Quantum Ethics

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A Spinozist Interpretation of Quantum Field Theory

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Preface

The birth of Quantum Physics in the 1920s has been marked by a long period of intense controversies about its interpretation, which has been recently reviewed by Juan Miguel Marin in his paper *'Mysticism' in quantum mechanics: the forgotten controversy* [11]. The Copenhagen interpretation which emerged from these debates has been seemingly dominating the scene in the 1950s-1960s, not because it was intrinsically better than others, but because it seemed to challenge the materialist world view of classical, 'everyday' physics as little as possible. Since the 1970s, however, numerous alternative interpretations have been proposed and older interpretations further developed: Pilot-Wave Theory, Many-Worlds interpretation, Many-Minds interpretation, Dynamical Collapse Theories, Decoherent Histories... All these attempts to give Quantum Physics a sound interpretation are facing the problem that the mathematical theory itself, in the form of the Standard Model of Quantum Field Theory (or of slight variants regarding the existence of the Higgs field, of neutrino masses...), is ill-defined, and that it is therefore impossible to assign a physical or metaphysical meaning to the fundamental mathematical entities of the theory. Of course, it is possible to choose a specific, mathematically well-defined regularization of the theory for this purpose, but since renormalization methods are leading, in the singular limit of the original theory, to the same results for different regularization schemes, we don't know which regularization is the right one, and as a consequence we don't know either which are the fundamental mathematical entities to be interpreted. Quite surprisingly, however, it seems that this issue has never been seriously accounted for yet. Existing interpretations of Quantum Physics are either restricted to non-relativistic Quantum Mechanics, which is no fundamental theory, or are formulated so vaguely that they are hardly more than the mere idea of an interpretation, which has made most 'serious' physicists doubt that such an interpretation is possible at all. This book will have reached his goal if it convinces the reader of the contrary and helps interpretation issues recovering again their place at the heart of the research on Quantum Field Theory. For this purpose, I shall take as an example a lattice regularization scheme, formulate in this well-defined framework a rather conservative interpretation inspired by Spinoza's philosophy and show that classical philosophical questions can be formulated as simple physical hypotheses in the frame of the resulting naturalistic metaphysics.

Introduction

Motivation

Most philosophers have ascribed a central role to the ethics in their work, as the answer to the question "How should I live?" requires a preliminary reflection about all the fields of our existence, from metaphysics via physics, psychology and morals up to politics. Up to the emergence of Quantum Field Theory in the late 1920s, philosophers have always been able to integrate the knowledge gathered in the field of physics into their world view: In 17th-century Europe, for instance, the Dutch philosopher Baruch Spinoza worked out in his *Ethics* [14] the deterministic materialism of classical mechanics and based his philosophy on the idea that everything in Nature happens according to the divine necessity, both at the material and at the spiritual level. In fact, from the Antics up to the Age of Enlightenment, physicists used to consider themselves primarily as Nature philosophers. In the modern ages, however, the scientific community began to split under the influence of industrial work organization into small groups of specialists lacking interdisciplinary skills. Nowadays, mainstream physicists even consider philosophical interpretations of physics as non-scientific and pointless. Needless to say, such a lack of intellectual rigor has had serious consequences for the conceptual and formal quality of physical theories. In the case of Quantum Field Theory, this attitude has resulted in the fact that, for the last eighty years, no consensus could be reached on its two major issues, known as the *measurement problem* and as the *main issue*. The latter is a formal issue consisting in conceiving a mathematically well-defined quantum field theory formally compatible to Special Relativity*, which is thought to be impossible, although this hasn't been proved definitely yet. The former is an interpretation issue concerning the relation between "mind", i.e. the form of our experience of the world, and "body", i.e. the physical world described in terms of quantum fields. There have been numerous propositions for this interpretation, from the very beginning of Quantum Field Theory in the 1920s and 1930s until now, but as long as the mathematical formalism of the theory is ill-defined, these interpretations cannot be formulated precisely either. Strictly speaking, however, the whole theory doesn't make any sense if this interpretation issue doesn't become precisely answered. In order to give a sound philosophical interpretation of Quantum Field Theory, it is therefore necessary in the first place to put the mathematical formalism on a well-defined basis, which cannot be formally compatible with Special Relativity if the main issue really cannot be solved; Relativity Theory should then emerge as an approximation at usual energy and distance scales. In this book, I shall propose an answer to both issues and thus lay the foundations of an ethic taking into account the world view sustained by Quantum Field Theory.

Abstract

In the formulation of Quantum Field Theory proposed in this book, Nature presents the two aspects of a physical and of a mental world in mutual interaction. The mental world can be adequately experienced by the collectivity of all sentient beings: A state of the mental world is given by a set of a various number of individual minds with definite conscious thoughts. Though we are experiencing this mental world directly, we only experience it partially under the aspect of a single individual mind (which we're used to call "our" mind), and must communicate with others in order to get closer to an adequate representation of this mental world. Communication happens over social interactions in the physical world, an aspect of Nature that we don't experience directly but only through its influence on our subjective mental experience. This physical world, best described in terms of quantum fields, is by nature holistic and doesn't involve precise boundaries of individual bodies as they exist for individual minds. A state of this physical world is given by a quantum superposition of so-called localized states, which are given by the number of particles of each kind (e.g. 1 spin down positron and 4 right-handed circularly polarized photons) at each point of space. Each physical state can be uniquely decomposed into a sum of components corresponding to each possible mental state, and this decomposition also defines a probability law on the set of all possible mental states. The joined temporal evolution of both aspects of Nature is a tree-steps process repeated indefinitely: First, the initial state of the physical world undergoes a deterministic, Hamiltonian evolution of a given, "elementary" duration. Then, the final physical state defines a probability law according to which a mental state is being selected and becomes collectively experienced. Finally, the component of the physical state corresponding to the selected mind state becomes the initial state of the next evolution process. In this world view, the mystery of consciousness consists in the fact that there is, to some ex-

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*Precisely, one requires that the classical Lagrangian used in the heuristical construction of the theory be Poincaré invariant and describes local (contact) interactions of point particles in the Minkowski space-time.

tent, an adequation between the conscious thoughts of individual minds and some physical processes happening in the corresponding physical states, e.g. the biological processes of consciousness within a human brain.

Overview

This book begins in chapter 1 with the formulation of a mathematically well-defined frame for any theory of mutually interacting quantum fields of point particles. Well-definedness is achieved by making sure that the Hamilton space of the quantum states is finite dimensional, so that the Hamiltonian evolution is trivially well-defined for any interaction Hamiltonian. The "ingredients" of this mathematical frame are already well-known in Quantum Field Theory: Space is supposed to have the structure of a finite three dimensional lattice, the definition of the kinetic energy Hamiltonian makes use of the SLAC derivative and the occupation number of single modes of the particle fields is supposed to be bounded for Bosons, too.

The Hamiltonian evolution of quantum fields is then defined very classically for an arbitrary interaction Hamiltonian in chapter 2, where general results of scattering theory are being derived.

A general model for the mental world is defined in chapter 3 and the joint stochastic evolution of the physical and mental worlds in chapter 4. The basic idea of this model – that "mind causes collapse" – isn't new, as it has first been formulated by John von Neumann [12] and was once knew as the "standard interpretation" of Quantum Mechanics. As far as I know, however, it is the first time with this book that a precise interpretation of a mathematically well-defined Quantum Field Theory has ever been given. This provides thus the first sound basis for a discussion of the philosophical implications of the theory, which is the main goal of this book.

The interaction Hamiltonian of Quantum Electrodynamics is then defined in chapter 7 and, as an example, the semi-classical cross-section of Coulomb scattering is calculated to the leading order in chapter 9.

Finally, some usual mathematical functions, notations and operators are being defined in the appendix.

Part I

Physical world

Chapter 1

Quantum fields

> The first simplification to be considered involves the very existence of the theory.
>
> John Collins,
> *Renormalization* [3]

1.1 Abstract

The aim of this chapter is to develop a well-defined, divergence-free mathematical formalism for the Standard Model of particle physics. To achieve this, we suppose that elementary particles are bounded to a finite set of world lines in the flat space-time (so that the particle field only has a finite number of modes), and that there is a maximum occupation number for a single mode of the field for bosons as well as for fermions, so that the Hilbert space of the states of the universe is finite dimensional. We will first develop a general formalism, valid for any set of elementary particles and for any form of the interaction Hamiltonian, and define the notations used in the rest of this paper.

1.2 Space

DEFINITION Space is a finite set of points of the form $[\![-\mathrm{N}, \mathrm{N}]\!]^3$, where the physical constant N is a positive integer.

REMARKS This constant is supposed to be a "huge" integer ($\gtrsim 10^{46}$) which hasn't been measured experimentally yet. The finiteness of space is one of the conditions of the finite dimensionality of the Hilbert space of the physical states, which will be defined in section 1.6. This is in turn a necessary condition of the well-definedness of the evolution equation 2.1 for an arbitrary Hamiltonian operator. It is therefore a theoretical necessity, which I shall assume although this fact hasn't been proved experimentally yet.

COMMENTARIES No notion of distance emerges from this definition of space. Indeed, according to the ideas developed in Einstein's vulgarization work *Relativity: The Special and General Theory* [5], we consider that distance and duration are actually no fundamental notions but have to be defined on an empirical basis. Distance and duration are measured using physical apparatus like rods or clocks, and their theoretical definition must rely on a theoretical modeling of these apparatus and of the observer making use of them. These concepts will emerge from the evolution equation 2.1 and from the expression of the Hamiltonian operator defined in sections 2.2 and 7.6. According to this expression, we will see that space has a toroidal structure, i.e. that opposite points on the boundary of the lattice $[\![-\mathrm{N},\mathrm{N}]\!]^3$ are actually nearest neighbors. This boundary is also a mere artifact, like the boundary of a world map, and doesn't represent in any way the "frontier of the universe". The physical constant a in the expression of the interaction Hamiltonian plays the role of the lattice step, i.e. of the distance between nearest neighbors. It is supposed to be very small ($\lesssim 10^{-20}$ m) and hasn't been measured experimentally yet.

COMPLEMENTS We could equivalently postulate that, in the Minkowski space-time $(\mathcal{E}, \boldsymbol{g})$, defined by:

$$\begin{aligned} \mathcal{E} &:= \mathbb{R}^4 \\ \boldsymbol{g} &:= diag(1,-1,-1,-1) \\ \boldsymbol{x}\cdot\boldsymbol{y} &:= g_{\mu\nu}x^\mu y^\nu \end{aligned}$$

elementary particles cannot occupy an arbitrary point of space but are bounded to a finite set of $(1+2\mathrm{N})^3$ world lines $\boldsymbol{x_n}$ forming in some reference frame a finite lattice of step a:

$$\begin{aligned} \boldsymbol{x_n}(\tau) &:= \begin{pmatrix} \mathrm{c}\tau \\ \mathrm{a}\boldsymbol{n} \end{pmatrix} \\ \boldsymbol{n} &\in [\![-\mathrm{N},\mathrm{N}]\!]^3 \end{aligned}$$

In a reference frame moving with a velocity $\boldsymbol{v}$ relative to the lattice, the space-time coordinates of these world lines would be given (up to a translation of the origin) by:

$$\begin{aligned} \boldsymbol{x}'_{\boldsymbol{n}}(t) &= \begin{pmatrix} \mathrm{c}t \\ \mathrm{a}\boldsymbol{n}_\perp + \gamma^{-1}\mathrm{a}\boldsymbol{n}_\parallel - \boldsymbol{v}t \end{pmatrix} \\ \gamma &:= \frac{1}{\sqrt{1-(v/\mathrm{c})^2}} \end{aligned}$$

where we use the notations $\boldsymbol{n}_\parallel := (\boldsymbol{n}\cdot\boldsymbol{v})\boldsymbol{v}/v^2$ and $\boldsymbol{n}_\perp := \boldsymbol{n} - \boldsymbol{n}_\parallel$.

The lattice reference frame itself as well as the physical constants N and a are free parameters of the theory. As a working hypothesis, we will assume that the lattice reference frame corresponds to a rest frame of the cosmic microwave background radiation. The relative velocity $\boldsymbol{v}$ of the sun relative to the lattice would then verify [9]:

$$v \approx 3.7\ 10^5\ \mathrm{m/s}$$

We will also assume that the lattice step is of the order of the Plank length:

$$\mathrm{a} \sim \sqrt{4\pi \mathrm{Gh}/\mathrm{c}^3} \approx 1.4\ 10^{-34}\ \mathrm{m}$$

and that the lattice size is of the order of the Hubble length [7]:

$$\begin{aligned} (1+2\mathrm{N})\mathrm{a} &\sim \mathrm{R_H} \approx 1.3\ 10^{26}\ \mathrm{m} \\ \mathrm{N} &\sim 4.6\ 10^{59} \end{aligned}$$

Incidentally, with this values, the cosmological constant of the Λ-Cold Dark Matter model of Big-Bang cosmology coincides numerically (with a relative error of only 8%) with [10]:

$$\rho_{vac} \sim 2\mathrm{N}\frac{\mathrm{hc}}{\mathrm{a}}\left((1+2\mathrm{N})\mathrm{a}\right)^{-3} \approx 5.6\ 10^{-10}\ \mathrm{J/m^3}$$

Deriving such a relation, however, isn't the goal of this paper.

1.3 One particle states

Definition The (hypothetical) physical state in which the universe only contains a single particle, of type ϕ, at point $\boldsymbol{n}$ and in the spin state λ, is written:

$$|\Psi\rangle = \left|1^{\phi}_{\boldsymbol{n},\lambda}\right\rangle$$

We postulate that a one particle state is given by any linear combination of the form:

$$|\Psi\rangle = \sum_{\phi,\boldsymbol{n},\lambda} \Psi(1^{\phi}_{\boldsymbol{n},\lambda})\ \left|1^{\phi}_{\boldsymbol{n},\lambda}\right\rangle$$

The set of all these vectors forms a finite dimensional Hilbert space given by:

$$\mathcal{H}_1 := \bigoplus_{\phi,\boldsymbol{n},\lambda}^{\perp} \mathbb{C}\ \left|1^{\phi}_{\boldsymbol{n},\lambda}\right\rangle$$

Momentum representation We postulate that the momentum $\boldsymbol{p}$ of a particle in the lattice reference frame can only take values of the form:

$$\begin{aligned} \boldsymbol{p} &= \frac{\mathrm{h}}{\mathrm{a}}\boldsymbol{q} \\ \boldsymbol{q} &\in \left(\frac{[\![-\mathrm{N},\mathrm{N}]\!]}{1+2\mathrm{N}}\right)^3 \end{aligned}$$

and that the (hypothetical) physical state in which the universe only contains a single particle, of type ϕ, in the spin state λ with the momentum $\mathrm{h}\boldsymbol{q}/\mathrm{a}$ in the lattice reference frame, is given by:

$$\left|1^{\phi}_{\boldsymbol{q},\lambda}\right\rangle := (1+2\mathrm{N})^{-3/2}\sum_{\boldsymbol{n}}\exp\left(\mathrm{i}2\pi\boldsymbol{n}\cdot\boldsymbol{q}\right)\,\left|1^{\phi}_{\boldsymbol{n},\lambda}\right\rangle$$

These vectors form an orthogonal basis of the Hilbert space $\mathcal{H}_1$ and we will use the notation:

$$|\Psi\rangle = \sum_{\phi,\boldsymbol{q},\lambda}\widetilde{\Psi}(1^{\phi}_{\boldsymbol{q},\lambda})\,\left|1^{\phi}_{\boldsymbol{q},\lambda}\right\rangle$$

In order to simplify the notations, when defining and using periodical functions on all $\boldsymbol{q}\in\mathbb{R}^3$, we will define $\underline{\boldsymbol{q}}\in\left]-\frac{1}{2},\frac{1}{2}\right]^3$ by the equivalence relation $\underline{\boldsymbol{q}}-\boldsymbol{q}\in\mathbb{Z}^3$. We have then in particular $\underline{\boldsymbol{q}}\in\left(\frac{[\![-\mathrm{N},\mathrm{N}]\!]}{1+2\mathrm{N}}\right)^3$ for all $\boldsymbol{q}\in\left(\frac{\mathbb{Z}}{1+2\mathrm{N}}\right)^3$.

1.4 Position and momentum operators

Definition In the lattice reference frame, we define on $\mathcal{H}_1$ the position and momentum operators by:

$$\begin{aligned}\widehat{\boldsymbol{r}}\left|1^{\phi}_{\boldsymbol{n},\lambda}\right\rangle &:= \mathrm{a}\boldsymbol{n}\left|1^{\phi}_{\boldsymbol{n},\lambda}\right\rangle\\ \widehat{\boldsymbol{p}}\left|1^{\phi}_{\boldsymbol{q},\lambda}\right\rangle &:= \frac{\mathrm{h}}{\mathrm{a}}\boldsymbol{q}\left|1^{\phi}_{\boldsymbol{q},\lambda}\right\rangle\end{aligned}$$

Remark This definition of the momentum operator follows the same principle as the SLAC derivative [13], but can be expressed as a proper eigenvalue equation, since momentum eigenstates are well-defined on a finite lattice.

Complements In another reference frame, moving with a velocity $\boldsymbol{v}$ relative to the lattice, these operators are given (up to a translation of the origin) by:

$$\begin{aligned}\widehat{\boldsymbol{r}}\left|1^{\phi}_{\boldsymbol{n},\lambda}\right\rangle &:= \left(\mathrm{a}\boldsymbol{n}_\perp+\gamma^{-1}\mathrm{a}\boldsymbol{n}_\parallel-\boldsymbol{v}t\right)\left|1^{\phi}_{\boldsymbol{n},\lambda}\right\rangle\\ \widehat{\boldsymbol{p}}\left|1^{\phi}_{\boldsymbol{q},\lambda}\right\rangle &:= \left(\frac{\mathrm{h}}{\mathrm{a}}\boldsymbol{q}_\perp+\gamma\frac{\mathrm{h}}{\mathrm{a}}\boldsymbol{q}_\parallel-\gamma\frac{E^{\phi}_{\boldsymbol{q}}}{\mathrm{c}^2}\boldsymbol{v}\right)\left|1^{\phi}_{\boldsymbol{q},\lambda}\right\rangle\end{aligned}$$

where $E^{\phi}_{\boldsymbol{q}}$ is the kinetic energy of the particle in the lattice reference frame, defined as a function of its (bare) rest mass m_ϕ by:

$$E^{\phi}_{\boldsymbol{q}} := \sqrt{(\mathrm{m}_\phi\mathrm{c}^2)^2+\left(\frac{\mathrm{hc}}{\mathrm{a}}\underline{\boldsymbol{q}}\right)^2}$$

Similarly, we define the relativistic parameters $\boldsymbol{\beta}^{\phi}_{\boldsymbol{q}}$ and $\gamma^{\phi}_{\boldsymbol{q}}$ by:

$$\boldsymbol{\beta}^{\phi}_{\boldsymbol{q}} := \frac{\underline{\boldsymbol{q}}}{\sqrt{(\mathrm{m}_{\phi}\mathrm{ac}/\mathrm{h})^2 + \underline{\boldsymbol{q}}^2}}$$

$$\gamma^{\phi}_{\boldsymbol{q}} := \sqrt{1 + \left(\frac{\mathrm{h}\underline{\boldsymbol{q}}}{\mathrm{m}_{\phi}\mathrm{ac}}\right)^2}$$

as well as the velocity by $\boldsymbol{v}^{\phi}_{\boldsymbol{q}} := \boldsymbol{\beta}^{\phi}_{\boldsymbol{q}}\mathrm{c}$.

1.5 Wave function

We can associate following wave function components to each one particle state:

$$\Psi^{\phi}_{\lambda}(\boldsymbol{x}) := (1+2\mathrm{N})^{-3/2}\sum_{\boldsymbol{q}} \widetilde{\Psi}(1^{\phi}_{\boldsymbol{q},\lambda})\exp\left(\mathrm{i}2\pi\frac{\boldsymbol{x}\cdot\boldsymbol{q}}{\mathrm{a}}\right)$$

Eigenstates of the momentum operator are thus associated with plane waves on $\mathbb{R}^3$. Equivalently, we can write:

$$\begin{aligned} \Psi^{\phi}_{\lambda}(\boldsymbol{x}) &= \sum_{\boldsymbol{n}} \Psi(1^{\phi}_{\boldsymbol{n},\lambda})\delta_{\boldsymbol{x}}(\boldsymbol{x} - \mathrm{a}\boldsymbol{n}) \\ \delta_{\boldsymbol{x}}(\boldsymbol{x}) &:= (1+2\mathrm{N})^{-3}\prod_{i} \frac{\sin(\pi x_i/\mathrm{a})}{\sin(\pi x_i/(1+2\mathrm{N})\mathrm{a})} \end{aligned}$$

We define thus an isomorphism between a finite set, indexed on (ϕ, λ), of complementary subspaces of $\mathcal{H}_1$, and a finite dimentional subspace of $\mathrm{C}^{\infty}\left(\mathbb{R}^3, \mathbb{C}\right)$ containing functions of period $(1+2\mathrm{N})\mathrm{a}$ along each coordinate. In that space, the (image of the) momentum operator acts according to:

$$\widehat{\boldsymbol{p}}\Psi^{\phi}_{\lambda}(\boldsymbol{x}) = \frac{\mathrm{h}}{\mathrm{i}2\pi}\boldsymbol{\nabla}\Psi^{\phi}_{\lambda}(\boldsymbol{x})$$

The dynamic of the free fields on the lattice is also identical to the usual dynamic of the free fields on the continuum in the box $\left]-(\mathrm{N}+\frac{1}{2})\mathrm{a}, (\mathrm{N}+\frac{1}{2})\mathrm{a}\right[^3$ with periodical boundary conditions.

1.6 Many particles states

The physical state in which each point $\boldsymbol{n}$ is being occupied by $N^{\phi}_{\boldsymbol{n},\lambda}$ particles of each type ϕ in each spin state λ is written:

$$|\Psi\rangle = \left|(N^{\phi}_{\boldsymbol{n},\lambda})\right\rangle$$

and is called a "localized state". We postulate that a many particles state is given by any linear combination of the form:

$$\begin{aligned}|\Psi\rangle &= \sum_{(N^{\phi}_{\boldsymbol{n},\lambda})} \Psi\left((N^{\phi}_{\boldsymbol{n},\lambda})\right) \left|(N^{\phi}_{\boldsymbol{n},\lambda})\right\rangle \\ N^{\phi}_{\boldsymbol{n},\lambda} &\in [\![0, \mathrm{M}_{\phi}]\!]\end{aligned}$$

where M_{ϕ} is the maximum occupation number of the field ϕ. For fermions, we have experimentally $\mathrm{M}_{\phi} = 1$. For bosons, no upper limit of the occupation number is experimentally known; a lower limit of about $\mathrm{M}_{\gamma} \gtrsim 10^{21}$ for photons has been reached experimentally by high intensity lasers.

The set of all these vectors forms a finite dimensional Hilbert space given by:

$$\mathcal{H} := \bigoplus_{(N^{\phi}_{\boldsymbol{n},\lambda})}^{\perp} \mathbb{C} \left|(N^{\phi}_{\boldsymbol{n},\lambda})\right\rangle$$

and the basis of the localized states is called "position basis".

1.7 Creation and annihilation operators

The annihilation operators are defined by:

$$\widehat{a^{\phi}}_{\boldsymbol{n},\lambda} \left|(N^{\phi}_{\boldsymbol{n},\lambda})\right\rangle := \begin{cases}\left|(N^{\phi}_{\boldsymbol{n},\lambda}) - 1^{\phi}_{\boldsymbol{n},\lambda}\right\rangle & \text{if } N^{\phi}_{\boldsymbol{n},\lambda} > 0 \\ 0 & \text{otherwise}\end{cases}$$

and the creation operators by:

$$\widehat{a^{\phi}}^{\dagger}_{\boldsymbol{n},\lambda} \left|(N^{\phi}_{\boldsymbol{n},\lambda})\right\rangle := \begin{cases}\left|(N^{\phi}_{\boldsymbol{n},\lambda}) + 1^{\phi}_{\boldsymbol{n},\lambda}\right\rangle & \text{if } N^{\phi}_{\boldsymbol{n},\lambda} < \mathrm{M}_{\phi} \\ 0 & \text{otherwise}\end{cases}$$

The (hypothetical) state of the universe in which no particles are present is written:

$$|\Psi\rangle = |\Omega\rangle := \left|(0^{\phi}_{\boldsymbol{n},\lambda})\right\rangle$$

The annihilation (resp. creation) operators form a (finite) set of generators of a commutative algebra $\mathcal{A}$ (resp. $\mathcal{A}^{\dagger}$). Any state of the universe can be obtained by applying creation operators on the vacuum according to:

$$\begin{aligned}|\Psi\rangle &= \widehat{\Psi}^{\dagger} \, |\Omega\rangle \\ \widehat{\Psi}^{\dagger} &:= \sum_{(N^{\phi}_{\boldsymbol{n},\lambda})} \Psi\left((N^{\phi}_{\boldsymbol{n},\lambda})\right) \prod_{\phi,\boldsymbol{n},\lambda} \left(\widehat{a^{\phi}}^{\dagger}_{\boldsymbol{n},\lambda}\right)^{N^{\phi}_{\boldsymbol{n},\lambda}}\end{aligned}$$

associating thus an operator $\widehat{\Psi}^{\dagger} \in \mathcal{A}^{\dagger}$ to each vector $|\Psi\rangle \in \mathcal{H}$ canonically.

1.8 Plane wave field modes

Creation and annihilation operators can also be defined for the plane wave modes of the field by:

$$\begin{aligned}
\widehat{a^\phi}_{\boldsymbol{q},\lambda} &:= (1+2\mathrm{N})^{-3/2} \sum_{\boldsymbol{n}} \exp\left(-\mathrm{i}2\pi \boldsymbol{n}\cdot\boldsymbol{q}\right) \widehat{a^\phi}_{\boldsymbol{n},\lambda} \\
\widehat{a^\phi}^\dagger_{\boldsymbol{q},\lambda} &:= (1+2\mathrm{N})^{-3/2} \sum_{\boldsymbol{n}} \exp\left(\mathrm{i}2\pi \boldsymbol{n}\cdot\boldsymbol{q}\right) \widehat{a^\phi}^\dagger_{\boldsymbol{n},\lambda}
\end{aligned}$$

Note that this definition can be extended to all $\boldsymbol{q} \in \mathbb{R}^3$. The plane wave states of the field are then defined by:

$$\left|(N^\phi_{\boldsymbol{q},\lambda})\right\rangle := \prod_{\phi,\boldsymbol{q},\lambda} \left(\widehat{a^\phi}^\dagger_{\boldsymbol{q},\lambda}\right)^{N^\phi_{\boldsymbol{q},\lambda}} |\Omega\rangle$$

These vectors form an orthogonal basis of the Hilbert space $\mathcal{H}$ and we will use the notation:

$$|\Psi\rangle = \sum_{(N^\phi_{\boldsymbol{q},\lambda})} \widetilde{\Psi}\left((N^\phi_{\boldsymbol{q},\lambda})\right) \left|(N^\phi_{\boldsymbol{q},\lambda})\right\rangle$$

The decomposition of the plane wave state $\left|(N'^\phi_{\boldsymbol{q},\lambda})\right\rangle$ on the position basis is given by:

$$\begin{aligned}
\left\langle (N^\phi_{\boldsymbol{n},\lambda}) | (N'^\phi_{\boldsymbol{q},\lambda}) \right\rangle &= \left[\prod_{\phi,\lambda} \delta\left(N'^\phi_\lambda - N^\phi_\lambda\right)\right] \psi\left((\boldsymbol{q}^{\phi,\lambda}_j), (\boldsymbol{n}^{\phi,\lambda}_j)\right) \\
\psi\left((\boldsymbol{q}^{\phi,\lambda}_j), (\boldsymbol{n}^{\phi,\lambda}_j)\right) &:= \prod_{\substack{\phi,\lambda \\ N^\phi_\lambda \neq 0}} \frac{(1+2\mathrm{N})^{-3N^\phi_\lambda/2}}{\prod_{\boldsymbol{n}} N^\phi_{\boldsymbol{n},\lambda}!} \sum_{\sigma\in\mathfrak{S}_{N^\phi_\lambda}} \prod_{j=1}^{N^\phi_\lambda} \exp\left(\mathrm{i}2\pi \boldsymbol{n}^{\phi,\lambda}_{\sigma_j} \cdot \boldsymbol{q}^{\phi,\lambda}_j\right)
\end{aligned}$$

where we use the notations $N'^\phi_\lambda := \sum_{\boldsymbol{q}} N'^\phi_{\boldsymbol{q},\lambda}$ and $N^\phi_\lambda := \sum_{\boldsymbol{n}} N^\phi_{\boldsymbol{n},\lambda}$, where $\mathfrak{S}_{N^\phi_\lambda}$ denotes the symmetric group of order N^ϕ_λ and where we have chosen for each mode (ϕ,λ) of the field the families $(\boldsymbol{n}^{\phi,\lambda}_j)$ and $(\boldsymbol{q}^{\phi,\lambda}_j)$ such as:

$$\begin{aligned}
\left|(N^\phi_{\boldsymbol{n},\lambda})\right\rangle &= \prod_{\phi,\lambda,j} \widehat{a^\phi}^\dagger_{\boldsymbol{n}^{\phi,\lambda}_j,\lambda} |\Omega\rangle \\
\left|(N'^\phi_{\boldsymbol{q},\lambda})\right\rangle &= \prod_{\phi,\lambda,j} \widehat{a^\phi}^\dagger_{\boldsymbol{q}^{\phi,\lambda}_j,\lambda} |\Omega\rangle
\end{aligned}$$

In the definition of $\psi\left((\boldsymbol{q}^{\phi,\lambda}_j), (\boldsymbol{n}^{\phi,\lambda}_j)\right)$, we used for convenience the symbols $N^\phi_{\boldsymbol{n},\lambda}$ and N^ϕ_λ, which can be defined as a function of $(\boldsymbol{n}^{\phi,\lambda}_j)$ with $N^\phi_{\boldsymbol{n},\lambda} := \left|\{j|\boldsymbol{n}^{\phi,\lambda}_j = \boldsymbol{n}\}\right|$.

1.9 Particle number operators

The particle number operators are defined by:

$$\begin{aligned}
\widehat{N^\phi}_{\boldsymbol{n},\lambda} \left|(N^\phi_{\boldsymbol{n},\lambda})\right\rangle &:= N^\phi_{\boldsymbol{n},\lambda} \left|(N^\phi_{\boldsymbol{n},\lambda})\right\rangle \\
\widehat{N^\phi}_{\boldsymbol{q},\lambda} \left|(N^\phi_{\boldsymbol{q},\lambda})\right\rangle &:= N^\phi_{\boldsymbol{q},\lambda} \left|(N^\phi_{\boldsymbol{q},\lambda})\right\rangle
\end{aligned}$$

The total particle number operator is defined as the (finite) sum:

$$\widehat{N} := \sum_{\phi,\boldsymbol{n},\lambda} \widehat{N^\phi}_{\boldsymbol{n},\lambda} = \sum_{\phi,\boldsymbol{q},\lambda} \widehat{N^\phi}_{\boldsymbol{q},\lambda}$$

Its eigenspace for the eigenvalue N is written $\mathcal{H}_N$ and its elements are called "N particle states" of the field. The Hilbert space can be decomposed as a (finite) sum of the form:

$$\mathcal{H} = \bigoplus_N^{\perp} \mathcal{H}_N$$

Chapter 2

Hamiltonian evolution

2.1 Schrödinger equation

We postulate that the state of the quantum field evolves according to an equation of the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\ |\Psi\rangle = -\mathrm{i}2\pi\frac{1}{\mathrm{h}}\widehat{\mathrm{H}}\ |\Psi\rangle$$

called "Schrödinger equation" where $\widehat{\mathrm{H}}$ is the (time independent) Hamiltonian operator of the field. This operator is supposed to be hermitian and is therefore diagonalizable (with real eigenvalues) on the finite dimensional Hilbert space $\mathcal{H}$. The equation can also be integrated as:

$$\begin{aligned}
|\Psi(t)\rangle &= \widehat{\mathrm{U}}(t,t_0)\ |\Psi(t_0)\rangle \\
\widehat{\mathrm{U}}(t,t_0) &:= \exp\left(-\mathrm{i}2\pi\frac{t-t_0}{\mathrm{h}}\widehat{\mathrm{H}}\right)
\end{aligned}$$

2.2 Kinetic energy Hamiltonian

The Hamiltonian operator of the field can be separated into a kinetic energy Hamiltonian depending only on the momentum of the particles and an interaction term as follows:

$$\widehat{\mathrm{H}} = \widehat{\mathrm{H}}_0 + \widehat{\mathrm{H}}'$$

In the lattice reference frame, the kinetic energy Hamiltonian is given by:

$$\widehat{\mathrm{H}}_0 := \sum_{\phi,\boldsymbol{q},\lambda} E^{\phi}_{\boldsymbol{q}}\widehat{N^{\phi}}_{\boldsymbol{q},\lambda}$$

In another reference frame, moving with a velocity $\boldsymbol{v}$ relative to the lattice, this operator is given by:

$$\widehat{\mathrm{H}}_0 := \sum_{\phi,\boldsymbol{q},\lambda} \gamma\left\{E^{\phi}_{\boldsymbol{q}} - \frac{\mathrm{h}}{\mathrm{a}}\boldsymbol{q}\cdot\boldsymbol{v}\right\}\widehat{N^{\phi}}_{\boldsymbol{q},\lambda}$$

2.3 Interaction picture

The kinetic energy Hamiltonian can be integrated as:

$$\widehat{\mathrm{U}}_0(t,t_0) := \exp\left(-\mathrm{i}2\pi\frac{t-t_0}{\mathrm{h}}\widehat{\mathrm{H}}_0\right)$$

The state of the quantum field in the interaction picture is defined in such a way that it would be a time constant if the interaction term $\widehat{\mathrm{H}}'$ vanishes:

$$|\Psi_I\rangle := \widehat{\mathrm{U}}_0(0,t)\ |\Psi\rangle$$

The Hamiltonian operator in the interaction picture is defined is such a way that the state of the quantum field in the interaction picture obeys following Schrödinger-like equation, where the Hamiltonian is time-dependant:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\ |\Psi_I\rangle &= -\mathrm{i}2\pi\frac{1}{\mathrm{h}}\widehat{\mathrm{H}}_I\ |\Psi_I\rangle \\ \widehat{\mathrm{H}}_I &:= \widehat{\mathrm{U}}_0(0,t)\widehat{\mathrm{H}}'\widehat{\mathrm{U}}_0(t,0)\end{aligned}$$

The integration of this equation yields to:

$$|\Psi_I(t)\rangle = \widehat{\mathrm{U}}_I(t,t_0)\ |\Psi_I(t_0)\rangle$$

where the evolution operator in the interaction picture is given by a series of the form (assuming $t > t_0$):

$$\begin{aligned}\widehat{\mathrm{U}}_I(t,t_0) &:= \mathbb{1} + \sum_{n=1}^{\infty}\widehat{\mathrm{U}}_I^{(n)}(t,t_0) \\ \widehat{\mathrm{U}}_I^{(n)}(t,t_0) &:= \left(\frac{-\mathrm{i}2\pi}{\mathrm{h}}\right)^n \int_{t>t_n>\cdots>t_1>t_0} \mathrm{d}t_1\cdots\mathrm{d}t_n \\ &\quad\ \widehat{\mathrm{U}}_0(0,t_n)\widehat{\mathrm{H}}'\widehat{\mathrm{U}}_0(t_n,t_{n-1})\cdots\widehat{\mathrm{U}}_0(t_2,t_1)\widehat{\mathrm{H}}'\widehat{\mathrm{U}}_0(t_1,0)\end{aligned}$$

The evolution operator in the interaction picture verifies:

$$\widehat{\mathrm{U}}_I(t,t_0) = \widehat{\mathrm{U}}_0(0,t)\widehat{\mathrm{U}}(t,t_0)\widehat{\mathrm{U}}_0(t_0,0)$$

The usual evolution operator can also be written too as a series of the form:

$$\begin{aligned}\widehat{\mathrm{U}}(t,t_0) &:= \sum_{n=0}^{\infty}\widehat{\mathrm{U}}^{(n)}(t,t_0) \\ \widehat{\mathrm{U}}^{(0)}(t,t_0) &:= \widehat{\mathrm{U}}_0(t,t_0) \\ \widehat{\mathrm{U}}^{(n)}(t,t_0) &:= \left(\frac{-\mathrm{i}2\pi}{\mathrm{h}}\right)^n \int_{t>t_n>\cdots>t_1>t_0} \mathrm{d}t_1\cdots\mathrm{d}t_n \\ &\quad\ \widehat{\mathrm{U}}_0(t,t_n)\widehat{\mathrm{H}}'\widehat{\mathrm{U}}_0(t_n,t_{n-1})\cdots\widehat{\mathrm{U}}_0(t_2,t_1)\widehat{\mathrm{H}}'\widehat{\mathrm{U}}_0(t_1,t_0)\end{aligned}$$

2.4 Transition amplitudes

In scattering experiments, the evolution operator in the interaction picture is often called "scattering operator". In this context, cross sections are usually calculated in the limit $t_0 \to -\infty$ and $t \to +\infty$, so the variables t_0 and t are implicit in the notation:

$$\widehat{\mathrm{S}} := \widehat{\mathrm{U}}_I(t, t_0)$$

Its matrix elements, called "scattering amplitudes" and written:

$$\begin{aligned}\mathrm{S}_{fi} &:= \langle\Psi_f|\ \widehat{\mathrm{S}}\ |\Psi_i\rangle \\ &= \langle\Psi_f|\ \widehat{\mathrm{U}}_I(t, t_0)\ |\Psi_i\rangle\end{aligned}$$

can be developed in a series of the form (assuming $t > t_0$):

$$\begin{aligned}\mathrm{S}_{fi} &= \sum_{n=0}^{\infty} \mathrm{S}_{fi}^{(n)} \\ \mathrm{S}_{fi}^{(0)} &:= \langle\Psi_f|\Psi_i\rangle \\ \mathrm{S}_{fi}^{(n)} &:= \left(\frac{-\mathrm{i}2\pi}{\mathrm{h}}\right)^n \int_{t>t_n>\cdots>t_1>t_0} \mathrm{d}t_1 \cdots \mathrm{d}t_n \\ &\quad \langle\Psi_f|\ \widehat{\mathrm{U}}_0(0, t_n)\widehat{\mathrm{H}}'\widehat{\mathrm{U}}_0(t_n, t_{n-1}) \cdots \widehat{\mathrm{U}}_0(t_2, t_1)\widehat{\mathrm{H}}'\widehat{\mathrm{U}}_0(t_1, 0)\ |\Psi_i\rangle\end{aligned}$$

For plane wave states $|\Psi_i\rangle = \left|(N_{i\,\boldsymbol{q},\lambda}^{\phi})\right\rangle$ and $|\Psi_f\rangle = \left|(N_{f\,\boldsymbol{q},\lambda}^{\phi})\right\rangle$, they are directly related to the matrix elements of the evolution operator, called "transition amplitudes", by:

$$\begin{aligned}\mathrm{U}_{fi}(t, t_0) &= \exp\left(-\mathrm{i}2\pi(tE_f - t_0E_i)/\mathrm{h}\right)\mathrm{S}_{fi} \\ \mathrm{U}_{fi}(t, t_0) &:= \langle\Psi_f|\ \widehat{\mathrm{U}}(t, t_0)\ |\Psi_i\rangle \\ E_i &:= \langle\Psi_i|\ \widehat{\mathrm{H}}_0\ |\Psi_i\rangle \\ E_f &:= \langle\Psi_f|\ \widehat{\mathrm{H}}_0\ |\Psi_f\rangle\end{aligned}$$

The transition amplitude from a plane wave state $|\Psi_i\rangle = \left|(N_{i\,\boldsymbol{q},\lambda}^{\phi})\right\rangle$ to a localized state $|\Psi_f\rangle = \left|(N_{f\,\boldsymbol{n},\lambda}^{\phi})\right\rangle$ can in turn be written as:

$$\mathrm{U}_{fi}(t, t_0) = \sum_{(N_{f\,\boldsymbol{q},\lambda}^{\phi})} \mathrm{S}_{fi}\exp\left(\mathrm{i}2\pi t_0E_i/\mathrm{h}\right)\psi\left((\boldsymbol{q}_j^{\phi,\lambda}), (\boldsymbol{n}_j^{\phi,\lambda}), t\right)$$

with:

$$\psi\left((\boldsymbol{q}_j^{\phi,\lambda}), (\boldsymbol{n}_j^{\phi,\lambda}), t\right) := \prod_{\substack{\phi,\lambda \\ N_{f\,\lambda}^{\phi}\neq 0}} \frac{(1+2\mathrm{N})^{-3N_{f\,\lambda}^{\phi}/2}}{\prod_{\boldsymbol{n}} N_{f\,\boldsymbol{n},\lambda}^{\phi}!} \sum_{\sigma\in\mathfrak{S}_{N_{f\,\lambda}^{\phi}}} \prod_{j=1}^{N_{f\,\lambda}^{\phi}} \exp\left(\mathrm{i}2\pi(\boldsymbol{n}_{\sigma_j}^{\phi,\lambda}\cdot\boldsymbol{q}_j^{\phi,\lambda} - E_{\boldsymbol{q}_j^{\phi,\lambda}}^{\phi}t/\mathrm{h}\right)$$

where the summation runs over plane wave states $({N_f}^{\phi}_{\boldsymbol{q},\lambda})$ such as $\sum_{\boldsymbol{q}} {N_f}^{\phi}_{\boldsymbol{q},\lambda} = \sum_{\boldsymbol{n}} {N_f}^{\phi}_{\boldsymbol{n},\lambda}$ for each mode (ϕ, λ) of the field, where we use the notations $\mathrm{S}_{fi} := \left\langle ({N_f}^{\phi}_{\boldsymbol{q},\lambda}) \right| \widehat{\mathrm{S}} \left| ({N_i}^{\phi}_{\boldsymbol{q},\lambda}) \right\rangle$ and ${N_f}^{\phi}_{\lambda} := \sum_{\boldsymbol{n}} {N_f}^{\phi}_{\boldsymbol{n},\lambda}$, where $\mathfrak{S}_{{N_f}^{\phi}_{\lambda}}$ denotes the symmetric group of order ${N_f}^{\phi}_{\lambda}$ and where we have chosen for each mode (ϕ, λ) of the field the families $(\boldsymbol{n}_j^{\phi,\lambda})$ and $(\boldsymbol{q}_j^{\phi,\lambda})$ such as:

$$\begin{aligned}
\left|({N_f}^{\phi}_{\boldsymbol{n},\lambda})\right\rangle &= \prod_{\phi,\lambda,j} \widehat{a^{\phi}}^{\dagger}_{\boldsymbol{n}_j^{\phi,\lambda},\lambda} \; |\Omega\rangle \\
\left|({N_f}^{\phi}_{\boldsymbol{q},\lambda})\right\rangle &= \prod_{\phi,\lambda,j} \widehat{a^{\phi}}^{\dagger}_{\boldsymbol{q}_j^{\phi,\lambda},\lambda} \; |\Omega\rangle
\end{aligned}$$

In the definition of $\psi\left((\boldsymbol{q}_j^{\phi,\lambda}), (\boldsymbol{n}_j^{\phi,\lambda}), t\right)$, we used for convenience the symbols ${N_f}^{\phi}_{\boldsymbol{n},\lambda}$ and ${N_f}^{\phi}_{\lambda}$, which can be defined as a function of $(\boldsymbol{n}_j^{\phi,\lambda})$ with ${N_f}^{\phi}_{\boldsymbol{n},\lambda} := \left|\{j | \boldsymbol{n}_j^{\phi,\lambda} = \boldsymbol{n}\}\right|$.

The transition amplitude from any initial state $|\Psi_i\rangle$ to a localized final state $|\Psi_f\rangle = \left|({N_f}^{\phi}_{\boldsymbol{n},\lambda})\right\rangle$ is finally given by:

$$\mathrm{U}_{fi}(t, t_0) = \sum_{({N_i}^{\phi}_{\boldsymbol{q},\lambda})} \sum_{({N_f}^{\phi}_{\boldsymbol{q},\lambda})} \mathrm{S}_{fi} \widetilde{\Psi_i}\left(({N_i}^{\phi}_{\boldsymbol{q},\lambda})\right) \exp\left(\mathrm{i}2\pi t_0 E_i/\mathrm{h}\right) \psi\left((\boldsymbol{q}_j^{\phi,\lambda}), (\boldsymbol{n}_j^{\phi,\lambda}), t\right)$$

with the same notations.

2.5 Scattering matrix

The scattering matrix can be developed quite easily on the basis of the plane wave states, i.e. by developing the initial and final states as:

$$\begin{aligned}
|\Psi_i\rangle &= \sum_{(N^{\phi}_{\boldsymbol{q},\lambda})} \widetilde{\Psi_i}\left((N^{\phi}_{\boldsymbol{q},\lambda})\right) \left|(N^{\phi}_{\boldsymbol{q},\lambda})\right\rangle \\
|\Psi_f\rangle &= \sum_{(N^{\phi}_{\boldsymbol{q},\lambda})} \widetilde{\Psi_f}\left((N^{\phi}_{\boldsymbol{q},\lambda})\right) \left|(N^{\phi}_{\boldsymbol{q},\lambda})\right\rangle
\end{aligned}$$

With these notations, we have to the zeroth order:

$$\mathrm{S}^{(0)}_{fi} = \sum_{({N_0}^{\phi}_{\boldsymbol{q},\lambda})} \overline{\widetilde{\Psi_f}\left(({N_0}^{\phi}_{\boldsymbol{q},\lambda})\right)} \widetilde{\Psi_i}\left(({N_0}^{\phi}_{\boldsymbol{q},\lambda})\right)$$

and to the n-th order:

$$\begin{aligned}
\mathrm{S}_{fi}^{(n)} &= \sum\nolimits_{\substack{k=0\\ (N_k{}_{\boldsymbol{q},\lambda}^{\phi})}}^{n} \overline{\widetilde{\Psi_f}\left((N_n{}_{\boldsymbol{q},\lambda}^{\phi})\right)}\widetilde{\Psi_i}\left((N_0{}_{\boldsymbol{q},\lambda}^{\phi})\right) \mathrm{S}_{n,\ldots,0}^{(n)} \\
\mathrm{S}_{n,\ldots,0}^{(n)} &:= \exp\left(\mathrm{i}2\pi\frac{t_0}{\mathrm{h}}(E_n - E_0)\right)\left(\prod_{k=1}^{n} H'_{k,k-1}\right)\mathrm{S}_{t-t_0}^{(n)}(E_n, \ldots, E_0)
\end{aligned}$$

$$\begin{aligned}
\mathrm{S}_{t-t_0}^{(n)}(E_n, \ldots, E_0) := \left(\frac{-\mathrm{i}2\pi}{\mathrm{h}}\right)^n \int_{t-t_0>t_n>\cdots>t_1>0} \\
\prod_{k=1}^{n}\exp\left(\mathrm{i}2\pi\frac{t_k}{\mathrm{h}}(E_k - E_{k-1})\right)\mathrm{d}t_n\cdots\mathrm{d}t_1
\end{aligned}$$

The functions $\mathrm{S}_{t-t_0}^{(n)}(E_n, \ldots, E_0)$ can be calculated recursively according to:

$$\mathrm{S}_{t-t_0}^{(1)}(E_1, E_0) = -\mathrm{i}2\pi\frac{t-t_0}{\mathrm{h}}\mathrm{esinc}\left(\frac{t-t_0}{\mathrm{h}}(E_1 - E_0)\right)$$

$$\begin{aligned}
\mathrm{S}_{t-t_0}^{(n+1)}(E_{n+1}, E_n, \ldots, E_0) = \frac{1}{E_{n+1}-E_n}\Big(\mathrm{S}_{t-t_0}^{(n)}(E_{n+1}, \ldots, E_0) \\
-\exp\left(\mathrm{i}2\pi\frac{t-t_0}{\mathrm{h}}(E_{n+1}-E_n)\right)\mathrm{S}_{t-t_0}^{(n)}(E_n, \ldots, E_0)\Big)
\end{aligned}$$

where the esinc function is defined as in A.2. To the second order, for instance, we have:

$$\begin{aligned}
\mathrm{S}_{t-t_0}^{(2)}(E_2, E_1, E_0) = -\mathrm{i}2\pi\frac{t-t_0}{\mathrm{h}}\exp\left(\mathrm{i}\pi\frac{t-t_0}{\mathrm{h}}(E_2 - E_0)\right)\frac{1}{E_2 - E_1} \\
\left[\mathrm{sinc}\left(\frac{t-t_0}{\mathrm{h}}(E_2 - E_0)\right) - \exp\left(\mathrm{i}\pi\frac{t-t_0}{\mathrm{h}}(E_2 - E_1)\right)\mathrm{sinc}\left(\frac{t-t_0}{\mathrm{h}}(E_1 - E_0)\right)\right]
\end{aligned}$$

where the sinc function is defined as in A.1.

2.6 Perturbative development

The explicit perturbative development of the transition amplitude between two plane wave states $|\Psi_i\rangle = \left|(N_i{}_{\boldsymbol{q},\lambda}^{\phi})\right\rangle$ and $|\Psi_f\rangle = \left|(N_f{}_{\boldsymbol{q},\lambda}^{\phi})\right\rangle$ is therefore given by:

$$\mathrm{U}_{fi}^{(0)}(t, t_0) = \exp\left(-\mathrm{i}2\pi\frac{t-t_0}{\mathrm{h}}E_f\right)\delta_{fi}$$

$$\mathrm{U}_{fi}^{(1)}(t, t_0) = -\mathrm{i}2\pi\frac{t-t_0}{\mathrm{h}}\exp\left(-\mathrm{i}\pi\frac{t-t_0}{\mathrm{h}}(E_f + E_i)\right)\mathrm{sinc}\left(\frac{t-t_0}{\mathrm{h}}(E_f - E_i)\right)H'_{f,i}$$

$$\mathrm{U}_{fi}^{(n)}(t,t_0) = \exp\left(-\mathrm{i}2\pi\frac{t-t_0}{\mathrm{h}}E_f\right)\sum_{\substack{k=0\\(N_{k}{}_{\boldsymbol{q},\lambda}^{\phi})}}^{n} \mathrm{S}_{t-t_0}^{(n)}\left(E_n,\ldots,E_0\right)\prod_{k=1}^{n} H'_{k,k-1}$$

where we take in the last sum $(N_0{}_{\boldsymbol{q},\lambda}^{\phi}) = (N_i{}_{\boldsymbol{q},\lambda}^{\phi})$ and $(N_n{}_{\boldsymbol{q},\lambda}^{\phi}) = (N_f{}_{\boldsymbol{q},\lambda}^{\phi})$ and where we have:

$$\mathrm{S}_{t-t_0}^{(n)}\left(E_n,\ldots,E_0\right) := \left(\frac{-\mathrm{i}2\pi}{\mathrm{h}}\right)^n \int_{t-t_0>t_n>\cdots>t_1>0} \prod_{k=1}^{n}\exp\left(\mathrm{i}2\pi\frac{t_k}{\mathrm{h}}(E_k-E_{k-1})\right)\mathrm{d}t_n\cdots\mathrm{d}t_1$$

More generally, the explicit perturbative development of the transition amplitude from any initial state $|\Psi_i\rangle$ to a localized final state $|\Psi_f\rangle = \left|(N_f{}_{\boldsymbol{n},\lambda}^{\phi})\right\rangle$ is given by:

$$\mathrm{U}_{fi}^{(0)}(t,t_0) = \sum_{(N_f{}_{\boldsymbol{q},\lambda}^{\phi})} \widetilde{\Psi_i}\left((N_f{}_{\boldsymbol{q},\lambda}^{\phi})\right)\psi\left((\boldsymbol{q}_j^{\phi,\lambda}),(\boldsymbol{n}_j^{\phi,\lambda})\right)\exp\left(-\mathrm{i}2\pi\frac{t-t_0}{\mathrm{h}}E_f\right)$$

$$\mathrm{U}_{fi}^{(1)}(t,t_0) = -\mathrm{i}2\pi\frac{t-t_0}{\mathrm{h}}\sum_{(N_i{}_{\boldsymbol{q},\lambda}^{\phi})} \widetilde{\Psi_i}\left((N_i{}_{\boldsymbol{q},\lambda}^{\phi})\right)\sum_{(N_f{}_{\boldsymbol{q},\lambda}^{\phi})}\psi\left((\boldsymbol{q}_j^{\phi,\lambda}),(\boldsymbol{n}_j^{\phi,\lambda})\right)$$
$$\exp\left(-\mathrm{i}\pi\frac{t-t_0}{\mathrm{h}}(E_f+E_i)\right)\operatorname{sinc}\left(\frac{t-t_0}{\mathrm{h}}(E_f-E_i)\right)H'_{f,i}$$

$$\mathrm{U}_{fi}^{(n)}(t,t_0) = \sum_{(N_i{}_{\boldsymbol{q},\lambda}^{\phi})} \widetilde{\Psi_i}\left((N_i{}_{\boldsymbol{q},\lambda}^{\phi})\right)\sum_{(N_f{}_{\boldsymbol{q},\lambda}^{\phi})}\psi\left((\boldsymbol{q}_j^{\phi,\lambda}),(\boldsymbol{n}_j^{\phi,\lambda})\right)$$
$$\exp\left(-\mathrm{i}2\pi\frac{t-t_0}{\mathrm{h}}E_f\right)\sum_{\substack{k=0\\(N_{k}{}_{\boldsymbol{q},\lambda}^{\phi})}}^{n} \mathrm{S}_{t-t_0}^{(n)}\left(E_n,\ldots,E_0\right)\prod_{k=1}^{n} H'_{k,k-1}$$

where the summation runs over plane wave states $(N_f{}_{\boldsymbol{q},\lambda}^{\phi})$ such as $\sum_{\boldsymbol{q}} N_f{}_{\boldsymbol{q},\lambda}^{\phi} = \sum_{\boldsymbol{n}} N_f{}_{\boldsymbol{n},\lambda}^{\phi}$ for each mode (ϕ,λ) of the field, and where we use the notation:

$$\psi\left((\boldsymbol{q}_j^{\phi,\lambda}),(\boldsymbol{n}_j^{\phi,\lambda})\right) := \prod_{\substack{\phi,\lambda\\ N_f{}_{\lambda}^{\phi}\neq 0}} \frac{(1+2\mathrm{N})^{-3N_f{}_{\lambda}^{\phi}/2}}{\prod_{\boldsymbol{n}} N_f{}_{\boldsymbol{n},\lambda}^{\phi}!}\sum_{\sigma\in\mathfrak{S}_{N_f{}_{\lambda}^{\phi}}}\prod_{j=1}^{N_f{}_{\lambda}^{\phi}}\exp\left(\mathrm{i}2\pi\boldsymbol{n}_{\sigma_j}^{\phi,\lambda}\cdot\boldsymbol{q}_j^{\phi,\lambda}\right)$$

Part II

Mental world

Chapter 3

Mental states

> On the other hand I think I can safely say that nobody understands quantum mechanics.
>
> Richard Feynman,
> *The Character of Physical Law* [6]

3.1 The mind-body problem

Since the end of the second World War and the translation of the intellectual center of the scientific community from Europe to the United States of America, materialism, i.e. the complete reduction of our experience of mind to physical-material processes, has become the philosophical conviction of mainstream physicists, although they still may have opposite religious beliefs as private persons. Of course, it doesn't make any doubt that the biological activity of human and similar animal brains is involved in the processing of external and internal stimuli, and it is reasonable to believe that conscious thinking consists, at the physical level, in a particularly long and intensive stimuli processing involving a kind of cross-validation through large parts of the neural network. Nevertheless, "mind", i.e. the form of our experience of the world, with our feelings, our body schema, memories seen with the mind's eye, melodies imagined in the mind's ear... is just not *the same* as the neural activity of some body being anyhow hardly identifiable quantum physically. Determining the relation between these two realities is the essence of the mind-body problem, which has become the most various answers over the ages. The usual divergence points arise about the questions: Do both realities exist at all, or is one of them a mere illusion? Are they independent of each other and just exist as parallel realities, or are there divergences and a mutual influence in the one, the other or both directions? In this old debate, Quantum Field Theory introduces the new idea that a mutual influence doesn't have to be a deterministic causal influence but also could be a probabilistic one, so that neither "mind" nor "body" have to be kind of a subordinated slave of its counterpart, but retain to some extent a form of "freedom" under its influence. I think this idea should have the potential to take some heat out of the debate.

3.2 Individual mind

Each of us has a direct experience of the reality of an individual mind, so I will only expose a few reflections in this place. I think that the state of an individual mind should be considered in its "organic" unity, that picking out single conscious thoughts and considering that the state of this mind is simply composed of these should be considered as an oversimplified and inadequate view. Within this "organic" unity, however, the intensity of consciousness may vary, focussing our awareness on some aspects rather than on others. The border between conscious and unconscious thoughts is therefore not really clear to ourselves, as there is a slow transition made up of more or less subconscious thoughts of decreasing intensity. When I refer to the state of an individual mind, I will mean in principle the unity of all conscious and subconscious thoughts, although we're not quite sure of where they end. They will, in general, contain among other things representations of a body, of its activity, of its environment, of past experiences... as well as representations of time, which make up our feeling of being continuously ourselves in the continuity of time. But I believe that this feeling of permanence of the individual mind is a mere illusion, for two reasons: First, this feeling is experienced in every single instant of consciousness; we could by no mean find out if we really have experienced other instants of consciousness "before" and if these instants of consciousness correspond to our current memories or not, so this feeling of permanence *could* be an illusion. In fact, if I would suddenly experience another individual mind (with its own memories and not mine), I wouldn't even notice it! Second, the experience of an individual mind seems to cease as "our" body is dreamless sleeping, swoon or eventually die, so I think its permanence is discarded by common experience. Therefore, I don't believe that an individual mind is a fundamental entity of the mental world, which would have an existence of its own and evolve in time, and I will only refer to instantaneous *states* of an individual mind, which are not related to each other across time in the form of a personal history at a fundamental level.

3.3 Collective mind

The states of the mental world are supposed to be experienced collectively by the set of all "sentient beings". A state $\mathfrak{M}$ of the mental world can also be described by the number $N_{\mathfrak{m}}$ of "sentient beings" experiencing each possible individual mind state $\mathfrak{m}$. An arbitrary sequence $(N_{\mathfrak{m}})$, however, doesn't necessarily correspond to a possible collective mind state $\mathfrak{M}$. In fact, as a consequence of the correspondence between mental and physical states defined subsequently and of the finite dimensionality of the Hilbert space of the physical states, there must be a finite number of possible collective and individual mind states. The set of all possible collective mind states is written $\mathcal{M}$.

3.4 Physical realization of mental states

The correspondence between mental and physical states is given by a Hilbert subspace $\mathcal{H}_{\mathfrak{M}}$ associated to each possible mental state $\mathfrak{M}$ in such a way that these subspaces verify:

$$\mathcal{H} = \bigoplus_{\mathfrak{M}}^{\perp} \mathcal{H}_{\mathfrak{M}}$$

Each vector $|\Psi\rangle \in \mathcal{H}_{\mathfrak{M}}$ is a physical state of the universe in which the mental state $\mathfrak{M}$ is being experienced. Knowing the correspondence between $\mathfrak{M}$ and $\mathcal{H}_{\mathfrak{M}}$ is in essence solving the mind-body problem. As a working hypothesis, I shall assume that a mental state $\mathfrak{M} = (N_{\mathfrak{m}})$ is being realized physically by any physical state describing a universe containing, for each individual mind state $\mathfrak{m}$, exactly $N_{\mathfrak{m}}$ human or animal brains presenting the specific activity pattern corresponding to $\mathfrak{m}$. The task of describing the possible individual mind states belongs in principle to psychology, whereas the characterization of the corresponding activity patterns of the brain is the aim of cognitive neuroscience.

In mathematical terms, this hypothesis can be modeled as follows. First, the mental state $\mathfrak{M}_\Omega := (0_{\mathfrak{m}})$, in which no "sentient being" is experiencing any individual mind state, is supposed to be possible, i.e. the corresponding subspace $\mathcal{H}_{\mathfrak{M}_\Omega}$ is supposed not to be null. Then, for each possible individual mind state $\mathfrak{m}$, there is supposed to be a finite family of brain creation operators $(\widehat{\Psi_{\mathfrak{m}}^{\alpha}}^{\dagger})$ in $\mathcal{A}^{\dagger}$, which are creating a single brain with an activity pattern corresponding to $\mathfrak{m}$, such as:

$$\mathcal{H}_{1_{\mathfrak{m}}} = \bigoplus_{\alpha}^{\perp} \widehat{\Psi_{\mathfrak{m}}^{\alpha}}^{\dagger} \mathcal{H}_{\mathfrak{M}_\Omega}$$

Finally, for every collective mental state $\mathfrak{M}$, noting $\mathfrak{M} + 1_{\mathfrak{m}}$ the collective mental state in which a single further "sentient being" is experiencing the individual mental state $\mathfrak{m}$, the corresponding subspaces are supposed to verify:

$$\mathcal{H}_{\mathfrak{M}+1_{\mathfrak{m}}} = \sum_{\alpha} \widehat{\Psi_{\mathfrak{m}}^{\alpha}}^{\dagger} \mathcal{H}_{\mathfrak{M}}$$

These relations define all the subspaces $\mathcal{H}_{\mathfrak{M}}$ recursively as a function of $\mathcal{H}_{\mathfrak{M}_{\Omega}}$ and of the operators $\widehat{\Psi_{\mathfrak{m}}^{\alpha}}^{\dagger}$. If a subspace defined in this way happens to be null (because of the existence of a maximum occupation number for single field modes), the corresponding mental state is impossible.

Given two collective mental states $\mathfrak{M} = (N_{\mathfrak{m}})$ and $\mathfrak{M}' = (N'_{\mathfrak{m}})$, we define the partial order relation $\mathfrak{M}' \geq \mathfrak{M}$ by $\forall \mathfrak{m}, N'_{\mathfrak{m}} \geq N_{\mathfrak{m}}$. The subspace $\mathcal{H}_{\mathfrak{M}}^{+}$ of the physical states corresponding to collective mental states where at least $N_{\mathfrak{m}}$ "sentient beings" are experiencing each individual mental state $\mathfrak{m}$ can be defined, with this notation, by:

$$\mathcal{H}_{\mathfrak{M}}^{+} := \bigoplus_{\mathfrak{M}' \geq \mathfrak{M}}^{\perp} \mathcal{H}_{\mathfrak{M}'}$$

Chapter 4

Stochastic evolution

4.1 Collapse and collective mind selection

In the joint evolution of the mental and physical states of the universe, I suppose that the physical state $|\Psi\rangle$ periodically becomes randomly projected into one of the subspaces $\mathcal{H}_{\mathfrak{M}}$, corresponding to a given mental state $\mathfrak{M}$, with a probability given by:

$$P(\mathfrak{M}) = \frac{\langle\Psi|\ \widehat{\Pi}_{\mathfrak{M}}\ |\Psi\rangle}{\langle\Psi|\Psi\rangle}$$

where $\widehat{\Pi}_{\mathfrak{M}}$ is the orthogonal projection operator on $\mathcal{H}_{\mathfrak{M}}$, and that this projection corresponds to the fact that, in the mental world, the collective mind state $\mathfrak{M} = (N_{\mathfrak{m}})$ is being experienced, i.e. each possible individual mind state $\mathfrak{m}$ is being experienced by $N_{\mathfrak{m}}$ individual minds. We call the physical part of this process "collapse" of the physical state of the universe and its mental counterpart "collective mind selection". The operators $\widehat{\Pi}_{\mathfrak{M}}$ are called "collapse operators". As a working hypothesis, we assume that the period τ of this process is of the order of the Plank time:

$$\tau \approx \sqrt{4\pi \mathrm{Gh}/\mathrm{c}^5} \approx 4.8\ 10^{-43}\ \mathrm{s}$$

4.2 Mental evolution

Fundamentally, Quantum Field Theory also defines the probability that any given succession of collective mind states be experienced, an initial physical and collective mind state being given. In the case where the actually experienced collective mind state has a relatively high probability, our mental state may give us some clues about the physics of the world we live in; on the opposite, if our collective mind state has a very low probability, our mental experience has very little to do with the laws of the physical world and we live in a mere illusion of knowing something about reality – without having any mean of noticing it. This dilemma is very well known of particle physicists, who have to accept they cannot make more precise statements about reality than, for instance, “in the context of the standard model and in the presence of a sequential fourth family of fermions with high masses [...] a Higgs boson with mass between 144 and 207 GeV/c^2 is ruled out at 95% confidence level” [2]. Any physical model can also be conventionally, but not definitely, “ruled out” if it predicts the observed results with a probability considered to be too low.

Given an initial physical state $|\Psi_0\rangle \in \mathcal{H}_{\mathfrak{M}_0}$ at $t = 0$, the probability $P(\mathfrak{M}_t, t)$ that a given collective mind state $\mathfrak{M}_t$ is being realized at time $t\tau$, where $t \in \mathbb{N}^*$, reads for $t = 1$:

$$P(\mathfrak{M}_1, 1) = \langle\Psi_0|\ \widehat{U}_\tau^\dagger \widehat{\Pi}_{\mathfrak{M}_1} \widehat{U}_\tau\ |\Psi_0\rangle \,/\, \langle\Psi_0|\Psi_0\rangle$$

where $\widehat{U}_\tau := \exp\left(-\mathrm{i}2\pi\widehat{\mathrm{H}}\tau/\mathrm{h}\right)$, and more generally for $t \geq 2$:

$$P(\mathfrak{M}_t, t) = \sum_{\mathfrak{M}_{t-1}} \cdots \sum_{\mathfrak{M}_1} \langle\Psi_0|\ \widehat{U}_\tau^\dagger \widehat{\Pi}_{\mathfrak{M}_1} \cdots \widehat{U}_\tau^\dagger \widehat{\Pi}_{\mathfrak{M}_t} \widehat{U}_\tau \cdots \widehat{\Pi}_{\mathfrak{M}_1} \widehat{U}_\tau\ |\Psi_0\rangle \,/\, \langle\Psi_0|\Psi_0\rangle$$

If the initial vector state isn't exactly known, averaging on an orthonormal basis of $\mathcal{H}_{\mathfrak{M}_0}$ leads to:

$$\langle P(\mathfrak{M}_t, t)\rangle_{\mathfrak{M}_0} = \sum_{\mathfrak{M}_{t-1}} \cdots \sum_{\mathfrak{M}_1} \mathrm{Tr}_{\mathcal{H}_{\mathfrak{M}_0}} \widehat{U}_\tau^\dagger \widehat{\Pi}_{\mathfrak{M}_1} \cdots \widehat{U}_\tau^\dagger \widehat{\Pi}_{\mathfrak{M}_t} \widehat{U}_\tau \cdots \widehat{\Pi}_{\mathfrak{M}_1} \widehat{U}_\tau / \dim\mathcal{H}_{\mathfrak{M}_0}$$

4.3 Transition probability

We consider, to simplify the discussion, a repeated experiment with a single possible outcome, which may have been realized or not after a given duration $t\tau$. Notice that this duration doesn't correspond to the instant at which some physical event occurs, but is a sufficiently long duration after which the experimenter can consciously remember of having (just) observed the expected outcome or not.

The possible states of the minds corresponding to the beginning of the experiment are written $\mathfrak{M}_i$ and the initial state of the quantum fields is also an element of the Hilbert subspace $\mathcal{H}_i$ given by:

$$\mathcal{H}_i = \bigoplus_{\mathfrak{M}_i}^{\perp} \mathcal{H}_{\mathfrak{M}_i}$$

The possible states of the minds corresponding to the measurement of the given outcome resp. of its absence are written $\mathfrak{M}_f^+$ resp. $\mathfrak{M}_f^-$. If the experiment works correctly, the final state of the quantum fields is, after measurement, an element of either of the Hilbert subspaces $\mathcal{H}_f^+$ or $\mathcal{H}_f^-$ given by:

$$\mathcal{H}_f^{\pm} = \bigoplus_{\mathfrak{M}_f^{\pm}}^{\perp} \mathcal{H}_{\mathfrak{M}_f^{\pm}}$$

If the experiment fails for some reason (e.g. if some measuring device is getting damaged during the experiment), the final state of the quantum fields is orthogonal to $\mathcal{H}_f^+ \oplus \mathcal{H}_f^-$.

The absolute probability of measuring the given outcome resp. its absence is given by:

$$\mathcal{P}\left(\mathcal{H}_i \to \mathcal{H}_f^{\pm}\right) = \sum_{\mathfrak{M}_{t-1}} \cdots \sum_{\mathfrak{M}_1} \mathrm{Tr}_{\mathcal{H}_i} \widehat{U}_\tau^\dagger \widehat{\Pi}_{\mathfrak{M}_1} \cdots \widehat{U}_\tau^\dagger \widehat{\Pi}_{f^\pm} \widehat{U}_\tau \cdots \widehat{\Pi}_{\mathfrak{M}_1} \widehat{U}_\tau / \mathrm{dim}\mathcal{H}_i$$

where $\widehat{\Pi}_{f^\pm} = \sum_{\mathfrak{M}_f^\pm} \widehat{\Pi}_{\mathfrak{M}_f^\pm}$. The conditional probability of measuring the given outcome if the experiment doesn't fail is then given by:

$$\mathcal{TP}\left(\mathcal{H}_i \to \mathcal{H}_f^{+}\right) = \frac{\mathcal{P}\left(\mathcal{H}_i \to \mathcal{H}_f^{+}\right)}{\mathcal{P}\left(\mathcal{H}_i \to \mathcal{H}_f^{+}\right) + \mathcal{P}\left(\mathcal{H}_i \to \mathcal{H}_f^{-}\right)}$$

and we call it “transition probability” from $\mathcal{H}_i$ to $\mathcal{H}_f^+$.

If the experiment is conceived in such a way that the studied system is isolated from the observer for the duration of the experiment until it interacts with some measurement apparatus, the experiment is considered to have failed if the observer has gained some information about the studied system before it interacts with this apparatus. An intermediate observation of the system, as it would leave a permanent trace in the memory of the observer, would lead with a vanishingly small probability to a final state of the minds in which the observer isn't conscious of having made this observation. The only intermediate states of the minds $\mathfrak{M}_1, \ldots, \mathfrak{M}_{t-1}$ to be considered in the above sums (i.e. which haven't a vanishingly small contribution to the transition probability) correspond also to projectors that don't affect the Hamiltonian evolution of the studied system. In that case, the absolute probability of measuring the given outcome resp. its absence can be approximated by:

$$\mathcal{P}\left(\mathcal{H}_i \to \mathcal{H}_f^{\pm}\right) \approx \mathrm{Tr}_{\mathcal{H}_i} \widehat{U}_{t\tau}^\dagger \widehat{\Pi}_{f^\pm} \widehat{U}_{t\tau} / \mathrm{dim}\mathcal{H}_i$$

and can be written as a sum resp. a mean on quantum states forming an orthogonal basis of $\mathcal{H}_f^\pm$ resp. $\mathcal{H}_i$:

$$\begin{aligned} \mathcal{P}\left(\mathcal{H}_i \to \mathcal{H}_f^{\pm}\right) &\approx \sum_f \left\langle \mathcal{P}\left(i \to f\right) \right\rangle_i \\ \mathcal{P}\left(i \to f\right) &:= \left|\mathrm{U}_{fi}(t\tau, 0)\right|^2 \end{aligned}$$

In this expression, the (absolute) transition probabilities $\mathcal{P}(i \to f)$ between two quantum states can be developed in series of the form:

$$\begin{aligned}\mathcal{P}(i \to f) &= \sum_{n=0}^{\infty} \mathcal{P}^{(n)}(i \to f) \\ \mathcal{P}^{(n)}(i \to f) &:= \sum_{n_1+n_2=n} \overline{\mathrm{U}_{fi}^{(n_1)}(t\tau, 0)} \mathrm{U}_{fi}^{(n_2)}(t\tau, 0)\end{aligned}$$

If i and f are plane wave states, these terms can be written using the scattering matrix as:

$$\mathcal{P}^{(n)}(i \to f) := \sum_{n_1+n_2=n} \overline{\mathrm{S}_{fi}^{(n_1)}} \mathrm{S}_{fi}^{(n_2)}$$

4.4 Leading order transition probability

The general form of the transition probability between plane wave modes of the field can be given without knowing much about the interaction term $\widehat{\mathrm{H}}'$. We assume here that the initial and final states of the field are plane waves of the form:

$$\begin{aligned}|\Psi_i\rangle &= \left|(N_i{}^{\phi}_{\boldsymbol{q},\lambda})\right\rangle \\ |\Psi_f\rangle &= \left|(N_f{}^{\phi}_{\boldsymbol{q},\lambda})\right\rangle\end{aligned}$$

The first interesting terms in the development of the transition probability are given in that case by:

$$\begin{aligned}\mathcal{P}^{(0)}(i \to f) &:= \overline{\mathrm{S}_{fi}^{(0)}} \mathrm{S}_{fi}^{(0)} = \delta_{f,i} \\ \mathcal{P}^{(1)}(i \to f) &:= \overline{\mathrm{S}_{fi}^{(0)}} \mathrm{S}_{fi}^{(1)} + \overline{\mathrm{S}_{fi}^{(1)}} \mathrm{S}_{fi}^{(0)} = 0\end{aligned}$$

$$\begin{aligned}\mathcal{P}^{(2)}(i \to f) &:= \overline{\mathrm{S}_{fi}^{(0)}} \mathrm{S}_{fi}^{(2)} + \overline{\mathrm{S}_{fi}^{(1)}} \mathrm{S}_{fi}^{(1)} + \overline{\mathrm{S}_{fi}^{(2)}} \mathrm{S}_{fi}^{(0)} \\ &= (2\pi)^2 \frac{t - t_0}{\mathrm{h}} \left|H'_{f,i}\right|^2 \delta_{t-t_0}^{(2)}(E_f - E_i) \\ &\quad - \delta_{f,i} (2\pi)^2 \frac{t - t_0}{\mathrm{h}} \sum_{(N_1{}^{\phi}_{\boldsymbol{q},\lambda})} \left|H'_{1,i}\right|^2 \delta_{t-t_0}^{(2)}(E_1 - E_i)\end{aligned}$$

where the nascent delta function $\delta_{t-t_0}^{(2)}(E)$ is defined as in A.3.

4.5 Higher order transition probability

To the order $n \geq 2$, the transition probability between plane wave states $\left|(N_{i\,\boldsymbol{q},\lambda}^{\phi})\right\rangle$ and $\left|(N_{f\,\boldsymbol{q},\lambda}^{\phi})\right\rangle$ is given by:

$$\begin{aligned}
\mathcal{P}^{(n)}\left(i \to f\right) \quad := & \quad \sum_{n_1+n_2=n} \overline{\mathrm{S}_{fi}^{(n_1)}}\mathrm{S}_{fi}^{(n_2)} \\
= & \quad \delta_{f,i} \sum_{\substack{k=1 \\ (N_{k\,\boldsymbol{q},\lambda}^{\phi})}}^{n-1} \left(\mathrm{S}_{i,\ldots,i}^{(n)} + \overline{\mathrm{S}_{i,\ldots,i}^{(n)}} \right) \\
& + \sum_{\substack{n_1+n_2=n \\ n_1,n_2 \geq 1}} \overline{\mathrm{S}_{fi}^{(n_1)}}\mathrm{S}_{fi}^{(n_2)}
\end{aligned}$$

The first term vanishes for $f \neq i$. The development of the last term involves a "closed loop" of length n from i to i over f, i.e. a summation over $n-2$ intermediate states $\left|(N_{k\,\boldsymbol{q},\lambda}^{\phi})\right\rangle$, where $k \in [\![-n_1, n_2]\!]$, $(N_{0\,\boldsymbol{q},\lambda}^{\phi}) = (N_{i\,\boldsymbol{q},\lambda}^{\phi})$ and $(N_{-n_1\,\boldsymbol{q},\lambda}^{\phi}) = (N_{n_2\,\boldsymbol{q},\lambda}^{\phi}) = (N_{f\,\boldsymbol{q},\lambda}^{\phi})$, and can be written as:

$$\sum_{\substack{n_1+n_2=n \\ n_1,n_2 \geq 1}} \sum_{\substack{k=-n_1 \\ (N_{k\,\boldsymbol{q},\lambda}^{\phi})}}^{n_2} \left(\prod_{k=-n_1}^{n_2-1} H'_{k+1,k} \right) \overline{\mathrm{S}_{t-t_0}^{(n_1)}\left(E_{-n_1}, \ldots, E_0\right)} \mathrm{S}_{t-t_0}^{(n_2)}\left(E_{n_2}, \ldots, E_0\right)$$

To the third order, for instance, the transition probability for $f \neq i$ reads:

$$\begin{aligned}
&\mathcal{P}^{(3)}\left(i \to f\right) = (2\pi)^3 \delta_{t-t_0}^{(1)}\left(E_f - E_i\right) \\
&\qquad \sum_{(N_{1\,\boldsymbol{q},\lambda}^{\phi})} \Big[\Im\left(H'_{i,f}H'_{f,1}H'_{1,i}\right) \delta_{t-t_0}^{(1)}\left(E_f - E_1\right) \delta_{t-t_0}^{(1)}\left(E_1 - E_i\right) \\
&+ \Re\left(H'_{i,f}H'_{f,1}H'_{1,i}\right) \frac{\delta_{t-t_0}^{(1)}\left(E_f - E_i\right) - \cos\left(\pi \frac{t-t_0}{\mathrm{h}}(E_f - E_1)\right) \delta_{t-t_0}^{(1)}\left(E_1 - E_i\right)}{\pi(E_f - E_1)} \Big]
\end{aligned}$$

where the nascent delta function $\delta_{t-t_0}^{(1)}\left(E\right)$ is defined as in A.3.

4.6 Ideal experimental setup

We consider a scattering experiment designed to produce n_f particles of type ϕ_j, of wave vector $\boldsymbol{q}_j$ and of spin state λ_j. To detect them all, a set of n_f particle detectors D_j is being used and we consider a single alternative: either all the detectors are activated or at least one of them isn't. The momentum of the detected particles is measured with an uncertainty given by the domain δP_j of the momentum space in which particle j could be found without changing the measurement result. The corresponding subset δQ_j of values of $\boldsymbol{q}_j$ is given in the lattice reference frame by:

$$\delta Q_j = \left(\frac{[\![-\mathrm{N}, \mathrm{N}]\!]}{1+2\mathrm{N}}\right)^3 \cap \frac{\mathrm{a}}{\mathrm{h}} \delta P_j$$

We assume that the corresponding subspace $\delta\mathcal{F}$ of $\mathcal{H}$ is given by:

$$\begin{aligned} \delta\mathcal{F} &= \bigoplus_{(\delta\boldsymbol{q}_j)}^{\perp} \mathbb{C}\widehat{\Psi}^{\dagger}_{f+(\delta\boldsymbol{q}_j)} \, |O\rangle \\ \widehat{\Psi}^{\dagger}_{f+(\delta\boldsymbol{q}_j)} &:= \prod_j \widehat{a^{\phi_j}}^{\dagger}_{\boldsymbol{q}_j+\delta\boldsymbol{q}_j,\lambda_j} \end{aligned}$$

where the summation goes over all the $\delta\boldsymbol{q}_j$ verifying $\boldsymbol{q}_j + \delta\boldsymbol{q}_j \in \delta Q_j$.

The probability that all the detectors are activated is then given by:

$$\mathcal{P}\left(i \to \delta\mathcal{F}\right) := \sum_{(\delta\boldsymbol{q}_j)} \mathcal{P}\left(i \to f + (\delta\boldsymbol{q}_j)\right)$$

If the transition probability $\mathcal{P}\left(i \to f + (\delta\boldsymbol{q}_j)\right)$, as a function of $(\delta\boldsymbol{q}_j)$, admits a continuation on $\mathbb{R}^{3n_f}$, an approximation of this sum can be obtained by taking the corresponding integral:

$$\mathcal{P}\left(i \to \delta\mathcal{F}\right) \approx \int_{\prod_j \delta P_j} \mathcal{P}\left(i \to f + (\delta\boldsymbol{q}_j)\right) \left((1+2\mathrm{N})\frac{\mathrm{a}}{\mathrm{h}}\right)^{3n_f} \mathrm{d}^3\boldsymbol{p}_1 \cdots \mathrm{d}^3\boldsymbol{p}_{n_f}$$

where $\delta\boldsymbol{q}_j := \frac{\mathrm{a}}{\mathrm{h}}\boldsymbol{p}_j - \boldsymbol{q}_j$ in the lattice reference frame.

In particular, if $\left|H'_{f+(\delta\boldsymbol{q}_j),i}\right|^2$ admits such a continuation and if i and f could be approximated by plane wave states, the leading order transition probability could be approximated, for $i \notin \delta\mathcal{F}$, by:

$$\begin{aligned} \mathcal{P}^{(2)}\left(i \to \delta\mathcal{F}\right) \approx{}& \int_{\prod_j \delta P_j} (2\pi)^2 \frac{t-t_0}{\mathrm{h}} \left|H'_{f+(\delta\boldsymbol{q}_j),i}\right|^2 \delta^{(2)}_{t-t_0}\left(E_{f+(\delta\boldsymbol{q}_j)} - E_i\right) \\ & \left((1+2\mathrm{N})\frac{\mathrm{a}}{\mathrm{h}}\right)^{3n_f} \mathrm{d}^3\boldsymbol{p}_1 \cdots \mathrm{d}^3\boldsymbol{p}_{n_f} \end{aligned}$$

Part III

Interpretation

Chapter 5

Metaphysics

As I have said so many times,
God doesn't play dice with the world.

Albert Einstein,
in *Einstein and the Poet* [8]

5.1 Spinoza's philosophy

Since the interpretation of Quantum Field Theory I am about to give has been inspiered by Spinoza's classical work *The Ethics* [14], I shall make here a short presentation of its basic ideas. According to the causalist world view of classical mechanics, each individual existent thing – an object, a thought – has necessarily a cause which explains its existence at a given moment. These things are considered to be alterations, or "modes", of some fundamental "substance" constitutive of Nature as a whole. Since this substance has some of the fundamental properties attributed to God by Judaic theology – self-caused, free, eternal, infinite (i.e. containing everything) –, it has been identified by Spinoza to God itself, confounding thus the concept of 'God' with what philosophers traditionally call 'Nature'. The human intellect conceives the substance, as well as every individual existent thing, under the two aspects, or "attributes", of an extended (physical) and of a thinking (mental) thing. This categorization, however, is nothing but a property of the human intellect and not an intrinsic property of the things themselves. Considered under its physical aspect, a human being, for instance, consists in a body extending in the substance, i.e. in God, whereas it consists in a mind thinking in God when considered under its mental aspect. Nevertheless, both are one and the same thing, so that the laws of Physics – considered to be part of the nature of God – could determine the laws of Psychology. The knowledge of God, which also encompasses the knowledge of the world in general and of Man in particular, is therefore considered to be the mind's highest good.

5.2 Quantum metaphysics

Interestingly, Spinoza's metaphysical concepts can be identified quite straightforwardly with the fundamental notions of Quantum Field Theory, thus providing them with a naturalistic basis. On the other hand, Quantum Field Theory, generally considered to be counter-intuitive, paradoxical and hardly understandable, becomes grounded in a very classical philosophical tradition and should thus become accessible to a broader range of Science philosophers.

The states (modes) of God (the substance) are evidently identified, under their physical aspect, with the physical states $|\Psi\rangle$ of the universe (the elements of the Hilbert space $\mathcal{H}$), and, under their mental aspect, with the collective mind states $\mathfrak{M}$ (the elements of $\mathcal{M}$). The relation between the physical and the mental aspects is given by the decomposition $\mathcal{H} = \bigoplus_{\mathfrak{M}}^{\perp} \mathcal{H}_{\mathfrak{M}}$ of the Hilbert space, or equivalently by the orthogonal projection operators $\widehat{\Pi}_{\mathfrak{M}}$, relating each mental state $\mathfrak{M}$ with the set of all corresponding physical states $\mathcal{H}_{\mathfrak{M}}$. Furthermore, the nature of God encompasses the laws of Physics, given by the Hamilton operator $\widehat{\mathrm{H}}$, or more precisely by the elementary evolution operator $\widehat{U}_\tau := \exp\left(-\mathrm{i}2\pi\widehat{\mathrm{H}}\tau/\mathrm{h}\right)$. God can finally be defined as a mathematical structure $\mathfrak{G}$ given by:

$$\mathfrak{G} := \left(\mathcal{H} \times \mathcal{M}, (\widehat{\Pi}_{\mathfrak{M}}), \widehat{U}_\tau\right)$$

The states of God, taking the general form $(\ |\Psi\rangle, \mathfrak{M})$, are said to be "real" if $|\Psi\rangle \in \mathcal{H}_{\mathfrak{M}}$ and "virtual" otherwise. An elementary evolution step of the state of God proceeds from a real state $(\ |\Psi_0\rangle, \mathfrak{M}_0)$, first evolving to a generally virtual state $(\widehat{U}_\tau\ |\Psi_0\rangle, \mathfrak{M}_0)$ and eventually collapsing to one of the real states $(\widehat{\Pi}_{\mathfrak{M}_1}\widehat{U}_\tau\ |\Psi_0\rangle, \mathfrak{M}_1)$ with a probability $\langle\Psi_0|\ \widehat{U}_\tau^\dagger\widehat{\Pi}_{\mathfrak{M}_1}\widehat{U}_\tau\ |\Psi_0\rangle / \langle\Psi_0|\Psi_0\rangle$.

Chapter 6

Philosophical issues

6.1 Skepticism

According to philosophical skepticism, in the form of Descartes' *Cogito Ergo Sum* argument in his *Discourse on the Method* [4] for instance, the one and only aspect of the world which we know beyond any doubt to be real is the experience of our present individual mind state, the 'Cogito'. Nothing can guarantee us that the representations of the world carried by this mind state – like our past experiences, the feeling of the permanence of our existence, the image of our body, of the outer world, of our relations to others – have or have had any physical reality. In particular, it cannot be taken for granted that experimental evidence can be accumulated over the ages: Experimental science *must* rely on the mere belief that the mental representations of what we consider to be accumulated experimental evidence are related to physical processes that really did happen in the past. Indeed, in physical terms, stating that I am experiencing some individual mind state $\mathfrak{m}_s$ only implies that the collective mind state $\mathfrak{M} = (N_\mathfrak{m})$ is such that $N_{\mathfrak{m}_s} \geq 1$ and that the physical state $|\Psi\rangle$ of the universe belongs to $\mathcal{H}^+_{\mathfrak{m}_s}$. It doesn't necessarily imply that the past evolution of $|\Psi\rangle$ corresponds to the mental representation of past experiences in the mind state $\mathfrak{m}_s$. The physical theory presented in this book belongs therefore to the long tradition of philosophical skepticism insofar as it doubts the very possibility of experimental science.

6.2 Materialism

Materialism is the doctrine according to which the subjective experience of consciousness can be completely reduced to the corresponding physical processes happening within our brains and thus can be explained without involving any other level of reality than the purely physical one. It is generally considered among philosophers as the daydream of a physicist absorbed by his study object and becoming blind for the reality of his own subjective experience. Nevertheless, it still has numerous supporters in today's scientific community. In the frame of the theory developed in this book, it could be formulated as the hypothesis that no individual mind state is possible, since this would be equivalent to denying the existence of the mental world, which is of another nature as the physical one. Mathematically, this hypothesis can be expressed simply as $\mathcal{H}_{\mathfrak{M}_\Omega} = \mathcal{H}$, so that no individual mind state is being experienced in any physical state. Equivalently, this could be expressed in terms of collapse operators by $\widehat{\Pi}_{\mathfrak{M}_\Omega} = \mathbb{1}$, so that there is no collapse of the physical state of the universe. Its evolution reduces therefore to its Hamiltonian part,

$$|\Psi(t)\rangle \quad = \quad \exp\left(-\mathrm{i}2\pi\frac{t-t_0}{\mathrm{h}}\widehat{\mathrm{H}}\right)\,|\Psi(t_0)\rangle$$

and the stochastic process of collective mind selection do not apply.

Materialism in this context is facing the problem that it cannot satisfactorily explain how it is supposed to "feel like" in physical states where brains happen to be in a quantum superposition of physical states corresponding to different states of consciousness. There are two well-known ways of trying to escape this issue. In the no collapse theory of Everett, each consciousness state in the quantum superposition of a brain is supposed to be equally real as the others and to be experienced on its own. More precisely, these consciousness states are supposed to be statistically "weighted" in some (mysterious) way by the square norm of the corresponding component of the physical state of the universe, so that we are supposed to be more likely to experience them if they correspond to a component with a greater square norm.

The second way of escaping the difficulties of materialism is to deny that there are "noticeable" quantum superpositions of consciousness states of the brain. This is basically the aim of all spontaneous collapse theories, which have been reviewed exhaustively by Angelo Bassi and GianCarlo Ghirardi in their report *Dynamical Reduction Models* [1]. Generally, the physical state of the universe is supposed to collapse in such a way that the center of mass of macroscopic objects is practically always localized in a small region of space, so that we cannot notice its quantum fluctuations with our naked senses. As a consequence, insofar as our consciousness state is being mostly driven by sensory experience only, the states of consciousness corresponding to the components of a quantum superposition of brains are most likely to differ very little from another, so that it shouldn't really mind if we don't know which one is being experienced.

6.3 Solipsism

The solipsist is convinced that she is (and must be) the only person in the universe who has a subjective mental experience. Solipsism makes thus unproblematic the fact that we are experiencing the mental world in the form of a single individual mind instead of experiencing the whole collective mind state directly. In the frame of the theory developed in this book, solipsism can be expressed as the hypothesis that the only possible collective mind states (apart from $\mathfrak{M}_\Omega$) are of the form $1_\mathfrak{m}$, or in physical terms, that:

$$\mathcal{H} = \mathcal{H}_{\mathfrak{M}_\Omega} \overset{\perp}{\oplus} \overset{\perp}{\bigoplus_{\mathfrak{m}}} \mathcal{H}_{1_\mathfrak{m}}$$

Of course, this hypothesis is logically perfectly correct, but it is utmost difficult to make it compatible with the idea that mind states are being realized physically by the presence of corresponding physical states of brains. Even if one supposes that the solipsist's brain has something special that makes it differ from other brains that aren't being experienced as individual mind states, one faces the problem that a physical state in which many "copies" of the solipsist's brain, corresponding to different individual mind states, would be present couldn't be related in a satisfactory way to a single individual mind state: It is unclear, for instance, if physical states in a subspace of the form $\widehat{\Psi^{\alpha'}_{\mathfrak{m}'}}^{\dagger} \widehat{\Psi^{\alpha}_{\mathfrak{m}}}^{\dagger} \mathcal{H}_{\mathfrak{M}_\Omega}$ should be experienced as the mental state $1_\mathfrak{m}$ or $1_{\mathfrak{m}'}$.

Part IV

Physical interactions

Chapter 7

Quantum Electrodynamics

7.1 Abstract

In this chapter, we will define the interaction Hamiltonian of Quantum Electrodynamics (QED), describing the photon mediated electromagnetic interaction between electrically charged fermions, and we will derive the composition of the corresponding dressed particles.

7.2 Electric charge operator

On each point $\boldsymbol{n}$ of space, the electric charge operator is defined by:

$$\widehat{Q}_{\boldsymbol{n}} := \mathrm{e} \sum_{\phi,\lambda',\lambda} \mathrm{Q}_{\phi} \left(\widehat{\overline{\psi}^{\phi}}_{\boldsymbol{n},\lambda'} + \widehat{\psi^{\overline{\phi}}}_{\boldsymbol{n},\lambda'} \right) \gamma^0 \left(\widehat{\psi^{\phi}}_{\boldsymbol{n},\lambda} + \widehat{\overline{\psi}^{\overline{\phi}}}_{\boldsymbol{n},\lambda} \right)$$

where e is the elementary electric charge (opposite electric charge of a bare electron) and Q_ϕ the electric charge number of fermions of type ϕ: $\mathrm{Q}_\phi = 0$ for the neutrinos $\phi \in \{\nu_e, \nu_\mu, \nu_\tau\}$, $\mathrm{Q}_\phi = -1$ for the charged leptons $\phi \in \{e, \mu, \tau\}$, $\mathrm{Q}_\phi = \frac{2}{3}$ for the quarks $\phi \in \{u, c, t\}$ and $\mathrm{Q}_\phi = -\frac{1}{3}$ for the quarks $\phi \in \{d, s, b\}$. In this expression, the creation and annihilation spinor operators are defined as in C.2 and the Dirac matrices as in B.2.

The anti-particle creation and annihilation spinor operators in this expression yield to a uniformly distributed mean electric charge in the vacuum state, given by:

$$\langle \Omega | \ \widehat{Q}_{\boldsymbol{n}} \ | \Omega \rangle = \mathrm{e} \sum_{\phi,\lambda} \mathrm{Q}_\phi = -4\mathrm{e}$$

Since the charge distribution is uniform, it doesn't have any contribution to the interaction Hamiltonian as defined in 7.6.

7.3 Electric current operator

On each point $\boldsymbol{n}$ of space, the electric current operator is defined by:

$$\widehat{\boldsymbol{J}}_{\boldsymbol{n}} := \mathrm{ec} \sum_{\phi,\lambda',\lambda} \mathrm{Q}_{\phi} \left(\widehat{\overline{\psi}^{\phi}}_{\boldsymbol{n},\lambda'} + \widehat{\psi^{\overline{\phi}}}_{\boldsymbol{n},\lambda'} \right) \boldsymbol{\gamma} \left(\widehat{\psi^{\phi}}_{\boldsymbol{n},\lambda} + \widehat{\overline{\psi}^{\overline{\phi}}}_{\boldsymbol{n},\lambda} \right)$$

where the summation goes over all fermions ϕ. In this expression, the creation and annihilation spinor operators are defined as in C.2 and the Dirac matrices as in B.2.

7.4 Electric potential operator

On each point $\boldsymbol{n}$ of space, the electric potential operator is defined by:

$$\widehat{V}_{\boldsymbol{n}} := \sum_{\boldsymbol{n}'} \frac{\widehat{Q}_{\boldsymbol{n}'}}{8\pi\varepsilon_0 \mathrm{a}} (1+2\mathrm{N})^{-3} \sum_{\boldsymbol{q}_\gamma \neq \boldsymbol{0}} \frac{\exp\left(\mathrm{i}2\pi \boldsymbol{q}_\gamma \cdot (\boldsymbol{n}-\boldsymbol{n}')\right)}{\pi q_\gamma^2}$$

where ε_0 is the permittivity of the bare vacuum. Its constant Fourier component has been set to 0 (which is consistent with the Coulomb gauge condition used in C.1), so that the zero-point electric charge in 7.2 doesn't have any contribution to the interaction Hamiltonian as defined in 7.6.

7.5 Magnetic potential operator

On each point $\boldsymbol{n}$ of space, the magnetic potential operator is defined by:

$$\widehat{\boldsymbol{A}}_{\boldsymbol{n}} := \sum_{\lambda_\gamma} \widehat{\boldsymbol{\psi}^{\gamma}}^{\dagger}_{\boldsymbol{n},\lambda_\gamma} + \widehat{\boldsymbol{\psi}^{\gamma}}_{\boldsymbol{n},\lambda_\gamma}$$

In this expression, the creation and annihilation spinor operators are defined as in C.1.

7.6 QED interaction Hamiltonian

The interaction Hamiltonian of QED is defined by:

$$\widehat{\mathrm{H}}'_{QED} := \sum_{\boldsymbol{n}} \widehat{\boldsymbol{J}}_{\boldsymbol{n}} \cdot \widehat{\boldsymbol{A}}_{\boldsymbol{n}} + \widehat{Q}_{\boldsymbol{n}} \widehat{V}_{\boldsymbol{n}}$$

Its development on the plane waves basis is given by:

$$\begin{aligned}\sum_{\boldsymbol{n}} \widehat{\boldsymbol{J}}_{\boldsymbol{n}} \cdot \widehat{\boldsymbol{A}}_{\boldsymbol{n}} \;=\;& \sqrt{\frac{\mathrm{e}^2\mathrm{hc}}{8\pi^2\varepsilon_0\mathrm{a}^2}}(1+2\mathrm{N})^{-3/2} \sum_{\phi,\boldsymbol{q},\lambda',\lambda} \mathrm{Q}_\phi \sum_{\boldsymbol{q}_\gamma\neq\boldsymbol{0},\lambda_\gamma} q_\gamma^{-1/2} \\ & \left[\left(\widehat{\overline{\psi}^{\phi}}_{\boldsymbol{q}-\boldsymbol{q}_\gamma,\lambda'} + \widehat{\psi^{\overline{\phi}}}_{-\boldsymbol{q}+\boldsymbol{q}_\gamma,\lambda'}\right)\boldsymbol{\gamma}\left(\widehat{\psi^{\phi}}_{\boldsymbol{q},\lambda} + \widehat{\overline{\psi}^{\overline{\phi}}}_{-\boldsymbol{q},\lambda}\right)\cdot\right. \\ & \boldsymbol{\varepsilon}^*_{\boldsymbol{q}_\gamma,\lambda_\gamma}\widehat{a^\gamma}^{\dagger}_{\boldsymbol{q}_\gamma,\lambda_\gamma}\sqrt{1+\widehat{N^\gamma}_{\boldsymbol{q}_\gamma,\lambda_\gamma}} + \\ & \left(\widehat{\overline{\psi}^{\phi}}_{\boldsymbol{q}+\boldsymbol{q}_\gamma,\lambda'} + \widehat{\psi^{\overline{\phi}}}_{-\boldsymbol{q}-\boldsymbol{q}_\gamma,\lambda'}\right)\boldsymbol{\gamma}\left(\widehat{\psi^{\phi}}_{\boldsymbol{q},\lambda} + \widehat{\overline{\psi}^{\overline{\phi}}}_{-\boldsymbol{q},\lambda}\right)\cdot \\ & \left.\boldsymbol{\varepsilon}_{\boldsymbol{q}_\gamma,\lambda_\gamma}\widehat{a^\gamma}_{\boldsymbol{q}_\gamma,\lambda_\gamma}\sqrt{\widehat{N^\gamma}_{\boldsymbol{q}_\gamma,\lambda_\gamma}}\right]\end{aligned}$$

$$\begin{aligned}\sum_{\boldsymbol{n}} \widehat{Q}_{\boldsymbol{n}}\widehat{V}_{\boldsymbol{n}} \;=\;& \frac{\mathrm{e}^2}{8\pi^2\varepsilon_0\mathrm{a}}(1+2\mathrm{N})^{-3} \sum_{\phi,\boldsymbol{q},\lambda',\lambda} \mathrm{Q}_\phi \sum_{\phi_0,\boldsymbol{q}_0,\lambda'_0,\lambda_0} \mathrm{Q}_{\phi_0} \sum_{\boldsymbol{q}_\gamma\neq\boldsymbol{0}} q_\gamma^{-2} \\ & \left(\widehat{\overline{\psi}^{\phi}}_{\boldsymbol{q}+\boldsymbol{q}_\gamma,\lambda'} + \widehat{\psi^{\overline{\phi}}}_{-\boldsymbol{q}-\boldsymbol{q}_\gamma,\lambda'}\right)\gamma^0\left(\widehat{\psi^{\phi}}_{\boldsymbol{q},\lambda} + \widehat{\overline{\psi}^{\overline{\phi}}}_{-\boldsymbol{q},\lambda}\right) \\ & \left(\widehat{\overline{\psi}^{\phi_0}}_{\boldsymbol{q}_0-\boldsymbol{q}_\gamma,\lambda'_0} + \widehat{\psi^{\overline{\phi_0}}}_{-\boldsymbol{q}_0+\boldsymbol{q}_\gamma,\lambda'_0}\right)\gamma^0\left(\widehat{\psi^{\phi_0}}_{\boldsymbol{q}_0,\lambda_0} + \widehat{\overline{\psi}^{\overline{\phi_0}}}_{-\boldsymbol{q}_0,\lambda_0}\right)\end{aligned}$$

7.7 Dressed states

As a consequence of the electromagnetic interaction between the photon and the charged fermion fields, an excitation of a single particle field like $\left|N^{\phi}_{\boldsymbol{q},\lambda}\right\rangle$ is unstable and is also a poor model for observed particles. In fact, these particles are always being observed “dressed”, i.e. forming a particle complex together with excitations of the other fields. As a consequence, the “bare” rest mass, electric charge and magnatic moment of these particles, as they appear in the QED model, do not correspond to the values observed by dressed particles. These renormalized values as well as the composition of dressed particles can be calculated in the frame of QED as a function of the bare values, which can also be indirectly determined experimentally.

We consider a bare state of the form $\left|(N^{\phi}_{\boldsymbol{q},\lambda})_0\right\rangle$ and will derive the corresponding dressed state $|\Psi\rangle$ as eigenstate of $\widehat{\mathrm{H}}_0 + \widehat{\mathrm{H}}'$ for an eigenvalue E to be determined. Assuming the bare and dressed states aren’t orthogonal to each other, we write the latter as:

$$|\Psi\rangle := \widetilde{\Psi}\left((N^{\phi}_{\boldsymbol{q},\lambda})_0\right) \sum_{(N^{\phi}_{\boldsymbol{q},\lambda})} \widetilde{\Phi}_0\left((N^{\phi}_{\boldsymbol{q},\lambda})\right)\,\left|(N^{\phi}_{\boldsymbol{q},\lambda})\right\rangle$$

using unnormalized coefficients $\widetilde{\Phi}_0$ verifying the condition:

$$\widetilde{\Phi}_0\left((N^{\phi}_{\boldsymbol{q},\lambda})_0\right) = 1$$

The eigenvalue equation, projected on $\left|(N^{\phi}_{\boldsymbol{q},\lambda})_0\right\rangle$ resp. on another plane wave state $\left|(N^{\phi}_{\boldsymbol{q},\lambda})_1\right\rangle$, reads:

$$\begin{aligned}
H'_{0,0} + \sum_{(N^{\phi}_{\boldsymbol{q},\lambda})_2 \neq (N^{\phi}_{\boldsymbol{q},\lambda})_0} H'_{0,2}\widetilde{\Phi}_{2,0} &= E - E_0 \\
H'_{1,0} + \sum_{(N^{\phi}_{\boldsymbol{q},\lambda})_2 \neq (N^{\phi}_{\boldsymbol{q},\lambda})_0} H'_{1,2}\widetilde{\Phi}_{2,0} &= (E - E_1)\widetilde{\Phi}_{1,0}
\end{aligned}$$

where we use the shorthand notations:

$$\begin{aligned}
H'_{b,a} &:= \left\langle (N^{\phi}_{\boldsymbol{q},\lambda})_b \right| \widehat{\mathrm{H}}' \left| (N^{\phi}_{\boldsymbol{q},\lambda})_a \right\rangle \\
E_a &:= \left\langle (N^{\phi}_{\boldsymbol{q},\lambda})_a \right| \widehat{\mathrm{H}}_0 \left| (N^{\phi}_{\boldsymbol{q},\lambda})_a \right\rangle \\
\widetilde{\Phi}_{a,0} &:= \widetilde{\Phi}_0\left((N^{\phi}_{\boldsymbol{q},\lambda})_a\right)
\end{aligned}$$

To solve this equation iteratively, we develop $\widehat{\mathrm{H}}'$, $\widetilde{\Phi}_0$ and E as power series in the elementary electric charge e:

$$\begin{aligned}
\widehat{\mathrm{H}}' &:= \widehat{\mathrm{H}}'^{(1)} + \widehat{\mathrm{H}}'^{(2)} \\
\widetilde{\Phi}_0 &:= \sum_{n=0}^{\infty} \widetilde{\Phi}_0^{(n)} \\
E &:= \sum_{n=0}^{\infty} E^{(n)}
\end{aligned}$$

where we take:

$$\begin{aligned}
\widehat{\mathrm{H}}'^{(1)} &:= \sum_{\boldsymbol{n}} \widehat{\boldsymbol{J}}_{\boldsymbol{n}} \cdot \widehat{\boldsymbol{A}}_{\boldsymbol{n}} \\
\widehat{\mathrm{H}}'^{(2)} &:= \sum_{\boldsymbol{n}} \widehat{Q}_{\boldsymbol{n}} \widehat{V}_{\boldsymbol{n}} \\
\widetilde{\Phi}^{(0)}_{a,0} &:= \delta_{a,0}
\end{aligned}$$

In the case of bare states which are nondegenerate with respect to $\widehat{\mathrm{H}}_0$, i.e. such as $E_2 \neq E_0$ for any $(N^{\phi}_{\boldsymbol{q},\lambda})_2 \neq (N^{\phi}_{\boldsymbol{q},\lambda})_0$, assuming that the eigenvalue equation should hold to each order separately, we have to the zeroth order:

$$E^{(0)} = E_0$$

to the first order:

$$\begin{aligned} E^{(1)} &= H'^{(1)}_{0,0} \\ \widetilde{\Phi}^{(1)}_{1,0} &= \frac{H'^{(1)}_{1,0}}{E_0 - E_1} \end{aligned}$$

to the second order:

$$\begin{aligned} E^{(2)} &= H'^{(2)}_{0,0} + \sum_{(N^{\phi}_{\boldsymbol{q},\lambda})_2 \neq (N^{\phi}_{\boldsymbol{q},\lambda})_0} \frac{H'^{(1)}_{0,2} H'^{(1)}_{2,0}}{E_0 - E_2} \\ \widetilde{\Phi}^{(2)}_{1,0} &= \frac{H'^{(2)}_{1,0}}{E_0 - E_1} + \sum_{(N^{\phi}_{\boldsymbol{q},\lambda})_2 \neq (N^{\phi}_{\boldsymbol{q},\lambda})_0} \frac{H'^{(1)}_{1,2} H'^{(1)}_{2,0}}{(E_0 - E_1)(E_0 - E_2)} - \frac{H'^{(1)}_{1,0} H'^{(1)}_{0,0}}{(E_0 - E_1)^2} \end{aligned}$$

and to the order $n > 2$:

$$\begin{aligned} E^{(n)} &= \sum_{(N^{\phi}_{\boldsymbol{q},\lambda})_2 \neq (N^{\phi}_{\boldsymbol{q},\lambda})_0} \left(H'^{(1)}_{0,2} \widetilde{\Phi}^{(n-1)}_{2,0} + H'^{(2)}_{0,2} \widetilde{\Phi}^{(n-2)}_{2,0} \right) \\ \widetilde{\Phi}^{(n)}_{1,0} &= \sum_{(N^{\phi}_{\boldsymbol{q},\lambda})_2 \neq (N^{\phi}_{\boldsymbol{q},\lambda})_0} \frac{H'^{(1)}_{1,2} \widetilde{\Phi}^{(n-1)}_{2,0} + H'^{(2)}_{1,2} \widetilde{\Phi}^{(n-2)}_{2,0}}{E_0 - E_1} - \sum_{m=1}^{n-1} \widetilde{\Phi}^{(m)}_{1,0} \frac{E^{(n-m)}}{E_0 - E_1} \end{aligned}$$

7.8 Dressed vacuum

The vacuum state itself isn't stable and would become populated by pair creation processes. Up to the first order, the dressed vacuum is composed of the bare vacuum $|\Omega\rangle$ as well as of states of the form $\left| 1^{\phi}_{\underline{\boldsymbol{q}-\boldsymbol{q}_\gamma},\lambda'} 1^{\overline{\phi}}_{-\boldsymbol{q},\lambda} 1^{\gamma}_{\boldsymbol{q}_\gamma,\lambda_\gamma} \right\rangle$, where ϕ is any electrically charged fermion and $\boldsymbol{q}_\gamma \neq \boldsymbol{0}$. The corresponding unnormalized coefficients are given by:

$$\widetilde{\Phi}^{(1)} = -\sqrt{\frac{\mathrm{e}^2}{4\pi\varepsilon_0 \mathrm{hc}}} (1+2\mathrm{N})^{-3/2} \mathrm{Q}_\phi \frac{u^{\phi\dagger}_{\boldsymbol{q}-\boldsymbol{q}_\gamma,\lambda'} \gamma^0 \boldsymbol{\gamma} u^{\overline{\phi}}_{-\boldsymbol{q},\lambda} \cdot \boldsymbol{\varepsilon}^*_{\boldsymbol{q}_\gamma,\lambda_\gamma}}{(2\pi q_\gamma)^{1/2} \left(E^{\phi}_{\boldsymbol{q}-\boldsymbol{q}_\gamma} + E^{\overline{\phi}}_{-\boldsymbol{q}} + E^{\gamma}_{\boldsymbol{q}_\gamma} \right) \mathrm{a/hc}}$$

The corresponding energy is of second order and can be written as:

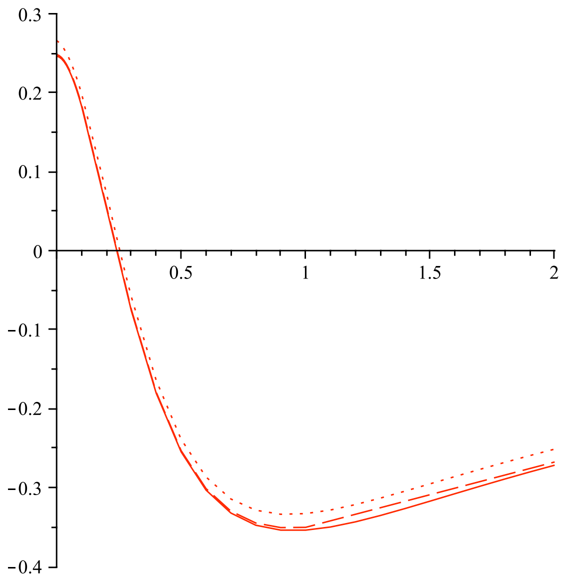
$$\begin{aligned} E^{(2)} &= (1+2\mathrm{N})^3 \frac{\mathrm{e}^2}{4\pi\varepsilon_0 \mathrm{a}} \sum_{\phi} \mathrm{Q}^2_\phi \kappa^{\Omega}_\phi \\ \kappa^{\Omega}_\phi &:= (1+2\mathrm{N})^{-6} \sum_{\boldsymbol{q},\boldsymbol{q}_\gamma \neq \boldsymbol{0}} \left[\frac{1}{2\pi q_\gamma^2} \sum_{\lambda',\lambda} \left| u^{\phi\dagger}_{\boldsymbol{q}-\boldsymbol{q}_\gamma,\lambda'} u^{\overline{\phi}}_{-\boldsymbol{q},\lambda} \right|^2 \right. \\ & \left. - \frac{\sum_{\lambda',\lambda,\lambda_\gamma} \left| u^{\phi\dagger}_{\boldsymbol{q}-\boldsymbol{q}_\gamma,\lambda'} \gamma^0 \boldsymbol{\gamma} u^{\overline{\phi}}_{-\boldsymbol{q},\lambda} \cdot \boldsymbol{\varepsilon}^*_{\boldsymbol{q}_\gamma,\lambda_\gamma} \right|^2}{2\pi q_\gamma \left(E^{\phi}_{\boldsymbol{q}-\boldsymbol{q}_\gamma} + E^{\overline{\phi}}_{-\boldsymbol{q}} + E^{\gamma}_{\boldsymbol{q}_\gamma} \right) \mathrm{a/hc}} \right] \end{aligned}$$

where the spin summations evaluate to:

$$\sum_{\lambda',\lambda} \left| u^{\phi\dagger}_{\boldsymbol{q}-\boldsymbol{q}_\gamma,\lambda'} u^{\overline{\phi}}_{-\boldsymbol{q},\lambda} \right|^2 = 1 - \frac{(\mathrm{m}_\phi \mathrm{ac/h})^2 + \underline{\boldsymbol{q}-\boldsymbol{q}_\gamma} \cdot \boldsymbol{q}}{E^{\phi}_{\boldsymbol{q}-\boldsymbol{q}_\gamma} E^{\overline{\phi}}_{-\boldsymbol{q}} (\mathrm{a/hc})^2}$$

$$\sum_{\lambda',\lambda,\lambda_\gamma} \left| u^{\phi\dagger}_{\boldsymbol{q}-\boldsymbol{q}_\gamma,\lambda'} \gamma^0 \boldsymbol{\gamma} u^{\overline{\phi}}_{-\boldsymbol{q},\lambda} \cdot \boldsymbol{\varepsilon}^*_{\boldsymbol{q}_\gamma,\lambda_\gamma} \right|^2 = 2\left(1 + \frac{(\mathrm{m}_\phi \mathrm{ac/h})^2 + (\underline{\boldsymbol{q}-\boldsymbol{q}_\gamma} \cdot \boldsymbol{q}_\gamma)(\boldsymbol{q}\cdot\boldsymbol{q}_\gamma)/q_\gamma^2}{E^{\phi}_{\boldsymbol{q}-\boldsymbol{q}_\gamma} E^{\overline{\phi}}_{-\boldsymbol{q}} (\mathrm{a/hc})^2}\right)$$

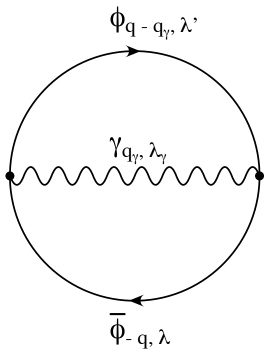
The numerical coefficients κ^{Ω}_{ϕ} only depend on the masses of the bare particles and are plotted below:



κ^{Ω}_{ϕ} as a function of $\mathrm{m}_\phi \mathrm{ac/h}$
for N = 1, 2 and 3 (dotted, dashed and solid lines)

In $\mathrm{m}_\phi \mathrm{ac/h} = 0$, we have $\kappa^{\Omega}_{\phi} \approx 0.266$, 0.248 and 0.246 for N = 1, 2 and 3 respectively; as $\mathrm{m}_\phi \mathrm{ac/h} \to \infty$, we have $\kappa^{\Omega}_{\phi} \to 0^-$. Since the result converges to an integral expression for $\mathrm{N} \to \infty$, I shall assume that the coefficients obtained by carrying out the computation for N = 3 are already a good approximation.

This energy is represented by following Feynman diagram:



where the Coulomb interaction term is conventionally represented by the case $\lambda_\gamma = 0$. Assumung $\kappa_\phi^\Omega \approx 0.25$ for electrically charged fermions, the energy of the electromagnetically dressed vacuum evaluates up to the second order to:

$$E^{(2)} \approx 0.25(1+2\mathrm{N})^3 \frac{14}{3} \frac{\mathrm{e}^2}{4\pi\varepsilon_0 \mathrm{a}}$$

7.9 Dressed charged fermion

We consider an electrically charged fermion of type f in the bare state $\left|1^f_{\boldsymbol{q}_f,\lambda_f}\right\rangle$. Up to the first order, the corresponding dressed state is composed of the bare state, of states of the form $\left|1^f_{\boldsymbol{q}_f,\lambda_f} 1^\phi_{\underline{\boldsymbol{q}-\boldsymbol{q}_\gamma},\lambda'} 1^{\overline{\phi}}_{-\boldsymbol{q},\lambda} 1^\gamma_{\boldsymbol{q}_\gamma,\lambda_\gamma}\right\rangle$, where ϕ is any electrically charged fermion, $\boldsymbol{q}_\gamma \neq \mathbf{0}$ and $(\phi, \boldsymbol{q}-\boldsymbol{q}_\gamma, \lambda') \neq (f, \boldsymbol{q}_f, \lambda_f)$, as well as of states of the form $\left|1^f_{\underline{\boldsymbol{q}_f-\boldsymbol{q}_\gamma},\lambda} 1^\gamma_{\boldsymbol{q}_\gamma,\lambda_\gamma}\right\rangle$, where $\boldsymbol{q}_\gamma \neq \mathbf{0}$. The corresponding unnormalized coefficients are given by:

$$\widetilde{\Phi}^{(1)} = -\sqrt{\frac{\mathrm{e}^2}{4\pi\varepsilon_0 \mathrm{hc}}}(1+2\mathrm{N})^{-3/2}\mathrm{Q}_\phi \frac{u^{\phi\dagger}_{\boldsymbol{q}-\boldsymbol{q}_\gamma,\lambda'} \gamma^0 \boldsymbol{\gamma} u^{\overline{\phi}}_{-\boldsymbol{q},\lambda} \cdot \boldsymbol{\varepsilon}^*_{\boldsymbol{q}_\gamma,\lambda_\gamma}}{(2\pi q_\gamma)^{1/2}\left(E^\phi_{\boldsymbol{q}-\boldsymbol{q}_\gamma} + E^{\overline{\phi}}_{-\boldsymbol{q}} + E^\gamma_{\boldsymbol{q}_\gamma}\right)\mathrm{a/hc}}$$

for states of the form $\left|1^f_{\boldsymbol{q}_f,\lambda_f} 1^\phi_{\underline{\boldsymbol{q}-\boldsymbol{q}_\gamma},\lambda'} 1^{\overline{\phi}}_{-\boldsymbol{q},\lambda} 1^\gamma_{\boldsymbol{q}_\gamma,\lambda_\gamma}\right\rangle$, and by:

$$\widetilde{\Phi}^{(1)} = -\sqrt{\frac{\mathrm{e}^2}{4\pi\varepsilon_0 \mathrm{hc}}}(1+2\mathrm{N})^{-3/2}\mathrm{Q}_f \frac{u^{f\dagger}_{\boldsymbol{q}_f-\boldsymbol{q}_\gamma,\lambda} \gamma^0 \boldsymbol{\gamma} u^f_{\boldsymbol{q}_f,\lambda_f} \cdot \boldsymbol{\varepsilon}^*_{\boldsymbol{q}_\gamma,\lambda_\gamma}}{(2\pi q_\gamma)^{1/2}\left(E^f_{\boldsymbol{q}_f-\boldsymbol{q}_\gamma} + E^\gamma_{\boldsymbol{q}_\gamma} - E^f_{\boldsymbol{q}_f}\right)\mathrm{a/hc}}$$

for states of the form $\left|1^f_{\underline{\boldsymbol{q}_f-\boldsymbol{q}_\gamma},\lambda} 1^\gamma_{\boldsymbol{q}_\gamma,\lambda_\gamma}\right\rangle$, respectively.

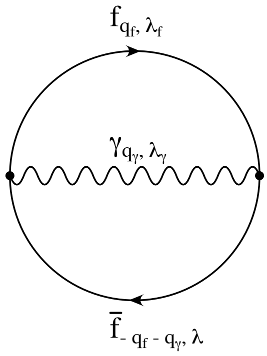
The corresponding energy is of second order and can be written as:

$$
\begin{aligned}
E^{(2)} &= E^{(2)}(\Omega) + \frac{\mathrm{e}^2}{4\pi\varepsilon_0 \mathrm{a}} \mathrm{Q}_f^2 \kappa^f_{\boldsymbol{q}_f,\lambda_f} \\
\kappa^f_{\boldsymbol{q}_f,\lambda_f} &:= (1+2\mathrm{N})^{-3} \sum_{\boldsymbol{q}_\gamma \neq \boldsymbol{0}} \left[\frac{1}{2\pi q_\gamma^2} \sum_\lambda \left| {u^f}^\dagger_{\boldsymbol{q}_f-\boldsymbol{q}_\gamma,\lambda} u^f_{\boldsymbol{q}_f,\lambda_f} \right|^2 \right. \\
&\quad \left. - \frac{\sum_{\lambda,\lambda_\gamma} \left| {u^f}^\dagger_{\boldsymbol{q}_f-\boldsymbol{q}_\gamma,\lambda} \gamma^0 \boldsymbol{\gamma} u^f_{\boldsymbol{q}_f,\lambda_f} \cdot \boldsymbol{\varepsilon}^*_{\boldsymbol{q}_\gamma,\lambda_\gamma} \right|^2}{2\pi q_\gamma \left(E^f_{\boldsymbol{q}_f-\boldsymbol{q}_\gamma} + E^\gamma_{\boldsymbol{q}_\gamma} - E^f_{\boldsymbol{q}_f} \right) \mathrm{a}/\mathrm{hc}} \right] \\
&\quad -(1+2\mathrm{N})^{-3} \sum_{\boldsymbol{q}_\gamma \neq \boldsymbol{0}} \left[\frac{1}{2\pi q_\gamma^2} \sum_\lambda \left| {u^f}^\dagger_{\boldsymbol{q}_f,\lambda_f} u^{\overline{f}}_{-\boldsymbol{q}_f-\boldsymbol{q}_\gamma,\lambda} \right|^2 \right. \\
&\quad \left. - \frac{\sum_{\lambda,\lambda_\gamma} \left| {u^f}^\dagger_{\boldsymbol{q}_f,\lambda_f} \gamma^0 \boldsymbol{\gamma} u^{\overline{f}}_{-\boldsymbol{q}_f-\boldsymbol{q}_\gamma,\lambda} \cdot \boldsymbol{\varepsilon}^*_{\boldsymbol{q}_\gamma,\lambda_\gamma} \right|^2}{2\pi q_\gamma \left(E^f_{\boldsymbol{q}_f} + E^{\overline{f}}_{-\boldsymbol{q}_f-\boldsymbol{q}_\gamma} + E^\gamma_{\boldsymbol{q}_\gamma} \right) \mathrm{a}/\mathrm{hc}} \right]
\end{aligned}
$$

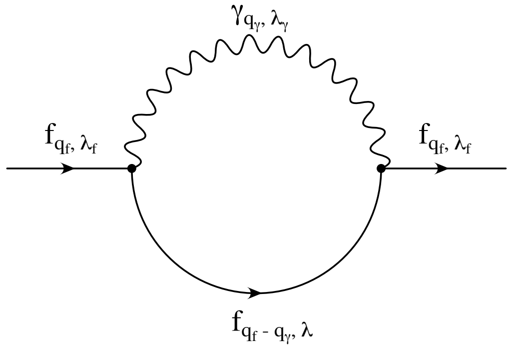
where $E^{(2)}(\Omega)$ is the second order energy of the dressed vacuum. The spin summations evaluate to:

$$
\begin{aligned}
\sum_\lambda \left| {u^f}^\dagger_{\boldsymbol{q}_f-\boldsymbol{q}_\gamma,\lambda} u^f_{\boldsymbol{q}_f,\lambda_f} \right|^2 &= \frac{1}{2} \left(1 + \frac{(\mathrm{m}_f \mathrm{ac}/\mathrm{h})^2 + \underline{\boldsymbol{q}_f - \boldsymbol{q}_\gamma} \cdot \boldsymbol{q}_f}{E^f_{\boldsymbol{q}_f-\boldsymbol{q}_\gamma} E^f_{\boldsymbol{q}_f} (\mathrm{a}/\mathrm{hc})^2} \right) \\
\sum_{\lambda,\lambda_\gamma} \left| {u^f}^\dagger_{\boldsymbol{q}_f-\boldsymbol{q}_\gamma,\lambda} \gamma^0 \boldsymbol{\gamma} u^f_{\boldsymbol{q}_f,\lambda_f} \cdot \boldsymbol{\varepsilon}^*_{\boldsymbol{q}_\gamma,\lambda_\gamma} \right|^2 &= 1 - \frac{(\mathrm{m}_f \mathrm{ac}/\mathrm{h})^2 + (\underline{\boldsymbol{q}_f - \boldsymbol{q}_\gamma} \cdot \boldsymbol{q}_\gamma)(\boldsymbol{q}_f \cdot \boldsymbol{q}_\gamma)/q_\gamma^2}{E^f_{\boldsymbol{q}_f-\boldsymbol{q}_\gamma} E^f_{\boldsymbol{q}_f} (\mathrm{a}/\mathrm{hc})^2} \\
\sum_\lambda \left| {u^f}^\dagger_{\boldsymbol{q}_f,\lambda_f} u^{\overline{f}}_{-\boldsymbol{q}_f-\boldsymbol{q}_\gamma,\lambda} \right|^2 &= \frac{1}{2} \left(1 - \frac{(\mathrm{m}_f \mathrm{ac}/\mathrm{h})^2 + \boldsymbol{q}_f \cdot \underline{\boldsymbol{q}_f + \boldsymbol{q}_\gamma}}{E^f_{\boldsymbol{q}_f} E^{\overline{f}}_{-\boldsymbol{q}_f-\boldsymbol{q}_\gamma} (\mathrm{a}/\mathrm{hc})^2} \right) \\
\sum_{\lambda,\lambda_\gamma} \left| {u^f}^\dagger_{\boldsymbol{q}_f,\lambda_f} \gamma^0 \boldsymbol{\gamma} u^{\overline{f}}_{-\boldsymbol{q}_f-\boldsymbol{q}_\gamma,\lambda} \cdot \boldsymbol{\varepsilon}^*_{\boldsymbol{q}_\gamma,\lambda_\gamma} \right|^2 &= 1 + \frac{(\mathrm{m}_f \mathrm{ac}/\mathrm{h})^2 + (\boldsymbol{q}_f \cdot \boldsymbol{q}_\gamma)(\underline{\boldsymbol{q}_f + \boldsymbol{q}_\gamma} \cdot \boldsymbol{q}_\gamma)/q_\gamma^2}{E^f_{\boldsymbol{q}_f} E^{\overline{f}}_{-\boldsymbol{q}_f-\boldsymbol{q}_\gamma} (\mathrm{a}/\mathrm{hc})^2}
\end{aligned}
$$

The vacuum energy diagram is also completed by subtracting following contribution:



and by adding following self-energy diagram:



for a given mode $(\boldsymbol{q}_f, \lambda_f)$ of the fermion field.

7.10 Dressed photon

We consider a photon in the bare state $\left|1^{\gamma}_{\boldsymbol{q}_\gamma,\lambda_\gamma}\right\rangle$, where $\boldsymbol{q}_\gamma \neq \boldsymbol{0}$. Up to the first order, the corresponding dressed state is composed of the bare state, of states of the form $\left|1^{\gamma}_{\boldsymbol{q}_\gamma,\lambda_\gamma} 1^{\phi}_{\underline{\boldsymbol{q}-\boldsymbol{q}'_\gamma},\lambda'} 1^{\overline{\phi}}_{-\boldsymbol{q},\lambda} 1^{\gamma}_{\boldsymbol{q}'_\gamma,\lambda'_\gamma}\right\rangle$, where ϕ is any electrically charged fermion, $\boldsymbol{q}'_\gamma \neq \boldsymbol{0}$ and $(\boldsymbol{q}'_\gamma, \lambda'_\gamma) \neq (\boldsymbol{q}_\gamma, \lambda_\gamma)$, of states of the form $\left|2^{\gamma}_{\boldsymbol{q}_\gamma,\lambda_\gamma} 1^{\phi}_{\underline{\boldsymbol{q}-\boldsymbol{q}_\gamma},\lambda'} 1^{\overline{\phi}}_{-\boldsymbol{q},\lambda}\right\rangle$ as well as of states of the form $\left|1^{\phi}_{\underline{\boldsymbol{q}+\boldsymbol{q}_\gamma},\lambda'} 1^{\overline{\phi}}_{-\boldsymbol{q},\lambda}\right\rangle$. The corresponding unnormalized coefficients are given by:

$$\widetilde{\Phi}^{(1)} = -\sqrt{\frac{\mathrm{e}^2}{4\pi\varepsilon_0 \mathrm{hc}}}(1+2\mathrm{N})^{-3/2}\mathrm{Q}_\phi \frac{u^{\phi\dagger}_{\boldsymbol{q}-\boldsymbol{q}'_\gamma,\lambda'}\gamma^0 \boldsymbol{\gamma} u^{\overline{\phi}}_{-\boldsymbol{q},\lambda} \cdot \boldsymbol{\varepsilon}^*_{\boldsymbol{q}'_\gamma,\lambda'_\gamma}}{(2\pi q'_\gamma)^{1/2}\left(E^{\phi}_{\boldsymbol{q}-\boldsymbol{q}'_\gamma} + E^{\overline{\phi}}_{-\boldsymbol{q}} + E^{\gamma}_{\boldsymbol{q}'_\gamma}\right)\mathrm{a/hc}}$$

for states of the form $\left|1^{\gamma}_{\boldsymbol{q}_\gamma,\lambda_\gamma} 1^{\phi}_{\underline{\boldsymbol{q}-\boldsymbol{q}'_\gamma},\lambda'} 1^{\overline{\phi}}_{-\boldsymbol{q},\lambda} 1^{\gamma}_{\boldsymbol{q}'_\gamma,\lambda'_\gamma}\right\rangle$, by:

$$\widetilde{\Phi}^{(1)} = -\sqrt{2\frac{\mathrm{e}^2}{4\pi\varepsilon_0\mathrm{hc}}}(1+2\mathrm{N})^{-3/2}\mathrm{Q}_\phi \frac{u^{\phi\dagger}_{\boldsymbol{q}-\boldsymbol{q}_\gamma,\lambda'}\gamma^0\boldsymbol{\gamma}u^{\overline{\phi}}_{-\boldsymbol{q},\lambda}\cdot\boldsymbol{\varepsilon}^*_{\boldsymbol{q}_\gamma,\lambda_\gamma}}{(2\pi q_\gamma)^{1/2}\left(E^{\phi}_{\boldsymbol{q}-\boldsymbol{q}_\gamma}+E^{\overline{\phi}}_{-\boldsymbol{q}}+E^{\gamma}_{\boldsymbol{q}_\gamma}\right)\mathrm{a/hc}}$$

for states of the form $\left|2^{\gamma}_{\boldsymbol{q}_\gamma,\lambda_\gamma} 1^{\phi}_{\underline{\boldsymbol{q}-\boldsymbol{q}'_\gamma},\lambda'} 1^{\overline{\phi}}_{-\boldsymbol{q},\lambda}\right\rangle$, and by:

$$\widetilde{\Phi}^{(1)} = -\sqrt{\frac{\mathrm{e}^2}{4\pi\varepsilon_0\mathrm{hc}}}(1+2\mathrm{N})^{-3/2}\mathrm{Q}_\phi \frac{u^{\phi\dagger}_{\boldsymbol{q}+\boldsymbol{q}_\gamma,\lambda'}\gamma^0\boldsymbol{\gamma}u^{\overline{\phi}}_{-\boldsymbol{q},\lambda}\cdot\boldsymbol{\varepsilon}_{\boldsymbol{q}_\gamma,\lambda_\gamma}}{(2\pi q_\gamma)^{1/2}\left(E^{\phi}_{\boldsymbol{q}+\boldsymbol{q}_\gamma}+E^{\overline{\phi}}_{-\boldsymbol{q}}-E^{\gamma}_{\boldsymbol{q}_\gamma}\right)\mathrm{a/hc}}$$

for states of the form $\left|1^{\phi}_{\underline{\boldsymbol{q}+\boldsymbol{q}_\gamma},\lambda'} 1^{\overline{\phi}}_{-\boldsymbol{q},\lambda}\right\rangle$, respectively.

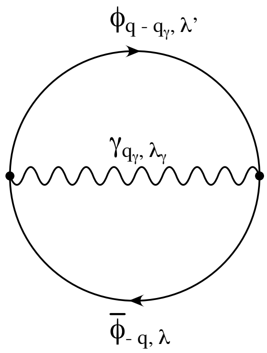
The corresponding energy is of second order and can be written as:

$$\begin{aligned}
E^{(2)} &= E^{(2)}(\Omega) - \frac{\mathrm{e}^2}{4\pi\varepsilon_0\mathrm{a}}\sum_\phi \mathrm{Q}^2_\phi \kappa^{\gamma}_{\boldsymbol{q}_\gamma,\lambda_\gamma,\phi} \\
\kappa^{\gamma}_{\boldsymbol{q}_\gamma,\lambda_\gamma,\phi} &:= (1+2\mathrm{N})^{-3}\sum_{\boldsymbol{q}} \frac{\sum_{\lambda',\lambda}\left|u^{\phi\dagger}_{\boldsymbol{q}-\boldsymbol{q}_\gamma,\lambda'}\gamma^0\boldsymbol{\gamma}u^{\overline{\phi}}_{-\boldsymbol{q},\lambda}\cdot\boldsymbol{\varepsilon}^*_{\boldsymbol{q}_\gamma,\lambda_\gamma}\right|^2}{2\pi q_\gamma\left(E^{\phi}_{\boldsymbol{q}-\boldsymbol{q}_\gamma}+E^{\overline{\phi}}_{-\boldsymbol{q}}+E^{\gamma}_{\boldsymbol{q}_\gamma}\right)\mathrm{a/hc}} \\
&\quad +(1+2\mathrm{N})^{-3}\sum_{\boldsymbol{q}} \frac{\sum_{\lambda',\lambda}\left|u^{\phi\dagger}_{\boldsymbol{q}+\boldsymbol{q}_\gamma,\lambda'}\gamma^0\boldsymbol{\gamma}u^{\overline{\phi}}_{-\boldsymbol{q},\lambda}\cdot\boldsymbol{\varepsilon}_{\boldsymbol{q}_\gamma,\lambda_\gamma}\right|^2}{2\pi q_\gamma\left(E^{\phi}_{\boldsymbol{q}+\boldsymbol{q}_\gamma}+E^{\overline{\phi}}_{-\boldsymbol{q}}-E^{\gamma}_{\boldsymbol{q}_\gamma}\right)\mathrm{a/hc}}
\end{aligned}$$

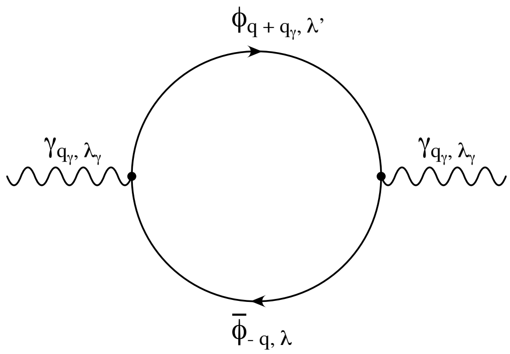
where the spin summations evaluate to:

$$\begin{aligned}
\sum_{\lambda',\lambda}\left|u^{\phi\dagger}_{\boldsymbol{q}-\boldsymbol{q}_\gamma,\lambda'}\gamma^0\boldsymbol{\gamma}u^{\overline{\phi}}_{-\boldsymbol{q},\lambda}\cdot\boldsymbol{\varepsilon}^*_{\boldsymbol{q}_\gamma,\lambda_\gamma}\right|^2 &= 1+\frac{(\mathrm{m}_\phi\mathrm{ac/h})^2+(\underline{\boldsymbol{q}-\boldsymbol{q}_\gamma}\cdot\boldsymbol{q}_\gamma)(\boldsymbol{q}\cdot\boldsymbol{q}_\gamma)/q_\gamma^2}{E^{\phi}_{\boldsymbol{q}-\boldsymbol{q}_\gamma}E^{\overline{\phi}}_{-\boldsymbol{q}}(\mathrm{a/hc})^2} \\
\sum_{\lambda',\lambda}\left|u^{\phi\dagger}_{\boldsymbol{q}+\boldsymbol{q}_\gamma,\lambda'}\gamma^0\boldsymbol{\gamma}u^{\overline{\phi}}_{-\boldsymbol{q},\lambda}\cdot\boldsymbol{\varepsilon}_{\boldsymbol{q}_\gamma,\lambda_\gamma}\right|^2 &= 1+\frac{(\mathrm{m}_\phi\mathrm{ac/h})^2+(\underline{\boldsymbol{q}+\boldsymbol{q}_\gamma}\cdot\boldsymbol{q}_\gamma)(\boldsymbol{q}\cdot\boldsymbol{q}_\gamma)/q_\gamma^2}{E^{\phi}_{\boldsymbol{q}+\boldsymbol{q}_\gamma}E^{\overline{\phi}}_{-\boldsymbol{q}}(\mathrm{a/hc})^2}
\end{aligned}$$

The vacuum energy diagram is also completed by adding following (negative) contribution:



and by adding following (negative) self-energy diagram:



for a given mode $(\boldsymbol{q}_\gamma, \lambda_\gamma)$ of the photon field.

Part V

Examples

Chapter 8

Wave packets

8.1 Gaussian wave packet

A well-known consequence of the Quantum formalism is the impossibility to describe a particle, like in classical mechanics, as a mass point having at each instant a well-defined position and velocity. In the Quantum mechanics of a single particle in continuous space-time, the movement of the wave packet defining its statistical position can still be described, like in classical fluid mechanics, by a probability current density (which is related to the phase gradient of the wave packet), but as soon as several particles are present or are even being created and annihilated like in Quantum Field Theory, the analogy to classical fluid mechanics becomes much more elusive. It is still possible, however, to describe approximate particle trajectories in the frame of Quantum Field Theory if one considers that proper quantum effects may remain beyond the reach of experimental precision in some situations. Gaussian wave packets are a typical model of such particles with a quasi-classical behavior, i.e. with a position and a velocity being well-defined to a good approximation.

A Gaussian wave packet of a particle of type ϕ and in the spin state λ, with a mean momentum $\boldsymbol{q}_0 \in (\mathbb{Z}/(1+2\mathrm{N}))^3$, a mean position $\boldsymbol{n}_0 \in \mathbb{R}^3$ and a width $w_0 \in \mathbb{R}_+^*$, is given by:

$$\begin{aligned}
|\Psi\rangle &= \widehat{G^{\phi}_{\lambda}}^{\dagger}(\boldsymbol{q}_0, \boldsymbol{n}_0, w_0)\ |\Omega\rangle \\
\widehat{G^{\phi}_{\lambda}}^{\dagger}(\boldsymbol{q}_0, \boldsymbol{n}_0, w_0) &:= C(\boldsymbol{q}_0, w_0)(2w_0)^{3/2}(1+2\mathrm{N})^{-3/2} \\
&\quad \sum_{\boldsymbol{q}} \exp\left(-2\pi w_0^2 \boldsymbol{q}^2 - \mathrm{i}2\pi \boldsymbol{q}\cdot\boldsymbol{n}_0\right) \widehat{a^{\phi}}^{\dagger}_{\lambda,\boldsymbol{q}_0+\boldsymbol{q}}
\end{aligned}$$

with the normalization factor:

$$C(\boldsymbol{q}_0, w_0) := \left[(2w_0)^3 (1+2\mathrm{N})^{-3} \sum_{\boldsymbol{q}} \exp\left(-4\pi w_0^2 \boldsymbol{q}^2\right) \right]^{-1/2}$$

In the usual case where $1 \ll w_0 \ll \mathrm{N}$, this normalization factor approximates to 1. On the position basis, the creation operator of the Gaussian wave packet can be expressed as:

$$\begin{aligned}\widehat{G^{\phi}_{\lambda}}^{\dagger}(\boldsymbol{q}_0, \boldsymbol{n}_0, w_0) &= C(\boldsymbol{q}_0, w_0) w_0^{-3/2} \sum_{\boldsymbol{n}} A(\boldsymbol{n} - \boldsymbol{n}_0'(\boldsymbol{n}), w_0) \\ &\qquad \exp\left(-\pi(\boldsymbol{n} - \boldsymbol{n}_0'(\boldsymbol{n}))^2/2w_0^2 + \mathrm{i}2\pi \boldsymbol{q}_0 \cdot \boldsymbol{n}\right) \widehat{a^{\phi}}^{\dagger}_{\lambda,\boldsymbol{n}}\end{aligned}$$

with the numerical factor:

$$A(\boldsymbol{n} - \boldsymbol{n}_0'(\boldsymbol{n}), w_0) := (2w_0^2)^{3/2} (1+2\mathrm{N})^{-3} \sum_{\boldsymbol{q}} \exp\left(-2\pi w_0^2 \left(\boldsymbol{q} - \mathrm{i}(\boldsymbol{n} - \boldsymbol{n}_0'(\boldsymbol{n}))/2w_0^2\right)^2\right)$$

where $\boldsymbol{n}_0'(\boldsymbol{n})$ can be chosen arbitrarily in $\boldsymbol{n}_0 + ((1+2\mathrm{N})\mathbb{Z})^3$. In the usual case where $1 \ll w_0 \ll \mathrm{N}$, this factor approximates to 1 if $\boldsymbol{n}_0'(\boldsymbol{n})$ can be chosen such as $\|\boldsymbol{n} - \boldsymbol{n}_0'(\boldsymbol{n})\| \ll \mathrm{N}$. To the zeroth order, the Hamiltonian evolution of the Gaussian wave packet $|\Psi_0\rangle = \widehat{G^{\phi}_{\lambda}}^{\dagger}(\boldsymbol{q}_0, \boldsymbol{n}_0, w_0)\ |\Omega\rangle$ is given by:

$$\begin{aligned}|\Psi_t\rangle &= C(\boldsymbol{q}_0, w_0)(2w_0)^{3/2}(1+2\mathrm{N})^{-3/2} \\ &\quad \sum_{\boldsymbol{q}} \exp\left(-2\pi w_0^2 \boldsymbol{q}^2 - \mathrm{i}2\pi \boldsymbol{q} \cdot \boldsymbol{n}_0 - \mathrm{i}2\pi E^{\phi}_{\boldsymbol{q}_0+\boldsymbol{q}}(t-t_0)/\mathrm{h}\right) \widehat{a^{\phi}}^{\dagger}_{\lambda,\boldsymbol{q}_0+\boldsymbol{q}}\ |\Omega\rangle\end{aligned}$$

If $w_0 \gg q_0^{-1}$, the saddle-point approximation $E^{\phi}_{\boldsymbol{q}_0+\boldsymbol{q}} \approx E^{\phi}_{\boldsymbol{q}_0} + \boldsymbol{q} \cdot \nabla_{\boldsymbol{q}} E^{\phi}_{\boldsymbol{q}_0}$ can be used and it follows:

$$\begin{aligned}|\Psi_t\rangle &\approx \exp\left(-\mathrm{i}2\pi E^{\phi}_{\boldsymbol{q}_0}(t-t_0)/\mathrm{h}\right) \widehat{G^{\phi}_{\lambda}}^{\dagger}(\boldsymbol{q}_0, \boldsymbol{n}_t, w_0)\ |\Omega\rangle \\ \boldsymbol{n}_t &:= \boldsymbol{n}_0 + \boldsymbol{v}^{\phi}_{\boldsymbol{q}_0}(t-t_0)/\mathrm{a} \\ \boldsymbol{v}^{\phi}_{\boldsymbol{q}_0} &:= \frac{\underline{\boldsymbol{q}_0}\mathrm{c}}{\sqrt{(\mathrm{m}_{\phi}\mathrm{ac}/\mathrm{h})^2 + \underline{\boldsymbol{q}_0}^2}}\end{aligned}$$

The mean position $\boldsymbol{n}_t$ of the particle follows therefore, in the toroidal space $(\mathbb{R}/(1+2\mathrm{N})\mathbb{Z})^3$, a classical trajectory at the constant velocity $\boldsymbol{v}^{\phi}_{\boldsymbol{q}_0}$ which would be attributed classically to a point mass of mass m_{ϕ} and of momentum $\mathrm{h}\underline{\boldsymbol{q}_0}/\mathrm{a}$.

Chapter 9

Coulomb scattering

9.1 Leading order calculation

We consider in this section the scattering of an electron by an atomic nucleus of atomic number Z. We model the nucleus by a classical point charge without magnetic moment, being at rest at the origin in the lattice reference frame and having a mass much higher as the mass of the electron. The corresponding electromagnetic field is described as a classical Coulomb potential V^{cl} given in terms of Fourier components by:

$$\begin{aligned} V^{cl}_{\boldsymbol{n}} &:= (1+2\mathrm{N})^{-3} \sum_{\boldsymbol{q}_\gamma \neq \boldsymbol{0}} \widetilde{V}^{cl}_{\boldsymbol{q}_\gamma} \exp\left(\mathrm{i}2\pi \boldsymbol{n}\cdot\boldsymbol{q}_\gamma\right) \\ \widetilde{V}^{cl}_{\boldsymbol{q}_\gamma} &:= \frac{Z\mathrm{e}}{4\pi^2\varepsilon_0 \mathrm{a} q_\gamma^2} \end{aligned}$$

The corresponding semi-classical interaction Hamiltonian takes the form:

$$\begin{aligned} \widehat{\mathrm{H}}' &:= \widehat{\mathrm{H}}'_{QED} + \widehat{\mathrm{H}}^{cl} \\ \widehat{\mathrm{H}}^{cl} &:= \sum_{\boldsymbol{n}} \widehat{Q}_{\boldsymbol{n}} V^{cl}_{\boldsymbol{n}} \end{aligned}$$

and its development on the plane wave basis is given by:

$$\begin{aligned} \widehat{\mathrm{H}}^{cl} = {} & \frac{Z\mathrm{e}^2}{4\pi^2\varepsilon_0\mathrm{a}}(1+2\mathrm{N})^{-3} \sum_{\phi,\boldsymbol{q},\lambda',\lambda} \mathrm{Q}_\phi \sum_{\boldsymbol{q}_\gamma\neq\boldsymbol{0}} q_\gamma^{-2} \\ & \left(\widehat{\overline{\psi}^{\phi}}_{\boldsymbol{q}+\boldsymbol{q}_\gamma,\lambda'} + \widehat{\psi^{\overline{\phi}}}_{-\boldsymbol{q}-\boldsymbol{q}_\gamma,\lambda'}\right)\gamma^0\left(\widehat{\psi^{\phi}}_{\boldsymbol{q},\lambda} + \widehat{\overline{\psi}^{\overline{\phi}}}_{-\boldsymbol{q},\lambda}\right) \end{aligned}$$

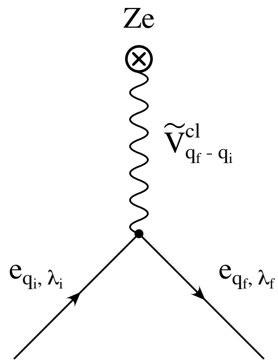
As initial and final states, we take:

$$\begin{aligned} |\Psi_i\rangle &:= \left|1^e_{\boldsymbol{q}_i,\lambda_i}\right\rangle \\ |\Psi_f\rangle &:= \left|1^e_{\boldsymbol{q}_f,\lambda_f}\right\rangle \end{aligned}$$

The matrix element of the interaction Hamiltonian is given for $\boldsymbol{q}_f \neq \boldsymbol{q}_i$ by:

$$H'_{f,i} = (1+2\mathrm{N})^{-3} \frac{-Ze^2}{4\pi^2\varepsilon_0 \mathrm{a}\|\boldsymbol{q}_f - \boldsymbol{q}_i\|^2} u^{e\dagger}_{\boldsymbol{q}_f,\lambda_f} u^{e}_{\boldsymbol{q}_i,\lambda_i}$$

and the leading order transition probability for this process is also represented by following diagram:



We consider a detector capturing the electrons having their momentum in the solid angle $\delta\Omega$. For $i \notin \delta\mathcal{F}$, the leading order transition probability takes the form:

$$\begin{aligned} \mathcal{P}^{(2)}(i \to \delta\mathcal{F}) \approx & \int_{\delta\Omega}\int_0^{\infty} (2\pi)^2 \frac{t-t_0}{\mathrm{h}} \left|H'_{f+\delta\boldsymbol{q},i}\right|^2 \delta^{(2)}_{t-t_0}\left(E_{f+\delta\boldsymbol{q}} - E_i\right) \\ & \left((1+2\mathrm{N})\frac{\mathrm{a}}{\mathrm{h}}\right)^3 p^2 \mathrm{d}p\mathrm{d}\Omega \end{aligned}$$

By taking following continuation for the factors of the integrand:

$$\begin{aligned} \left|H'_{f+\delta\boldsymbol{q},i}\right|^2 &\approx ((1+2\mathrm{N})\mathrm{a})^{-6} \frac{Z^2\mathrm{e}^4\mathrm{h}^4}{16\pi^4\varepsilon_0^2\|\boldsymbol{p}-\boldsymbol{p}_i\|^4} \left|u^{e\dagger}_{\boldsymbol{q},\lambda_f} u^{e}_{\boldsymbol{q}_i,\lambda_i}\right|^2 \\ \delta^{(2)}_{t-t_0}\left(E_{f+\delta\boldsymbol{q}} - E_i\right) &\approx \delta^{(2)}_{t-t_0}\left(\sqrt{(\mathrm{m}_e\mathrm{c}^2)^2 + (p\mathrm{c})^2} - E_i\right) \end{aligned}$$

the integration over p yields to:

$$\mathcal{P}^{(2)}(i \to \delta\mathcal{F}) \approx (t-t_0) j_i \sigma^{(2)}(i \to \delta\mathcal{F})$$

where $\boldsymbol{j}_i$ is the incident particle flux, given by:

$$\begin{aligned} \boldsymbol{j}_i &:= ((1+2\mathrm{N})\mathrm{a})^{-3}\, \boldsymbol{v}_i \\ \boldsymbol{v}_i &:= \frac{\boldsymbol{p}_i}{E_i/\mathrm{c}^2} \end{aligned}$$

and $\sigma^{(2)}(i \to \delta\mathcal{F})$ the leading order cross section, given for $i \notin \delta\mathcal{F}$ by:

$$\sigma^{(2)}(i \to \delta\mathcal{F}) \approx \int_{\delta\Omega} \left(\frac{Z\mathrm{e}^2}{8\pi\varepsilon_0 v_i p_i}\right)^2 \left| {u^e}^\dagger_{\boldsymbol{q},\lambda_f} u^e_{\boldsymbol{q}_i,\lambda_i} \right|^2 \frac{\mathrm{d}\Omega}{\sin(\theta/2)^4}$$

where θ is the deviation angle of the electron.

If the incident electron beam isn't polarized and if the polarization of the scattered electron isn't being measured, the cross section is obtained by adding the cross sections corresponding to the final spin states λ_f and averaging over the cross sections corresponding to the initial spin states λ_i:

$$\left\langle \sigma^{(2)}(i \to \delta\mathcal{F}) \right\rangle \approx \int_{\delta\Omega} \left(\frac{Z\mathrm{e}^2}{8\pi\varepsilon_0 v_i p_i}\right)^2 \frac{1}{2} \sum_{\lambda_f,\lambda_i} \left| {u^e}^\dagger_{\boldsymbol{q},\lambda_f} u^e_{\boldsymbol{q}_i,\lambda_i} \right|^2 \frac{\mathrm{d}\Omega}{\sin(\theta/2)^4}$$

The spin summation is given for $E = E_i$ by:

$$\frac{1}{2} \sum_{\lambda_f,\lambda_i} \left| {u^e}^\dagger_{\boldsymbol{q},\lambda_f} u^e_{\boldsymbol{q}_i,\lambda_i} \right|^2 = 1 - \beta_i^2 \sin(\theta/2)^2$$

and the mean cross section takes also the form:

$$\left\langle \sigma^{(2)}(i \to \delta\mathcal{F}) \right\rangle \approx \int_{\delta\Omega} \left(\frac{Z\mathrm{e}^2}{8\pi\varepsilon_0 v_i p_i}\right)^2 \left(1 - \beta_i^2 \sin(\theta/2)^2\right) \frac{\mathrm{d}\Omega}{\sin(\theta/2)^4}$$

The total mean cross section for deviation angles $\theta \geq \theta_m$ is also given by:

$$\left\langle \sigma^{(2)}(i \to \mathcal{F}) \right\rangle \approx \left(\frac{Z\mathrm{e}^2}{8\pi\varepsilon_0 v_i p_i}\right)^2 \left(\frac{4\pi}{\tan(\theta_m/2)^2} + 8\pi\beta_i^2 \ln(\sin(\theta_m/2))\right)$$

Part VI

Appendix

Appendix A

Usual functions

A.1 The sinc function

In this document, the sinc function is defined by:

$$\mathrm{sinc}\left(X\right) := \begin{cases} 1 & \text{for } X = 0 \\ \sin\left(\pi X\right)/\left(\pi X\right) & \text{otherwise} \end{cases}$$

This function admits following integral expression:

$$\mathrm{sinc}\left(X\right) = \frac{1}{X}\int_{-X/2}^{X/2} \exp\left(\mathrm{i}2\pi x\right)\mathrm{d}x$$

and is normalized by:

$$\int_{-\infty}^{+\infty} \mathrm{sinc}\left(X\right)\mathrm{d}X = 1$$

A.2 The esinc function

In this document, the esinc function is defined by:

$$\mathrm{esinc}\left(X\right) := \exp\left(\mathrm{i}\pi X\right)\mathrm{sinc}\left(X\right)$$

where the sinc function is defined as in A.1. This function admits following integral expression:

$$\mathrm{esinc}\left(X\right) = \frac{1}{X}\int_{0}^{X} \exp\left(\mathrm{i}2\pi x\right)\mathrm{d}x$$

can be written:

$$\mathrm{esinc}\left(X\right) = \frac{\sin\left(2\pi X\right)}{2\pi X} + \mathrm{i}\frac{1-\cos\left(2\pi X\right)}{2\pi X}$$

and verifies:

$$\mathrm{esinc}\left(-X\right) = \overline{\mathrm{esinc}\left(X\right)}$$

A.3 Nascent delta functions

In this document, we make use of following nascent delta functions, which converge to the delta energy distribution for $t - t_0 \to \infty$:

$$\begin{aligned} \delta^{(1)}_{t-t_0}(E) &:= \frac{t-t_0}{\mathrm{h}} \mathrm{sinc}\left(\frac{t-t_0}{\mathrm{h}} E\right) \\ \delta^{(2)}_{t-t_0}(E) &:= \frac{t-t_0}{\mathrm{h}} \mathrm{sinc}\left(\frac{t-t_0}{\mathrm{h}} E\right)^2 \end{aligned}$$

where the sinc function is defined as in A.1. The square of the first one can be expressed in terms of the second one as:

$$\delta^{(1)}_{t-t_0}(E)^2 = \frac{t-t_0}{\mathrm{h}} \delta^{(2)}_{t-t_0}(E)$$

We also make use of following family of functions converging to a distribution as $t - t_0 \to \infty$:

$$\delta^{(\mathrm{P.V.})}_{t-t_0}(E) := 2\frac{t-t_0}{\mathrm{h}} \mathrm{esinc}\left(\frac{t-t_0}{\mathrm{h}} E\right)$$

where the esinc function is defined as in A.2. Its limit is given by:

$$\lim_{t-t_0\to\infty} \delta^{(\mathrm{P.V.})}_{t-t_0}(E) = \delta(E) + \frac{\mathrm{i}}{\pi}\mathrm{P.V.}\left(\frac{1}{E}\right)$$

where the Cauchy principal value of $1/E$ is defined by its action on any test function $\phi(E)$ by:

$$\begin{aligned} \left(\mathrm{P.V.}\left(\frac{1}{E}\right), \phi(E)\right) &:= \mathrm{P.V.}\int_{-\infty}^{+\infty} \frac{\phi(E)}{E}\mathrm{d}E \\ &= \lim_{\varepsilon\to 0^+}\left(\int_{-\infty}^{-\varepsilon} \frac{\phi(E)}{E}\mathrm{d}E + \int_{\varepsilon}^{+\infty} \frac{\phi(E)}{E}\mathrm{d}E\right) \end{aligned}$$

Appendix B

Dirac and Pauli matrices

B.1 Pauli matrices

In this document, the Pauli matrices, which act canonically as endomorphisms of $\mathcal{H}^2$, are represented by:

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These matrices verify the anticommutation relations:

$$\{\sigma_a, \sigma_b\} := \sigma_a\sigma_b + \sigma_b\sigma_a = 2\delta_{a,b}I_2$$

B.2 Dirac matrices

In this document, the Dirac matrices, which act canonically as endomorphisms of $\mathcal{H}^4$, are represented by:

$$\gamma^0 := \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \quad \gamma^1 := \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}$$

$$\gamma^2 := \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \quad \gamma^3 := \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}$$

These matrices verify the anticommutation relations:

$$\{\gamma^\mu, \gamma^\nu\} := \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}I_4$$

We will make use of the condensed vectorial notation:

$$\boldsymbol{\gamma} := \begin{pmatrix} \gamma^1 \\ \gamma^2 \\ \gamma^3 \end{pmatrix}$$

Appendix C

Spinor operators

C.1 Photon spinor operators

In this document, we use following conventions for the polarization vectors of photons in the lattice reference frame:

$$
\begin{aligned}
\boldsymbol{\varepsilon}_{\boldsymbol{q},1} &:= -\frac{1}{\sqrt{2}}\frac{1}{\sqrt{q_1^2+q_2^2}}\frac{1}{q}\begin{pmatrix} q_1q_3 - \mathrm{i}q_2q \\ q_2q_3 + \mathrm{i}q_1q \\ -(q_1^2+q_2^2) \end{pmatrix} \\
\boldsymbol{\varepsilon}_{\boldsymbol{q},-1} &:= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{q_1^2+q_2^2}}\frac{1}{q}\begin{pmatrix} q_1q_3 + \mathrm{i}q_2q \\ q_2q_3 - \mathrm{i}q_1q \\ -(q_1^2+q_2^2) \end{pmatrix}
\end{aligned}
$$

For the special case of wave vectors $\boldsymbol{q}$ parallel to the third axis, we use the conventions:

$$
\begin{aligned}
\boldsymbol{\varepsilon}_{\boldsymbol{q},1} &:= -\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \mathrm{i}q_3/q \\ 0 \end{pmatrix} \\
\boldsymbol{\varepsilon}_{\boldsymbol{q},-1} &:= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -\mathrm{i}q_3/q \\ 0 \end{pmatrix}
\end{aligned}
$$

For the special case of the wave vector $\boldsymbol{q} = \mathbf{0}$, we take $\boldsymbol{\varepsilon}_{\boldsymbol{q},\lambda} := \mathbf{0}$. We extend this definition periodically to all $\boldsymbol{q} \in \left(\frac{\mathbb{Z}}{1+2\mathrm{N}}\right)^3$ by $\boldsymbol{\varepsilon}_{\boldsymbol{q},\lambda} := \boldsymbol{\varepsilon}_{\underline{\boldsymbol{q}},\lambda}$.

The polarization vectors of photons verify the Coulomb gauge conditions:

$$
\begin{aligned}
\boldsymbol{q}\cdot\boldsymbol{\varepsilon}_{\boldsymbol{q},\lambda} &= 0 \\
\boldsymbol{\varepsilon}_{\mathbf{0},\lambda} &= \mathbf{0}
\end{aligned}
$$

as well as, for $\boldsymbol{q} \neq \mathbf{0}$, the orthogonality relations:

$$
\boldsymbol{\varepsilon}^*_{\boldsymbol{q},\lambda'}\cdot\boldsymbol{\varepsilon}_{\boldsymbol{q},\lambda} = \delta_{\lambda',\lambda}
$$

One can also notice the relations:

$$\begin{aligned}\boldsymbol{\varepsilon}_{-\boldsymbol{q},\lambda} &= \boldsymbol{\varepsilon}^*_{\boldsymbol{q},\lambda}\\ \boldsymbol{\varepsilon}_{\boldsymbol{q},-\lambda} &= -\boldsymbol{\varepsilon}^*_{\boldsymbol{q},\lambda}\end{aligned}$$

The photon annihilation and creation spinor operators, which act canonically as homomorphisms from $\mathcal{H}$ to $\mathcal{H}^3$, are defined for $\boldsymbol{q} \neq \mathbf{0}$ by:

$$\begin{aligned}\widehat{\boldsymbol{\psi}^\gamma}_{\boldsymbol{q},\lambda} &:= ((1+2\mathrm{N})\mathrm{a})^{-3/2}\sqrt{\frac{\mathrm{ha}}{8\pi^2\varepsilon_0\mathrm{c}q}}\boldsymbol{\varepsilon}_{\boldsymbol{q},\lambda}\widehat{a^\gamma}_{\boldsymbol{q},\lambda}\sqrt{\widehat{N^\gamma}_{\boldsymbol{q},\lambda}}\\ \widehat{\boldsymbol{\psi}^\gamma}^\dagger_{\boldsymbol{q},\lambda} &:= ((1+2\mathrm{N})\mathrm{a})^{-3/2}\sqrt{\frac{\mathrm{ha}}{8\pi^2\varepsilon_0\mathrm{c}q}}\boldsymbol{\varepsilon}^*_{\boldsymbol{q},\lambda}\widehat{a^\gamma}^\dagger_{\boldsymbol{q},\lambda}\sqrt{1+\widehat{N^\gamma}_{\boldsymbol{q},\lambda}}\end{aligned}$$

where ε_0 is the permittivity of the bare vacuum. For $\boldsymbol{q} = \mathbf{0}$, we take $\widehat{\boldsymbol{\psi}^\gamma}_{\boldsymbol{q},\lambda} = \mathbf{0}$ and $\widehat{\boldsymbol{\psi}^\gamma}^\dagger_{\boldsymbol{q},\lambda} = \mathbf{0}$. The spinor operators can also be defined on the position basis by:

$$\begin{aligned}\widehat{\boldsymbol{\psi}^\gamma}_{\boldsymbol{n},\lambda} &:= \sum_{\boldsymbol{q}} \exp\left(\mathrm{i}2\pi\boldsymbol{n}\cdot\boldsymbol{q}\right)\widehat{\boldsymbol{\psi}^\gamma}_{\boldsymbol{q},\lambda}\\ \widehat{\boldsymbol{\psi}^\gamma}^\dagger_{\boldsymbol{n},\lambda} &:= \sum_{\boldsymbol{q}} \exp\left(-\mathrm{i}2\pi\boldsymbol{n}\cdot\boldsymbol{q}\right)\widehat{\boldsymbol{\psi}^\gamma}^\dagger_{\boldsymbol{q},\lambda}\end{aligned}$$

We extend these definitions periodically to all $\boldsymbol{q} \in \left(\frac{\mathbb{Z}}{1+2\mathrm{N}}\right)^3$ by $\widehat{\boldsymbol{\psi}^\gamma}_{\boldsymbol{q},\lambda} := \widehat{\boldsymbol{\psi}^\gamma}_{\underline{\boldsymbol{q}},\lambda}$ and $\widehat{\boldsymbol{\psi}^\gamma}^\dagger_{\boldsymbol{q},\lambda} := \widehat{\boldsymbol{\psi}^\gamma}^\dagger_{\underline{\boldsymbol{q}},\lambda}$.

We will make use following condensed notation, representing a matrix acting canonically as an endomorphism of $\mathcal{H}^4$:

$$\boldsymbol{\gamma}\cdot\boldsymbol{\varepsilon}_{\boldsymbol{q},\lambda} := \left(\boldsymbol{\varepsilon}_{\boldsymbol{q},\lambda}\right)_1\gamma^1 + \left(\boldsymbol{\varepsilon}_{\boldsymbol{q},\lambda}\right)_2\gamma^2 + \left(\boldsymbol{\varepsilon}_{\boldsymbol{q},\lambda}\right)_3\gamma^3$$

C.2 Dirac spinor operators

In this document, we use following conventions for the Dirac spinors in the lattice reference frame (for charged leptons $\phi \in \{e, \mu, \tau\}$, neutrinos $\phi \in \{\nu_e, \nu_\mu, \nu_\tau\}$ and quarks $\phi \in \{u, c, t, d, s, b\}$):

$$\begin{aligned}u^\phi_{\boldsymbol{q},1/2} &:= \sqrt{\frac{1}{2}\left(1+\frac{\mathrm{m}_\phi\mathrm{c}^2}{E}\right)}\begin{pmatrix}1\\0\\p_3/\left(\mathrm{m}_\phi\mathrm{c}+E/\mathrm{c}\right)\\\left(p_1+ip_2\right)/\left(\mathrm{m}_\phi\mathrm{c}+E/\mathrm{c}\right)\end{pmatrix}\\ u^\phi_{\boldsymbol{q},-1/2} &:= \sqrt{\frac{1}{2}\left(1+\frac{\mathrm{m}_\phi\mathrm{c}^2}{E}\right)}\begin{pmatrix}0\\1\\\left(p_1-ip_2\right)/\left(\mathrm{m}_\phi\mathrm{c}+E/\mathrm{c}\right)\\-p_3/\left(\mathrm{m}_\phi\mathrm{c}+E/\mathrm{c}\right)\end{pmatrix}\end{aligned}$$

$$u^{\overline{\phi}}_{\boldsymbol{q},1/2} := \sqrt{\frac{1}{2}\left(1+\frac{\mathrm{m}_\phi \mathrm{c}^2}{E}\right)} \begin{pmatrix} (p_1 - ip_2)/(\mathrm{m}_\phi \mathrm{c} + E/\mathrm{c}) \\ -p_3/(\mathrm{m}_\phi \mathrm{c} + E/\mathrm{c}) \\ 0 \\ 1 \end{pmatrix}$$

$$u^{\overline{\phi}}_{\boldsymbol{q},-1/2} := \sqrt{\frac{1}{2}\left(1+\frac{\mathrm{m}_\phi \mathrm{c}^2}{E}\right)} \begin{pmatrix} p_3/(\mathrm{m}_\phi \mathrm{c} + E/\mathrm{c}) \\ (p_1 + ip_2)/(\mathrm{m}_\phi \mathrm{c} + E/\mathrm{c}) \\ 1 \\ 0 \end{pmatrix}$$

In these expressions, energy and momentum are defined by:

$$\begin{aligned} E &:= \sqrt{(\mathrm{m}_\phi \mathrm{c}^2)^2 + \left(\frac{\mathrm{hc}}{\mathrm{a}}\boldsymbol{q}\right)^2} \\ \boldsymbol{p} &:= \frac{\mathrm{h}}{\mathrm{a}}\boldsymbol{q} \end{aligned}$$

We extend these definitions periodically to all $\boldsymbol{q} \in \left(\frac{\mathbb{Z}}{1+2\mathrm{N}}\right)^3$ by $u^{\phi}_{\boldsymbol{q},\lambda} := u^{\phi}_{\underline{\boldsymbol{q}},\lambda}$ and $u^{\overline{\phi}}_{\boldsymbol{q},\lambda} := u^{\overline{\phi}}_{\underline{\boldsymbol{q}},\lambda}$.

These spinors verify the orthogonality relations:

$$\begin{aligned} {u^{\phi}}^{\dagger}_{\boldsymbol{q},\lambda'} u^{\phi}_{\boldsymbol{q},\lambda} &= \delta_{\lambda',\lambda} \\ {u^{\overline{\phi}}}^{\dagger}_{\boldsymbol{q},\lambda'} u^{\overline{\phi}}_{\boldsymbol{q},\lambda} &= \delta_{\lambda',\lambda} \end{aligned}$$

as well as the Dirac equations:

$$\begin{aligned} \gamma^\mu p_\mu u^{\phi}_{\boldsymbol{q},\lambda} &= \mathrm{m}_\phi \mathrm{c}\; u^{\phi}_{\boldsymbol{q},\lambda} \\ \gamma^\mu p_\mu u^{\overline{\phi}}_{\boldsymbol{q},\lambda} &= -\mathrm{m}_\phi \mathrm{c}\; u^{\overline{\phi}}_{\boldsymbol{q},\lambda} \end{aligned}$$

where we use the condensed notation:

$$\gamma^\mu p_\mu := \frac{E}{\mathrm{c}}\gamma^0 - p_1\gamma^1 - p_2\gamma^2 - p_3\gamma^3$$

The annihilation and creation spinor operators of these particles and of their anti-particles, which act canonically as homomorphisms between $\mathcal{H}$ and $\mathcal{H}^4$, are defined by:

$$\begin{aligned} \widehat{\psi^{\phi}}_{\boldsymbol{q},\lambda} &:= u^{\phi}_{\boldsymbol{q},\lambda}\widehat{a^{\phi}}_{\boldsymbol{q},\lambda} \\ \widehat{\overline{\psi}^{\phi}}_{\boldsymbol{q},\lambda} &:= {u^{\phi}}^{\dagger}_{\boldsymbol{q},\lambda}\gamma^0 \widehat{a^{\phi}}^{\dagger}_{\boldsymbol{q},\lambda} \\ \widehat{\psi^{\overline{\phi}}}_{\boldsymbol{q},\lambda} &:= {u^{\overline{\phi}}}^{\dagger}_{\boldsymbol{q},\lambda}\gamma^0 \widehat{a^{\overline{\phi}}}_{\boldsymbol{q},\lambda} \\ \widehat{\overline{\psi}^{\overline{\phi}}}_{\boldsymbol{q},\lambda} &:= u^{\overline{\phi}}_{\boldsymbol{q},\lambda}\widehat{a^{\overline{\phi}}}^{\dagger}_{\boldsymbol{q},\lambda} \end{aligned}$$

and can also be defined on the position basis by:

$$
\begin{aligned}
\widehat{\psi^{\phi}}_{\boldsymbol{n},\lambda} &:= (1+2\mathrm{N})^{-3/2} \sum_{\boldsymbol{q}} \exp\left(\mathrm{i}2\pi \boldsymbol{n}\cdot\boldsymbol{q}\right) \widehat{\psi^{\phi}}_{\boldsymbol{q},\lambda} \\
\widehat{\overline{\psi}^{\phi}}_{\boldsymbol{n},\lambda} &:= (1+2\mathrm{N})^{-3/2} \sum_{\boldsymbol{q}} \exp\left(-\mathrm{i}2\pi \boldsymbol{n}\cdot\boldsymbol{q}\right) \widehat{\overline{\psi}^{\phi}}_{\boldsymbol{q},\lambda} \\
\widehat{\psi^{\overline{\phi}}}_{\boldsymbol{n},\lambda} &:= (1+2\mathrm{N})^{-3/2} \sum_{\boldsymbol{q}} \exp\left(\mathrm{i}2\pi \boldsymbol{n}\cdot\boldsymbol{q}\right) \widehat{\psi^{\overline{\phi}}}_{\boldsymbol{q},\lambda} \\
\widehat{\overline{\psi}^{\overline{\phi}}}_{\boldsymbol{n},\lambda} &:= (1+2\mathrm{N})^{-3/2} \sum_{\boldsymbol{q}} \exp\left(-\mathrm{i}2\pi \boldsymbol{n}\cdot\boldsymbol{q}\right) \widehat{\overline{\psi}^{\overline{\phi}}}_{\boldsymbol{q},\lambda}
\end{aligned}
$$

We extend these definitions periodically to all $\boldsymbol{q} \in \left(\frac{\mathbb{Z}}{1+2\mathrm{N}}\right)^3$ by $\widehat{\psi^{\phi}}_{\boldsymbol{q},\lambda} := \widehat{\psi^{\phi}}_{\underline{\boldsymbol{q}},\lambda}$, $\widehat{\overline{\psi}^{\phi}}_{\boldsymbol{q},\lambda} := \widehat{\overline{\psi}^{\phi}}_{\underline{\boldsymbol{q}},\lambda}$, $\widehat{\psi^{\overline{\phi}}}_{\boldsymbol{q},\lambda} := \widehat{\psi^{\overline{\phi}}}_{\underline{\boldsymbol{q}},\lambda}$ and $\widehat{\overline{\psi}^{\overline{\phi}}}_{\boldsymbol{q},\lambda} := \widehat{\overline{\psi}^{\overline{\phi}}}_{\underline{\boldsymbol{q}},\lambda}$.

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