Ex 12.11

$\mathcal{E}$ = Trace-preserving q. op.

$$\chi(\mathcal{E}) = \max_{\{p_j, \rho_j\}} \left[S\Big(\mathcal{E}\big(\textstyle\sum_j p_j \rho_j\big)\Big) - \sum_j p_j S(\mathcal{E}(\rho_j)) \right]$$

Pure-state ensemble: $\{p_j, \rho_j\}$, $\rho_j = |\phi_j\rangle\langle\phi_j|$

Generic ensemble: $\{\alpha_j, \gamma_j\}$, $\gamma_j = \sum_i c_i^{(j)} |\psi_i^{(j)}\rangle\langle\psi_i^{(j)}|$

$$\boxed{\begin{array}{c}\text{Assume } \mathcal{E} \text{ linear} \\ \mathcal{E}\big(\sum_j p_j \rho_j\big) = \sum_j p_j \mathcal{E}(\rho_j)\end{array}}$$

Define: $\rho_j' \triangleq \mathcal{E}(\rho_j)$, $\gamma_j' = \mathcal{E}(\gamma_j)$

Claim: $\forall \{\alpha_j, \gamma_j\}$,
$\exists$ pure state ensemble $\{p_j, \rho_j\}$ s.t. $S\big(\sum_j p_j \rho_j'\big) - \sum_j p_j S(\rho_j') \geq S\big(\sum \alpha_j \gamma_j'\big) - \sum_j \alpha_j S(\gamma_j')$

Proof: Can find $\{p_k, \rho_k\}$ s.t. $\boxed{\sum_k p_k \rho_k = \sum_j \alpha_j \gamma_j'}$ and $\boxed{-\sum_k p_k S(\rho_k') \geq -\sum_j \alpha_j S(\gamma_j')}$

<u>PART 1</u>

$$\sum_j \alpha_j \gamma_j' = \sum_j \alpha_j \sum_i c_i^{(j)} \mathcal{E}(|\psi_i^{(j)}\rangle\langle\psi_i^{(j)}|)$$

Let $\rho_{ij} = |\psi_i^{(j)}\rangle\langle\psi_i^{(j)}|$, $p_{ij} = \alpha_j c_i^{(j)}$. Clearly $\sum_{ij} p_{ij} \rho_{ij}' = \sum_j \alpha_j \gamma_j'$, as desired.

$(i,j) = k$

<u>PART 2</u>

$$-\sum_{(ij)} p_{ij} S(\rho_{ij}') = -\sum_j \alpha_j \sum_i c_i^{(j)} S(\mathcal{E}(|\psi_i^{(j)}\rangle\langle\psi_i^{(j)}|))$$

$$-\sum_j \alpha_j S(\gamma_j') = -\sum_j \alpha_j S\Big(\mathcal{E}\big(\textstyle\sum_i c_i^{(j)} |\psi_i^{(j)}\rangle\langle\psi_i^{(j)}|\big)\Big)$$

$$= -\sum_j \alpha_j S\Big(\textstyle\sum_i c_i^{(j)} \mathcal{E}(|\psi_i^{(j)}\rangle\langle\psi_i^{(j)}|)\Big)$$

$$\leq -\sum_j \alpha_j \sum_i c_i^{(j)} S(\mathcal{E}(|\psi_i^{(j)}\rangle\langle\psi_i^{(j)}|))$$, as desired!