IE 400 2018-2019 Fall **Midterm Study Set Solutions**

1. (30 points) A cargo plane has three compartments for storing cargo: front, center and rear. These compartments have the following limits on both weight and space:

Compartment	Weight capacity (tons) w_J	Space capacity (cubic meters) s_J
Front	10	6800
Center	16	8700
Rear	8	5300

The following four cargoes are available for shipment on the next flight:

Cargo	Weight (tons) wt_i	Volume (cubic meters/ton) V_i	Profit (TL/ton) P_i
C1	18	480	310
C2	15	650	380
C3	23	580	350
C4	12	390	285

We assume

- that each cargo can be split into whatever proportions/fractions we desire for shipment
- that each cargo can be split between 2 or more compartments if we so desire

The aim is to determine **how much** (if any) of each cargo C1, C2, C3 and C4 should be accepted and **how to distribute** each among the compartments so that the total profit for the flight is maximized.

a) (~~8 pts~~) 12pts Let us define the utilization rate of a cargo compartment as the ratio of the total cargo weight carried in that compartment to the weight capacity of that compartment (for example, if a total of 8 tons of cargo is carried in the front compartment, the utilization rate will be 8/10=0.8). Formulate this problem as a linear programming problem under the constraint that the utilization rate of each compartment must be the same to maintain the balance of the plane.

Decision Variables:

x_{iJ} = amount of cargo i distributed to compartment J in tons

$i = 1,2,3,4$ (c1 c2 c3 c4) $J = 1, 2, 3$ (F C R)

Model:

$$\max \sum_{i=1}^{4}\sum_{J=1}^{3} P_i x_{iJ}$$

st

(a) $\sum_{J=1}^{3} x_{iJ} \leq wt_i \quad \forall\, i = 1,2,3,4$ (available cargoes)

(b) $\sum_{i=1}^{4} V_i x_{iJ} \leq s_J \quad \forall\, J = 1,2,3$ (space capacity for each compartment)

(c) $\sum_{i=1}^{4} x_{iJ} \leq w_J \quad {}^2 \quad \forall\, J = 1,2,3$ (weight capacity for each compartment)

(d) $\dfrac{\sum_{i=1}^{4} x_{i1}}{w_1} = \dfrac{\sum_{i=1}^{4} x_{i2}}{w_2} = \dfrac{\sum_{i=1}^{4} x_{i3}}{w_3}$ (utilization rate of each compartment is equal)

$x_{iJ} \geq 0 \quad \forall\, i = 1,2,3,4 \quad \forall\, J = 1,2,3$

b) (8 pts) Formulate this problem as a linear programming problem under the constraint that the differences between the utilization rates of any two compartments should not be more than 0.1 to maintain the balance of the plane.

Decision Variables:

Model: Replace (d) with

$$\left| \frac{\sum_{i=1}^{4} x_{i1}}{w_1} - \frac{\sum_{i=1}^{4} x_{i2}}{w_2} \right| \leq 0.1$$

$$\left| \frac{\sum_{i=1}^{4} x_{i1}}{w_1} - \frac{\sum_{i=1}^{4} x_{i3}}{w_3} \right| \leq 0.1$$

$$\left| \frac{\sum_{i=1}^{4} x_{i2}}{w_2} - \frac{\sum_{i=1}^{4} x_{i3}}{w_3} \right| \leq 0.1$$

linearize =>

$$\frac{\sum x_{i1}}{w_1} - \frac{\sum x_{i2}}{w_2} \leq 0.1$$

$$\frac{\sum x_{i2}}{w_2} - \frac{\sum x_{i}}{w_1} \leq 0.1$$

$$\frac{\sum x_{i1}}{w_1} - \frac{\sum x_{i3}}{w_3} \leq 0.1$$

$$\frac{\sum x_{i3}}{w_3} - \frac{\sum x_{i1}}{w_1} \leq 0.1$$

or

$$-0.1 \leq \frac{\sum x_{i2}}{w_2} - \frac{\sum x_{i3}}{w_3} \leq 0.1$$

c) (4 pts) Consider the two LPs in parts a) and b). Which one is more likely to have a larger optimal value? Please justify your answer.

(b) is more likely to have a larger optimal value since feasible region in (b) includes feasible region in (a),

d) (~~10~~ pts) If cargo can be delivered only in integer amounts (in tons) and if it is not possible to split the cargo between compartments, how would your model in part b) change? Your new model should still be a linear one.

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Decision Variables:

x_{ij} = amount shipped in tons from cargo i in compartment j

$y_{ij} = \begin{cases} 1 & \text{if cargo } i \text{ is shipped in compartment } j \\ 0 & \text{o.w.} \end{cases}$

Model:

All constrainst in (b) +

$x_{ij} \leq w_j y_{ij} \quad \forall i=1,..4 \quad j=1,2,3$

$\sum_{j=1}^{3} y_{ij} \leq 1 \quad \forall i=1,2,3,4$

$y_{ij} \in \{0,1\}$

x_{ij} integer $\quad \forall i,j$

Q. 2)

Parameters : d_J, L_k, c_k are given

Dec. Var : $x_k = \begin{cases} 1 & \text{if truck } k \text{ is used} \\ 0 & \text{o.w} \end{cases}$ $k=1..4$

$y_{Jk} = \begin{cases} 1 & \text{if cust } J \text{ is visited by truck } k \\ 0 & \text{o.w} \end{cases}$ $J=1..10$, $k=1..4$

z_{Jk} : amount of good is carried by truck k for cust J, $J=1..10$, $k=1..4$

Model
$$\min \sum_{k=1}^{4} c_k x_k$$
s.t
$$\sum_{k=1}^{4} z_{Jk} = d_J \quad \forall J=1..10 \quad \text{(demand satisfaction)}$$
$$\sum_{J=1}^{10} z_{Jk} \leq L_k x_k \quad \forall k=1..4 \quad \text{(truck capacity)}$$
$$z_{Jk} \leq d_J y_{Jk} \quad \forall J=1..10,\ k=1..4$$
$$y_{Jk} \leq x_k \quad \forall J=1..10,\ k=1..4$$
(relation constraints)
$$x_k \in \{0,1\} \quad \forall k=1..4$$
$$y_{Jk} \in \{0,1\} \quad z_{Jk} \geq 0 \quad \forall J=1..10;\ k=1..4$$

(a) $\sum_{k=1}^{10} y_{Jk} \leq 5 \quad \forall k=1..4$

(b) $y_{1k} + y_{7k} \leq 1 \quad \forall k=1..4$

(c) $y_{3k} \leq y_{4k} + y_{5k} \quad \forall k=1..4$

(d) $y_{2k} = y_{6k} \quad \forall k=1..4$

(e) either $y_{14} + y_{24} + y_{34} \geq 1$ (I)
or $y_{54} + y_{64} + y_{74} \leq 2$ (II)

Not linear
You must linearize in your model.

* Add a binary var $w = \begin{cases} 1 & \text{if (I) holds} \\ 0 & \text{if (II) holds} \end{cases}$

$$y_{14} + y_{24} + y_{34} \geq w$$
$$y_{54} + y_{64} + y_{74} \leq 2(1-w) + 3w$$
$$w \in \{0,1\}$$

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Q. 3)

5) Dec. Var

$y_i = \begin{cases} 1 & \text{if } i^{th} \text{ constraint is satisfied} \\ 0 & \text{o.w} \end{cases}$

$z_1 = \begin{cases} 1 & \text{if at least one of } 7,8,9 \text{ is satisfied} \\ 0 & \text{o.w} \end{cases}$

$z_2 = \begin{cases} 1 & \text{if at least two of } 10,11,12,13 \text{ is satisfied} \\ 0 & \text{o.w} \end{cases}$

(since we only searching for feasibility of the problem. i.e if there exists a feasible solution then the objective will be zero)

Model min 0

s.t

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i + M(1-y_i) \qquad \forall i = 1..20$$

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i - My_i \qquad \forall i = 1..20$$

$$\sum_{i=1}^{20} y_i \geq 10$$

$$y_1 \leq y_4$$

$$y_2 + y_3 \leq 1$$

$$y_5 = y_6$$

$$y_7 + y_8 + y_9 \geq z_1$$

$$y_{10} + y_{11} + y_{12} + y_{13} \geq 2z_2$$

$$z_1 + z_2 \geq 1$$

$$z_1, z_2, y_i \in \{0,1\} \quad \forall i = 1..20$$

if $y_i = 1$

$\sum a_{ij}x_j \leq b_i$ (const is satisfied)

$\sum a_{ij}x_j \geq b_i - M$ → (adds $-M$ to no to restrict: redundant)

if $y_i = 0$

$\sum a_{ij}x_j \leq b_i + M$ → redundant

$\sum a_{ij}x_j \geq b_i$ → (const. not satisfied)

Q. 4)

Solution:

x_i: # workers employed on line i

y_i: $\begin{cases} 1 & \text{if line i is used} \\ 0 & \text{o.w} \end{cases}$

IP is:

$$\min \quad z = 1000y_1 + 2000y_2 + 500x_1 + 900x_2$$

s.t.

$$20x_1 + 50x_2 \geq 120$$
$$30x_1 + 35x_2 \geq 150$$
$$40x_1 + 45x_2 \geq 200$$
$$x_1 \leq 7y_1 \quad \text{(each line can have 7 workers at a time)}$$
$$x_2 \leq 7y_2$$
$$x_1, x_2 \geq 0; \quad y_1, y_2 \in \{0,1\}$$

Q. 5)

Soln:

let x_i: Amount of product produced on machine i. where $i = 1,2,3,4$.

y_i: $\begin{cases} 1 & \text{if machine i is used for production.} \\ 0 & \text{o.w} \end{cases}$

$$\min \quad 1000y_1 + 920y_2 + 800y_3 + 700y_4 + 20x_1 + 24x_2 + 16x_3 + 28x_4$$

s.t.

$$x_1 + x_2 + x_3 + x_4 \geq 2000 \quad \text{(demand const.)}$$
$$x_1 \leq 900y_1$$
$$x_2 \leq 1000y_2$$
$$x_3 \leq 1200y_3$$
$$x_4 \leq 1600y_4$$

(Capacity constraints)

$$x_1, x_2, x_3, x_4 \geq 0, \quad y_1, y_2, y_3, y_4 \in \{0,1\}$$

x_1, x_2, x_3, x_4 are integers.

Q. 6)

Let C be the set of countries, N_i be the set of neighbors of country i. Say there are m countries and m colors.

Define $x_{ij} = \begin{cases} 1 & \text{if country } i \text{ is colored by } j \\ 0 & \text{o/w} \end{cases}$

$y_j = \begin{cases} 1 & \text{if color } j \text{ is used} \\ 0 & \text{o/w} \end{cases}$

$$\min \sum_{j=1}^{m} y_j$$

st

$$\sum_{j=1}^{m} x_{ij} = 1 \quad \forall i \in C$$

$$x_{ij} + x_{kj} \leq y_j \quad \forall i \in C, k \in N_i \quad j = 1, \dots, m$$

$$x_{ij} \in \{0,1\} \quad \forall i \in C, j = 1..m$$

$$y_j \in \{0,1\} \quad j = 1..m$$

Q. 7) Let's first convert it to the standard form:

$$\begin{array}{llllllll}
\min & 3x_1 & + 2x_2 & + 4x_3 & & & & \\
\text{s.t.} & & & & & & & \\
& 2x_1 & + x_2 & + 3x_3 & & & = & 60 \\
& 3x_1 & + 3x_2 & + 5x_3 & - & e_1 & = & 120 \\
& x_1 & , \; x_2 & , \; x_3 & , & e_1 & \geq & 0
\end{array}$$

Adding the artificial variables for each row, we will end up with following LP:

$$\begin{array}{llllllllll}
\min & 3x_1 & + 2x_2 & + 4x_3 & & & & & & \\
\text{s.t.} & & & & & & & & & \\
& 2x_1 & + x_2 & + 3x_3 & & & + a_1 & & = & 60 \\
& 3x_1 & + 3x_2 & + 5x_3 & - & e_1 & & + a_2 & = & 120 \\
& x_1 & , \; x_2 & , \; x_3 & , & e_1 & , \; a_1 & , \; a_2 & \geq & 0
\end{array}$$

Two-Phase Method:

The phase-I problem is the following:

$$
\begin{array}{lllllllll}
\min & a_1 & + a_2 & & & & & & \\
\text{s.t.} & & & & & & & & \\
& 2x_1 & + x_2 & + 3x_3 & & + a_1 & & = & 60 \\
& 3x_1 & + 3x_2 & + 5x_3 & - e_1 & & + a_2 & = & 120 \\
& x_1, & x_2, & x_3, & e_1, & a_1, & a_2 & \geq & 0
\end{array}
$$

Now we need to solve phase-I LP:

Tableau 0:

z	x_1	x_2	x_3	e_1	a_1	a_2	RHS
1	0	0	0	0	-1	-1	0
0	2	1	3	0	1	0	60
0	3	3	5	-1	0	1	120

The simplex tableau above is not in proper format since the row zero coefficients of basic variables a_1 and a_2 are not zero. We add the first and second row to row zero to make these coefficients zero:

Tableau 0:

z	x_1	x_2	x_3	e_1	a_1	a_2	RHS	
1	5	4	8	-1	0	0	180	
0	2	1	3	0	1	0	60	60/3
0	3	3	5	-1	0	1	120	120/5

Now we can start simplex iterations: since it is a minimization problem we are looking for positive coefficients to decide on entering variables.

x_3 enters and a_1 leaves:

Tableau 1:

z	x_1	x_2	x_3	e_1	a_1	a_2	RHS	
1	-1/3	4/3	0	-1	-8/3	0	20	
0	2/3	1/3	1	0	1/3	0	20	20/(1/3)
0	-1/3	4/3	0	-1	-5/3	1	20	20/(4/3)

x_2 enters and a_2 leaves:

Tableau 2:

z	x_1	x_2	x_3	e_1	a_1	a_2	RHS
1	0	0	0	0	-1	-1	0
0	3/4	0	1	1/4	3/4	-1/4	15
0	-1/4	1	0	-3/4	-5/4	3/4	15

Both a_1 and a_2 left the basis so we can delete their columns from the tableau and move onto phase-II. We write the original objective function coefficients to the tableau:

Tableau 0:

z	x_1	x_2	x_3	e_1	RHS
1	-3	-2	-4	0	0
0	3/4	0	1	1/4	15
0	-1/4	1	0	-3/4	15

Again, notice that this tableau is not in proper format, we need to make row zero coefficients of basic variables x_2 and x_3 zero:

Tableau 0:

z	x_1	x_2	x_3	e_1	RHS
1	-1/2	0	0	-1/2	90
0	3/4	0	1	1/4	15
0	-1/4	1	0	-3/4	15

Stop. The tableau is optimal since there is no positive entry in row zero.

The optimal solution is $x_1^*=0$, $x_2^*=x_3^*=15$ with objective value 90.

Now let's solve the same problem with Big-M method.

<u>Big-M method:</u>

The LP is the following:

$$\begin{array}{llllllll}
\min & 3x_1 & + 2x_2 & + 4x_3 & + Ma_1 & + Ma_2 & & \\
\text{s.t.} & & & & & & & \\
& 2x_1 & + x_2 & + 3x_3 & & + a_1 & & = 60 \\
& 3x_1 & + 3x_2 & + 5x_3 & - e_1 & & + a_2 & = 120 \\
& x_1, & x_2, & x_3, & e_1, & a_1, & a_2 & \geq 0
\end{array}$$

Tableau 0:

z	x_1	x_2	x_3	e_1	a_1	a_2	RHS
1	-3	-2	-4	0	-M	-M	0
0	2	1	3	0	1	0	60
0	3	3	5	-1	0	1	120

Let's put the tableau in the proper format first:

Tableau 0:

z	x_1	x_2	x_3	e_1	a_1	a_2	RHS	
1	5M-3	4M-2	8M-4	-M	0	0	180M	
0	2	1	3	0	1	0	60	→ 60/3
0	3	3	5	-1	0	1	120	120/5

x_3 enters and a_1 leaves:

Tableau 1:

z	x_1	x_2	x_3	e_1	a_1	a_2	RHS	
1	-(M+1)/3	(4M-2)/3	0	-M	(4-8M)/3	0	80 +20M	
0	2/3	1/3	1	0	1/3	0	20	20/(1/3)
0	-1/3	4/3	0	-1	-5/3	1	20	→ 20/(4/3)

x_2 enters and a_2 leaves:

Tableau 2:

z	x_1	x_2	x_3	e_1	a_1	a_2	RHS
1	-1/2	0	0	-1/2	1/2-M	1/2-M	90
0	3/4	0	1	1/4	3/4	-1/4	15
0	-1/4	1	0	-3/4	-5/4	3/4	15

Stop. The tableau is optimal since there is no positive entry in row zero.

The optimal solution is $x_1^*=0$, $x_2^*=x_3^*=15$ with objective value 90.

Q. 8) First Tableau:

x_1	x_2	e_1	e_2	x_3	x_4	RHS
-2	-3	0	0	-M	-M	0
2	1	-1	0	1	0	4
1	-1	0	-1	0	1	1

x_3 and x_4 are basic variables but their row zero coefficients are not zero so we need to add the first and the second row to row zero after multiplying by M:

x_1	x_2	e_1	e_2	x_3	x_4	RHS
3M-2	-3	-M	-M	0	0	5M
2	1	-1	0	1	0	4
1	-1	0	-1	0	1	1

Then, x_1 enters x_4 leaves.

Second Tableau:

x_1	x_2	s_1	e_1	x_3	x_4	RHS
0	0	0	0	-1	-1	0
1	1	1	0	0	0	3
2	1	0	-1	1	0	4
1	1	0	0	0	1	3

x_3 and x_4 are basic variables but their row zero coefficients are not zero so we need to add the second and the third row to row zero:

x_1	x_2	s_1	e_1	x_3	x_4	RHS
3	2	0	-1	0	0	7
1	1	1	0	0	0	3
2	1	0	-1	1	0	4
1	1	0	0	0	1	3

Then, x_1 enters x_3 leaves.

Third Tableau:

x_1	x_2	x_3	s_1	s_2	e_1	RHS
-2	-1	-1	0	0	0	0
1/3	2/3	0	1	0	1/3	1
2	1	0	0	1	0	3
2/3	1/3	1	0	0	-1/3	1

x_3 is a basic variable but its row zero coefficient is not zero so we need to add the third row to row zero:

x_1	x_2	x_3	s_1	s_2	e_1	RHS
-4/3	-2/3	0	0	0	-1/3	1
1/3	2/3	0	1	0	1/3	1
2	1	0	0	1	0	3
2/3	1/3	1	0	0	-1/3	1

Then, x_1 enters and s_2 leaves (or x_3 leaves).

Q. 9)

a) (5 pts) The tableau is final and there exists a unique optimal solution.

$A > 0$ $\quad$ B, C, D any

$E \geq 0$

b) (5 pts) The simplex method determines an unbounded solution from this tableau.

$A < 0$ Ratio test does not return a finite value.

$\Rightarrow$ $B, C \leq 0$ $\quad$ $E \geq 0$ $\quad$ D any

c) (5 pts) The current bfs is degenerate (not necessarily optimal).

$E = 0$ $\quad$ A, B, C, D anything

d) (5 pts) The current solution is optimal, there are alternative optimal solutions but no alternative optimal bfs.

$A = 0$ Ratio test is not finite

$E \geq 0$ $\quad$ $B, C \leq 0$ $\quad$ D any

e) (10 pts) Find a specific set of values for unknowns A-E satisfying the general conditions in part d). Give the current optimal bfs and an alternative optimal solution that is not a bfs.

$A = 0$ $\quad$ $B = 0$ $\quad$ $C = 0$ $\quad$ $D = 0$ $\quad$ $E = 0$

x_1 enters no leaving variable

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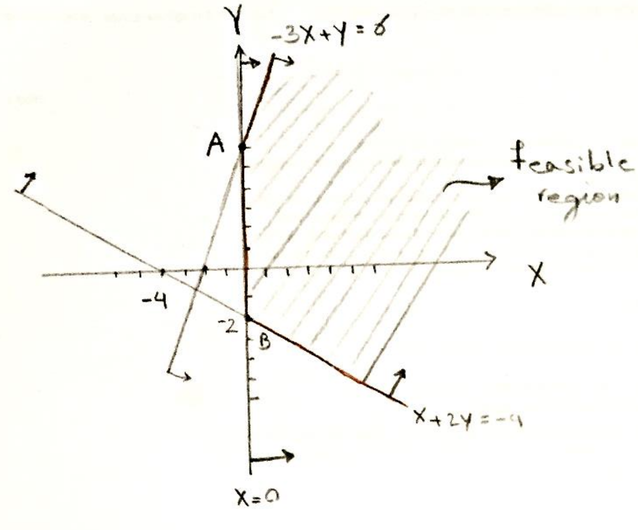
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \\ 1 \\ 0 \end{pmatrix} + x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$ is an alternate optimal solution for any $x_1 > 0$

Q.10)

a)



b) Let assume we want that point $A = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$ be unique optimal Solution, then $-C = (-a, -b)$ should lie in the following region $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$, e.g. $-C = (-4, 1) \Rightarrow a = 4$, $b = -1$

the optimal solution is $\begin{pmatrix} 0 \\ 6 \end{pmatrix}$ and the optimal solution value is $4 \times 0 + (-1) \times 6 = -6$

c) $-C$ should lie in the following region $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$

e.g. $-C = (1, 1) \Rightarrow a = -1$, $b = -1$

d) There are two possible cases $\Rightarrow$ Case 1: $-C = (-3, 1)$
Case 2: $-C = (-1, -2)$

!! for Case 1, point A is the only optimal extreme point and for Case 2, point B is the only optimal extreme point

e) $\min \; ax + b(y'' - y')$

$-x - 2(y'' - y') + s_1 = 4$

$-3x + (y'' - y') + s_2 = 6$

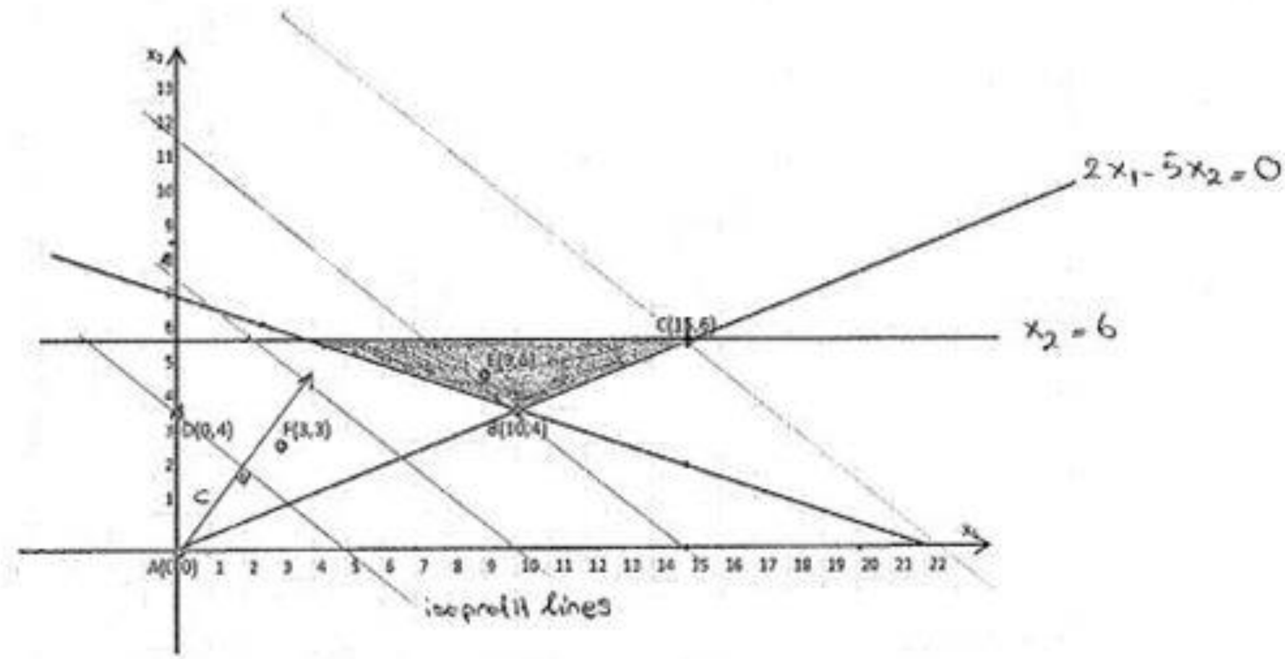
$x, y'', y', s_1, s_2 \geq 0$

f) $\begin{pmatrix} x \\ y' \\ y'' \\ s^1 \\ s^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 4 \\ 6 \end{pmatrix}$ bfs

x, y', y'' are nonbasic

s_1, s_2 are basic

Q. 11)



The shaded area is the feasible region, the initial bfs can determined using this region.

Standard form:

$$z = 4x_1 + 5x_2$$

$$x_1 + 3x_2 - e_1 + s_1 = 22$$

$$x_2 + s_2 = 6$$

$$2x_1 - 5x_2 + s_3 = 0$$

$$x_1, x_2, e_1, s_1, s_2, s_3 \geq 0$$

Simplex tableau:

Basic Var.	x1	x2	e1	s1	s2	s3	RHS
z	-4	-5	0	0	0	0	0
s1	1	3	-1	1	0	0	22
s2	0	1	0	0	1	0	6
s3	2	-5	0	0	0	1	0
z	-4	0	0	0	5	0	30
s1	1	0	-1	1	-3	0	4
x2	0	1	0	0	1	0	6
s3	2	0	0	0	5	1	30
z	0	0	-4	4	-7	0	46
x1	1	0	-1	1	-3	0	4
x2	0	1	0	0	1	0	6
s3	0	0	2	-2	11	1	22
z	0	0	-30/11	30/11	0	7/11	60
x1	1	0	-5/11	5/11	0	3/11	10
x2	0	1	-2/11	2/11	0	-1/11	4
s2	0	0	2/11	-2/11	1	1/11	2
z	0	0	0	0	15	2	90
x1	1	0	0	0	5/2	1/2	15
x2	0	1	0	0	1	0	6
e1	0	0	1	-1	11/2	1/2	11

Tableau stops with optimality.

x* = (x1=15, x2=6)   z*=90