CS-473/573: Algorithms I Example

Quiz 5: Greedy Choice

Name: Sample Solution

Ms. Redcrane would like to drive from mile 0 (source) on the highway to mile m (destination). Starting with a full tank, she can drive up to k miles. She knows that there are n gas stations on her way, at miles $0 = s_0 < s_1 < s_2 < ... < s_n = m$. She would like to select the stations to stop in such a way that the number of stops she makes will be minimized.

State and prove the greedy choice property for this problem.

Note: We call a set of stations "feasible" if it is possible to complete the trip by stopping in all stations in the set, i.e., each station in the set is at most k miles away from the previous one. Observe that an optimal set of stations has to be feasible. You can assume that there is always at least one feasible solution.

Greedy choice property:

Let s_j be the last station before mile k. There is an optimal set of stations that contains s_j.

Proof:

Take any optimal set S of stations.

If $s_j \in S$, the proof is completed. If not, we will construct another solution, S' that contains s_j.

First observe that, since S is a feasible set of stations, there is at least one station in S that comes before mile k. Let that station be s_i.

Now since s_j is the last station before mile k and s_i comes before mile k, we must have $s_i \leq s_j \leq k$.

Let $S' = (S - \{s_i\}) \cup \{s_j\}$ (i.e., we create S' by replacing s_i with s_j in S). S' is feasible, since:

- $s_j \leq k$,
- the second station in S' (same as that in S) is at most k miles away from s_j (since it is at most k miles away from s_i and $s_i \leq s_j$), and
- the rest of the stops are identical to those in S, which is a feasible solution.

S' is also optimal since $|S'| = |S|$.

Thus, there is always an optimal set of stations that contains s_j.