Systems Theory Exercises - State Feedback Control
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Exercise 1. Consider the following single input (discrete time) state space models, and design (if possible) a stabilizing state feedback control law that attributes to the closed-loop system the given characteristic polynomial:

(a) $F = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -1 \\ 2 & 1 & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad p(z) = \left(z + \frac{1}{2}\right)^3;$

(b) $F = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -1 \\ 2 & 1 & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad p(z) = \left(z - \frac{1}{2}\right) z^2;$

(c) $F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad p(z) = z^3 - z^2 + \frac{1}{4}z.$

Exercise 2. Consider the following (continuous time) state space models with two inputs, and design (if possible) a stabilizing state feedback control law that attributes to the closed-loop system the given characteristic polynomial:

(a) $F = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -1 \\ 2 & 1 & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \qquad p(s) = (s+1)^3;$

(b) $F = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}, \qquad G = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad p(s) = \left(s + \frac{1}{2}\right)^2 (s+1).$

Exercise 3. Consider the following discrete time system

$$\begin{aligned} x(t+1) &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -\frac{3}{4} & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(t). \end{aligned}$$

(1) Design, if possible, a matrix K such that $F + GK$ has $(z - \frac{1}{2})^4$ as characteristic polynomial.

(2) Design, if possible, an output feedback control law $u(t) = Ky(t)$ such that the characteristic polynomial of the closed loop system is $(z - \frac{1}{2})^4$.

Exercise 4. Consider the continuous time system described by the following equations:

$$\dot{x}(t) = Fx(t) + Gu(t) = \begin{bmatrix} a+1 & 0 & 0 \\ -1 & 1 & -a \\ 0 & a & -a^2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u(t), \qquad t \geq 0,$$

where a is a real parameter.

i) Study, as a varies in $\mathbb{R}$, the system stability and specify its modes and their behaviour;

ii) for $a = 0$ design, if possible, a stabilizing controller for the system;

iii) for $a = -1$ design, if possible, a controller acting only on the first input, such that the closed-loop system exhibits only the modes 1 and e^{-t}.

Exercise 5. Consider the following continuous time dynamic linear system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u(t) = Fx(t) + Gu(t).$$

i) Design, if possible, a feedback control law from the first input that attributes to the closed-loop system the characteristic polynomial s^3.

ii) Design, if possible, a feedback control law from the second input that attributes to the closed-loop system the characteristic polynomial s^3.

iii) Design, if possible, a feedback control that attributes to the closed-loop system the characteristic polynomial $(s+2)s^2$ and the minimal polynomial $(s+2)s$.

The answers must be provided with an adequate explanation.

Exercise 6. Given the system

$$\dot{x}(t) = Fx(t) + Gu(t), \quad \text{with } F = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ and } G = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

let Σ_K be the closed-loop system obtained corresponding to the state feedback law of matrix K. Design, if possible, feedback matrices K (one for each of the following requests) such that Σ_K fulfils the following constraints:

i) Σ_K has just periodic modes;

ii) for every initial condition $x(0)$, the unforced state evolution of Σ_K is entirely contained in a straight line passing through the origin;

iii) the system Σ_K has the property that, if $x(0) \in \langle e_1, e_2 \rangle$, then the unforced state evolution is a linear combination of the modes 1 and e^t only, while if $x(0) \in \langle e_3 \rangle$ then the unforced state evolution is of the type $x(t) = x(0)e^{-t}$ (Recall that e_i represents the i-th vector of the canonical basis, whose components are all zero except for the i-th one that is unitary).

Exercise 7. Consider the following discrete time system

$$\begin{aligned} x(t+1) &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -\frac{3}{4} & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(t). \end{aligned}$$

(1) Design, if possible, a matrix K such that $F + GK$ has $(z - \frac{1}{2})^4$ as minimal polynomial.

(2) Design, if possible, an output feedback control $u(t) = Ky(t)$ such that the minimal polynomial of the closed-loop system is $(z - \frac{1}{2})^4$.

The answers must be provided with an adequate explanation.

Exercise 8. Consider the continuous time system described by the following equations:

$$
\begin{aligned}
\dot{x}(t) &= Fx(t) + [\, g_1 \mid g_2 \,]\, u(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \\
y(t) &= Hx(t) = [\, 1 \quad 2 \quad 1 \,]\, x(t).
\end{aligned}
$$

i) Design, if possible, a feedback control law from the first input such that the closed-loop system has all and only the following modes:

$$e^{-t}, te^{-t}, t^2e^{-t}.$$

ii) Design, if possible, a feedback control law from the first input such that the closed-loop system has all and only the following modes:

$$e^{-t}, te^{-t}.$$

iii) Design, if possible, a feedback control law such that the closed-loop system has all and only the following modes:

$$e^{-t}, te^{-t}.$$

The answers must be provided with an adequate explanation.

SOLUTIONS OF SOME EXERCISES

Exercise 1. In the following it is assumed

$$K = [\, a \quad b \quad c \,]\,,$$

with a, b, c real parameter to be suitably chosen.

(a) The examined pair is not reachable since rank$\mathcal{R} = 2$. The characteristic polynomial of the matrix F is

$$\Delta_F(z) = \left(z + \frac{1}{2}\right)[(z-1)z+1] = \left(z + \frac{1}{2}\right)(z^2 - z + 1).$$

The only possibility for the problem to be solvable is that the only eigenvalue located in $-\frac{1}{2}$ is the eigenvalue of the non-reachable subsystem (if there were no eigenvalues located in $-\frac{1}{2}$, due to the non-reachability of the system, the problem could not be solvable). The computation of the PBH reachability matrix at $z = -\frac{1}{2}$ confirms such hypothesis, and hence the problem is solvable. Among all the possible strategies, the simplest one consists of imposing on the matrix $F + GK$ the following structure

$$F + GK = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 1+b & -1+c \\ 2 & 1+b & c \end{bmatrix}$$

with the sub-matrix 2×2

$$\begin{bmatrix} 1+b & -1+c \\ 1+b & c \end{bmatrix}$$

with characteristic polynomial

$$z^2 - (1+b+c)z + (1+b) = \left(z + \frac{1}{2}\right)^2 = z^2 + z + \frac{1}{4}.$$

This corresponds to the choice $b = -3/4, c = -5/4$ and leads to the feedback matrix

$$K = [\, 0 \quad -3/4 \quad -5/4 \,]\,.$$

However the number of controllers solving this problem is infinite. Such controllers are all and only the ones expressed in the form

$$K = [\, a \quad -3/4 \quad -5/4 \,]\,,$$

as a varies in $\mathbb{R}$.

(b) The examined system is the same as the one of the previous point, for which we have already checked that the only eigenvalue of the non-reachable subsystem is $-\frac{1}{2}$. Therefore, although the system is stabilizable, it is not possible to attribute to it the given characteristic polynomial.

(c) It is immediate to notice that the pair is in standard reachability form and that the non-reachable subsystem matrix is $F_{22} = 0$. By reasoning on the reachable subsystem

$$F_{11} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \qquad G_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

we can choose $K_1 = [\, a \quad b\,]$ such that the matrix in companion form

$$F_{11} + G_1 K_1 = \begin{bmatrix} 0 & 1 \\ a & 1+b \end{bmatrix},$$

has characteristic polynomial $\Delta_{F_{11}+G_1K_1}(z) = z^2 - z + \frac{1}{4}$. In this way, one gets

$$K_1 = [\, -\tfrac{1}{4} \quad 0\,]$$

and hence

$$K = [\, -\tfrac{1}{4} \quad 0 \quad c\,],$$

with c arbitrary in $\mathbb{R}$.

Exercise 2. In the following it is assumed

$$K = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix},$$

with $a_i, b_i, c_i, i = 1, 2$, real parameters to be suitably chosen.

(a) The pair (F, G) is reachable, thus the problem is solvable. By writing the matrix $F + GK$ in parametric form one gets:

$$F + GK = \begin{bmatrix} -\frac{1}{2} + a_1 & b_1 & c_1 \\ a_2 & 1 + b_2 & -1 + c_2 \\ 2 + a_2 & 1 + b_2 & c_2 \end{bmatrix}.$$

A block triangular structure can be imposed on $F + GK$, by assuming $b_1 = c_1 = a_2 = 0$. In this way one gets

$$F + GK = \begin{bmatrix} -\frac{1}{2} + a_1 & 0 & 0 \\ 0 & 1 + b_2 & -1 + c_2 \\ 2 & 1 + b_2 & c_2 \end{bmatrix}.$$

At this point it is sufficient to impose:

– $\left(-\frac{1}{2} + a_1\right) = -1$, namely $a_1 = -\frac{1}{2}$;

– the characteristic polynomial of $\begin{bmatrix} 1 + b_2 & -1 + c_2 \\ 1 + b_2 & c_2 \end{bmatrix}$ must be $(s + 1)^2$,

so that one obtains the desired results. Therefore one gets $b_2 = 0$ and $c_2 = -3$, from which it follows

$$K = \begin{bmatrix} -1/2 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

(b) In this case the system turns out to be non-reachable and the reachable subsystem has dimension 2, hence it is necessary to evaluate what is the eigenvalue of the non-reachable subsystem. The structure of the matrices F and G allows to immediately notice that the eigenvalue of the matrix F_{22} is $-1/2$. Therefore the desired feedback control exists. The matrix $F + GK$ is of the type

$$F + GK = \begin{bmatrix} 1 + a_1 & b_1 & -1 + c_1 \\ 0 & -\frac{1}{2} & 0 \\ \frac{1}{2} + a_2 & b_2 & \frac{1}{2} + c_2 \end{bmatrix}.$$

It is immediate to notice that a simple way to solve this problem is to impose

$$F + GK = \begin{bmatrix} 1 + a_1 & b_1 & -1 + c_1 \\ 0 & -\frac{1}{2} & 0 \\ \frac{1}{2} + a_2 & b_2 & \frac{1}{2} + c_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

This corresponds to choose

$$K = \begin{bmatrix} -\frac{3}{2} & 0 & 1 \\ \\ -\frac{1}{2} & 0 & -\frac{3}{2} \end{bmatrix}.$$

Exercise 3. i) We preliminary observe that the direct computation of the reachability matrix $\mathcal{R}$ leads to deduce that its rank is 4 and thus the problem is solvable. By assuming

$$K = [\, k_0 \quad k_1 \quad k_2 \quad k_3 \,]$$

one can write $F + gK$ in the form

$$F + gK = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 + k_0 & -\frac{3}{4} + k_1 & k_2 & k_3 \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}$$

and set it equal to the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/8 & -\frac{3}{4} & \frac{3}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}$$

having $(z - 1/2)^4$ as characteristic polynomial. This solution corresponds to

$$K = [\, -7/8 \quad 0 \quad 3/2 \quad 0 \,]$$

and it is also the only possible solution.

ii) The output feedback leads to a closed-loop system assuming the following structure: $F + g\tilde{K}H$. Since at the previous point we have found the only possible solution to the problem, now we need to understand if the matrix K can be expressed in the form $\tilde{K}H$ for an appropriate $\tilde{K}$. If $\tilde{K} = [\, a \quad b \,]$, then

$$\tilde{K}H = [\, a \quad 0 \quad b \quad 0 \,]$$

and imposing

$$K = [\, -7/8 \quad 0 \quad 3/2 \quad 0 \,] = [\, a \quad 0 \quad b \quad 0 \,]$$

one gets $\tilde{K} = [\, -7/8 \quad 3/2 \,]$.

Exercise 4. i) The matrix F has a block triangular structure and its eigenvalues are $\sigma(F) = \{a+1, 1-a^2, 0\}$. One needs to evaluate for which values of the parameter a the matrix F has non simple eigenvalues and one gets:

- if $a = -1$: $a + 1 = 1 - a^2 = 0 \Rightarrow \sigma(F) = \{0, 0, 0\}$;
  the geometric multiplicity of the eigenvalue 0 is 1, namely there is only one miniblock in the Jordan form of the matrix F; the modes turns out to be $m_1(t) = 1$, bounded, $m_2(t) = t$ and $m_3(t) = t^2$, divergent; the system turns out to be unstable;

- if $a = 0$: $a + 1 = 1 - a^2 \Rightarrow \sigma(F) = \{1, 1, 0\}$;
  the geometric multiplicity of the eigenvalue 1 is 1, i.e. there is only one miniblock in the Jordan form of the matrix F; the modes turns out to be $m_1(t) = 1$, bounded, $m_2(t) = e^t$ and $m_3(t) = te^t$, divergent; the system turns out to be unstable;

- if $a = +1$: $1 - a^2 = 0 \Rightarrow \sigma(F) = \{0, 0, 2\}$;
  the geometric multiplicity of the eigenvalue 0 is 1, i.e. there is only one miniblock in the Jordan form of the matrix F; the modes turns out to be $m_1(t) = 1$, bounded, $m_2(t) = t$ and $m_3(t) = e^{2t}$, divergent; the system turns out to be unstable;

- otherwise: $\sigma(F) = \{a+1, 1-a^2, 0\}$;
  the eigenvalues are distinct and the modes are $m_1(t) = e^{(a+1)t}$, convergent for $a < -1$ (divergent otherwise), $m_2(t) = e^{(1-a^2)t}$, convergent for $a < -1$ or $a > 1$ (divergent otherwise), and $m_3(t) = 1$, bounded; one gets simple stability for $a < -1$, instability otherwise.

ii) For $a = 0$ the state matrix turns out to be $F = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, that has as spectrum $\sigma(F) = \{1, 1, 0\}$. The stability requirement for the system hence implies the allocation of all three eigenvalues of F.
The computation of the reachability matrix leads to:

$$\mathcal{R} = [\, G \quad FG \quad F^2G \,] = \left[\begin{array}{cc|cc|cc} 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 2 & 0 & 3 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

that is of full rank; notice that instead, the reachability matrix from the first or second input is not full rank. Therefore, the existence of a stabilizing control is guaranteed only if both inputs are considered.
By posing

$$K = \begin{bmatrix} k_1 & k_2 & k_3 \\ k_4 & k_5 & k_6 \end{bmatrix},$$

the closed-loop system matrix turns out to be:

$$F + GK = \begin{bmatrix} 1-k_4 & -k_5 & -k_6 \\ -1+k_4 & 1+k_5 & k_6 \\ k_1 & k_2 & k_3 \end{bmatrix}.$$

By giving a structure to the matrix one can get a block diagonal matrix by posing $k_1 = k_2 = k_6 = 0$, that represents a 2×2 matrix with two degrees of freedom (k_4 and k_5) and a matrix 1×1 with one degree of freedom (k_3): therefore the problem turns out to be solvable. One chooses for example as characteristic polynomial $p(s) = (s+1)^3$, from which it follows $k_3 = -1$, $k_4 = 0$ and $k_5 = -4$, and the overall K matrix is equal to:

$$K = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -4 & 0 \end{bmatrix}.$$

iii) For $a = -1$ the state matrix turns out to be $F = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix}$, having as spectrum $\sigma(F) = \{0, 0, 0\}$.
The computation of the reachability matrix from the first input, g_1, leads to:

$$\mathcal{R}_1 = [\, g_1 \quad Fg_1 \quad F^2g_1 \,] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix},$$

that has rank 2. Certainly $\lambda = 0$ is a non-reachable eigenvalue, that is compatible with the requirement of a constant mode. Since the other two eigenvalues can be arbitrary located, the problem is solvable.
By posing

$$K = [\, k_1 \quad k_2 \quad k_3 \,],$$

the closed-loop state matrix:

$$F + g_1K = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ k_1 & -1+k_2 & -1+k_3 \end{bmatrix} = \begin{bmatrix} F_{11} & 0 \\ F_{21} & F_{22} \end{bmatrix},$$

is lower block triangular, with F_{11} of dimension 1 and F_{22} of dimension 2.
The characteristic polynomials compatible with the problem requirements are:

- $p(s) = s(s+1)^2$: the matrix F_{22} must have as characteristic polynomial $(s+1)^2$, that is obtained for $k_2 = -3$ and $k_3 = -2$; the geometric multiplicity of the eigenvalue -1 turns out to be 1 for any value of the free parameter k_1: it follows that the minimal polynomial is $\psi(s) = p(s)$ and it is not possible to fulfil the requirements on the modes;
- $p(s) = s^2(s+1)$: the matrix F_{22} must have as characteristic polynomial $s(s+1)$, that is obtained for $k_2 = k_3 = -1$; the geometric multiplicity of the eigenvalue 0 turns out to be 2 if $k_1 = 2$: in this case, the minimal polynomial is $\psi(s) = s(s+1)$ and the modes of the system are the required ones.

Therefore, the feedback matrix is:

$$K = [\,2 \quad -1 \quad -1\,].$$

Exercise 5. i) By observing the pair (F, g_1), where g_1 is the first column of G, it is immediate to notice that it is in standard reachability form. Therefore, the non-reachable subsystem matrix, F_{22}, coincides with -1. Since the eigenvalues of the non-reachable system are not modified by the state feedback, it is evident that it does not exist a feedback control that allows to place all the system eigenvalues in 0.

ii) Let us consider the pair (F, g_2), where g_2 is the second column of G. Since the reachability matrix of such pair is

$$\mathcal{R}_2 = [\,g_2 \quad Fg_2 \quad F^2g_2\,] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix},$$

the system is reachable, therefore the answer is certainly positive. The simplest way to solve the problem is to assume

$$K = \begin{bmatrix} 0 & 0 & 0 \\ k_0 & k_1 & k_2 \end{bmatrix},$$

to write $\det(sI_3 - F - GK)$ in parametric form and to impose it to be equal to s^3. By following this procedure one gets $\det(sI_3 - F - gK) = s^3 - k_2 s^2 + (k_2 - k_1 - 1)s - k_0$, and it coincides with s^3 if and only if $k_0 = 0$, $k_1 = -1$ e $k_2 = 0$, from which it follows

$$K = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

iii) It is immediate to verify that the pair (F, G) is in multivariable controllable canonical form, hence in order to get the desired result it is sufficient to choose K such that $F + GK$ is block diagonal with the first block (in companion form) of dimension 2 and characteristic polynomial $s(s+2)$ and the second block, of dimension 1, coinciding with 0. Therefore, by writing

$$K = \begin{bmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{bmatrix},$$

and imposing

$$F + GK = \begin{bmatrix} 0 & 1 & 0 \\ a_0 & a_1 + 1 & a_2 + 1 \\ b_0 & b_1 & b_2 - 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

one gets

$$K = \begin{bmatrix} 0 & -3 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Exercise 6. i) Since the system has dimension 3 the only way to attribute it 3 periodic modes is to attribute it two sinusoidal modes and one constant mode. This is equivalent to attribute to the matrix $F + GK$ of the closed loop system the eigenvalues $0, +j\omega, -j\omega$. Let us take, for instance $\sigma(F + GK) = (0, +j, -j)$. The system is reachable, therefore the problem is certainly solvable. Taken into account that for a generic feedback matrix

$$K = \begin{bmatrix} k_1 & k_2 & k_3 \\ k_4 & k_5 & k_6 \end{bmatrix}$$

the matrix $F + GK$ takes the form

$$F + GK = \begin{bmatrix} 0 & 1 & 0 \\ 1 + k_1 & 2 + k_2 & 3 + k_3 \\ 4 + k_4 & 5 + k_5 & 6 + k_6 \end{bmatrix},$$

it is immediate to notice that by imposing

$$F + GK = \begin{bmatrix} 0 & 1 & 0 \\ 1 + k_1 & 2 + k_2 & 3 + k_3 \\ 4 + k_4 & 5 + k_5 & 6 + k_6 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

corresponding to

$$K = \begin{bmatrix} -2 & -2 & -3 \\ -4 & -5 & -6 \end{bmatrix},$$

one gets the desired result. Another solution (among the infinite ones available) consists of attributing to $F + GK$ the structure of a matrix in companion form with characteristic polynomial $s(s^2 + 1)$.

ii) We preliminary observe that requiring that for a *specific* initial state the corresponding state evolution is contained in a straight line passing through the origin is equivalent to require that for any $t \geq 0$ the vector $\dot{x}(t)$ and the vector $x(t)$ are parallel. This implies, in particular, that $\dot{x}(0) = (F + GK)x(0)$ must be parallel to $x(0)$, but this is possible if and only if $x(0)$ is an eigenvector of $F + GK$ related to a real eigenvalue, and if this occurs the condition keeps holding for all the time instants $t \geq 0$.

In order to make this condition satisfied for *every* initial condition it is necessary and sufficient that every state $x(0) \in \mathbb{R}^3$ is an eigenvector of $F + GK$. It is immediate to notice that (1) if $F + GK$ had two distinct eigenvalues then the linear combination of two eigenvectors related to two distinct eigenvalues would not give an eigenvector, therefore all the eigenvalues must be equal ($F + GK$ must have only one eigenvalue of geometric multiplicity $n = 3$); (2) if such eigenvalue had geometric multiplicity smaller than $n = 3$ then there would exist vectors that are generalized eigenvectors and not eigenvectors. But then the matrix $F + GK$ has all vectors as eigenvectors if and only if its Jordan form is a "scalar" matrix λI_3. And since it is immediate to check that a matrix similar to a scalar matrix is already a scalar matrix itself, it follows that $F + GK = \lambda I_3$.

Without resorting to Rosenbrock's theorem (that would lead to the same conclusions) it is immediate to notice, from the structure of the matrix $F + GK$, corresponding to the generic K matrix, that $F + GK$ never coincides with a scalar matrix.

iii) The constraint that if $x(0) \in \langle e_3 \rangle$ then the unforced state evolution is of the type $x(t) = x(0)e^{-t}$, implies that the third column of $F + GK$ must be $-\mathbf{e}_3$. The constraint that if $x(0) \in \langle e_1, e_2 \rangle$ then the unforced state evolution is linear combination of the modes 1 and e^t only, can be fulfilled by imposing

$$F + GK = \begin{bmatrix} F_{11} & 0 \\ 0 & -1 \end{bmatrix},$$

with $\sigma(F_{11}) = (0, 1)$. Then, if we impose

$$F + GK = \begin{bmatrix} 0 & 1 & 0 \\ 1 + k_1 & 2 + k_2 & 3 + k_3 \\ 4 + k_4 & 5 + k_5 & 6 + k_6 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

corresponding to

$$K = \begin{bmatrix} -1 & -1 & -3 \\ -4 & -5 & -7 \end{bmatrix},$$

we get the desired result.

Exercise 7. (1) We note that the pair (F, g) is reachable. Indeed the reachability matrix is

$$\mathcal{R} = [\, g \quad Fg \quad F^2 g \quad F^3 g \,] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3/4 \\ 1 & 0 & -3/4 & 1 \\ 0 & 1 & 1/2 & -1/2 \end{bmatrix}$$

and it is nonsingular. Therefore there exists a matrix $K \in \mathbb{R}^{1\times 4}$ such that the characteristic polynomial of $F + gK$ is $(z - \frac{1}{2})^4$. On the other hand, since also the pair $(F + gK, g)$ is reachable (from a single input), the matrix $F+gK$ is cyclic and this ensures that also the minimal polynomial of $F + gK$ will coincide with $(z - \frac{1}{2})^4$. If we assume $K = [\, a \quad b \quad c \quad d \,]$ then

$$F + gK = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a+1 & b - \frac{3}{4} & c & d \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}.$$

It is just a matter of calculations to verify that there is only one possible solution, namely

$$K = [\, -7/8 \quad 0 \quad 3/2 \quad 0 \,].$$

(2) We note that if $K = [\, a \quad b \,]$ is the matrix involved in the output feedback control $u(t) = Ky(t)$, then it corresponds to the state feedback matrix $\tilde{K} = KH = [\, a \quad 0 \quad b \quad 0 \,]$. Since the matrix obtained in part (1) has exactly this same structure, by imposing

$$\tilde{K} = KH = [\, a \quad 0 \quad b \quad 0 \,] = [\, -7/8 \quad 0 \quad 3/2 \quad 0 \,],$$

we obtain that the output feedback matrix

$$K = [\, -7/8 \quad 3/2 \,]$$

allows to obtain the desired minimal polynomial.

Exercise 8. i) Since the pair (F, g_1) is clearly a controllable canonical form, it is possible to attribute to the system matrix of the closed-loop system obtained through a feedback from the first input, an arbitrary polynomial, in this case $(s + 1)^3 = s^3 + 3s^2 + 3s + 1$. Since the resulting matrix will necessarily be in companion form, and hence, cyclic, the minimal polynomial of the resulting matrix will coincide with the characteristic polynomial and hence the system modes will coincide with the required ones. The matrix k_1 such that $F + g_1 k_1$ has characteristic and minimal polynomial equal to $(s + 1)^3$ is obtained by imposing

$$F + g_1 k_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}.$$

Therefore, one gets $k_1 = [\, -2 \quad -3 \quad -2 \,]$ and, hence

$$K = \begin{bmatrix} -2 & -3 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

ii) By resorting to the same reasoning as in point 1), the matrix $F + g_1 k_1$ must necessarily be cyclic, therefore it is not possible to attribute to such matrix the minimal polynomial $(s + 1)^2$, having degree strictly smaller than the one of the characteristic polynomial.

iii) Being the system reachable from the first input, obviously it is reachable from both inputs. The computation of the control invariants leads to $k_1 = 2$ and $k_2 = 1$, therefore, as consequence of Rosenbrock's theorem, the problem is solvable. The solution is obtained by imposing $\psi_1(s) = (s+1)^2$ and $\psi_2(s) = s+1$. Since the system is in multivariable controllable canonical form it suffices to attribute to the matrix $F + GK$ the following form:

$$F + GK = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

This is possible by choosing

$$K = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}.$$