Systems Theory Exercises - Dead-Beat Control
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Exercise 1. Given the discrete time system

$$\mathbf{x}(t+1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u}(t),$$

design, if possible, a feedback matrix K such that the resulting matrix $F + GK$ of the closed-loop system has the following invariant polynomials:

i) $\psi_1(z) = z^4$;

ii) $\psi_1(z) = z^2$, $\psi_2(z) = \psi_3(z) = z$;

iii) $\psi_1(z) = z^2$, $\psi_2(z) = z^2$.

The answers must be provided with an adequate explanation.

Exercise 2. Consider the discrete time linear system

$$x(t+1) = \begin{bmatrix} 1 & 2 & 1 \\ 1/2 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} u(t), \qquad y(t) = \begin{bmatrix} -2 & 1 & 1 \end{bmatrix} x(t).$$

Determine, if possible, state feedback matrices K (one for each of the following requests) such that:

i) the eigenvalues of the closed loop system are $0, \frac{1}{2}, \frac{1}{4}$;

ii) the unforced state evolution goes to zero in at most one step for every initial condition $x(0)$;

iii) the unforced state evolution goes to zero in at most two steps (and two steps are necessary for at least one $x(0)$) for every initial condition $x(0)$;

iv) the unforced state evolution goes to zero in at most three steps (and three steps are necessary for at least one $x(0)$) for every initial condition $x(0)$;

v) the impulse response of the closed-loop system is $\delta(t-1)$.

Exercise 3. Given the linear discrete time system

$$\mathbf{x}(t+1) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u}(t),$$

Design, if possible, a dead-beat controller that acts in as few steps as possible and satisfies the following additional requirement

i) acts on the first input only;

ii) acts on the second input only;

iii) acts on both inputs;

iv) acts on both inputs, with the further constraint that $u_1(t) = u_2(t)$ for every $t \geq 0$.

Exercise 4. Consider the discrete time linear dynamic system, described by the following equations:

$$x(t+1) = \begin{bmatrix} a & 0 & 0 \\ 0 & 2a & 0 \\ 0 & 0 & 1/2 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u(t) \qquad\qquad t \geq 0,$$

with a a real parameter.

i) Demonstrate that for every $a \in \mathbb{R}$ the system is controllable to zero, and

ii) design, for each value of a, a dead-beat controller that makes the closed-loop system state equal to zero in as few steps as possible.

Exercise 5. Consider the following discrete time system:

$$x(t+1) = Fx(t) + gu(t) = \begin{bmatrix} 1 & a & 1 \\ 1 & 1 & a \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(t), \qquad t \geq 0$$

where a is a real parameter.

i) As a varies in $\mathbb{R}$, compute, if it exists, a state feedback controller K that attributes to the matrix $F + gK$ of the resulting closed-loop system the following spectrum $\{0, 2, -1\}$.

ii) Determine for what values of a the system is controllable to zero (equivalently it admits a dead-beat controller), and

iii) for any such value determine, if possible, the family of dead-beat controllers that make the closed-loop system matrix $F + gK$ nilpotent with minimal nilpotency index.

Exercise 6. Consider the following discrete time system:

$$x(t+1) \quad = \quad Fx(t) + gu(t) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & a & a \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(t), \qquad t \geq 0,$$

where a is a real parameter.

i) Compute, as a varies in $\mathbb{R}$, the k-steps reachable and controllable subspaces for every $k \in \mathbb{N}$;

ii) for every value of a such that the system is controllable to zero, determine the expression (as a function of a) of a dead-beat controller for the system;

iii) among the controllers identified at point ii), indicate for $a = 0$ the one that attributes to the matrix $F + gK$ of the closed-loop system minimal nilpotency index.

Exercise 7. Consider the following discrete time model:

$$\begin{aligned} x(t+1) \quad &= \quad Fx(t) + gu(t) = \begin{bmatrix} 1-a & 0 & 1 \\ 2a & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} u(t), \\ y(t) \quad &= \quad Hx(t) = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} x(t), \qquad t \geq 0 \end{aligned}$$

where a is a real parameter.

i) Compute, as a varies in $\mathbb{R}$, the k-steps controllable subspaces for every $k \in \mathbb{N}$.

By assuming $a = 1$ in the following

ii) determine, if possible, a dead-beat controller that makes the matrix $F + gK$ of the closed-loop system nilpotent with minimal nilpotency index;

iii) design, if possible, a state feedback controller that attributes to the closed-loop system the transfer function

$$w_K(z) = \frac{2}{z-1}.$$

[Suggestion: the system is not reachable, however the transfer function only depends on the reachable subsystem for which, being the system SISO ($p = m = 1$), the usual considerations regarding the structure of the closed-loop system transfer function hold].

Exercise 8. Consider the discrete time linear system

$$\begin{aligned} x(t+1) &= Fx(t) + gu(t) = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1-a}{2} & a-1 & 0 \\ 0 & 1 & \frac{1}{2} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(t), \\ y(t) &= Hx(t) = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} (t), \qquad t \geq 0, \end{aligned}$$

where a is a real parameter.

i) Determine, as a varies in $\mathbb{R}$, the Jordan form of the matrix F and study the asymptotic, simple and BIBO stability of the system.

In the following, assume $a = \frac{1}{2}$:

ii) design, if possible, a state feedback control such that the impulse response of the closed-loop system has finite support (namely there exists $\ell \in \mathbb{Z}_+$ such that $W_K(t) = 0$ for every $t > \ell$);

iii) set $x(0) = 0$ and design, if possible, an input $u(t)$, zero for $t \geq 2$, such that the output satisfies $y(t) = c\left(\frac{1}{2}\right)^t$ for every $t \geq 2$, for some $c \in \mathbb{R} \setminus \{0\}$.

SOLUTIONS OF SOME EXERCISES

Exercise 1. We preliminary notice that the pair (F, G) is in multivariable controllable canonical form. This allows both to say that the system is reachable and to determine the control invariants: $k_1 = k_2 = 2$. From the synthesis part of Rosenbrock's theorem, we know that every time one has $c \leq q = 2$ monic polynomials $\psi_1(z), \psi_2(z), \ldots, \psi_c(z)$ such that $\psi_c(z) \mid \psi_{c-1}(z) \mid \ldots \mid \psi_2(z) \mid \psi_1(z)$, satisfying the constraints

$$\deg \psi_1(s) \geq k_1, \qquad \deg \psi_1(z) + \deg \psi_2(z) \geq k_1 + k_2 ...$$

there exists K such that $F + GK$ has exactly them as invariant polynomials. Case 2) is not solvable since $c = 3 > k_1$. The other two cases satisfy all the constraints and therefore they have solution. Once we set

$$K = \begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix},$$

to solve the problem in case 1) it is sufficient to impose

$$F + GK = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a+e & -1+b+f & 2+c+g & 1+d+h \\ 0 & 0 & 0 & 1 \\ e & f & 1+g & 1+h \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and one gets

$$K = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix},$$

while for case 3) it is sufficient to impose

$$F + GK = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a+e & -1+b+f & 2+c+g & 1+d+h \\ 0 & 0 & 0 & 1 \\ e & f & 1+g & 1+h \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and one gets

$$K = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}.$$

Exercise 2. If we observe the discrete time linear system

$$x(t+1) = \begin{bmatrix} 1 & 2 & 1 \\ 1/2 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} u(t), \quad y(t) = [\,-2 \quad 1 \quad 1\,]\, x(t),$$

we may think that it is in standard reachability form, but it is not because the pair

$$(F_{11}, g_1) = \left(\begin{bmatrix} 1 & 2 \\ 1/2 & 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)$$

is not reachable. Indeed, the reachability matrix of the pair is

$$\mathcal{R}_1 = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix},$$

that clearly has rank 1. This implies that surely $F_{22} = [0]$ is an eigenvalue of the non-reachable subsystem but it is not the only one. On the other hand, it is easy to see that $\Delta_{F_{11}}(z) = z(z-2)$ and that

$$\text{rank}[0I_2 - F_{11} \mid g_1] = 1 < 2.$$

This implies that 0 is the only eigenvalue of the non-reachable subsystem and it has algebraic multiplicity equal to 2.
Based on this fact, we can claim that we can attribute to $F+GK$, as K varies in $\mathbb{R}^{1\times 3}$, all and only the characteristic polynomials taking the form $p(z) = z^2(z-\lambda)$, where λ is real. Consequently, as far as point i) is concerned, no state feedback matrix K that attributes to the closed loop system the eigenvalues $0, \frac{1}{2}, \frac{1}{4}$ exists.
As far as point ii) is concerned, the request amounts to imposing that $F+GK$ is nilpotent with nilpotency index equal to 1. This amounts to saying that $F+GK = 0$. To answer a priori we should identify $\max\{\text{reachability index of the reachable subsystem, nilpotency index of the matrix of the non}-\text{reachable subsystem}\}$. As the reachable subsystem has dimension 1, and hence reachability index equal to 1, $F+GK$ can have nilpotency index equal to 1 if and only if the nilpotency index of the matrix of the non-reachable subsystem is 1. It is easy to see that there is no matrix K such that $F+GK=0$. This also implies that the nilpotency index of the 2×2 matrix of the non-reachable subsystem is necessarily 2.
In point iii), on the other hand, we are requested to impose that $F+GK$ is nilpotent with nilpotency index equal to 2. This is clearly feasible because: the reachable subsystem has reachability index equal to 1, while the matrix of the unreachable subsystem is nilpotent with nilpotency index equal to 2, and therefore $\max\{\text{reachability index of the reachable subsystem, nilpotency index of the matrix of the non}-\text{reachable subsystem}\} = \max\{1,2\} = 2$.
If we choose $K = [\,-1/2 \quad -1 \quad -1/2\,]$, we get

$$F+GK = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 0 & 0 \end{bmatrix},$$

that has nilpotency index equal to 2.
Finally, to impose (point iv)) that $F+GK$ is nilpotent with nilpotency index equal to 3, we need to set $K = [\,a \quad b \quad c\,]$ and first impose that the characteristic polynomial of $F+GK$ is equal to z^3. Clearly, this is equivalent to impose that the characteristic polynomial of

$$F_{11} + g_1 [\,a \quad b\,],$$

namely $z^2-(2+b+2a)z$ is z^2. This implies that all DBCs can be expressed as $K = [\,a \quad -2a-2 \quad c\,]$, with $a, c \in \mathbb{R}$ arbitrary. The corresponding $F+GK$ becomes

$$F+GK = \begin{bmatrix} 1+2a & -2-4a & 1+2c \\ 1/2+a & -1-2a & 1+c \\ 0 & 0 & 0 \end{bmatrix}.$$

It has nilpotency index 3 if and only if its Jordan form is

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and this happens if and only if a and c ensure that $F+GK$ has rank 2. For instance, assuming $a = 0 = c$, which leads to

$$F+GK = \begin{bmatrix} 1 & -2 & 1 \\ 1/2 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

To conclude, as far as part v) is concerned, the impulse response of the closed-loop system is

$$W_K(t) = \begin{cases} Hg, & t=1; \\ H(F+gK)g, & t=2; \\ H(F+gK)^2 g, & t=3; \\ \dots & \dots \end{cases}$$

So, it is clear that $W_K(1) = Hg = -3$ independently of K and the problem is not solvable.

Exercise 5. i) We observe that the pair (F, G) seems to be always in standard reachability form:

$$F = \left[\begin{array}{cc|c} 1 & a & 1 \\ 1 & 1 & a \\ \hline 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{bmatrix} \quad G = \left[\begin{array}{c} 0 \\ 1 \\ \hline 0 \end{array}\right] = \begin{bmatrix} G_1 \\ 0 \end{bmatrix},$$

but this is the case if and only if the pair (F_{11}, G_1) is reachable, and this is true if and only if $a \neq 0$. So, for $a \neq 0$, the pair is in standard reachability form with $F_{22} = [0]_{1\times 1}$ and hence $\Delta_{F_{22}}(z) = z$ divides $p(z) := z(z+1)(z-2)$, which means that the eigenvalue allocation problem is solvable. For $a = 0$ the the pair (F_{11}, G_1) is not reachable, and hence F_{22} has additional eigenvalues and not only 0. Since for $a = 0$ the spectrum of F is $\sigma(F) = (0, 1, 1)$, this means that 1 is also an eigenvalue of F_{22} and hence $\Delta_{F_{22}}(z)$ is not compatible with $p(z)$. This implies that the eigenvalue allocation problem is not solvable.
To solve the problem for $a \neq 0$, we can assume $K = [\, K_1 \quad K_2 \,] = [\, K_1 \quad 0 \,]$ and $K_1 = [\, \alpha \quad \beta \,]$. Then

$$F_{11} + G_1 K_1 = \begin{bmatrix} 1 & a \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [\, \alpha \quad \beta \,] = \begin{bmatrix} 1 & a \\ 1+\alpha & 1+\beta \end{bmatrix}$$

has characteristic polynomial

$$\Delta_{F_{11}+G_1K_1}(z) = z^2 - (2+\beta)z + [1 + \beta - a(1+\alpha)]. \tag{1}$$

If we impose $\Delta_{F_{11}+G_1K_1}(z) = p(z)$ we obtain

$$K_1 = [\, \tfrac{2}{a} - 1 \quad -1 \,],$$

and hence the solution for every $a \neq 0$ is

$$K = [\, \tfrac{2}{a} - 1 \quad -1 \quad 0 \,].$$

ii) The previous analysis tells us that the system is controllable to zero (equivalently it admits a dead-beat controller) if and only if F_{22} has only eigenvalues in 0 and this is true if and only if $a \neq 0$.

iii) For every $a \neq 0$, the pair (F_{11}, G_1) is reachable with reachability index 2, while the matrix F_{22} is nilpotent with nilpotency index 1. Therefore the minimum nilpotency index we can attribute to $F + GK$ is the maximum between these two numbers and hence 2. To solve the problem for $a \neq 0$, we can assume $K = [\, K_1 \quad K_2 \,]$ with $K_1 = [\, \alpha \quad \beta \,]$. Then

$$F_{11} + G_1 K_1 = \begin{bmatrix} 1 & a \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [\, \alpha \quad \beta \,] = \begin{bmatrix} 1 & a \\ 1+\alpha & 1+\beta \end{bmatrix}$$

has characteristic polynomial as in (1). If we impose $\Delta_{F_{11}+G_1K_1}(z) = z^2$ we obtain

$$K_1 = [\, -\tfrac{1}{a} - 1 \quad -2 \,].$$

If we replace these values in $F + GK$ we obtain

$$F + GK = \left[\begin{array}{cc|c} 1 & a & 1 \\ -\frac{1}{a} & -1 & a + K_2 \\ \hline 0 & 0 & 0 \end{array}\right].$$

$F + GK$ has nilpotency index 2 if and only if its Jordan form is

$$J_{F+GK} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and this is the case if and only if $F+GK$ has rank 1. This is possible if and only if $a+K_2=-\frac{1}{a}$, namely if and only if $K_2=-\frac{1}{a}-a$. Therefore the solution for every $a\neq 0$ is

$$K=[\,-\tfrac{1}{a}-1 \quad -2 \quad -\tfrac{1}{a}-a\,].$$

Exercise 6. i) We first evaluate the k-steps reachable subspaces. One gets

$$\begin{aligned}
X_1^R &= \operatorname{Im} g = \operatorname{Im}\begin{bmatrix}0\\1\\0\end{bmatrix} = \langle \begin{bmatrix}0\\1\\0\end{bmatrix} \rangle \\
X_2^R &= \operatorname{Im}[\,g \quad Fg\,] = \operatorname{Im}\begin{bmatrix}0 & 1\\1 & -1\\0 & a\end{bmatrix} = \langle \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\a\end{bmatrix} \rangle \\
X_3^R &= \operatorname{Im}[\,g \quad Fg \quad F^2g\,] = \operatorname{Im}\begin{bmatrix}0 & 1 & a-1\\1 & -1 & 1\\0 & a & a(a-1)\end{bmatrix}.
\end{aligned}$$

The reachability matrix $\mathcal{R}=\mathcal{R}_3$ has rank 2 for every $a\in\mathbb{R}$. Consequently, it is immediate to check that

$$X^R = X_2^R = \langle \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\a\end{bmatrix} \rangle,$$

for every $a\in\mathbb{R}$, and hence the system is never reachable.

Let us evaluate now the controllable subspaces.

$$\begin{aligned}
X_1^C &= \{\mathbf{x} : F\mathbf{x}\in \operatorname{Im} g\} = \left\{ \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} : \begin{bmatrix}x_2+x_3\\-x_2\\a(x_2+x_3)\end{bmatrix} \in \langle \begin{bmatrix}0\\1\\0\end{bmatrix} \rangle \right\} \\
&= \left\{ \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} : x_2+x_3=0 \right\} = \langle \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix} \rangle.
\end{aligned}$$

The 2-steps controllable subspace is

$$\begin{aligned}
X_2^C &= \{\mathbf{x} : F^2\mathbf{x}\in \operatorname{Im}[\,g \quad Fg\,]\} \\
&= \left\{ \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} : \begin{bmatrix}a(x_2+x_3)-x_2\\x_2\\a^2(x_2+x_3)-ax_2\end{bmatrix} \in \langle \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\a\end{bmatrix} \rangle \right\} \\
&= \left\{ \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} : a\,[a(x_2+x_3)-x_2] = a^2(x_2+x_3)-ax_2 \right\} = \mathbb{R}^3.
\end{aligned}$$

Therefore the system is controllable to zero (in two steps) for every value of $a\in\mathbb{R}$.

ii) The system is controllable for every value of a, therefore a dead-beat controller exists for every value of a. If we impose

$$K=[\,k_0 \quad k_1 \quad k_2\,],$$

the matrix $F+gK$ becomes, in parametric form:

$$F+gK = \begin{bmatrix}0 & 1 & 1\\k_0 & k_1-1 & k_2\\0 & a & a\end{bmatrix}.$$

Its characteristic polynomial is

$$\Delta_{F+gK}(z) = z^3 + (1-a-k_1)z^2 + [a(k_1-1) - k_0 - ak_2]z$$

and by imposing that it coincides with z^3 one gets the system of linear equations:

$$\begin{aligned} 1-a-k_1 &= 0, \\ a(k_1-1) - k_0 - ak_2 &= 0. \end{aligned}$$

They correspond to the controller

$$K = [\,(-a^2 - ak_2) \quad (1-a) \quad k_2\,], \qquad k_2 \in \mathbb{R}.$$

iii) For $a = 0$ we get the family of controllers

$$K = [\,0 \quad 1 \quad k_2\,], \qquad k_2 \in \mathbb{R}.$$

The matrix of the closed-loop system takes the form

$$F + gK = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & k_2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Such nilpotent matrix can never have nilpotency index 1 and has nilpotency index 2 if and only if it has rank 1 and this occurs if and only if $k_2 = 0$. Therefore the desired dead beat controller is

$$K = [\,0 \quad 1 \quad 0\,].$$

Exercise 7. i) Let us first evaluate the reachable subspaces. One gets

$$\begin{aligned} X_1^R &= \mathrm{Im} G = \mathrm{Im} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \langle \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \rangle \\ X_2^R &= \mathrm{Im}\,[\,G \quad FG\,] = \mathrm{Im} \begin{bmatrix} 1 & 1-a \\ -1 & 2a-1 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

By referring to X_2^R we can distinguish two cases: 1) for $a = 0$ the two columns of the matrix $[\,G \quad FG\,]$ are linearly dependent and therefore

$$X_2^R = X_1^R = \langle \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \rangle.$$

2) For $a \neq 0$ the two columns are linearly independent and hence

$$X_2^R = \langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rangle.$$

Clearly it makes sense to explore the case of reachability in three steps only for $a \neq 0$ and in that case one gets

$$X_3^R = \mathrm{Im}\,[\,G \quad FG \quad F^2G\,] = \langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rangle = X_2^R.$$

Therefore the system is never reachable, however for $a = 0$ the reachable subspace has dimension 1, while for $a \neq 0$ the subspace $X^R = X_2^R$ has dimension 2.

Let us evaluate now the controllable subspaces.

$$
\begin{aligned}
X_1^C &= \{\mathbf{x} : F\mathbf{x} \in \mathrm{Im} G\} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : \begin{bmatrix} (1-a)x_1 + x_3 \\ 2ax_1 + x_2 - x_3 \\ 0 \end{bmatrix} \in \langle \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \rangle \right\} \\
&= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : (1-a)x_1 + x_3 = -(2ax_1 + x_2 - x_3) \right\} = \left\{ \begin{bmatrix} x_1 \\ -(1+a)x_1 \\ x_3 \end{bmatrix} : x_1, x_3 \in \mathbb{R} \right\} \\
&= \langle \begin{bmatrix} 1 \\ -1-a \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rangle.
\end{aligned}
$$

For the computation of the 2-steps controllable subspace we distinguish the case $a = 0$ and the case $a \neq 0$. For $a = 0$ one gets

$$
\begin{aligned}
X_2^C &= \{\mathbf{x} : F^2\mathbf{x} \in \mathrm{Im}\,[\,G \quad FG\,]\} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : \begin{bmatrix} x_1 + x_3 \\ x_2 - x_3 \\ 0 \end{bmatrix} \in \langle \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \rangle \right\} \\
&= \langle \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rangle = X_1^C \subset \mathbb{R}^3.
\end{aligned}
$$

Therefore, for $a = 0$ the system is not controllable to zero. For $a \neq 0$, instead,

$$
\begin{aligned}
X_2^C &= \{\mathbf{x} : F^2\mathbf{x} \in \mathrm{Im}\,[\,G \quad FG\,]\} \\
&= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : \begin{bmatrix} (1-a)^2 x_1 + (1-a)x_3 \\ 2a(1-a)x_1 + 2ax_3 + 2ax_1 + x_2 - x_3 \\ 0 \end{bmatrix} \in \langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rangle \right\} \\
&= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : \begin{bmatrix} (1-a)^2 x_1 + (1-a)x_3 \\ 2a(2-a)x_1 + (2a-1)x_3 + x_2 \\ 0 \end{bmatrix} \in \langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rangle \right\} = \mathbf{R}^3 \\
X_3^C &= X_2^C.
\end{aligned}
$$

Therefore the system (that is never reachable) is controllable to zero in two steps for $a \neq 0$.

ii) For $a = 1$ the pair (F, G) becomes

$$
F = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad G = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},
$$

and according to the analysis made at point i) we know that such pair is not reachable but it is controllable to zero. Moreover we can easily check that the pair is in standard reachability form, since

$$
\mathcal{R}_1 = [\,G_1 \quad F_{11}G_1\,] = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}
$$

is non singular. By introducing the matrix $K = [\,k_0 \quad k_1 \quad k_2\,]$ in parametric form, we notice (due to the standard form structure) that $K = [\,K_1 \quad K_2\,]$ with $K_1 = [\,k_0 \quad k_1\,]$ that is (uniquely) chosen in order to make the matrix $F_{11} + G_1K_1$ (and hence $F + GK$) nilpotent. While k_2 is a design parameter that can be chosen in order to minimize the nilpotency index of the matrix $F+GK$. Imposing that

$$F_{11}+G_1K_1=\begin{bmatrix} k_0 & k_1 \\ 2-k_0 & 1-k_1 \end{bmatrix}$$

has characteristic polynomial

$$\Delta_{F_{11}+G_1K_1}(z)=z^2-(k_0-k_1+1)z+(k_0-2k_1)=z^2,$$

one gets $k_0=-2, k_1=-1$. If we now evaluate the matrix $F+GK$ corresponding to those two values of the parameters k_0 and k_1, for k_2 arbitrary, one gets

$$F+GK=\begin{bmatrix} -2 & -1 & 1+k_2 \\ 4 & 2 & -1-k_2 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is immediate to notice that this matrix will never be null and therefore it will never have nilpotency index 1. However, by imposing $1+k_2=0$ the matrix becomes block diagonal, with nilpotent diagonal blocks, one with index 1 and one with index 2. Therefore for $k_2=-1$, namely

$$K=[\,-2 \quad -1 \quad -1\,],$$

$F+GK$ has nilpotency index 2 that is the minimum one.

iii) As a first step, we observe that the transfer function of the system depends only on the reachable subsystem and hence, in order to modify the system transfer function via feedback, it makes sense to consider only the reachable subsystem. Moreover, it is known, from the theory that the state feedback for SISO systems (as the system (F_{11},G_1,H_1) is) leads to a transfer function whose numerator remains unchanged and whose denominator changes arbitrarily (as long as one assumes as description for the transfer function the one, without simplifications, that can be obtained by the three system matrices). The transfer function of the triple (F_{11},G_1,H_1) (and hence of the overall system) is

$$w(z)=w_1(z)=[\,1 \quad -1\,]\begin{bmatrix} z & 0 \\ -2 & z-1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ -1 \end{bmatrix}=\frac{2z-3}{z(z-1)}.$$

The transfer function of the closed-loop system $(F_{11}+G_1K_1,G_1,H_1)$ (and hence of the overall closed-loop system), becomes, as $K_1=[\,k_0 \quad k_1\,]$ varies, an expression of the type:

$$w_K(z)=w_{1,K_1}(z)=2\frac{z-\frac{3}{2}}{z^2+a_1z+a_0}.$$

Therefore it is evident that by imposing $\Delta_{F_{11}+G_1K_1}(z)=(z-1)\left(z-\frac{3}{2}\right)$ we get the desired result. This is to impose

$$\Delta_{F_{11}+G_1K_1}(z)=z^2-(k_0-k_1+1)z+(k_0-2k_1)=z^2-\frac{5}{2}z+\frac{3}{2},$$

that leads to

$$K=[\,K_1 \quad K_2\,]=[\,\tfrac{3}{2} \quad 0 \quad k_2\,], \qquad k_2\in\mathbb{R}.$$

Exercise 8. i) The matrix F has a triangular structure and its eigenvalues are $\sigma(F)=(1/2,1,a-1)$. Notice that the presence of an eigenvalue in 1 automatically excludes the possibility to obtain asymptotic stability for every value of the parameter a. Therefore, in the following, we will only consider simple and BIBO stability. Clearly, one needs to evaluate for what values of the parameter

a the matrix F has a double eigenvalue. This occurs if and only if $a-1=1$, namely $a=2$, or $a-1=1/2$, namely $a=3/2$. In the first case one gets

$$F=\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 1 & \frac{1}{2} \end{bmatrix}.$$

It is immediate to notice that the rank of (I_3-F) is 2 and hence the geometric multiplicity of 1 is 1. Therefore the Jordan form of F is

$$J=\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Clearly such matrix is not simply stable. For $a=3/2$ one gets

$$F=\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} \end{bmatrix}.$$

Also in this case 1/2 has geometric multiplicity 1 and hence the Jordan form of F is

$$J=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1 \\ 0 & 0 & 1/2 \end{bmatrix}.$$

Such matrix is simply stable. Finally for $a\neq 3/2, 2$ the three eigenvalues are distinct and hence the Jordan form is

$$J=\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & a-1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Such matrix is simply stable if and only if $|a-1|<1$ or $a-1=-1$. This corresponds to the condition $a\in[0,2)$.

Let us evaluate now BIBO stability. The transfer matrix of the system is

$$W(z)=[0 \quad 1 \quad 2]\begin{bmatrix} z-1 & 0 & 0 \\ -\frac{1-a}{2} & z-a+1 & 0 \\ 0 & -1 & z-\frac{1}{2} \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}=\frac{z+\frac{3}{2}}{(z-a+1)(z-1/2)}.$$

It must be said that one could have derived this result more quickly by noticing that the pair (F,g) is in a “reversed” (or “upside-down”) standard reachability form (namely with the reachable part described by the pair (F_{22},g_2)) for every value of a. Certainly the system is BIBO stable if $|a-1|<1$ namely $0<a<2$. Since the only zero at the numerator has modulus greater than 1, one needs to evaluate if there exists any value of a for which a cancellation between numerator and denominator occurs. The cancellation occurs if and only if $1-a=\frac{3}{2}$, namely $a=-\frac{1}{2}$, and in that case one gets

$$W(z)=\frac{1}{z-1/2}$$

that is BIBO stable. Therefore one gets BIBO stability if and only if $a\in(0,2)\cup\{-1/2\}$.

ii) First we observe that the requirement on the closed-loop system impulse response corresponds to imposing that the closed-loop system transfer function is

$$W_k(z)=\frac{n(z)}{z^\ell}$$

for some $\ell \in \mathbb{Z}_+$ and $n(z) \in \mathbb{R}[z]$ non zero, of degree smaller than ℓ. In other words, we have to place all the poles of the closed-loop system in 0. For $a = 1/2$ the matrix F becomes

$$F = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} \end{bmatrix}.$$

As already observed, the pair (F, g) is not reachable, but it is in standard reachability form. Since the transfer function coincides with the transfer function of the only reachable subsystem, a possible solution is the one of attributing to the reachable subsystem matrix (in this case F_{22}) null eigenvalues. Posed

$$K = [\, K_1 \quad K_2 \,] = [\, a \quad | \quad b \quad c \,]$$

one gets

$$F_{22} + g_2 K_2 = \begin{bmatrix} b - \frac{1}{2} & c \\ 1 & \frac{1}{2} \end{bmatrix},$$

whose characteristic polynomial is $z^2 - bz + \left(\frac{1}{2}b - \frac{1}{4} - c\right)$. By imposing that it coincides with z^2 one gets $b = 0$ and $c = -\frac{1}{4}$. Hence the general solution to the problem is

$$K = [\, a \quad 0 \quad -\tfrac{1}{4} \,]$$

with a an arbitrary parameter.

iii) The solution consists in noticing that if we manage to bring the state $x(2)$ in $\langle \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rangle$, all the subsequent dynamics will remain in that subspace, that is nothing but the eigenspace associated with the eigenvalue $1/2$, and hence from the time instant $t = 2$ onward one sees as output only $y(t) = y_\ell(t) = 2x_3(t) = 2\left(\frac{1}{2}\right)^{t-2} x_3(2)$, where $x_3(\cdot)$ is the third component of the state vector $x(2)$. The set of states that are reachable in two steps is given by

$$X_2^R = \mathrm{Im}\,[\, g \quad Fg \,] = \mathrm{Im} \begin{bmatrix} 0 & 0 \\ 1 & -1/2 \\ 0 & 1 \end{bmatrix},$$

therefore $[\, 0 \quad 0 \quad 1 \,]^\top$ is certainly reachable in two steps. Posed

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x(2) = \mathcal{R}_2 \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -1/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix},$$

one gets $u(0) = 1, u(1) = 1/2$ and, correspondingly, it holds $y(t) = 2\left(\frac{1}{2}\right)^{t-2} = 8\left(\frac{1}{2}\right)^t, \forall\, t \geq 2$.