EEE 391

Basics of Signals and Systems
Midterm Exam
25 March 2010, Thursday
closed book and notes
no calculators
Given Time: 120 min
Instructors: Billur Barshan and Haldun Özaktaş

Last Name : ....

First Name : ....

Section : ....

ID number : ....

Signature : ....

Exam Part	Total Points	Points Received
Q1	25	
Q2	25	
Q3	20	
Q4	30	
Total	100	

Allocation of points:

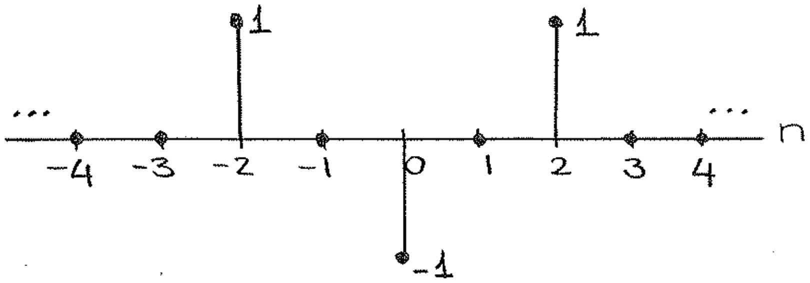
1) 25 pts (a) 6 pts (b) 6 pts (c) 6 pts (d) 7 pts
2) 25 pts (a) 4 pts (b) 8 pts (c) 9 pts (d) 4 pts
3) 20 pts (a) 5 pts (b) 5 pts (c) 5 pts (d) 5 pts
4) 30 pts (a) 3 pts (b) 4 pts (c) 5 pts (d) 10 pts (e) 8 pts

Attention:

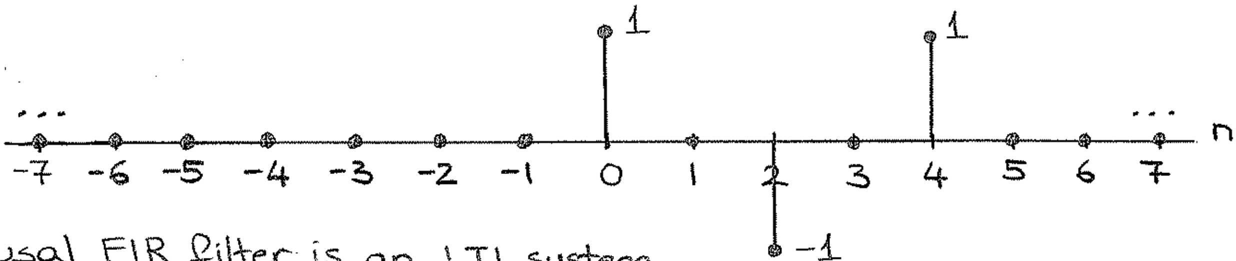
Read all the questions carefully and <u>show your work</u> or <u>justify your answer</u> for full or partial credit.

Please put your answer for each part <u>in the given spaces</u>.

① A causal FIR filter produces the following output sequence when $\delta[n+2]$ is given to it as input:



a) Find the impulse response of the filter and plot it below: $(h[n])$:



Causal FIR filter is an LTI system. → (This was shown in class.)
Therefore, $\delta[n+2] \mapsto h[n+2]$

$\delta[n] \mapsto h[n]$ (shift the given output by 2 units to the right)

b) Identify and list all of the filter coefficients $\{b_k\}$.

$$\boxed{\{b_k\} = \{1, 0, -1, 0, 1\}}$$

What is the order of the filter?

$$\boxed{\text{order} = 4}$$

c) Write the difference equation of the filter.

$$\boxed{y[n] = x[n] - x[n-2] + x[n-4]}$$

$$y[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + b_3 x[n-3] + b_4 x[n-4]$$
$$= x[n] - x[n-2] + x[n-4]$$

$b_0 = b_4 = 1$ $\quad b_2 = -1$

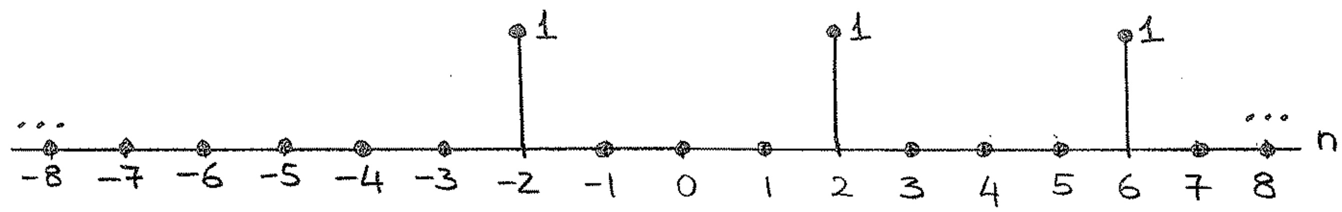
$b_1 = b_3 = 0$

① d) Find the output of the filter when the input is:

$$x[n] = \delta[n+2] + \delta[n] + \delta[n-2]$$

and plot it below:

$$\boxed{y[n] = \delta[n+2] + \delta[n-2] + \delta[n-6]}$$



Since the system is LTI, you can use superposition to solve this part:

$\delta[n+2] \longmapsto h[n+2]$ (given in the question statement)

$\delta[n] \longmapsto h[n]$ (found in part a))

$\delta[n-2] \longmapsto h[n-2]$ (shift the result of part a) by two units to the right <u>or</u> shift the output given in the question statement by four units to the right.)

$$\delta[n+2] + \delta[n] + \delta[n-2] \longmapsto h[n+2] + h[n] + h[n-2]$$

$$h[n+2] = \delta[n+2] - \delta[n] + \delta[n-2]$$

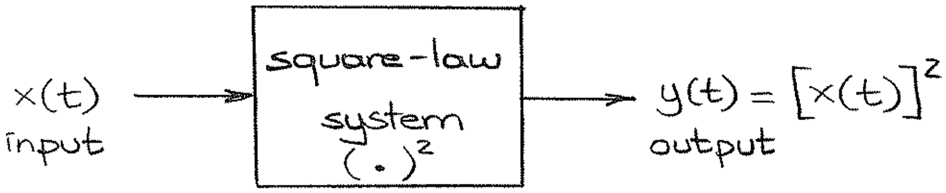
$$h[n] = \delta[n] - \delta[n-2] + \delta[n-4]$$

$$h[n-2] = \delta[n-2] - \delta[n-4] + \delta[n-6]$$

$$y[n] = h[n+2] + h[n] + h[n-2]$$

$$= \delta[n+2] + \delta[n-2] + \delta[n-6]$$

(2)



The signal $x(t) = 2\cos(\omega_0 t) + \sin(2\omega_0 t)$ is given as input to the square-law system shown above.

a) What is the average value of $x(t)$?

$$\text{average (dc) value} = \frac{1}{T_0}\int_0^{T_0}\left[2\cos(\omega_0 t) + \sin(2\omega_0 t)\right]dt = \left.\frac{2\sin(\omega_0 t)}{\omega_0 T_0}\right|_0^{T_0} - \left.\frac{\cos(2\omega_0 t)}{2\omega_0 T_0}\right|_0^{T_0}$$
$$= 0$$

b) Find the output of the system for the given input and express it in the form:

$$y(t) = A_0 + \sum_{k=1}^{N} A_k \cos(\omega_0 k t + \phi_k)$$

$$y(t) = [x(t)]^2$$
$$= \left[2\cos(\omega_0 t) + \sin(2\omega_0 t)\right]^2$$
$$= 4\cos^2(\omega_0 t) + 4\cos(\omega_0 t)\sin(2\omega_0 t) + \sin^2(2\omega_0 t)$$
$$= \overset{2}{\cancel{4}}\left[\frac{1+\cos(2\omega_0 t)}{\cancel{2}}\right] + \overset{2}{\cancel{4}}\left[\frac{\sin(3\omega_0 t) + \sin(\omega_0 t)}{\cancel{2}}\right] + \left[\frac{1-\cos(4\omega_0 t)}{2}\right]$$
$$= \frac{5}{2} + 2\cos(2\omega_0 t) + 2\sin(3\omega_0 t) + 2\sin(\omega_0 t) - \frac{1}{2}\cos(4\omega_0 t)$$
$$= \frac{5}{2} + 2\sin(\omega_0 t) + 2\cos(2\omega_0 t) + 2\sin(3\omega_0 t) - \frac{1}{2}\cos(4\omega_0 t)$$
$$= \frac{5}{2} + 2\cos\left(\omega_0 t - \frac{\pi}{2}\right) + 2\cos(2\omega_0 t) + 2\cos\left(3\omega_0 t - \frac{\pi}{2}\right) + \frac{1}{2}\cos(4\omega_0 t + \pi)$$

0:	ω_0:	$2\omega_0$:	$3\omega_0$:	$4\omega_0$:
$(k=0)$	$(k=1)$	$(k=2)$	$(k=3)$	$(k=4)$
$A_0 = \frac{5}{2}$	$A_1 = 2$	$A_2 = 2$	$A_3 = 2$	$A_4 = \frac{1}{2}$
$\phi_0 = 0$	$\phi_1 = -\frac{\pi}{2}$	$\phi_2 = 0$	$\phi_3 = -\frac{\pi}{2}$	$\phi_4 = \pi$

② c) Is $y(t)$ a periodic signal? If so find all of its Fourier series coefficients. If not, explain why.

$y(t)$ is a periodic signal with fundamental period $T_0 = \frac{2\pi}{\omega_0}$.

Fourier series coefficients:

$$y(t) = \frac{5}{2} + 2\cos\left(\omega_0 t - \frac{\pi}{2}\right) + 2\cos(2\omega_0 t) + 2\cos\left(3\omega_0 t - \frac{\pi}{2}\right) + \frac{1}{2}\cos(4\omega_0 t + \pi)$$

$$= \frac{5}{2} + \cancel{2}\left[\frac{e^{j(\omega_0 t - \frac{\pi}{2})} + e^{-j(\omega_0 t - \frac{\pi}{2})}}{\cancel{2}}\right] + \cancel{2}\left[\frac{e^{j2\omega_0 t} + e^{-j2\omega_0 t}}{\cancel{2}}\right] + \cancel{2}\left[\frac{e^{j(3\omega_0 t - \frac{\pi}{2})} + e^{-j(3\omega_0 t - \frac{\pi}{2})}}{\cancel{2}}\right]$$

$$+ \frac{1}{2}\left[\frac{e^{j(4\omega_0 t + \pi)} + e^{-j(4\omega_0 t + \pi)}}{2}\right]$$

$$= \frac{1}{4}e^{-j\pi} \cdot e^{-j4\omega_0 t} + e^{j\frac{\pi}{2}} e^{-j3\omega_0 t} + e^{-j2\omega_0 t} + e^{j\frac{\pi}{2}} \cdot e^{-j\omega_0 t} + \frac{5}{2}$$

$$+ \frac{1}{4}e^{j\pi} e^{j4\omega_0 t} + e^{-j\frac{\pi}{2}} e^{j3\omega_0 t} + e^{j2\omega_0 t} + e^{-j\frac{\pi}{2}} e^{j\omega_0 t}$$

$a_{-4} = \frac{1}{4}e^{-j\pi} = -\frac{1}{4}$ $\quad$ $a_4 = \frac{1}{4}e^{j\pi} = -\frac{1}{4}$

$a_{-3} = e^{j\frac{\pi}{2}} = j$ $\quad$ $a_3 = e^{-j\frac{\pi}{2}} = -j$

$a_{-2} = 1$ $\quad$ $a_2 = 1$

$a_{-1} = e^{j\frac{\pi}{2}} = j$ $\quad$ $a_1 = e^{-j\frac{\pi}{2}} = -j$

$a_0 = \frac{5}{2}$

d) What is the average value of $y(t)$?

$$\boxed{y_{av}(t) = a_0 = \frac{5}{2}}$$

Note that the average value of the input signal is zero whereas the average value of the output signal is not. This occurs because of the squaring operation involved.

③ A system is defined by the following equation where $x[n]$ is the input to the system and $y[n]$ is its output:

$$y[n]=\begin{cases} x[n]+(n+1)^2 & \text{for } 0\leq n\leq 10 \\ 0 & \text{otherwise} \end{cases}$$

a) Is the system linear? Justify.
b) Is the system time invariant? Justify.
c) Is the system causal? Justify.
d) Find the (unit) impulse response of the system.

a) The system is nonlinear.

$$x[n] \mapsto y[n]$$

$$\alpha x[n] \mapsto \begin{cases} \alpha x[n]+(n+1)^2 & \text{for } 0\leq n\leq 10 \\ 0 & \text{otherwise} \end{cases}$$

$$\neq \alpha y[n] \quad \text{(scalability fails)}$$

b) The system is not time invariant (it is time varying)

$$x[n] \mapsto y[n]$$

$$x[n-n_0] \mapsto \begin{cases} x[n-n_0]+(n+1)^2 & \text{for } 0\leq n\leq 10 \\ 0 & \text{otherwise} \end{cases} = w[n]$$

$$y[n-n_0] = \begin{cases} x[n-n_0]+(n-n_0+1)^2 & \text{for } 0\leq n-n_0\leq 10 \\ & n_0 \leq n \leq n_0+10 \\ 0 & \end{cases}$$

$$y[n-n_0] \neq w[n]$$

c) The system is causal. It does not use any future values of the input.

d) $$h[n]= \begin{cases} \delta[n]+(n+1)^2 & \text{for } 0\leq n\leq 10 \\ 0 & \text{otherwise} \end{cases}$$

(4) The signal $x(t) = -3 + 2\cos(80\pi t + \frac{\pi}{2})$ is sampled uniformly at a rate of $f_s = 60$ Hz to obtain a discrete-time signal $x[n]$.

a) What is the Nyquist rate of the given signal?

$$\boxed{f_{Nyquist} = 80 \text{ Hz.}}$$

$\omega_0 = 80\pi$ rad/s.

$f_0 = 40$ Hz $= f_{max}$

$f_{Nyquist} = 2 f_{max} = 80$ Hz.

b) At the given sampling rate, what is the maximum frequency that can be reconstructed perfectly by an ideal reconstructor?

$$\boxed{f = 30 \text{ Hz.}}$$

$w = \hat{w} f_s = \frac{\hat{w}}{T_s}$

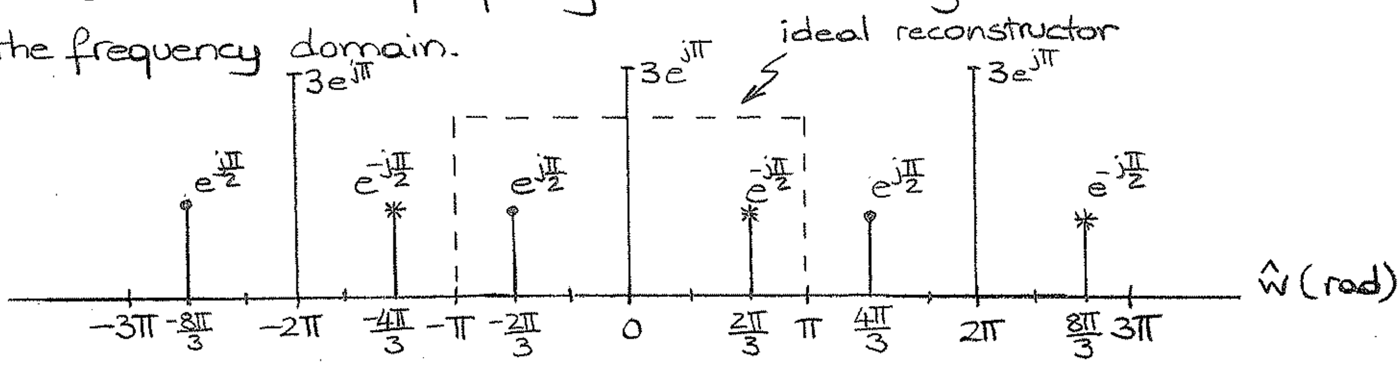
$= \pm\pi . f_s$

$f = \frac{\pm \pi f_s}{2\pi} = \pm \frac{f_s}{2}$

c) Find the sampled signal $x[n]$.

$$\boxed{\begin{aligned} x[n] = x(nT_s) &= -3 + 2\cos\left(\tfrac{4\pi}{3}n + \tfrac{\pi}{2}\right) \\ = x(n/f_s) & \end{aligned}}$$

d) Plot the digital spectrum of the signal $x[n]$ over the interval $-3\pi \leq \hat{w} \leq 3\pi$. Include all aliases in this interval and also their complex amplitudes. Draw a rectangular window using dashed lines to indicate the frequency window used by the ideal reconstructor in the frequency domain.



e) $x[n]$ is converted to an analog signal by an ideal D-to-C converter with $f_s = 60$ Hz. Find the signal at the output of the reconstructor. Indicate whether this is a case of undersampling or oversampling. If undersampling, indicate which components are aliased and whether folding occurs or not.

e) $y[n] = 3e^{j\pi} + e^{-j\frac{\pi}{2}} e^{j\frac{2\pi}{3}n} + e^{j\frac{\pi}{2}} e^{-j\frac{2\pi}{3}n}$

$$= -3 + 2\cos\left(\frac{2\pi}{3}n - \frac{\pi}{2}\right)$$

$$\hat{\omega} = \omega T_s = \frac{\omega}{f_s}$$

$$y(t) = -3 + 2\cos\left(\frac{2\pi}{3} f_s t - \frac{\pi}{2}\right)$$

$$= -3 + 2\cos\left(\frac{2\pi}{3} 60t - \frac{\pi}{2}\right)$$

$$= -3 + 2\cos\left(40\pi t - \frac{\pi}{2}\right) \quad \text{(reconstructed signal)}$$

The d.c. component of $x(t)$ is not aliased and can be reconstructed perfectly. The 40 Hz sinusoidal component is aliased because it requires a minimum sampling rate of 80 Hz. Therefore, this is a case of <u>undersampling</u> of the second component. Since $\hat{\omega} = \pm\frac{4\pi}{3}$ rad, $\pi < |\hat{\omega}| < 2\pi$, <u>folding</u> occurs, resulting in the <u>phase reversal</u> of the reconstructed signal. The reconstructed signal has two components: a d.c. component and a 20 Hz component.