CS-473/573: Algorithms I Example

Quiz 4: Optimal Substructure

Name: Solution

The 0-1 Knapsack Problem is defined as follows: We are given n items, each with integer weight w_i and integer value v_i for $1 \leq i \leq n$. We are also given an integer capacity c. The problem is to find a subset of items with maximum total value such that their total weight does not exceed the capacity, i.e., we want to find a subset $S \subseteq \{1, 2, .., n\}$ such that $V(S) = \sum_{i \in S} v_i$ is maximized, subject to $W(S) = \sum_{i \in S} w_i \leq c$.

For example, if $n = 4$, $v =< 4, 8, 5, 2 >$, $w =< 3, 5, 3, 2 >$, and $c = 6$, then the optimal solution is $S = \{1, 3\}$ with $V(S) = 4 + 5 = 9$ and $W(S) = 3 + 3 = 6 \leq 6$.

State and prove the optimal substructure property for this problem.

Solution:

Optimal Substructure: Let $S = \{i_1, i_2, ..., i_k\}$ be an optimal set of items such that $i_1 < i_2 < ... < i_k$. Then, $S' = \{i_1, i_2, ..., i_{k-1}\}$ is an optimal set of items for the problem with item set $\{1, 2, ..., i_{k-1}\}$ and capacity $c - w_k$.

Proof: Assume not. Then there is an item set $S'' \subseteq \{1, 2, ..., i_{k-1}\}$ with $W(S'') \leq c - w_k$ and $V(S'') > V(S')$.

Let $S''' = S'' \cup \{i_k\}$.

S''' is a feasible solution for the problem with item set $\{1, 2, ..., n\}$ and capacity c, since $i_k \notin S''$, $i_k \in \{1, 2, ..., n\}$, and $W(S''') = W(S'') + w_k \leq c - w_k + w_k = c$.

Furthermore, since $V(S'') > V(S')$, we have $V(S''') = V(S'') + v_k > V(S') + v_k = V(S)$.

It follows that S is not an optimal solution for the problem with item set $\{1, 2, ..., n\}$ and capacity c, which is a contradiction.

Thus, S' has to be optimal for the problem with item set $\{1, 2, ..., i_{k-1}\}$ and capacity $c - w_k$.