Systems Theory Exercises - Equilibrium points and linearisation A.Y. 2025/26

Exercise 1. Given the continuous time nonlinear autonomous state-space model

$$\begin{aligned}\dot{x}_1(t) &= x_1^2(t) + x_2(t)\\ \dot{x}_2(t) &= -2x_1^3(t) - 2x_1(t)x_2(t), \qquad t \geq 0,\end{aligned}$$

i) determine the set of all equilibrium points of the systems;

ii) for each equilibrium point $\mathbf{x}_e$, determine the linearised system around $\mathbf{x}_e$, and

iii) evaluate, if possible, the asymptotic stability of $\mathbf{x}_e$ as an equilibrium point of the above nonlinear system, by resorting to the linearisation method.

Exercise 2. Given the continuous time nonlinear autonomous state-space model

$$\begin{aligned}\dot{x}_1(t) &= (a-1)x_2(t) + a(x_1(t) - x_2(t))^3\\ \dot{x}_2(t) &= x_1(t) - ax_2(t) + ax_1^3(t) + a(x_1(t) - x_2(t))^3\,, \qquad t \geq 0,\end{aligned}$$

with $a \in \mathbb{R}$,

i) prove that the origin is an equilibrium point for every value of a;

ii) determine, for every value of a, the linearised system around $\mathbf{x}_e = 0$, and

iii) evaluate, if possible, the asymptotic stability of $\mathbf{x}_e = 0$ as an equilibrium point of the above nonlinear system, by resorting to the linearisation method.

Exercise 3. Given the discrete time nonlinear autonomous state-space model

$$\begin{aligned}x_1(t+1) &= (1+a)^2 x_1(t) - x_1(t)x_2^2(t),\\ x_2(t+1) &= (1-a^2)x_2(t) - x_1^2(t)x_2(t), \qquad t \geq 0,\end{aligned}$$

with $a \in \mathbb{R}$,

i) prove that the origin is an equilibrium point for every value of a;

ii) determine, for every value of a, the linearised system around $\mathbf{x}_e = 0$, and

iii) evaluate, if possible, the asymptotic stability of $\mathbf{x}_e = 0$ as an equilibrium point of the above nonlinear system, by resorting to the linearisation method.

Exercise 4. Given the continuous time nonlinear autonomous state-space model

$$\begin{aligned}\dot{x}_1(t) &= ax_2(t),\\ \dot{x}_2(t) &= -\sin x_1(t) - ax_2(t),\end{aligned} \qquad t \geq 0,$$

with $a \in \mathbb{R}$,

i) determine, for every value of a, the equilibrium points of the system, and

ii) evaluate, if possible, the asymptotic stability of each $\mathbf{x}_e$ as an equilibrium point of the above nonlinear system, by resorting to the linearisation method.

Exercise 5. Given the discrete time nonlinear autonomous state-space model

$$\begin{aligned} x_1(t+1) &= (1-a)x_1(t), \\ x_2(t+1) &= x_1^2(t) + (1-a^2)x_2(t), \end{aligned} \qquad t \geq 0,$$

with $a \in \mathbb{R}$,

i) determine, for every value of a, the equilibrium points of the system, and

ii) evaluate, if possible, the asymptotic stability of each $\mathbf{x}_e$ as an equilibrium point of the above nonlinear system, by resorting to the linearisation method.

SOLUTIONS OF SOME EXERCISES

Exercise 1. i) To find the equilibrium points we need to find the solutions of the algebraic equations:

$$\begin{aligned} 0 &= x_1^2 + x_2, & (1)\\ 0 &= -2x_1^3 - 2x_1x_2 = -2x_1(x_1^2 + x_2). & (2) \end{aligned}$$

Equation (2) has two solutions: (a) $x_1 = 0$ and (b) $x_1^2 + x_2 = 0$.
Corresponding to (a) equation (1) becomes $x_2 = 0$, and hence the only solution is $\mathbf{x}_e = (0, 0)$. Solution (b) is also a solution of (1), and hence identifies the set of equilibrium points

$$\mathcal{E} = \{(x_1, -x_1^2) : x_1 \in \mathbb{R}\}.$$

Note that the set $\mathcal{E}$ includes the equilibrium point $\mathbf{x}_e = (0, 0)$, previously determined. Therefore, the set of all equilibrium points of the system is $\mathcal{E}$.

ii) The nonlinear differential equations describing the system can be thought as

$$\begin{aligned} \dot{x}_1(t) &= f_1(x_1(t), x_2(t)),\\ \dot{x}_2(t) &= f_2(x_1(t), x_2(t)). \end{aligned}$$

To determine the linearised system around each equilibrium point, we need to determine the following partial derivatives:

$$\begin{aligned} \frac{\partial f_1}{\partial x_1} &= 2x_1, & \frac{\partial f_1}{\partial x_2} &= 1,\\ \frac{\partial f_2}{\partial x_1} &= -6x_1^2 - 2x_2, & \frac{\partial f_2}{\partial x_2} &= -2x_1. \end{aligned}$$

Therefore, the linearised model around each equilibrium point $(x_1, -x_1^2)$ is

$$\frac{d}{dt}\Delta\mathbf{x}(t) = \begin{bmatrix} 2x_1 & 1\\ -4x_1^2 & -2x_1 \end{bmatrix} \Delta\mathbf{x}(t).$$

iii) We preliminarily observe that since none of the equilibrium points is isolated, namely for each $\mathbf{x}_e \in \mathcal{E}$ and for each $\varepsilon > 0$, the ball of center $\mathbf{x}_e$ and radius ε includes (infinitely many) other equilibrium points, none of these points can be asymptotically stable. If we observe the matrix

$$F_{\mathbf{x}_e} = \begin{bmatrix} 2x_1 & 1\\ -4x_1^2 & -2x_1 \end{bmatrix}$$

of the linearised state-space model, we observe that its characteristic polynomial is

$$\det(sI_2 - F_{\mathbf{x}_e}) = s^2,$$

and hence $\sigma(F_{\mathbf{x}_e}) = \{0, 0\}$. This implies that the linearised method cannot be used to decide about the (simple) stability or instability of the equilibrium points.

Exercise 2. i) To prove that $\mathbf{x}_e = (0, 0)$ is an equilibrium point of the nonlinear state-space model for every value of a, we need to show that $(0, 0)$ is a solution of the algebraic equations:

$$\begin{aligned} 0 &= (a-1)x_2 + a(x_1 - x_2)^3\\ 0 &= x_1 - ax_2 + ax_1^3 + a(x_1 - x_2)^3, \end{aligned}$$

for every value of a, which is obvious.

ii) The nonlinear differential equations describing the system can be thought as

$$\begin{aligned} \dot{x}_1(t) &= f_1(x_1(t), x_2(t)), \\ \dot{x}_2(t) &= f_2(x_1(t), x_2(t)). \end{aligned}$$

To determine the linearised system around $\mathbf{x}_e = (0,0)$ for each value of a, we first need to determine the following partial derivatives:

$$\frac{\partial f_1}{\partial x_1} = 3a(x_1 - x_2)^2, \qquad \frac{\partial f_1}{\partial x_2} = (a-1) - 3a(x_1 - x_2)^2,$$
$$\frac{\partial f_2}{\partial x_1} = 1 + 3ax_1^2 + 3a(x_1 - x_2)^2, \qquad \frac{\partial f_2}{\partial x_2} = -a - 3a(x_1 - x_2)^2.$$

Therefore, the linearised model around $\mathbf{x}_e = (0,0)$ is

$$\frac{d}{dt}\Delta\mathbf{x}(t) = \begin{bmatrix} 0 & a-1 \\ 1 & -a \end{bmatrix} \Delta\mathbf{x}(t).$$

iii) If we observe the matrix

$$F_{\mathbf{0},a} = \begin{bmatrix} 0 & a-1 \\ 1 & -a \end{bmatrix}$$

of the linearised state-space model, we observe that its characteristic polynomial is

$$\det(sI_2 - F_{\mathbf{0},a}) = s^2 + as + 1 - a.$$

By Descartes' rule of signs, this polynomial is Hurwitz if and only if $0 < a < 1$. So, $\mathbf{x}_e = (0,0)$ is an asymptotically stable equilibrium point of the nonlinear state-space model if and only if $0 < a < 1$. For $a < 0$ or $a > 1$ the matrix $F_{\mathbf{0},a}$ has (at least) one eigenvalue with positive real part and hence the origin is an unstable equilibrium point of the nonlinear state-space model. For $a = 0$ and $a = 1$ the linearised method cannot be used to decide about the stability of the origin.

Exercise 3. i) To prove that $\mathbf{x}_e = (0,0)$ is an equilibrium point of the nonlinear state-space model for every value of a, we need to show that $(0,0)$ is a solution of the algebraic equations:

$$\begin{aligned} x_1 &= (1+a)^2 x_1 - x_1 x_2^2, \\ x_2 &= (1-a^2) x_2 - x_1^2 x_2, \end{aligned}$$

for every value of a, which is obvious.

ii) The nonlinear differential equations describing the system can be thought as

$$\begin{aligned} x_1(t+1) &= f_1(x_1(t), x_2(t)), \\ x_2(t+1) &= f_2(x_1(t), x_2(t)). \end{aligned}$$

To determine the linearised system around $\mathbf{x}_e = (0,0)$ for each value of a, we first need to determine the following partial derivatives:

$$\frac{\partial f_1}{\partial x_1} = (1+a)^2 - x_2^2, \qquad \frac{\partial f_1}{\partial x_2} = -2x_1x_2,$$
$$\frac{\partial f_2}{\partial x_1} = -2x_1x_2, \qquad \frac{\partial f_2}{\partial x_2} = (1-a^2) - x_1^2.$$

Therefore, the linearised model around $\mathbf{x}_e = (0,0)$ is

$$\frac{d}{dt}\Delta\mathbf{x}(t) = \begin{bmatrix} (1+a)^2 & 0 \\ 0 & 1-a^2 \end{bmatrix} \Delta\mathbf{x}(t).$$

iii) If we observe the matrix

$$F_{\mathbf{0},a} = \begin{bmatrix} (1+a)^2 & 0 \\ 0 & 1-a^2 \end{bmatrix}$$

of the linearised state-space model, we observe that its eigenvalues are $(1+a)^2$ and $1-a^2$. So, $\mathbf{x}_e = (0,0)$ is an asymptotically stable equilibrium point of the nonlinear state-space model if and only if both $|(1+a)^2| < 1$ and $|1-a^2| < 1$, which is possible if and only if $-\sqrt{2} < a < 0$. For $a > 0$ or $a < -\sqrt{2}$ the matrix $F_{\mathbf{0},a}$ has (at least) one eigenvalue with modulus greater than 1 and hence the origin is an unstable equilibrium point of the nonlinear state-space model. For $a = 0$ and $a = -\sqrt{2}$ the linearised method cannot be used to decide about the stability of the origin.

Exercise 5. i) The equilibrium points of the discrete time nonlinear autonomous state-space model are the solutions (x_{1e}, x_{2e}) of the following algebraic systems:

$$\begin{aligned} x_{1e} &= (1-a)x_{1e}, \\ x_{2e} &= x_{1e}^2 + (1-a^2)x_{2e}, \end{aligned}$$

which is equivalent to

$$\begin{aligned} 0 &= ax_{1e}, \\ x_{1e}^2 &= a^2 x_{2e}. \end{aligned}$$

If $a = 0$ then the solutions are $x_e = (0, x_{2e})$, with x_{2e} arbitrary in $\mathbb{R}$ (this implies that none of these points can be asymptotically stable, since in every neighbourhood of each of them there are always other equilibrium points). If $a \neq 0$, the only solution is $x_e = (0,0)$.

ii) The Iacobian of the function f that expresses $x(t+1)$ as a function of $x(t)$ is

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 1-a & 0 \\ 2x_1 & 1-a^2 \end{bmatrix}.$$

If we evaluate the Iacobian for $a = 0$ corresponding to any equilibrium point $x_e = (0, x_{2e})$, with x_{2e} arbitrary in $\mathbb{R}$, we get $F = I_2$, which is clearly a case undecidable by linearisation. If we evaluate the Iacobian for $a \neq 0$ corresponding to the equilibrium point $x_e = (0,0)$, we get

$$F = \begin{bmatrix} 1-a & 0 \\ 0 & 1-a^2 \end{bmatrix}$$

which is asymptotically stable if and only if both $|1-a| < 1$ and $|1-a^2| < 1$. This happens if and only if $0 < a < \sqrt{2}$. For $a = \sqrt{2}$ it is undecidable. For $a < 0$ and $a > \sqrt{2}$ the equilibrium is unstable.