:

$$\begin{aligned} x(t+1) &= Fx(t) + gu(t) = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(t) \\ y(t) &= Hx(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} x(t), \qquad t \geq 0. \end{aligned}$$

i) Design, if possible, an asymptotic state observer from the first output only.

ii) Design, if possible, a dead-beat state observer such that the estimation error goes to zero in as few steps as possible.

iii) Design, if possible, a dead-beat regulator for the given system.

Exercise 11. Consider a continuous time state model $\Sigma = (F, g, H)$, of dimension 3, with one input and one output, reachable, with

$$\begin{aligned} H\mathrm{adj}(sI_3 - F)g &= s^2 - 1 \\ \Delta_F(s) &= s^3 + 3s^2 + 2s. \end{aligned}$$

i) Discuss the existence of asymptotic observers for the system and determine bounds on the estimation error convergence speed.

ii) Determine a possible triple of matrices F, g and H compatible with the hypothesis on the system.

iii) By referring to the triple determined at point ii), determine, if possible, the parametric description of the state observers whose estimation error is a linear combination of only the elementary modes related to the eigenvalue -1.

Exercise 12. Consider the following continuous time system:

$$\begin{aligned} \dot{x}(t) &= Fx(t) + gu(t) = \begin{bmatrix} 0 & -a & 0 \\ 1 & -1-a & 0 \\ 0 & 1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(t), \\ y(t) &= Hx(t) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix} x(t), \qquad t \geq 0, \end{aligned}$$

where a is a real parameter.

i) Design, if possible, as a varies in $\mathbb{R}$ an asymptotic observer from the first output only;

ii) for $a = 1$, design, if possible, an asymptotic observer from the first output such that the estimation error $e(t)$ is a linear combination of only the modes e^{-2t}, te^{-2t} and e^{-t};

iii) design, if possible, as a varies in $\mathbb{R}$ an asymptotic state observer such that for any $e(0)$, the estimation error satisfies $e(t) = e^{-2t}e(0)$, for any $t \geq 0$.

SOLUTIONS OF SOME EXERCISES

Exercise 1. Let
$$L = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{bmatrix}^\top .$$
Then
$$F + LH = \begin{bmatrix} a_1 & 1 & a_2 & 0 \\ b_1 & 0 & 1+b_2 & 0 \\ 1+c_1 & -\frac{3}{4} & c_2 & 0 \\ d_1 & 0 & 1+d_2 & \frac{1}{2} \end{bmatrix}.$$
To ensure that (1) condition $e(0) \in \operatorname{span}\{e_1, e_4\}$ implies $e(t) = \frac{e(0)}{2^t}$ we need to impose that $(F+LH)e_1 = \frac{1}{2}e_1$ and $(F+LH)e_4 = \frac{1}{2}e_4$. This implies that the first and the fourth columns of $F+LH$ must be $\frac{1}{2}e_1$ and $\frac{1}{2}e_4$, respectively, namely
$$F + LH = \begin{bmatrix} 1/2 & 1 & a_2 & 0 \\ 0 & 0 & 1+b_2 & 0 \\ 0 & -\frac{3}{4} & c_2 & 0 \\ 0 & 0 & 1+d_2 & \frac{1}{2} \end{bmatrix}.$$
This corresponds to
$$L = \begin{bmatrix} \frac{1}{2} & 0 & -1 & 0 \\ a_2 & b_2 & c_2 & d_2 \end{bmatrix}^\top .$$
It is easy to see that the remaining eigenvalues of $F + LH$ coincide with the eigenvalues of the submatrix
$$\begin{bmatrix} 0 & 1+b_2 \\ -\frac{3}{4} & c_2 \end{bmatrix}.$$
So, if we impose $b_2 = -1$ and $c_2 = 1/2$ (and for instance $a_2 = 0, d_2 = -1$), the eigenvalues of $F + LH$ are $0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$ and hence it is automatically true that (2) for every of $e(0)$, $e(t)$ goes to zero faster than $\left(\frac{2}{3}\right)^t$. So, (a possible) final solution is
$$L = \begin{bmatrix} \frac{1}{2} & 0 & -1 & 0 \\ 0 & -1 & \frac{1}{2} & -1 \end{bmatrix}^\top .$$

Exercise 4. If $e(t)$ satisfies the equation $e(t+2) = 1/4\ e(t)$ from a certain $t \in \mathbb{Z}_+$ onwards it means that the corresponding Z-transform satisfies $z^2E(z) - 1/4E(z) = 0$ and hence the characteristic polynomial of $F + LH$ must be a multiple of $z^2 - 1/4$. Since the pair (F, H) is in standard observability form with $F_{22} = [0]$, one deduces that the only possibility is that the characteristic polynomial of $F + LH$ is $z(z^2 - 1/4)$. This solution corresponds to impose on the observer the structure
$$L = [-7/4 \ \ 1 \ \ c]^\top ,$$
with c an arbitrary real number.
Once it is assumed
$$F + LH = \begin{bmatrix} 0 & 1/4 & 0 \\ 1 & 0 & 0 \\ 0 & 2+c & 0 \end{bmatrix},$$

by explicitly computing $e(2)$ starting from a generic $e(0)$ one gets

$$e(2) = \begin{bmatrix} 1/4e_1(0) \\ 1/4e_2(0) \\ (c+2)e_1(0) \end{bmatrix}.$$

Clearly, this demonstrates that regardless the chosen value for c it is not possible to guarantee that $e(t+2) = 1/4\ e(t)$ for $t \geq 0$ for **every** choice of $e(0)$.
On the other hand, if one assumes $c = -2$ then, regardless of the choice of $e(0)$, one gets

$$e(1) = \begin{bmatrix} 1/4e_2(0) \\ e_1(0) \\ (c+2)e_2(0) \end{bmatrix} = \begin{bmatrix} e_1(1) \\ e_2(1) \\ 0 \end{bmatrix}$$

and one notices that

$$e(3) = (F + LH)^2 e(1) = \begin{bmatrix} 1/4e_1(1) \\ 1/4e_2(1) \\ 0 \end{bmatrix} = 1/4\ e(t),$$

and the same relation recursively holds by ensuring that $e(t+2) = 1/4\ e(t)$ holds for any $t \geq 1$.

Exercise 5. i) It is easy to see that the pair (F, H) is in standard observability form with matrix of the unobservable subsystem which is $F_{22} = [0]$. This implies that by properly choosing the matrix $L \in \mathbb{R}^{3\times 1}$ we can attribute to $F + LH$ all and only the characteristic polynomials that are multiple of s. If we want that the elementary modes of the error dynamics are periodic (of arbitrary period), we can attribute to $F + LH$ a characteristic polynomial of the form $p(s) = s(s^2 + \omega^2)$ for some $\omega > 0$. For instance, we can attribute to $F + LH$ the characteristic polynomial $p(s) = s(s^2 + 1)$. To this end it is sufficient to assume

$$L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix},$$

and impose that

$$\Delta_{F_{11}+L_1H_1}(s) = \det \begin{bmatrix} s+1-a & 2+a \\ 1-b & s+b \end{bmatrix} = s^2 + (1+b-a)s + (3b-a-2) = s^2 + 1.$$

This leads to

$$L = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}.$$

Notice that for such an observer the estimation error will remain bounded, but it will not converge to zero and hence it is not an asymptotic state observer.

ii) Since the matrix of the unobservable subsystem is $F_{22} = [0]$, it is not possible to design an asymptotic state observer whose estimation error evolves as a linear combination of the modes e^{-t}, te^{-t}, e^{-2t}.

Exercise 7. i) It is immediate to notice that the pair (F, h_1), where h_1 represents the first row of the matrix H, is in standard observability form. Consequently, the only eigenvalue of the non observable subsystem turns out to be 0, and hence a dead-beat observer from the first output exists. By indicating the matrix of the observer from the first output, l_1, in the form $l_1 = [a\ b\ c]^T$, then in order to get a dead-beat observer it suffices to impose

$$\Delta_{F+l_1h_1}(z) = \det\left(zI_3 - \begin{bmatrix} 1+a & 2+a & 0 \\ 2+b & b & 0 \\ 1+c & c-1 & 0 \end{bmatrix}\right) \equiv z^3.$$

One gets $\Delta_{F+l_1h_1}(z) = z^2 - (1+a+b)z + [b(1+a) - (2+a)(2+b)]$ and hence $a = -3$, $b = 2$ and c arbitrary. Let us choose c in order to minimize the nilpotency index of the matrix $F + l_1h_1$. Obviously, for the aforementioned values of a and b the matrix $F + l_1h_1$ is different from the null matrix, and hence the minimum nilpotency index is greater than or equal to 2. By computing $(F + l_1h_1)^2$ for $a = -3$, $b = 2$ and c arbitrary one gets:

$$(F + l_1h_1)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2c-6 & c-3 & 0 \end{bmatrix}.$$

Therefore it is immediate to verify that for $c = 3$ the matrix $F + l_1h_1$ has nilpotency index 2, while for $c \neq 3$ it has nilpotency index 3. Therefore the seeked solution is $l_1 = [-3\ 2\ 3]^T$.

ii) So that the observer estimation error, whose dynamics is determined by the equation $e(t+1) = (F + LH)e(t)$, $t \geq 0$, asymptotically behaves as $\frac{1}{2^t}$, it suffices that the matrix $F + LH$ has two eigenvalues of modulus less than $1/2$ and an eigenvalue (real and simple, and hence dominant) in $1/2$. It is immediate to notice that also the pair (F, H) is in standard observability form, and hence the only eigenvalue of the non observable subsystem is still 0. By attributing, for example, the spectrum $(0, 0, 1/2)$ to the matrix $F + LH$, we are able to solve the problem. To this aim it is sufficient to choose

$$L = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \\ 0 & 0 \end{bmatrix},$$

and impose

$$F + LH = \begin{bmatrix} 1+l_{11} & 2+l_{11}+l_{12} & 0 \\ 2+l_{21} & l_{21}+l_{22} & 0 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1/2 & 0 \\ 1 & -1 & 0 \end{bmatrix}.$$

Then, one gets $l_{11} = -1$, $l_{12} = 0$, $l_{21} = -2$ and $l_{22} = 3/2$.

Exercise 8. i) It is immediate to check that, since the sub-matrix of the observability matrix, given by

$$\begin{bmatrix} H \\ HF \end{bmatrix} = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 2 & 1 \\ \hline 1 & 2 & 0 \\ 2 & 4 & -3 \end{array}\right]$$

is already of full column rank, the system is observable. Therefore, the system is observable (and hence reconstructable) and a reduced order dead-beat observer for the system there exists. By completing the matrix H to a nonsingular square matrix

$$T^{-1} = \begin{bmatrix} V \\ H \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix},$$

one gets

$$\bar{x}(t) = T^{-1}x(t) = \begin{bmatrix} w(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ y(t) \end{bmatrix}.$$

Let us write the system equations with respect to the new basis:

$$\begin{aligned} w(t+1) &= A_{11}w(t) + A_{12}y(t) + B_1u(t) = [0]w(t) + [\,1 \quad 0\,]\,y(t) + [0]u(t) \\ y(t+1) &= A_{21}w(t) + A_{22}y(t) + B_2u(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} w(t) + \begin{bmatrix} 2 & 0 \\ 10 & -3 \end{bmatrix} y(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t) \end{aligned}$$

where

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = T^{-1}FT \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = T^{-1}G.$$

Then it is evident that, being $A_{11} = 0$, in order to get a dead-beat observer of reduced order it is sufficient to assume $L = 0$ and hence $v(t) = w(t)$ and

$$\hat{v}(t+1) = \hat{w}(t+1) = w(t+1) = A_{11}w(t) + A_{12}y(t) + B_1u(t) = [\,1 \quad 0\,]\,y(t).$$

ii) One needs to design K and L so that both $F + gK$ and $F + LH$ have as only eigenvalue, the eigenvalue -3. Since the system is observable, a matrix L such that $F + LH$ has only the eigenvalue in -3 exists. Posed

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \\ L_{31} & L_{32} \end{bmatrix},$$

one gets

$$F + LH = \begin{bmatrix} 0 & 1 + L_{11} + 2L_{12} & 0 \\ 1 & 2 + L_{21} + 2L_{22} & 0 \\ 0 & L_{31} + 2L_{32} & -3 \end{bmatrix}.$$

Due to the matrix structure, we can immediately impose that the second column of the matrix L is null and that

$$F + LH = \begin{bmatrix} 0 & 1 + L_{11} & 0 \\ 1 & 2 + L_{21} & 0 \\ 0 & L_{31} & -3 \end{bmatrix} = \begin{bmatrix} 0 & -9 & 0 \\ 1 & -6 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

From this it follows that

$$L = \begin{bmatrix} -10 & 0 \\ -8 & 0 \\ 0 & 0 \end{bmatrix}.$$

As concerns the matrix $F + gK$, it is evident that the given system is in standard reachability form and -3 is the only eigenvalue of the non reachable subsystem. Therefore a matrix K such that $F + gK$ has just the eigenvalue -3 exists. Once again, by exploiting the matrix structure, we can impose

$$F + gK = \begin{bmatrix} 0 & 1 & 0 \\ -9 & -6 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

Thus getting $K = [\,-10 \quad -8 \quad 0\,]$.

Exercise 9. The matrix H is of full row rank. By completing it to a non singular square matrix, one gets, for example,

$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

and hence

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

By exploiting the change of basis associated to the transformation matrix T, and hence by assuming as vector of coordinates of the state with respect to the new basis:

$$T^{-1}x(t) =: \begin{bmatrix} w(t) \\ y(t) \end{bmatrix},$$

one gets for the system the following description:

$$\begin{aligned} w(t+1) &= A_{11}w(t) + A_{12}y(t) = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} w(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} y(t) \\ y(t+1) &= A_{21}w(t) + A_{22}y(t) = [\,1 \quad 0\,]\, w(t) + [\,0\,]\, y(t). \end{aligned}$$

We observe that the pair (A_{11}, A_{21}) is in standard observability form and it is immediate to notice that for $L = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$, one gets $A_{11} + LA_{21} = 0$. Therefore the dead-beat observer of reduced order that attributes minimum nilpotency index to the matrix describing the estimation error is the given one, corresponding to which the estimation error goes to zero in one step.

Exercise 10. i) It is immediate to notice that the pair (F, h_1), where h_1 represents the first row of the matrix H, is a pair in standard observability form, with matrix F_{22} of the non observable subsystem given by $F_{22} = [1/2]$. Therefore for any choice of the observer matrix L_1 (from the first output only), the matrix $F + L_1h_1$, that determines the estimation error dynamics, will have $1/2$ as eigenvalue. Since such eigenvalue has modulus less than 1, the pair (F, h_1) is detectable, namely there exists an asymptotic observer from the first output only. In order to determine such observer, it is sufficient to choose $L_1 = [\,a \quad b \quad 0\,]^T$ and to impose on the matrix

$$F + L_1h_1 = \begin{bmatrix} 1 & -1+a & 0 \\ 1 & 3+b & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

a characteristic polynomial of the type $(z-1/2)(z-\lambda_1)(z-\lambda_2)$, with $|\lambda_i| < 1$ for $i = 1, 2$. By ensuring, for example, $\lambda_1 = \lambda_2 = 0$, from the identity

$$\Delta_{F+L_1h_1}(z) = (z-1/2)[z^2 - (4+b)z + (4+b-a)] \equiv (z-1/2)z^2,$$

one gets $b = -4$ and $a = 0$, namely

$$L_1 = [\,0 \quad -4 \quad 0\,]^T.$$

ii) Let us consider the pair (F, H) and its dual version (F^T, H^T). The pair (F^T, H^T) is reachable, with Kronecker indices $k_1 = 2$ and $k_2 = 1$. Therefore there exists L such that

$F + LH$ has nilpotency index 2 and such value for the nilpotency index is the minimum obtainable. Due to the block diagonal structure of the matrix F and the fact that the matrix $F + LH$ has the second two columns completely arbitrary we can choose L such that

$$F + LH = \begin{bmatrix} 1 & -1+a & 0 \\ 1 & 3+b & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

with

$$\begin{bmatrix} 1 & -1+a \\ 1 & 3+b \end{bmatrix}$$

nilpotent matrix. This means imposing on the matrix L the following structure:

$$L = \begin{bmatrix} a & 0 \\ b & 0 \\ 1/2 & -1/2 \end{bmatrix}.$$

Repeating the same calculations made at point i), one gets $a = 0$ and $b = -4$, from which

$$L = \begin{bmatrix} 0 & 0 \\ -4 & 0 \\ 1/2 & -1/2 \end{bmatrix}.$$

iii) For the regulator separation principle it suffices to find (if possible) a matrix K such that $F + gK$ is nilpotent and a matrix L such that $F + LH$ is nilpotent. The matrix L exists and we can choose, for example, the one determined at point ii). As concerns the matrix K representing a stabilizing controller, instead, it is immediate to notice that the pair (F, g) is in standard reachability form, with matrix of the non reachable subsystem $F_{22} = [1/2]$. Therefore a dead-beat controller does not exists and so neither does a dead-beat regulator.

Exercise 12. i) The pair (F, h_1), where h_1 stands for the first row of H, is in standard observability form. However, according to the value of a we deal with two different standard observability forms, with two different dimensions for F_{22}. In fact the observability matrix associated with the pair $(F_{11}, h_{11}) = \left(\begin{bmatrix} 0 & -a \\ 1 & -1-a \end{bmatrix}, [\,1 \;\; 0\,]\right)$ is $\begin{bmatrix} 1 & 0 \\ 0 & -a \end{bmatrix}$ and hence it is non singular for $a \neq 0$ and in that case the matrix F_{22} of the non observable subsystem is $F_{22} = -2$ (and hence $\Delta_{F_{22}}(s) = s + 2$). For $a = 0$, instead, the observable pair is $(F_{11}, h_{11}) = ([\,0\,], [\,1\,])$ and the matrix F_{22} of the non observable subsystem is

$$F_{22} = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}$$

(and $\Delta_{F_{22}}(s) = (s + 1)(s + 2)$). In both cases the pair (F, h_1) turns out to be detectable (since the eigenvalues of the non observable subsystem have negative real part) and hence an asymptotic observer exists. We choose a solution compatible with both cases, namely we search for a gain L_1 such that $\Delta_{F+L_1 h_1}(s) = (s + 1)^2(s + 2)$. By giving to the gain L_1 the parametric structure $L_1 = [\,\ell_0 \;\; \ell_1 \;\; \ell_2\,]^T$, and by imposing that the characteristic polynomial of the observable part is

$$\frac{(s + 1)^2(s + 2)}{\Delta_{F_{22}}(s)}$$

one gets

- for the case $a \neq 0$:

$$[F + L_1 h_1]_{11} = \begin{bmatrix} \ell_0 & -a \\ 1+\ell_1 & -1-a \end{bmatrix}$$

whose characteristic polynomial is

$$\Delta_{[F+L_1h_1]_{11}}(s) = s^2 + (1 + a - \ell_0)s + a(\ell_1 + 1) - \ell_0(1 + a).$$

By equating it to $(s + 1)^2$ one gets $\ell_0 = a - 1 = \ell_1$, corresponding to $L_1 = [\, a-1 \quad a-1 \quad \ell_2 \,]^T$, $\ell_2 \in \mathbb{R}$.

- for the case $a = 0$:

$$[F + L_1 h_1]_{11} = [\, \ell_0 \,]$$

whose characteristic polynomial is

$$\Delta_{[F+L_1h_1]_{11}}(s) = s - \ell_0.$$

By equating it to $(s + 1)$ one gets $\ell_0 = -1$, corresponding to $L_1 = [\, -1 \quad \ell_1 \quad \ell_2 \,]^T$, $\ell_1, \ell_2 \in \mathbb{R}$.

ii) For $a = 1$ the pair (F, h_1) is in standard observability form with $F_{22} = -2$. The requirement on the elementary modes of the matrix $F + L_1 h_1$ amounts to impose that the minimal polynomial of $F + L_1 h_1$, and hence its characteristic polynomial as well, is $(s+2)^2(s+1)$. This solution is compatible with $\Delta_{F_{22}}(s) = (s+2)$. By giving to the gain L_1 the parametric structure $L_1 = [\, \ell_0 \quad \ell_1 \quad \ell_2 \,]^T$, and by imposing on the characteristic polynomial of the observable part

$$\Delta_{[F+L_1h_1]_{11}}(s) = s^2 + (2 - \ell_0)s + (\ell_1 + 1 - 2\ell_0)$$

to coincide with $(s+1)(s+2)$, one gets $L_1 = [\, -1 \quad -1 \quad \ell_2 \,]^T$, $\ell_2 \in \mathbb{R}$. For such choice of the gain one gets

$$F + L_1 h_1 = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -2 & 0 \\ \ell_2 & 1 & -2 \end{bmatrix}.$$

The condition on the modes amounts to impose that the geometric multiplicity of the eigenvalue -2 is 1, namely

$$\operatorname{rank}\left((F + L_1 h_1) + 2I_3\right) = \operatorname{rank} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ \ell_2 & 1 & 0 \end{bmatrix} = 2.$$

This occurs for any value of ℓ_2 except for $\ell_2 = -1$. Therefore, we choose $L_1 = [\, -1 \quad -1 \quad 0 \,]^T$.

iii) In order to solve this problem it is necessary and sufficient to impose that for the matrix $F + LH$ any initial condition is an eigenvector associated with the eigenvalue -2. This is to say that the Jordan form of $F + LH$ (and hence, necessarily, $F + LH$ itself) is $(-2)I_3$. By giving to the gain L the parametric structure

$$L = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \\ \gamma_1 & \gamma_2 \end{bmatrix},$$

one gets

$$F + LH = \begin{bmatrix} \alpha_1 - \alpha_2 & -a + \alpha_2 & 0 \\ 1 + \beta_1 - \beta_2 & -1 - a + \beta_2 & 0 \\ \gamma_1 - \gamma_2 & 1 + \gamma_2 & -2 \end{bmatrix}.$$

It is evident that, by means of an appropriate choice of the parameters of L, one can attribute to the elements of the first two columns of $F + LH$ arbitrary values. Therefore, by imposing

$$F + LH = \begin{bmatrix} \alpha_1 - \alpha_2 & -a + \alpha_2 & 0 \\ 1 + \beta_1 - \beta_2 & -1 - a + \beta_2 & 0 \\ \gamma_1 - \gamma_2 & 1 + \gamma_2 & -2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix},$$

namely by assuming

$$L = \begin{bmatrix} a - 2 & a \\ a - 2 & a - 1 \\ -1 & -1 \end{bmatrix},$$

we obtain the desired result.