IE 400 2018-2019 Fall Study Set 2 Solutions

Q1)

Parameters f_i : fixed cost of plant i, $i=1,..,4$

c_{ij} : cost of producing car j at plant i $i=1..4$, $j=1,2,3$

Dec. Var $y_i : \begin{cases} 1 & \text{if plant } i \text{ is used} \\ 0 & \text{o.w} \end{cases}$ $i=1..4$

$x_{ij} : \begin{cases} 1 & \text{if car } j \text{ is produced at plant } i \\ 0 & \text{o.w} \end{cases}$ $i=1..4$, $j=1,2,3$

Model:

$$\min \sum_{i=1}^{4} f_i y_i + \sum_{i=1}^{4}\sum_{j=1}^{3} c_{ij} \cdot x_{ij}$$

s.t

$$\sum_{j=1}^{3} x_{ij} \leq 1 \quad \forall i=1..4$$

$$\sum_{i=1}^{4} x_{ij} = 1 \quad \forall j=1..3$$

$$y_3 + y_4 \leq 1 + y_1$$

$$x_{ij} \leq y_i \quad \forall i=1..4, j=1..3$$

$$x_{ij} \in \{0,1\} \quad \forall i=1..4, j=1..3$$

$$y_j \in \{0,1\} \quad \forall j=1..3$$

Q2)

(4) Parameters

cb_i : cost of producing a ton of steam in boiler i, $i=1,2,3$

ct_J : " " processing " " " " turbine J, $J=1,2,3$

$minb_i$: min amount of steam that can be produced in boiler i

$maxb_i$: max " " " " " " $i=1,2,3$

$mint_J$: min " " " " " " processed in turbine J

$maxt_J$: max " " " " " " $J=1,2,3$ " "

Dec. Var

$x_i = \begin{cases} 1 & \text{if boiler } i \text{ is used} \\ 0 & \text{o.w} \end{cases}$ $i=1,2,3$

y_i : amount of steam that is produced in boiler i, $i=1,2,3$

$t_J = \begin{cases} 1 & \text{if turbine } J \text{ is used} \\ 0 & \text{o.w} \end{cases}$ $J=1,2,3$

z_J : amount of steam that is processed in turbine $J=1,2,3$

Model

$$\min \sum_{i=1}^{3} cb_i y_i + \sum_{J=1}^{3} ct_J z_J$$

s.t

$minb_i x_i \leq y_i \leq maxb_i x_i \quad \forall i=1,2,3$ (production limitations of boilers)

$mint_J t_J \leq z_J \leq maxt_J t_J \quad \forall J=1,2,3$ (process limitations of turbines)

$4z_1 + 5z_2 + 6z_3 = 800$ (required production)

$\sum_{i=1}^{3} y_i \geq \sum_{J=1}^{3} z_J$ (produced steam should be larger than processed)

$x_i \in \{0,1\} \quad y_i \geq 0 \quad \forall i=1,2,3$

$t_J \in \{0,1\} \quad z_J \geq 0 \quad \forall J=1,2,3$

Q3)

If we assume the extra translator can handle both tourist groups simultaneously:

x_i: the number of Italian translators who start to work at period i, i=1,2,3,4,5,6

y_i: the number of German translators who start to work at period i, i=1,2,3,4,5,6

z_i: the number of extra translators who work at period i, i=1,2,3,4,5,6

min $320\, x_i + 280\, y_i + 240\, z_i$

st.

$x_6 + x_1 + z_1 \geq 5$

$x_1 + x_2 + z_2 \geq 3$

$x_2 + x_3 + z_3 \geq 8$

$x_3 + x_4 + z_4 \geq 8$

$x_4 + x_5 + z_5 \geq 11$

$x_5 + x_6 + z_6 \geq 4$

$y_6 + y_1 + z_1 \geq 4$

$y_1 + y_2 + z_2 \geq 4$

$y_2 + y_3 + z_3 \geq 7$

$y_3 + y_4 + z_4 \geq 7$

$y_4 + y_5 + z_5 \geq 13$

$y_5 + y_6 + z_6 \geq 4$

$x_i\ y_i\ z_i \geq 0$ for i=1,2,3,4,5,6 and integer

If we assume the extra translator is needed for each group separately:

x_{ij}: the number of regular translators who start to work at period i and translates language j, i=1,2,3,4,5,6 j=1,2 (1 Italian, 2 German)

y_{ij}: the number of extra translators who work at period i and translates language j, i=1,2,3,4,5,6 j=1,2 (1 Italian, 2 German)

min $320\, x_{i1} + 280\, x_{i2} + 240\, y_{ij}$

st.

$x_{61} + x_{11} + y_{11} \geq 5$

$x_{11} + x_{21} + y_{21} \geq 3$

$x_{21} + x_{31} + z_{31} \geq 8$

$x_{31} + x_{41} + y_{41} \geq 8$

$x_{41} + x_{51} + y_{51} \geq 11$

$x_{51} + x_{61} + y_{61} \geq 4$

$x_{62} + x_{12} + y_{12} \geq 4$

$x_{12} + x_{22} + y_{22} \geq 4$

$x_{22} + x_{32} + y_{32} \geq 7$

$x_{32} + x_{42} + y_{42} \geq 7$

$x_{42} + x_{52} + y_{52} \geq 13$

$x_{52} + x_{62} + y_{62} \geq 4$

$x_{ij}\ y_{ij} \geq 0$ for i=1,2,3,4,5,6 j=1,2 and integer

Q4)

Decision Variables:

$P_{N,i}$: Units produced at month i under normal shift

$P_{E,i}$: Units produced at month i under extended shift

X_i: $\begin{cases} 1 & \text{if } P_{E,i} > 0 \\ 0 & \text{if otherwise} \end{cases}$ $(i=1,\dots,6)$

I_i: inventory level at the end of month i

Parameters:

C_N: Cost of normal shift (£100,000 per month)

C_E: Cost of extended shift (£180,000 per month)

U_N: max capacity of normal shift (5,000 units per month)

U_E: " " " extended " (7,500 units per month)

C_c: cost of changing from normal to extended shift (£15,000)

h: holding cost (£2 per unit per month)

I_0: initial stock (3,000 units)

D_i: Demand for month i $(i=1,\dots,6)$

Model:

$$\min \sum P_{N,i} C_N + P_{E,i} C_E X_i + C_c X_i + h I_i$$

s.t. $P_{N,i} + P_{E,i} + I_{i-1} \geq D_i$ $\quad \forall i=1..6$

$P_{E,i} \leq M X_i$

$P_{E,i} + P_{N,i} \geq 2{,}000$

$P_{N,6} + P_{E,6} - D_6 + I_5 \geq 2{,}000$

$0 \leq P_{N,i} \leq U_N$

$0 \leq P_{E,i} \leq U_E$ $\qquad P_{N,i}, P_{E,i}, X_i, I_i$ integer

$0 \leq X_i \leq 1$

$I_i \geq 0$ $\qquad I_i = P_{N,i} + P_{E,i} + I_{i-1} - D_i$

Q5)

Dec. Var

$x_{iJ} = \begin{cases} 1 & \text{if } i^{th} \text{ ast. is assigned to } J^{th} \text{ task} \\ 0 & \text{o.w} \end{cases}$ $i=1..4,\ J=1..4$

$$\max \sum_{i=1}^{4}\sum_{J=1}^{4} p_{iJ} \cdot x_{iJ}$$

s.t

$\sum_{i=1}^{4} x_{iJ} = 1 \quad \forall J=1..4$ (each post should have a single ast.)

$\sum_{J=1}^{4} x_{iJ} = 1 \quad \forall i=1..4$ (each ast. should be assigned to a single post)

$x_{iJ} \in \{0,1\} \quad \forall i=1..4,\ \forall J=1..4$

Q6)

Dec. Var : given

Model : $\min \sum_{i=1}^{M} x_i$

s.t

$\sum_{i=1}^{M} y_{ik} = d_k \quad \forall k=1..n$

$\sum_{k=1}^{n} y_{ik} \cdot l_k \leq l \cdot x_i \quad \forall i=1..M$

$x_i \in \{0,1\} \quad \forall i=1..M$

$y_{ik} \geq 0$ and integer $\forall i=1..M,\ k=1...n$

Q7)

(4) Convert into standard form and add artificial variable.

$$\begin{aligned} \min\ & -3x_1 + 4x_2 \\ \text{s.t.}\ & x_1 + x_2 + s_1 = 4 \\ & 2x_1 + 3x_2 - e_1 + a_1 = 18 \\ & x_1, x_2, s_1, e_1, a_1 \geq 0 \end{aligned}$$

Phase I: change obj. function with $\min a_1$ s.t. (same const.)

	z	x_1	x_2 ↓	s_1	e_1	a_1	RHS
	1	0	0	0	0	-1	0
	1	2	3	0	-1	0	18
← s_1	0	1	1	1	0	0	4
a_1	0	2	3	0	-1	1	18

×1

	z	x_1	x_2	s_1	e_1	a_1	RHS
	1	-1	0	-3	-1	0	6
x_2	0	1	1	1	0	0	4
a_1	0	-1	0	-3	-1	1	6

There does not exist any positive value in row zero, so the tableau is optimal for Phase I.

Since a_1 is still basic, the original problem is infeasible.

Q8)

a) $A < 0$ $B = 0$ C any $D \geq 0$ $E = 1$ F any

b) $A > 0$ $B = 0$ $C \leq 0$ $D \geq 0$ $E = 1$ F any

c) A any $B = 0$ C any $D = 0$ $E = 1$ F any

d) $A = 0$ $B = 0$ $C \leq 0$ $D \geq 0$ $E = 1$ F any

Q9)

1) $x_1 - x_2 + 3x_3 + 5x_4 = 15$

$x_1 + x_2 + x_4 = 9$

$x_i \geq 0 \quad i=1,2,3,4$

$n=4 \quad m=2 \qquad \binom{n}{m} = \binom{4}{2} = 6$ # of basic solutions

- x_1, x_2 : basic
  x_3, x_4 : nonbasic $\Rightarrow$ $\begin{matrix} x_1 - x_2 = 15 \\ x_1 + x_2 = 9 \end{matrix}$ $\Rightarrow$ $x = \begin{pmatrix} 12 \\ -3 \\ 0 \\ 0 \end{pmatrix}$
  $(x_3 = 0 \quad x_4 = 0)$
  This is a basic solution but not feasible since $x_2 = -3 < 0$

- x_1, x_3 : basic
  x_2, x_4 : nonbasic $\Rightarrow$ $\begin{matrix} x_1 + 3x_3 = 15 \\ x_1 = 9 \end{matrix}$ $\Rightarrow$ $x = \begin{pmatrix} 9 \\ 0 \\ 2 \\ 0 \end{pmatrix}$ This's a bfs.
  $(x_2 = 0 \quad x_4 = 0)$

- x_1, x_4 : basic
  x_2, x_3 : nonbasic $\Rightarrow$ $\begin{matrix} x_1 + 5x_4 = 15 \\ x_1 + x_4 = 9 \end{matrix}$ $\Rightarrow$ $x = \begin{pmatrix} 15/2 \\ 0 \\ 0 \\ 3/2 \end{pmatrix}$ It's a bfs.
  $(x_2 = 0, \ x_3 = 0)$

- x_2, x_3 : basic
  x_1, x_4 : nonbasic $\Rightarrow$ $\begin{matrix} -x_2 + 3x_3 = 15 \\ x_2 = 9 \end{matrix}$ $\Rightarrow$ $x = \begin{pmatrix} 0 \\ 9 \\ 8 \\ 0 \end{pmatrix}$ It's a bfs.
  $(x_1 = 0, \ x_4 = 0)$

- x_2, x_4 : basic
  x_1, x_3 : nonbasic $\Rightarrow$ $\begin{matrix} -x_2 + 5x_4 = 15 \\ x_2 + x_4 = 9 \end{matrix}$ $\Rightarrow$ $x = \begin{pmatrix} 0 \\ 5 \\ 0 \\ 4 \end{pmatrix}$ It's a bfs.
  $(x_1 = 0, \ x_3 = 0)$

- x_3, x_4 : basic
  x_1, x_2 : nonbasic $\Rightarrow$ $\begin{matrix} 3x_3 + 5x_4 = 15 \\ x_4 = 9 \end{matrix}$ $\Rightarrow$ $x = \begin{pmatrix} 0 \\ 0 \\ -10 \\ 9 \end{pmatrix}$
  $(x_1 = 0, x_2 = 0)$
  This is basic but not feasible since $x_3 < 0$

Q10)

(4) First convert to standard form

$$x_2 = -x_2'$$
$$x_3 = x_3' - x_3''$$

$$\max \; -x_1 - 3x_2' + (x_3' - x_3'') - Ma_2$$

s.t

$$x_1 + x_2' + (x_3' - x_3'') + s_1 = 4$$
$$2x_1 - x_2' + (x_3' - x_3'') - e_2 + a_2 = 10$$
$$x_1, x_2', x_3', x_3'', s_1, e_2 \geq 0$$

	z	x_1	x_2'	x_3'	x_3''	s_1	e_2	a_2	RHS
	~~1~~	~~1~~	~~+3~~	~~-1~~	~~1~~	~~0~~	~~0~~	~~M~~	0
	1	-2M+1	M+3	-M-1	M+1	0	M	0	-10M
s_1	0	1	1	1	-1	1	0	0	4
a_1	0	2	-1	1	-1	0	-1	1	10

$\times(-M)$

MRT $\begin{pmatrix} 4/1 \\ 10/2 \end{pmatrix}$

	z	x_1	x_2'	x_3'	x_3''	s_1	e_2	a_2	RHS
	1	0	3M+2	M-2	-M+2	2M-1	M	0	-2M-4
x_1	0	1	1	1	-1	1	0	0	4
a_1	0	0	-3	-1	1	-2	-1	1	2

MRT $\begin{pmatrix} X \\ 2/1 \end{pmatrix}$

	z	x_1	x_2'	x_3'	x_3''	s_1	e_2	a_2	RHS
	1	0	8	0	0	3	2	~~M-2~~	-8
x_1	0	1	-2	0	0	-1	-1	~~1~~	6
x_3''	0	0	-3	-1	1	-2	-1	~~1~~	2

$$\rightarrow x^* = \begin{pmatrix} x_1 \\ x_2' \\ x_3' \\ x_3'' \\ s_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{pmatrix} \quad z^* = -8$$

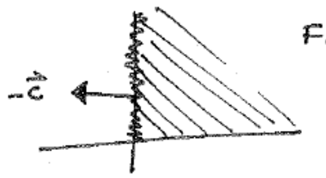
∃ no neg row 0 value. So the tableau is final. Since a_2 is not in the basis, remove the column a_2. the remaining tableau is for optimal soln.

$$x_2 = -x_2' = 0$$
$$x_3 = x_3' - x_3'' = 0 - 2 = -2$$

Q11)

2 **a)** Every LP with an unbounded feasible region is unbounded.

FALSE,



For the following obj. funct.

min x_1

$z^* = 0$

2 **b)** A linear programming problem cannot have exactly one feasible solution.

FALSE

min 0

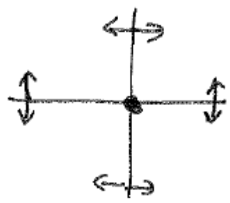
s.t

$x_1 \leq 0$

$x_1 \geq 0$

$x_2 \leq 0$

$x_2 \geq 0$



(0,0) is the feasible region for this LP

c) TRUE: Since the objective value of phase-I (w) is bounded, phase-I will always yield an optimal tableau.

d) FALSE: Since Big-M and 2-Phase methods make the same sequence of pivots, if Big-M stops with infeasibility (at least one artificial variable is positive and there is no entering variable) then phase-I will also stop with infeasibility and won't move to phase-II.

e) f) FALSE: counter example:

A Degenerate LP

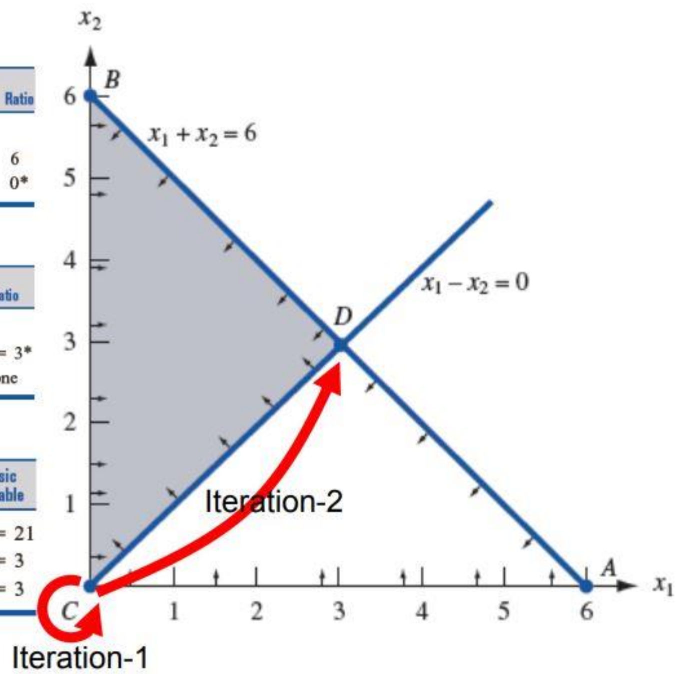
z	x_1	x_2	s_1	s_2	rhs	Basic Variable	Ratio
1	−5	−2	0	0	0	$z = 0$	
0	1	1	1	0	6	$s_1 = 6$	6
0	1	−1	0	1	0	$s_2 = 0$	0*

First Tableau for (16)

z	x_1	x_2	s_1	s_2	rhs	Basic Variable	Ratio
1	0	−7	0	5	0	$z = 0$	
0	0	2	1	−1	6	$s_1 = 6$	$\frac{6}{2} = 3$*
0	1	−1	0	1	0	$x_1 = 0$	None

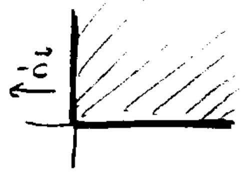
Optimal Tableau for (16)

z	x_1	x_2	s_1	s_2	rhs	Basic Variable
1	0	0	3.5	1.5	21	$z = 21$
0	0	1	0.5	−0.5	3	$x_2 = 3$
0	1	0	0.5	0.5	3	$x_1 = 3$



g) TRUE: By definition, feasible region of an LP is convex. Again by definition convex combination of any two points in a convex set is also convex.

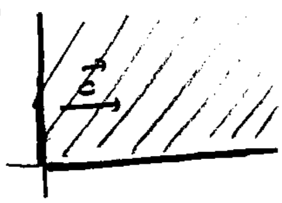
h) TRUE: Consider the feasible region below for a minimization problem.



min cx

s.t $Ax \geq b$

by multiplying c with -1



unbounded

Q12)

(4) First convert into standard form

max $5x_1+3x_2+x_3$

s.t.

$x_1+x_2+3x_3+s_1=6$

$5x_1+3x_2+6x_3+s_2=15$

$x_i\ge 0\ i=1..3\quad s_i\ge 0\ i=1,2$

	z	x_1	x_2	x_3	s_1	s_2	RHS
	1	-5	-3	-1	0	0	0
s_1	0	1	1	3	1	0	6
← s_2	0	5	3	6	0	1	15

x_1 enters (since $-5<0$) (problem is max)

Apply MRT and $\min\{\frac{6}{1},\frac{15}{5}\}$ so s_2 leaves

	z	x_1	x_2	x_3	s_1	s_2	RHS
	1	0	0	5	0	1	15
s_1	0	0	2/5	9/5	1	-1/5	3
← x_1	0	1	3/5	6/5	0	1/5	3

Since there is no negative row 0 coefficient, the solution is optimal

$x^*=\begin{pmatrix}3\\0\\0\end{pmatrix}$ with opt soln value $=15$

If there exists "0" in the row 0, there may be alternative optimal soln. (0 is for nonbasic variables)

So x_2 enters, apply MRT $\Rightarrow \min\{\frac{3}{3/5},\frac{3}{2/5}\}$, x_1 leaves

	z	x_1	x_2	x_3	s_1	s_2	RHS
	1	0	0	5	0	1	15
s_1	0	-2/3	0	1	1	-1/3	1
x_2	0	5/3	1	2	0	1/3	5

$\Rightarrow$ optimal soln

$x^*=\begin{pmatrix}0\\5\\0\end{pmatrix}$ with opt soln value $=15$

If you again enter x_1, you will turn back to previous tableau!

Q13)

(a) FALSE; this cannot be true for a maximization problem because a newly added constraint either does not change the feasible region or it makes it narrower. Hence the objective function may stay the same or decrease but there cannot be a constraint to improve the optimal solution.

(b) TRUE; $\binom{m+1}{m} = m+1$

(c) FALSE; it can have "at most" m positive variables. If it is less then m variables then that is called degeneracy.

(d) FALSE; because the set P is a convex set and the points on the convex combination of two optimal solutions also become optimal.

(e) FALSE; due to degeneracy, an extreme point of P may correspond to more than one basis.

Q14)

Converting the solution into standard form:

$$\begin{array}{lccccccccc}
\max & 2x_1 & +x_2 & +6x_3 & -4x_4 & & & & \\
\text{st.} & x_1 & +2x_2 & +4x_3 & -x_4 & +x_5 & & & =6 \\
 & 2x_1 & +3x_2 & -\ x_3 & +x_4 & & +x_6 & & =12 \\
 & x_1 & & +x_3 & +x_4 & & & +x_7 & =6 \\
 & x_1, & x_2, & x_3, & x_4, & x_5, & x_6, & x_7 & \geq 0
\end{array}$$

$$A=\begin{bmatrix} 1 & 2 & 4 & -1 \\ 2 & 3 & -1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

For basic variables x_1,x_2,x_4 the basis $B=\begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

$$B^{-1}=\begin{bmatrix} 3/4 & -1/2 & 5/4 \\ -1/4 & 1/2 & -3/4 \\ -3/4 & 1/2 & -1/4 \end{bmatrix}$$

$c_B=[2\ 1\ -4]$ $\qquad b=\begin{bmatrix} 6 \\ 12 \\ 6 \end{bmatrix}$ $\qquad c=[2\ 1\ 6\ -4]$

Remember the matrix notation

	z	x	x_s	
z	1	$c_BB^{-1}A-c$	c_BB^{-1}	$c_BB^{-1}b$
x_B	0	$B^{-1}A$	B^{-1}	$B^{-1}b$

$$B^{-1}A=\begin{bmatrix} 1 & 0 & 19/4 & 0 \\ 0 & 1 & -9/4 & 0 \\ 0 & 0 & -15/4 & 1 \end{bmatrix}$$

$c_BB^{-1}A=[2 \quad 1 \quad 89/4 \quad -4]$

$c_BB^{-1}A - c=[0 \quad 0 \quad 65/4 \quad 0]$

$c_BB^{-1}=[17/4 \quad -10/4 \quad 11/4]$ $\qquad c_BB^{-1}b=[12]$ $\qquad B^{-1}b=\begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$

Putting these values into the matrix:

	z	x_1	x_2	x_3	x_4	x_5	x_6 ↓	x_7	RHS	
z	1	0	0	65/4	0	17/4	-10/4	11/4	12	
x_1	0	1	0	19/4	0	3/4	-2/4	5/4	6	
x_2	0	0	1	-9/4	0	-1/4	2/4	-3/4	0	→
x_4	0	0	0	-15/4	1	-3/4	2/4	-1/4	0	

Since we have negative entry in Row 0, this tableau is not optimal.

x_6 enters and x_2 leaves the basis. New tableau: