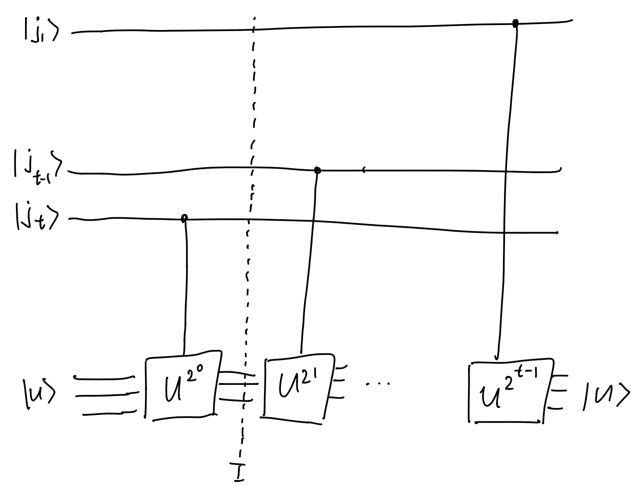
Exercise 5.7

Suppose j has t digits, $j = j_1 \cdots j_t$



$U|u\rangle = e^{2\pi i \varphi}|u\rangle$

U^{2^k} is controlled by the $(t-k)$th qubit in the 1st register

State at point I:

$$\left.\begin{array}{ll} \text{If } j_t = 0: & |j_1 \cdots j_{t-1}\rangle\, |0\rangle_{j_t} |u\rangle \\ \text{If } j_t = 1: & |j_1 \cdots j_{t-1}\rangle\, |1\rangle_{j_t}\, e^{2\pi i \varphi \cdot 2^0} |u\rangle \end{array}\right\} = |j_1 \cdots j_{t-1} j_t\rangle\, U^{j_t}|u\rangle = |j\rangle\, U^{j_t}|u\rangle$$

Using similar logic, the state after application of U^{2^k} is

$$|j\rangle\, U^{j_{t-k}\cdot 2^k} \cdots U^{j_t \cdot 2^0}|u\rangle = |j\rangle\, U^{\overbrace{j_{t-k} \cdots j_t}^{\text{last } k+1 \text{ digits of } j}}|u\rangle$$

Taking $k = t-1$ (final unitary) we get

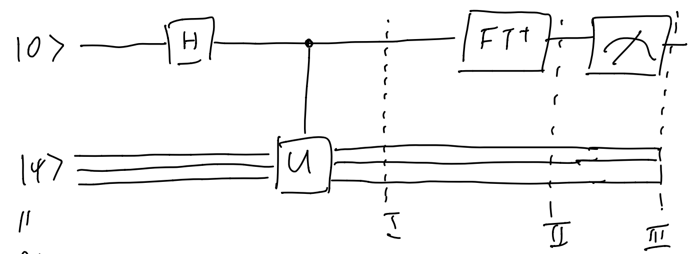
$$|\Psi\rangle_{final} = |j\rangle\, U^j |u\rangle$$

Exercise 5.8

Eigenvalues $e^{2\pi i \varphi_u}$: $\left[\begin{array}{l} 1 \to \text{associated evec} : u_1 \longrightarrow \varphi_{u_1} = 0 \\ -1 \to \text{associated evec} : u_{-1} \longrightarrow \varphi_{u_{-1}} = \frac{1}{2} = 0.1 \text{ (in binary).} \end{array}\right.$

(Phase)

The following circuit implements this:



$|\psi\rangle = \alpha|u_1\rangle + \beta|u_{-1}\rangle$

Unlike in phase estimation algo ancilla not guaranteed to be an eigenstate

Intuition: what happens here is a superposition of what happens in regular phase estimation.

I): $\alpha\left[\frac{1}{2}\left(|0\rangle + e^{2\pi i \varphi_{u_1}}|1\rangle\right)|u_1\rangle\right] + \beta\left[\frac{1}{2}\left(|0\rangle + e^{2\pi i \varphi_{u_{-1}}}|1\rangle\right)|u_{-1}\rangle\right] = \alpha\left[\frac{1}{2}\left(|0\rangle + e^{2\pi i \, 0/2}|1\rangle\right)|u_1\rangle\right] + \beta\left[\frac{1}{2}\left(|0\rangle + e^{2\pi i \, 1/2}|1\rangle\right)|u_{-1}\rangle\right]$

$$= \alpha\left(\frac{1}{2}\sum_{k=0}^{1} e^{2\pi i \, 0k/2}|k\rangle\right)|u_1\rangle + \beta\left(\frac{1}{2}\sum_{k=0}^{1} e^{2\pi i \, 1k/2}|k\rangle\right)|u_{-1}\rangle$$

II) $\alpha|0\rangle|u_1\rangle + \beta|1\rangle|u_{-1}\rangle$

III) Measurement projects 1st register onto basis states: $\{|0\rangle, |1\rangle\}$.

Resulting state is $|0\rangle|u_1\rangle$ w/ prob $|\alpha|^2$ → measurement outcome '0'

$|1\rangle|u_{-1}\rangle$ w/ prob $|\beta|^2$ → " " '1'