Systems Theory Exercises - Reachability and Controllability – A.Y. 2025/26

Exercise 1. Consider the following discrete time model

$$x(t+1) = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(t), \qquad t \geq 0.$$

1) Study the system reachability and determine the set of reachable states in exactly one step, the set of reachable states in exactly two steps and, finally, the set of reachable states in exactly three steps.

2) Design, if possible, a control input $u(t)$ that brings the system from the state $x(0) = [1 \quad 0 \quad 1]^T$ to the state $x(2) = [0 \quad 1 \quad 8]^T$. If it is not possible, give an adequate explanation.

3) Design, if possible, a control input $u(t)$ that brings the system from the state $x(0) = [1 \quad 0 \quad 1]^T$ to the state $x(3) = [0 \quad 1 \quad 8]^T$. If it is not possible, give an adequate explanation.

Exercise 2. Consider the following discrete time model

$$x(t+1) = Fx(t) + gu(t) = \begin{bmatrix} 0 & 1 & 1 \\ -1/4 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t).$$

i) Determine the Jordan form of the matrix F and the elementary modes of the system specifying their behaviour (convergent, bounded, divergent);

ii) Determine the system reachability subspaces X_k^R, $k = 1, 2, \ldots$.

iii) Determine the system controllability subspaces X_k^C, $k = 1, 2, \ldots$.

iv) For each of the states $x_0 \neq 0$ in X^C determine a control law of minimum duration k that brings the system state from x_0 for $t = 0$ to the null state for $t = k$.

Exercise 3. Consider the following continuous time dynamic linear system:

$$\dot{x}(t) = Fx(t) + gu(t) = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t), \qquad t \geq 0.$$

i) Determine, if possible, a control input that brings the system state from $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ at the time $t = 0$ to $\begin{bmatrix} 2 \\ 2 + e^2 \end{bmatrix}$ at the time $t = 1$.

ii) Determine, if possible, a control input that brings the system state from $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ at the time $t = 0$ to $\begin{bmatrix} 2 \\ e^4 \end{bmatrix}$ at the time $t = 2$.

Exercise 4. Consider the following discrete time dynamic system:

$$x(t+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) = Fx(t) + gu(t),$$

that is supposed to be characterized by three distinct real eigenvalues $\lambda_1 > \lambda_2 > \lambda_3 > 0$.

i) Suppose to apply, starting from $x(0) = 0$, the input $u(0) = 1$, $u(t) = 0$ for $t > 0$, and determine the asymptotic vector

$$v := \lim_{t \to +\infty} \frac{x(t)}{\|x(t)\|}.$$

ii) Determine the input $u(\cdot)$ that allows to reach $x(t) = v$ starting from $x(0) = 0$ in the least possible number of steps.

iii) Generalize the previous result to a completely reachable generic system of dimension n, with only one input, by proving that if it is endowed with a dominant eigenvalue then v can be reached starting from 0 always in exactly n steps.

Exercise 5. Consider the discrete time system

$$\begin{aligned} x(t+1) &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(t), \\ y(t) &= [\,-1 \quad 1 \quad 1\,]\, x(t) \qquad\qquad t \geq 0. \end{aligned}$$

i) Determine, if possible, an input $u(\cdot)$, zero for $t \geq 2$, such that, by choosing $x(0) = 0$ one gets $y(t) = a2^t$, for every $t \geq 2$.

ii) Determine, as a varies in $\mathbb{R}$, if possible, a control input that brings the initial state $x(0) = [\,a \quad 1 \quad a(a-1)\,]^T$ to zero in the least possible number of steps.

Exercise 6. Consider the continuous time model

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \\ y(t) &= [\,2 \quad -1\,]\, x(t) \qquad\qquad t \geq 0 \\ x(0) &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} \end{aligned}$$

i) Compute an input $u(\cdot)$ that makes the state $x(0)$ at the time $t = 1$ equal to zero.

ii) Compute, among all the inputs able to make the output y at the time $t = 1$ equal to zero, the one that minimizes the following integral:

$$\int_0^1 u^2(t)dt.$$

Exercise 7. Consider the following discrete time model

$$\begin{aligned} x(t+1) &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(t), \qquad\qquad t \geq 0. \\ y(t) &= [1 \quad 1 \quad 1]\, x(t), \end{aligned}$$

i) Determine, if it exists, an input sequence $u(0), u(1)$ that brings the initial state $x(0) = [1 \quad 1 \quad 1]^T$ to the final state $x(2) = [0 \quad 0 \quad 8]^T$.

ii) Determine, if it exists, an input sequence $u(0), u(1), u(2)$ that brings the initial state $x(0) = [1 \quad 1 \quad 1]^T$ to the final state $x(3) = [0 \quad 0 \quad 8]^T$.

Exercise 8. Consider the following continuous time dynamic linear system:

$$\dot{x}(t) = Fx(t) + gu(t) = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t), \qquad\qquad t \geq 0.$$

i) Determine, if possible, a control input that brings the system state from $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ at the time $t = 0$ to $\begin{bmatrix} 2 \\ 2 + e^2 \end{bmatrix}$ at the time $t = 1$.

ii) Determine, if possible, a control input that brings the state of the system from $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ at the time $t = 0$ to $\begin{bmatrix} 2 \\ e^4 \end{bmatrix}$ at the time $t = 2$.

Exercise 9. Given the continuous time system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(t),$$

i) Study its reachability and controllability to zero;

ii) Determine, if it exists, an input signal that brings the system state from $\mathbf{x}(0) = [1 \quad 0 \quad 0]^T$ to $\mathbf{x}(1) = [1 \quad 0 \quad -1]^T$.

Exercise 10. Consider the discrete time system, with m inputs,

$$x(t+1) = Fx(t) + Gu(t), \qquad t \geq 0,$$

where

$$F = \begin{bmatrix} 3 & 1 & & & \\ & 3 & & & \\ & & 3 & & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}.$$

i) What is the minimum value of m such that the system turns out to be, for an appropriate choice of G, reachable? For such value, determine a matrix G satisfying such requirement.

ii) Given

$$G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

study the system reachability and controllability.

Exercise 11. Consider the continuous time system

$$\dot{\mathbf{x}}(t) = F\mathbf{x}(t) + G\mathbf{u}(t), \qquad t \geq 0,$$

with

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Design if possible, a control input $\mathbf{u}(t)$, of <u>minimum norm</u> that brings the system from the given initial state to the given final state.

$$\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{x}(1) = \begin{bmatrix} \frac{e^2+1}{2} \\ \frac{5-e^{-2}}{2} \\ 1 \end{bmatrix}.$$

If this is not possible, give an adequate explanation.

SOLUTIONS OF SOME EXERCISES

Exercise 1. 1) The one step reachability subspace is

$$X_1^R = \mathrm{Im} g = \mathrm{Im} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

therefore the set of reachable states in exactly one step is $X_1^R \setminus \{0\} = \{[a \;\; a \;\; 0]^T : a \neq 0\}$. The two steps reachability subspace is

$$X_2^R = \mathrm{Im}[g \mid Fg] = \mathrm{Im} \left[\begin{array}{c|c} 1 & 1 \\ 1 & 2 \\ 0 & 0 \end{array}\right],$$

therefore the set of states that are reachable in exactly two steps is $X_2^R \setminus X_1^R = \{[a \;\; b \;\; 0]^T : a \neq b\}$.

Since $X_3^R = X_2^R$, the set of states that are reachable in exactly three steps is the empty set. Moreover, since X_3^R does not coincide with $X = \mathbb{R}^3$, the system is not reachable.

2) and 3) Since the system is not reachable, a control input satisfying such requirements does not necessarily exist. Due to the structure of the matrices F and g it is clear that, regardless of the choice of the control input $u(t)$, it holds $x_3(t) = 2^t x_3(0)$. Therefore it is not possible to ensure that $x_3(2) = 2^2 x_3(0) = 4$ is equal to 8 in two steps, and hence the answer to the second question is negative. In $t = 3$ steps, this turns out to be possible. One needs then to check if it is possible to chose $u(0), u(1)$ and $u(2)$ such that

$$x(3) = F^3 x(0) + [\, g \;\mid\; Fg \;\mid\; F^2 g \,] \begin{bmatrix} u(2) \\ u(1) \\ u(0 \end{bmatrix}$$

coincides with the given value. Since the reachable subspace X^R coincides with the subspace of $\mathbb{R}^3$ generated by the canonical basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, and hence the components $x_1(3)$ and $x_2(3)$ can be freely controlled, the answer is positive. In detail, by replacing matrices and vectors with their numerical values, one gets

$$\begin{bmatrix} 0 \\ 1 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 14 \\ 3 & 1 & 1 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u(2) \\ u(1) \\ u(0 \end{bmatrix}.$$

From which one gets the system of equations:

$$\begin{bmatrix} -15 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u(2) \\ u(1) \\ u(0 \end{bmatrix},$$

whose solutions are

$$u(0) = u_0 \qquad u(1) = 12 - 2u_0 \qquad u(2) = u_0 - 27,$$

where u_0 is an arbitrary real number.

Exercise 2. i) The characteristic polynomial of the matrix F is

$$\Delta_F(z) = z\left(z^2 - z + \frac{1}{4}\right) = z\left(z - \frac{1}{2}\right)^2.$$

Which means that we have two distinct eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 1/2$, the first one with unitary multiplicity and the second one with multiplicity 2. Corresponding to the simple null eigenvalue we have a Jordan miniblock of dimension 1 and, as elementary mode, the discrete unitary impulse located in 0. To understand if two distinct modes or only one mode (and hence only one Jordan miniblock of dimension 2 or two Jordan miniblocks of dimension 1) are associated with the eigenvalue λ_2, one needs to check the geometric multiplicity of λ_2. If it is unitary, then to such eigenvalue one associates, in the Jordan form, only one miniblock of dimension 2 and hence two distinct modes:

$$\left\{\frac{1}{2^t}\right\}_{t\in\mathbb{Z}_+} \qquad \text{and} \qquad \left\{\binom{t}{1}\frac{1}{2^{t-1}}\right\}_{t\in\mathbb{Z}_+},$$

If instead, it is 2, then to such eigenvalue one associates, in the Jordan form, two miniblocks of dimension 2 and hence only one mode:

$$\left\{\frac{1}{2^t}\right\}_{t\in\mathbb{Z}_+}.$$

Since

$$\ker\left(\begin{bmatrix} 0 & 1 & 1 \\ -1/4 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{1}{2}I_3\right) = \ker\begin{bmatrix} -\frac{1}{2} & 1 & 1 \\ -1/4 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} = \langle \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \rangle,$$

it follows that the first case occurs and hence the overall system modes are

$$\delta(t), \qquad \left\{\frac{1}{2^t}\right\}_{t\in\mathbb{Z}_+} \qquad \text{e} \qquad \left\{\binom{t}{1}\frac{1}{2^{t-1}}\right\}_{t\in\mathbb{Z}_+},$$

and all of them are convergent. Therefore the Jordan form of the matrix F is

$$J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 1 \\ 0 & 0 & 1/2 \end{bmatrix}.$$

ii) One gets

$$\begin{aligned}
X_1^R &= \operatorname{Im} g = \operatorname{Im}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
X_2^R &= \operatorname{Im}[\, g \quad Fg\,] = \operatorname{Im}\begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{4} \\ 0 & 0 \end{bmatrix} = \langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rangle \\
X_3^R &= \operatorname{Im}[\, g \quad Fg \quad F^2 g\,] = \operatorname{Im}\begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix} = \langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rangle = X_2^R = X^R.
\end{aligned}$$

iii) One gets

$$\begin{aligned}
X_1^C &= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : \begin{bmatrix} 0 & 1 & 1 \\ -1/4 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in X_1^R \right\} \\
&= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : \begin{bmatrix} x_2 + x_3 \\ -\frac{1}{4}x_1 + x_2 \\ 0 \end{bmatrix} \in \operatorname{Im} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \\
&= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : -\frac{1}{4}x_1 + x_2 = 0 \right\} = \langle \begin{bmatrix} 1 \\ \frac{1}{4} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rangle \\
X_2^C &= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : \begin{bmatrix} 0 & 1 & 1 \\ -1/4 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}^2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in X_2^R \right\} \\
&= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : \begin{bmatrix} -\frac{1}{4} & 1 & 0 \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in X_2^R \right\} \\
&= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : \begin{bmatrix} x_2 - \frac{1}{4}x_1 \\ -\frac{1}{4}x_1 + \frac{3}{4}x_2 - \frac{1}{4}x_3 \\ 0 \end{bmatrix} \in \operatorname{Im} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \right\} = \mathbb{R}^3 \\
X_3^C &= X_2^C = X^C.
\end{aligned}$$

iv) Since $\mathbf{x}_0 \neq 0$ and $X^C = X_3^C = X_2^C$, it suffices to distinguish two cases: one for the states belonging to X_1^C and one for the states belonging to $X_2^C \setminus X_1^C$. If

$$\mathbf{x}_0 \in X_1^C = \langle \begin{bmatrix} 1 \\ \frac{1}{4} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rangle,$$

then it suffices to set the equation

$$0 = \begin{bmatrix} 0 & 1 & 1 \\ -1/4 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \frac{x_1}{4} \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(0) = \begin{bmatrix} \frac{x_1}{4} + x_3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(0),$$

from which it follows

$$u(0) = -\left(\frac{x_1}{4} + x_3 \right).$$

If, instead, $\mathbf{x}_0 \in X_2^C = \mathbb{R}^3$, then it suffices to set the equation

$$\begin{aligned}
0 &= \begin{bmatrix} 0 & 1 & 1 \\ -1/4 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}^2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{4} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} \\
&= \begin{bmatrix} x_2 - \frac{1}{4}x_1 \\ -\frac{1}{4}x_1 + \frac{3}{4}x_2 - \frac{1}{4}x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{4} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix}
\end{aligned}$$

from which it follows

$$u(0) = -x_1 + 3x_2 - x_3 \qquad u(1) = -\left(x_2 - \frac{1}{4}x_1\right).$$

Exercise 3. i) and ii) The system is not reachable. In fact, the reachability matrix is

$$\mathcal{R} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

and it has rank 1. In order to verify if there exists any input corresponding to the desired control, one needs to check if, for the given values of t, $x(t)$ and $x(0)$, condition $x(t) - e^{Ft}x(0) \in \mathrm{Im}\mathcal{R}$ holds. This test preliminary requires to find the general expression of the exponential matrix e^{Ft}, as t varies in $\mathbb{R}$. Since $e^{Ft} = \sum_{i=0}^{+\infty} F^i \frac{t^i}{i!}$, we first compute the generic expression of F^i. It is immediate to check that (it can be done, for instance, by induction)

$$F^i = \begin{bmatrix} 1 & 0 \\ -2^i + 1 & 2^i \end{bmatrix},$$

therefore

$$e^{Ft} = \sum_{i=0}^{+\infty} \begin{bmatrix} 1 & 0 \\ -2^i + 1 & 2^i \end{bmatrix} \frac{t^i}{i!} = \begin{bmatrix} \sum_{i=0}^{+\infty} \frac{t^i}{i!} & 0 \\ -\sum_{i=0}^{+\infty} 2^i \frac{t^i}{i!} + \sum_{i=0}^{+\infty} \frac{t^i}{i!} & \sum_{i=0}^{+\infty} 2^i \frac{t^i}{i!} \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ e^t - e^{2t} & e^{2t} \end{bmatrix}.$$

Then one notices that

$$x(1) - e^{F1}x(0) = \begin{bmatrix} 2 \\ 2 + e^2 \end{bmatrix} - \begin{bmatrix} e & 0 \\ e - e^2 & e^2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \in \mathrm{Im}\mathcal{R},$$

while

$$x(2) - e^{F2}x(0) = \begin{bmatrix} 2 \\ e^4 \end{bmatrix} - \begin{bmatrix} e^2 & 0 \\ e^2 - e^4 & e^4 - e^2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 - e^2 \\ e^4 - e^2 \end{bmatrix} \notin \mathrm{Im}\mathcal{R}.$$

To calculate the control input, in the first case, one resorts to the known formulas: let us evaluate a vector v such that

$$x(1) - e^{F1}x(0) = W_1 v,$$

where W_1 represents the reachability Gramian at time $t = 1$, thus we find as control input

$$u(t) = g^T e^{F^T(1-t)}v, \qquad t \in [0, 1].$$

The Gramian computation leads to the following result

$$\begin{aligned} W_1 &= \int_0^1 e^{F(1-\tau)} g g^T e^{F^T(1-\tau)} d\tau \\ &= \int_0^1 \begin{bmatrix} e^{(1-\tau)} & 0 \\ e^{(1-\tau)} - e^{2(1-\tau)} & e^{2(1-\tau)} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} e^{(1-\tau)} & e^{(1-\tau)} - e^{2(1-\tau)} \\ 0 & e^{2(1-\tau)} \end{bmatrix} d\tau \\ &= \int_0^1 \begin{bmatrix} e^{(1-\tau)} \\ e^{(1-\tau)} \end{bmatrix} \begin{bmatrix} e^{(1-\tau)} & e^{(1-\tau)} \end{bmatrix} d\tau = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \frac{e^2 - 1}{2}. \end{aligned}$$

It follows that $v = \frac{2}{e^2-1}\begin{bmatrix}1\\1\end{bmatrix}$ satisfies the previous equation, and hence

$$u(t) = [\,1 \quad 1\,]\begin{bmatrix} e^{(1-t)} & e^{(1-t)} - e^{2(1-t)} \\ 0 & e^{2(1-t)} \end{bmatrix}\frac{2}{e^2-1}\begin{bmatrix}1\\1\end{bmatrix} = [\,e^{(1-t)} \quad e^{(1-t)}\,]\,\frac{2}{e^2-1}\begin{bmatrix}1\\1\end{bmatrix} = \frac{4e^{(1-t)}}{e^2-1},$$

$t \in [0,1]$, is a control input satisfying the requirements.

Exercise 4. We preliminarily observe that the characteristic polynomial of the matrix F, in companion form, is

$$\Delta_F(z) = z^3 - cz^2 - bz - a,$$

and the assumption that F has eigenvalues $\lambda_1 > \lambda_2 > \lambda_3 > 0$ implies that

$$\Delta_F(\lambda_i) = \lambda_i^3 - c\lambda_i^2 - b\lambda_i - a = 0, \quad i = 1,2,3. \tag{1}$$

We also observe that by imposing $F\mathbf{v}_i = \lambda_i \mathbf{v}_i$ we obtain that eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ associated with the three distinct and real eigenvalues λ_1, λ_2 and λ_3, respectively, take the form

$$\mathbf{v}_i = [\,1 \quad \lambda_i \quad \lambda_i^2\,]^\top, \quad i = 1,2,3,$$

which is consistent with (1). Since these three eigenvectors are linearly independent, they represent a basis of $\mathbb{R}^3$.

i) If we apply, starting from $x(0) = 0$, the input $u(0) = 1$, $u(t) = 0$ for $t > 0$, we substantially impose $x(1) = \mathbf{e}_3$ and then we let the system evolve in unforced evolution. So, $x(t) = F^{t-1}x(1) = F^{t-1}\mathbf{e}_3$ for every $t \geq 1$ and the vector v represents the vector of unitary norm describing the long term behavior of the state trajectory. Each vector $x \in \mathbb{R}^3$, and hence in particular $x(1) = \mathbf{e}_3$, can be expressed as a linear combination of $\{v_1, v_2, v_3\}$. Therefore,

$$x(1) = \mathbf{e}_3 = c_1 v_1 + c_2 v_2 + c_3 v_3.$$

We let w_1 be a left eigenvector of F corresponding to λ_1, i.e. such that $w_1^\top F = \lambda_1 w_1^\top$. Then one can verify that

$$w_1^\top = [\,a\lambda_1 \quad a + b\lambda_1 \quad \lambda_1^2\,].$$

Moreover it is always true that $w_1 \perp v_2, v_3$ (since a left eigenvector corresponding to an eigenvalue is always orthogonal to the (right) eigenvectors corresponding to other eigenvalues), then

$$\lambda_1^2 = \langle w_1, \mathbf{e}_3 \rangle = c_1(w_1^\top v_1),$$

and hence $c_1 \neq 0$. But this implies that

$$\begin{aligned} x(t) &= F^{t-1}x(1) = F^{t-1}\mathbf{e}_3 = c_1 F^{t-1}v_1 + c_2 F^{t-1}v_2 + c_3 F^{t-1}v_3 = c_1\lambda_1^{t-1}v_1 + c_2\lambda_2^{t-1}v_2 + c_3\lambda_3^{t-1}v_3 \\ &= c_1\lambda_1^{t-1}\left[v_1 + \frac{c_2}{c_1}\left(\frac{\lambda_2}{\lambda_1}\right)^{t-1} v_2 + \frac{c_3}{c_1}\left(\frac{\lambda_3}{\lambda_1}\right)^{t-1} v_3\right] \end{aligned}$$

and hence

$$\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} c_1\lambda_1^{t-1}v_1.$$

Consequently

$$v := \lim_{t\to+\infty} \frac{x(t)}{\|x(t)\|} = \frac{v_1}{\|v_1\|}.$$

ii) We observe that the reachability matrix of the pair (F, g) is

$$\mathcal{R} = [\, g \quad Fg \quad F^2 g \,] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ 1 & c & b + c^2 \end{bmatrix},$$

and it is nonsingular, therefore v, having the first entry which is nonzero, can be reached in 3 steps but not later and the input sequence is obtained by solving

$$\begin{bmatrix} u(2) \\ u(1) \\ u(0) \end{bmatrix} = \mathcal{R}^{-1} \frac{v_1}{\|v_1\|},$$

namely

$$\begin{bmatrix} u(2) \\ u(1) \\ u(0) \end{bmatrix} = \begin{bmatrix} -b & -c & 1 \\ -c & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \lambda_1 \\ \lambda_1^2 \end{bmatrix} \frac{1}{\sqrt{1 + \lambda_1^2 + \lambda_1^4}}.$$

iii) For a generic n, we can repeat the reasoning at points i) and ii), with

$$\mathbf{v}_i = [\, 1 \quad \lambda_i \quad \dots \quad \lambda_i^{n-1} \,]^\top, \quad i = 1, 2 \dots, n,$$

$x(1) = g = \mathbf{e}_n$, $v = \frac{v_1}{\|v_1\|}$ and the least number of steps required to reach v is n. The input sequence is obtained by solving

$$\begin{bmatrix} u(n-1) \\ \vdots \\ u(1) \\ u(0) \end{bmatrix} = \mathcal{R}^{-1} \frac{v_1}{\|v_1\|}.$$

Exercise 5. i) Since we assume $x(0) = 0$, the expression of the generic state at the time $t = 2$ is

$$x(2) = [\, G \quad FG \,] \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} = \begin{bmatrix} u(1) \\ u(1) + 2u(0) \\ 0 \end{bmatrix}.$$

On the other hand, the fact that the input is zero from $t = 2$ onward ensures that the state at the generic time instant $t > 2$ is

$$x(t) = F^{t-2} x(2) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2^{t-2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u(1) \\ u(1) + 2u(0) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2^{t-2}[u(1) + 2u(0)] \\ 0 \end{bmatrix},$$

and thus the output is

$$\begin{aligned} y(2) &= Hx(2) = 2u(0), \\ y(t) &= Hx(t) = 2^{t-2}[u(1) + 2u(0)], \qquad t > 2. \end{aligned}$$

Therefore, to guarantee that there exists $a \in \mathbb{R}$ such that $y(t) = a2^t$, for any $t \geq 2$, it is necessary and sufficient to impose $u(1) = 0$. Hence, for any choice of the input sequence $u(0) \in \mathbb{R}, u(1) = 0$, one gets $y(t) = 2^{t-1}u(0) = \frac{u(0)}{2} \cdot 2^t, t \geq 2$.

An intuitive solution could have been obtained much faster: the expression imposed for the unforced evolution from $t = 2$ onward reveals that the only mode of the system output which is "visible" is the mode 2^t. Therefore, if we bring the state at the time $t = 2$ to an eigenvector of F related to $\lambda = 2$ and from that moment onward we leave it evolving in unforced evolution, for sure the system output takes the form $y(t) = a2^t$, for any $t \geq 2$. Hence we can impose $x(2) = [0\ 1\ 0]^\top$ that is clearly a reachable state in two steps by taking $u(0) = 1/2, u(1) = 0$, and from that moment onward the output unforced evolution has the desired expression.

ii) The system is not reachable. In fact it is in standard reachability form and the matrix F_{22} of the non-reachable subsystem is not even controllable to zero. Let us determine the controllability subspaces in k steps. One gets:

$$
\begin{aligned}
X_1^C &= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in X_1^R = \langle \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \rangle \right\} \\
&= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : \begin{bmatrix} 0 \\ 2x_2 \\ x_3 \end{bmatrix} \in \operatorname{Im} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\} \\
&= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : x_2 = x_3 = 0 \right\} = \langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rangle \\
X_2^C &= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2x_2 \\ x_3 \end{bmatrix} \in X_2^R \right\} \\
&= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : \begin{bmatrix} 0 \\ 4x_2 \\ x_3 \end{bmatrix} \in \langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rangle \right\} \\
&= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : x_3 = 0 \right\} = \langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rangle \\
X_3^C &= X_2^C = X^C.
\end{aligned}
$$

Obviously the initial state $x(0) = [\, a \quad 1 \quad a(a-1) \,]^T$ is controllable to zero if and only if $a(a-1) = 0$ namely if and only if either $a = 0$ or $a = 1$ and in both cases it is controllable to zero in 2 steps and not less than 2. For $a = 0$ one gets $x(0) = [\, 0 \quad 1 \quad 0 \,]^T$ and by imposing

$$
0 = x(2) = F^2 x(0) + [\, G \quad FG \,] \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix},
$$

one gets $u(1) = 0, u(0) = -2$.

For $a = 1$ one gets $x(0) = [1 \quad 1 \quad 0]^T$ and by imposing

$$0 = x(2) = F^2 x(0) + [G \quad FG] \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix},$$

one gets $u(1) = 0, u(0) = -2$ once again. (Notice that, in fact, even if the two initial states are distinct, the values of $F^2 x(0)$ coincide and hence the solution is the same).

Exercise 6. i) We note that

$$e^{Ft} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix},$$

and hence

$$x(1) - e^{F1} x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}.$$

On the other hand, the pair (F, g) is reachable since it is in controllable canonical form, and hence $x(1) - e^{F1} x(0) \in \mathrm{Im}\mathcal{R} = \mathbb{R}^2$. To solve the problem we evaluate the reachability Grammian at time $t = 1$ namely

$$\begin{aligned} W_1 &= \int_0^1 e^{F(1-\tau)} g g^\top e^{F^\top (1-\tau)} d\tau \\ &= \int_0^1 \begin{bmatrix} 1 & 1-\tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} [0 \quad 1] \begin{bmatrix} 1 & 0 \\ 1-\tau & 1 \end{bmatrix} d\tau \\ &= \begin{bmatrix} 1/3 & 1/2 \\ 1/2 & 1 \end{bmatrix}. \end{aligned}$$

Then one gets

$$\begin{bmatrix} -3 \\ -1 \end{bmatrix} = \mathbf{x}(1) - e^F \mathbf{x}(0) = W_1 \mathbf{v}_1 = \begin{bmatrix} 1/3 & 1/2 \\ 1/2 & 1 \end{bmatrix} \mathbf{v}_1,$$

from which

$$\mathbf{v}_1 = [-30 \quad 14]^\top.$$

Therefore an input $u(\cdot)$ that drives the state $x(0)$ at the time $t = 1$ to zero is

$$u(t) = g^\top e^{F^\top (1-t)} \mathbf{v}_1 = [0 \quad 1] \begin{bmatrix} 1 & 0 \\ 1-t & 1 \end{bmatrix} \begin{bmatrix} -30 \\ 14 \end{bmatrix} = -16 + 30t, t \in [0, 1].$$

ii) The states $x(1)$ such that $y(1) = Hx(1) = 0$ are all and only those expressed as $x(1) = [a \quad 2a]^\top$ for $a \in \mathbb{R}$. By mimiking the procedure in i), we can express the input that leads the state from $x(0)$ to $x(1) = [a \quad 2a]^\top$ as

$$u(t) = g^\top e^{F^\top (1-t)} W_1^{-1} [[a \quad 2a]^\top - e^{F1} x(0)] = (2a - 16) + 30t, \quad t \in [0, 1].$$

If we now evaluate the integral, we find

$$\int_0^1 u^2(t) dt = \int_0^1 ((2a - 16) + 30t)^2 \, dt = (2a - 1)^2 + 75.$$

So, it is clear that the minimum value is obtained for $a = 1/2$ (namely for $x(1) = [1/2 \;\; 1]^\top$) and it corresponds to the input

$$u(t) = -15 + 30t, \quad t \in [0, 1].$$

Exercise 8. We first note that

$$e^{Ft} = \begin{bmatrix} e^t & 0 \\ e^t - e^{2t} & e^{2t} \end{bmatrix}$$

and $\text{Im}\mathcal{R} = \langle \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rangle$.

i) If we evaluate

$$x(1) - e^F x(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

we immediately realise that $x(1) - e^F x(0) \in \text{Im}\mathcal{R}$ and hence the problem is solvable. In order to solve the problem we first consider the equation

$$\begin{aligned} x(1) - e^F x(0) &= \int_0^1 e^{F(1-t)} g u(t) dt = \int_0^1 \begin{bmatrix} e^{1-t} & 0 \\ e^{1-t} - e^{2(1-t)} & e^{2(1-t)} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) dt \\ &= \int_0^1 \begin{bmatrix} e^{1-t} \\ e^{1-t} \end{bmatrix} u(t) dt. \end{aligned}$$

Due to the simple form of the previous integral equation we can immediately see that the problem can be solved by adopting a constant input in $[0, 1]$. Indeed

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = x(1) - e^F x(0) = \int_0^1 \begin{bmatrix} e^{1-t} \\ e^{1-t} \end{bmatrix} dt \cdot \bar{u} = \int_0^1 \begin{bmatrix} e^t \\ e^t \end{bmatrix} dt \cdot \bar{u} = \begin{bmatrix} e - 1 \\ e - 1 \end{bmatrix} \cdot \bar{u}.$$

This leads to

$$u(t) = \bar{u} = \frac{2}{e - 1}, \qquad t \in [0, 1].$$

Alternatively, we can evaluate the reachability Grammian at time $t = 1$ namely

$$\begin{aligned} W_1 &= \int_0^1 e^{F(1-\tau)} g g^\top e^{F^\top(1-\tau)} d\tau \\ &= \int_0^1 \begin{bmatrix} e^{1-t} & 0 \\ e^{1-t} - e^{2(1-t)} & e^{2(1-t)} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} [1 \quad 1] \begin{bmatrix} e^{1-t} & e^{1-t} - e^{2(1-t)} \\ 0 & e^{2(1-t)} \end{bmatrix} d\tau \\ &= \int_0^1 \begin{bmatrix} e^t \\ e^t \end{bmatrix} [e^t \quad e^t] \, dt \\ &= \int_0^1 e^{2t} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} dt = \frac{e^2 - 1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \end{aligned}$$

Clearly, W_1 is singular, since the system is not reachable. We find one possible solution of the equation

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = x(1) - e^F x(0) = W_1 v_1 = \frac{e^2 - 1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} v_1,$$

for instance

$$v_1 = \frac{4}{e^2-1}\begin{bmatrix}1\\0\end{bmatrix}.$$

Therefore an input $u(\cdot)$ that solves the given problem is

$$u(t) = g^\top e^{F^\top(1-t)}\mathbf{v}_1 = [\,1 \quad 1\,]\begin{bmatrix}e^{1-t} & e^{1-t}-e^{2(1-t)}\\ 0 & e^{2(1-t)}\end{bmatrix}\begin{bmatrix}\frac{4}{e^2-1}\\ 0\end{bmatrix} = \frac{4e^{1-t}}{e^2-1}, t \in [0,1].$$

ii) To check whether this second problem is solvable, we need to evaluate if

$$x(1) - e^F x(0) = \begin{bmatrix}2-e^2\\ e^4-e^2\end{bmatrix} \in \mathrm{Im}\mathcal{R},$$

and the answer is clearly negative. Therefore, this problem has no solution.

Exercise 11. We preliminary observe that the k-th power of the matrix F, for $k \geq 1$, is:

$$F^k = \begin{bmatrix}1 & 0 & 0\\ 0 & (-1)^k & -(-1)^k\\ 0 & 0 & 0\end{bmatrix}$$

and hence

$$e^{Ft} = \begin{bmatrix}e^t & 0 & 0\\ 0 & e^{-t} & 1-e^{-t}\\ 0 & 0 & 1\end{bmatrix}.$$

The problem is solvable if and only if

$$\mathbf{x}(1) - e^F\mathbf{x}(0) = \begin{bmatrix}\frac{e^2+1}{2}\\ \frac{5-e^{-2}}{2}\\ 1\end{bmatrix} - \begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}\frac{e^2+1}{2}\\ \frac{3-e^{-2}}{2}\\ 0\end{bmatrix} \in \mathrm{Im}\,[\,G \quad FG \quad F^2G\,] = \langle \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \rangle.$$

Clearly the problem admits solution. To determine the minimum norm solution one computes the reachability Gramian at the time $t = 1$, namely

$$\begin{aligned}
W_1 &= \int_0^1 e^{F(1-\tau)}GG^\top e^{F^\top(1-\tau)}d\tau\\
&= \int_0^1 \begin{bmatrix}e^{1-\tau} & 0 & 0\\ 0 & e^{-1+\tau} & 1-e^{-1+\tau}\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}[\,1 \quad 1 \quad 0\,]\begin{bmatrix}e^{1-\tau} & 0 & 0\\ 0 & e^{-1+\tau} & 0\\ 0 & 1-e^{-1+\tau} & 1\end{bmatrix}d\tau\\
&= \int_0^1 \begin{bmatrix}e^{2-2\tau} & 1 & 0\\ 1 & e^{-2+2\tau} & 0\\ 0 & 0 & 0\end{bmatrix}d\tau = \begin{bmatrix}\frac{e^2-1}{2} & 1 & 0\\ 1 & \frac{1-e^{-2}}{2} & 0\\ 0 & 0 & 0\end{bmatrix}.
\end{aligned}$$

Then one gets

$$\begin{bmatrix}\frac{e^2+1}{2}\\ \frac{3-e^{-2}}{2}\\ 0\end{bmatrix} = \mathbf{x}(1) - e^F\mathbf{x}(0) = W_1\mathbf{v}_1 = \begin{bmatrix}\frac{e^2-1}{2} & 1 & 0\\ 1 & \frac{1-e^{-2}}{2} & 0\\ 0 & 0 & 0\end{bmatrix}.\mathbf{v}_1,$$

from which

$$\mathbf{v}_1 = [\,1 \quad 1 \quad c\,]^\top,$$

with $c \in \mathbb{R}$, arbitrary. Therefore the minimum norm solution is:

$$u(t) = G^\top e^{F^\top(1-t)}\mathbf{v}_1 = [\,1 \quad 1 \quad 0\,]\begin{bmatrix}e^{1-t} & 0 & 0\\ 0 & e^{-1+t} & 0\\ 0 & 1-e^{-1+t} & 1\end{bmatrix}\begin{bmatrix}1\\1\\c\end{bmatrix} = e^{1-t} + e^{-1+t}, t \in [0,1].$$