8

Constrained Optimization

Finally equipped with the tool of Lagrangian duality from the previous chapter, we can now finally tackle constrained optimization, first in the (simpler) case of equality constraints only, and then in the general case.

8.1 Equality constrained optimization

In this section, we consider optimization problems of the form:

$$\begin{aligned} \min & f(x) \\ & Ax = b \end{aligned}$$

where we assume f to be convex and of class C^2, and, w.l.o.g., the matrix A of full row rank $p < n$. As usual, we assume that an optimal solution x^* exists, with optimal value $f(x^*) = p^*$. Under those assumptions, the problem is convex, it trivially satisfies Slater's conditions, and all the involved functions are differentiable, so the KKT system gives both necessary and sufficient conditions for optimality. In addition, the KKT system simplifies significantly, as there are no inequalities (and no corresponding multipliers):

$$\begin{aligned} Ax^* &= b && \text{(primal feas)} \\ \nabla f(x^*) + A^\top \pi^* &= 0 && \text{(dual feas)} \end{aligned}$$

Being this a nonlinear system, we cannot solve it analytically, but resort to some iterative algorithm converging to a solution (x^*, π^*) to the system. In order to understand the corresponding method, let us start with the simpler case of a convex quadratic function:

Yet important on its own.

$$\begin{aligned} \min \ f(x) = \frac{1}{2} x^\top Q x + q^\top x + r \\ Ax = b \end{aligned}$$

In this case, the KKT system corresponds to the *linear* system:

$$\begin{aligned} Qx^* + A^\top \pi^* &= -q \\ Ax^* &= b \end{aligned}$$

which can be rewritten in matrix form as:

$$\begin{bmatrix} Q & A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} x^* \\ \pi^* \end{bmatrix} = \begin{bmatrix} -q \\ b \end{bmatrix}$$

The matrix on the left is called the *KKT matrix*: if it is nonsingular, then the system of $n+p$ equations in $n+p$ unknowns has a unique solution, the only optimal primal-dual pair (x^*, π^*). If the matrix is singular, then if the system is still solvable, any solution gives an optimal primal-dual pair. Otherwise, if the system is non solvable, the quadratic optimization problem is either infeasible or unbounded. Indeed, if $Ax = b$ itself is unsolvable, then of course the problem is infeasible. Otherwise, assume there a feasible solution $\overline{x}$ but the KKT system as a whole is unsolvable. By the standard properties of linear systems, this means that there are vectors of multipliers (v, w) such that $v^\top Q + w\top A = 0$ and $Av = 0$, but $-v^\top q + w^\top b \neq 0$. We can now consider the line $\overline{x} + tv$ and evaluate $f(\overline{x} + tv)$:

$$\begin{aligned} f(\overline{x} + tv) &= \frac{1}{2}(\overline{x} + tv)^\top Q(\overline{x} + tv) + q^\top(\overline{x} + tv) + r \\ &= \underbrace{\frac{1}{2}\overline{x}^\top Q\overline{x} + q^\top\overline{x} + r}_{f(\overline{x})} + \frac{1}{2}t^2 v^\top Qv + t(v^\top Q\overline{x} + q^\top v) \\ &= f(\overline{x}) - \frac{1}{2}t^2 w^\top \underbrace{Av}_{=0} + t(-w^\top \underbrace{A\overline{x}}_{=b} + q^\top v) \\ &= f(\overline{x}) + t(\underbrace{q^\top v - w^\top b}_{\neq 0}) \end{aligned}$$

Thus the function goes to $-\infty$ for $t \to \infty$. Note that since $\overline{x}$ is feasible and $Av = 0$, we have that the whole line is feasible w.r.t. the equations defining the feasible region of the optimization problem.

In the general case, we can just the quadratic approximation of the function at every iteration, computing the direction solving the KKT system. In details, let $x^{(0)}$ be the starting point, which we assume to be feasible $Ax^{(0)} = b$. Then, at each iteration, we compute the direction Δx as:

$$\begin{bmatrix} \nabla^2 f(x) & A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \pi \end{bmatrix} = \begin{bmatrix} -\nabla f(x) \\ 0 \end{bmatrix}$$

which is well-defined if the KKT matrix is non-singular. The above method can we interpret as a variant of the Newton method, where we also force the Newton step to be feasible w.r.t. the linear system, i.e., $A\Delta x = 0$, in order to maintain feasibility throughout, and, indeed, it can be shown to share the same convergence properties (and affine invariance) of the regular Newton method for the unconstrained case.

Finally, the algorithm can be generalized to *not* require a feasible starting point. In details, the idea is to try to satisfy the system *after* the step is taken, i.e., to impose the condition $A(x + \Delta x) = b$ instead of $A\Delta x = 0$. The resulting system has the same KKT matrix as before, just a different right-hand side:

$$\begin{bmatrix} \nabla^2 f(x) & A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \pi \end{bmatrix} = - \begin{bmatrix} \nabla f(x) \\ Ax - b \end{bmatrix}$$

Note that once a full step is taken ($t = 1$), feasibility is achieved and the iterates will stay feasible from then on.

8.2 Interior-point methods

We are finally ready to tackle a convex optimization problem with inequalities:

$$\begin{aligned} &\min f(x) \\ &g_i(x) \leq 0 \quad i = 1, \dots, m \\ &Ax = b \end{aligned}$$

As usual, we assume that the problem admits an optimal solution x^*, that the Slater's condition are satisfied (let $\overline{x}$ be a strictly feasible point), and the functions are differentiable. Under those conditions, we know that an optimal primal-dual pair can be obtained by solving the KKT system. However, this is a again a nonlinear system that cannot be solved directly, but rather approximated via an iterative method.

8.2.1 *Logarithmic barriers*

The first step consists, again, in reducing the problem to a simpler form that we already know how to solve: a problem with only linear equations. We obtain this goal by moving the inequality constraints to the objective, via an *indicator* function:

$$\begin{aligned} &\min f(x) + \sum_{i=1}^{m} I(g_i(x)) \\ &Ax = b \end{aligned}$$

The indicator function $I(u)$ is formally defined as:

$$I(u) = \begin{cases} 0 & u \leq 0 \\ +\infty & u > 0 \end{cases}$$

Clearly, this is just an *equivalent reformulation* of the original problem, but the objective function is nondifferentiable, and thus unsuited for the Newton's method of the previous section. They main idea of the *barrier* method is to replace the indicator functions with a convex *logarithmic barrier*, i.e., a function of the form:

$$I_t(u) = -\frac{1}{t} \log(-u)$$

which is defined for any negative argument u, and is parametrized by the scalar multiplier $t > 0$. Note that a logarithmic barrier gives *only* an approximation of the real indicator function, but this approximations gets better and better as t increases. In addition, for any $t > 0$, the function is convex and differentiable in its domain. So we can write the approximation:

$$\begin{aligned} &\min f(x) + \sum_{i=1}^{m} -\frac{1}{t} \log(-g_i(x)) \\ &Ax = b \end{aligned}$$

Scaling the objective by a factor $t > 0$ clearly has no effect on the optimal solutions, so we can equivalently rewrite the approximation as:

$$\begin{aligned} \min\ & tf(x) + \phi(x) \\ & Ax = b \end{aligned}$$

where $\phi(x) = -\sum_{i=1}^{m} \log(-g_i(x))$ collects all the logarithmic barrier terms. The domain of $\phi(x)$ is the set of all points that satisfy all the inequality constraints of the problem strictly, and it goes to $+\infty$ as we approach even a single of them. For future reference, let us note that the gradient and Hessian of $\phi(x)$ can be obtained by the standard chain rules in closed form as:

$$\begin{aligned} \nabla\phi(x) &= \sum_{i=1}^{m} \frac{1}{-g_i(x)} \nabla g_i(x) \\ \nabla^2\phi(x) &= \sum_{i=1}^{m} \frac{1}{g_i^2(x)} \underbrace{\nabla g_i(x) \nabla g_i(x)^\top}_{\text{rank-1 matrices}} + \sum_{i=1}^{m} \underbrace{\frac{1}{-g_i(x)} \nabla^2 g_i(x)}_{\text{weighted sums of Hessians}} \end{aligned}$$

Let us assume we can solve the approximation for any $t > 0$, and let $x^*(t)$ be the corresponding optimal solution. Note that $x^*(t)$ is strictly feasible and for each $x^*(t)$ we have a matching $\overline{\pi}(t)$ such that:

$$t\nabla f(x^*(t)) + \nabla\phi(x^*(t)) + A^\top \overline{\pi}(t) = 0 \tag{1}$$

Now, let us define the dual multipliers:

$$\lambda_i^*(t) = -\frac{1}{t g_i(x^*(t))} \quad \text{and} \quad \pi^*(t) = \frac{\overline{\pi}(t)}{t}$$

Clearly, $(\lambda^*(t), \pi^*(t))$ is a dual feasible pair for the original problem, and the equation (1) can be rewritten as:

$$\nabla f(x^*(t)) + \sum_{i=1}^{m} \lambda_i^*(t) \nabla g_i(x^*(t)) + A^\top \pi^*(t) = 0$$

Again, this implies that $x^*(t)$ is a minimizer of $L(x, \lambda^*(t), \pi^*(t))$, and thus:

$$\begin{aligned} l(\lambda^*(t), \pi^*(t)) &= f(x^*(t)) + \sum_{i=1}^{m} \underbrace{\lambda_i^*(t) g_i(x^*(t))}_{-1/t} + \underbrace{\pi^*(t)^\top (Ax^*(t) - b)}_{=0} \\ &= f(x^*(t)) - \frac{m}{t} \end{aligned}$$

In other words, $x^*(t)$ and $(\lambda^*(t), \pi^*(t))$ are a primal-dual pair for the original problem with an optimality gap of at most m/t. As we increase $t \to +\infty$, the gap goes to zero and $x^*(t) \to x^*$. Notice that, by construction, the solution $(x^*(t), \lambda^*(t), \pi^*(t))$ basically satisfies a modified version of the KKT system, where the complementary slackness conditions are set to:

$$\lambda_i^*(t) g_i(x^*(t)) = -\frac{1}{t}$$

The considerations above suggest the scheme depicted in Algorithm 5.

Algorithm 5: Basic barrier algorithm.

let x be a strictly feasible starting point, $t > 0$, $\mu > 1$, $\varepsilon > 0$
while $m/t > \varepsilon$ **do**
  Compute $x^*(t)$ from x via the Newton method
  $x = x^*(t)$
  $t \leftarrow \mu t$
end
return x

Notice that we do not just solve a single approximation for a large enough value of t, as for large values of t the function is difficult to optimize with the Newton method, since its Hessian changes too rapidly at the boundary of the feasible set. To the contrary, we solve a sequence of problems, starting for moderate values of t, and use the optimal solution of the previous problem as starting point for the next. This is also called a *path following* method, as the sequence of optimal solutions $x^*(t)$ follow the so-called *central path*. The computation of $x^*(t)$ for a given value of t is called the *centering step*, and we refer to each iteration of the process as an *outer iteration*, to distinguish them from the *inner iterations* in the Newton method of each centering step. Note that (strict) primal feasibility is maintained throughout the solution process, and at the end of each outer iteration we can also compute the dual solution $(\lambda^*(t), \pi^*(t))$ to certificate the suboptimality of the current primal solution $x^*(t)$. We finally note that it is not really necessary to solve each centering step to high accuracy, at least until the very last iterations, as we are going to use each $x^*(t)$ only as a starting point for the next outer iteration.

What about convergence? Clearly, after k outer iterations, we have a guaranteed accuracy of $m/t^{(0)}\mu^k$, so the method will converge as long as each centering step can be solved by the Newton method, for $t \geq t^{(0)}$. This is easy to obtain with the standard assumptions.

8.2.2 *Primal-dual interior point methods*

In practice, the current state of the art is not achieved by barrier methods like the one just described, but rather by a related class of algorithms called *primal-dual interior point methods*. The main idea is to *not* distinguish between outer and inner iterations, but rather to update both the primal and dual variables together at each iteration, still via a Newton-like scheme. The advantages of primal-dual methods are their improved practical convergence and their ability to work with primal and dual infeasible iterates. Here we present a brief description of such methods, without convergence analysis.

As in the barrier method, the idea is to consider a modified KKT system, parametrized by t:

$$\begin{aligned} Ax &= b \\ g(x) &\leq 0 \\ \lambda &\geq 0 \\ \lambda_i g_i(x) &= -\frac{1}{t} \quad i = 1, \dots, m \\ \nabla f(x) + \lambda^\top \nabla g(x) + A^\top \pi &= 0 \end{aligned}$$

The starting point (x, λ, π) is not necessarily feasible, but it needs to satisfy strictly the KKT inequalities $g(x) < 0$ and $\lambda > 0$. As for the remaining equations, those are not necessarily satisfied, so we can consider the three *residual* terms:

$$
\begin{aligned}
r_{\text{primal}} &= Ax - b \\
r_{\text{dual}} &= \nabla f(x) + \lambda^\top \nabla g(x) + A^\top \pi \\
r_{\text{center}} &= -\text{diag}(\lambda) g(x) - \frac{1}{t}\mathbf{1}
\end{aligned}
$$

We can combine the three residual terms into a single vector function $r_t(x, \lambda, \pi)$, from $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$ to $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$:

$$
r_t = (r_{\text{dual}}, r_{\text{center}}, r_{\text{primal}})
$$

For ease of notation, let us denote the solution vector (x, λ, π) with y. The idea is to minimize this residual function with the Newton method, computing at each direction an update direction $\Delta y = (\Delta x, \Delta\lambda, \Delta\pi)$, computed via:

$$
r_t(y + \Delta y) \approx r_t(y) + \nabla r_t(y) \Delta y = 0
$$

In other words, the Newton step is computed by solving, at each iteration, the linear system $\nabla r_t(y)\Delta y = -r_t(y)$. By applying the usual derivation rules, we can obtain precise expressions for the different terms of $\nabla r_t(y)$, and eventually:

$$
\begin{bmatrix}
\nabla^2 f(x) + \sum_{i=1}^m \lambda_i \nabla^2 g_i(x) & \nabla g(x) & A^\top \\
-\text{diag}(\lambda)\nabla g(x) & -\text{diag}(g(x)) & 0 \\
A & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\Delta x \\ \Delta\lambda \\ \Delta\pi
\end{bmatrix}
= -
\begin{bmatrix}
r_{\text{dual}}(x, \lambda, \pi) \\
r_{\text{center}}(x, \lambda, \pi) \\
r_{\text{primal}}(x, \lambda, \pi)
\end{bmatrix}
$$

Note that the three directions are coupled together by the system, so that the primal and dual solution are *not* updated independently. A difficulty of the method is that, until convergence is reached, neither x is primal feasible nor (λ, π) is dual feasible, so we do *not* have a valid optimality gap to measure progress. The solution is to use the so-called *surrogate duality gap*:

$$
\eta(x, \lambda) = -\lambda^\top g(x)
$$

Note that this only requires that $g(x) < 0$ and $\lambda > 0$, a condition that is maintained throughout the algorithm via the logarithmic barrier for the first condition, and by backtracking line search for the second condition. Note also that it *would* be the standard duality gap, *if* the respective solutions were primal and dual feasible.

Algorithm 6: Primal-dual interior point algorithm.

let x s.t. $g(x) < 0$, $\lambda > 0$, $\mu > 1$, $\varepsilon_{\text{feas}} > 0$, $\varepsilon > 0$
repeat
  Compute $t \leftarrow \mu m/\eta$
  Compute search direction Δy
  Update step with backtracking line search $y \leftarrow y + s\Delta y$
until $\|r_{primal}\| \leq \varepsilon_{feas}, \|r_{dual}\| \leq \varepsilon_{feas}, \eta \leq \varepsilon$
return x

A pseudo-code of the primal-dual interior point method is depicted in Algorithm 6. At each iteration, the value of t associated with the current surrogate duality gap, m/η, is multiplied by the factor μ. Then a Newton iteration is performed, by first computing the step Δy and then choosing the step length s via backtracking line search, with the only additional condition that we need to maintain $\lambda > 0$.