Notes on implementing b-splines

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This is an attempt to clean-up the notation in Carl de Boor's b-spline notes, The m-th B-spline of order $K = 1$ is defined as,

$$X_m^1(t) \equiv \begin{cases} 1 & \text{if } \tau_m \leq t < \tau_{m+1}, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

All higher order B-splines are defined by recursion,

$$X_m^K(t) \equiv \frac{t - t_m}{t_{m+K-1} - t_m} X_m^{K-1}(t) + \frac{t_{m+K} - t}{t_{m+K} - t_{m+1}} X_{m+1}^{K-1}(t). \tag{2}$$

and a path is represented as $x(t) \equiv X_m^K(t)\xi^m$ where ξ^m are the coefficients.

The j-th derivative of this path is,

$$\left(\frac{d}{dt}\right)^j \left(\xi_{(1)}^m X_m^K(t)\right) = \xi_{(j+1)}^m X_m^{K-j}(t) \tag{3}$$

where

$$\xi_{(j+1)}^m \equiv \begin{cases} \xi_{(1)}^m & \text{for } j = 0, \\ \frac{\xi_{(j)}^m - \xi_{(j)}^{m-1}}{(t_{m+K-j} - t_m)/(K-j)} & \text{otherwise.} \end{cases} \tag{4}$$

So you compute the coefficients of the higher order derivatives, by differencing the coefficients. I find his notation a little confusing. I would have used j instead of $j + 1$ to denote the order of the coefficients.

So, de Boor's algorithm is the following sum over $K-j$ non-zero splines for position t,

$$x^{(j)}(t) = \sum_{m=1}^{K-j} \xi^m_{(j+1)} X^{K-j}_m(t) \tag{5}$$

$$= \sum_{m=1}^{K-j} \xi^m_{(j+1)} \left[\frac{t-t_m}{t_{m+K-j-1}-t_m} X^{K-j-1}_m(t) + \frac{t_{m+K-j}-t}{t_{m+K-j}-t_{m+1}} X^{K-j-1}_{m+1}(t) \right] \tag{6}$$

$$= \sum_{m=1}^{K-j} \xi^m_{(j+1)} \left[\frac{t-t_m}{t_{m+K-j-1}-t_m} X^{K-j-1}_m(t) \right] + \sum_{m=0}^{K-j-1} \xi^{m-1}_{(j+1)} \left[\frac{t_{m+K-j-1}-t}{t_{m+K-j-1}-t_m} X^{K-j-1}_m(t) \right]$$

One needs to think of the sum as going from $m = m_i$ to $m = m_i + K - j$ where m_i is the lowest non-zero spline at that point. Thus, there is no reason one can't add a few other splines that are zero at that point. So, it's perfectly reasonable to rewrite the sum as,

$$x^{(j)}(t) = \sum_{m=m_i-1}^{m_i+K-1-j} \left[\frac{\xi^m_{(j+1)}(t-t_m) + \xi^{m-1}_{(j+1)}(t_{m+K-j-1}-t)}{t_{m+K-j-1}-t_m} \right] X^{K-j-1}_m(t) \tag{7}$$

So we just expressed the value of $x^{(j)}(t)$ as a sum of lower order splines (one order less).

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$$= \sum_{m=1}^{K-j} \xi^m_{j+1} \left[\frac{t-t_m}{t_{m+K-1}-t_m} X^{K-j-1}_m(t) \right] + \sum_{m=0}^{K-j-1} \xi^{m-1}_{j+1} \left[\frac{t_{m+K-1}-t}{t_{m+K-1}-t_m} X^{K-j-1}_m(t) \right] \tag{10}$$

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