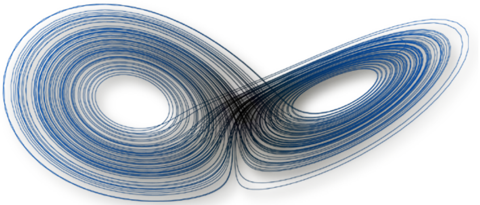

Nonlinear physics, dynamical systems and chaos theory

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Overview from last time

Given the non-linear dynamical system

$$\dot{\mathbf{X}} = \boldsymbol{\mathcal{F}}\left(\mathbf{X}\right),$$

we have seen in the previous lectures how to :

↪ Compute fixed points $\mathbf{X}^*$ of the system, i.e. solutions to

$$\boldsymbol{\mathcal{F}}\left(\mathbf{X}\right) = 0.$$

↪ Derive the linearized the equations governing the dynamics of a perturbation $\mathbf{x}$:

$$\dot{\mathbf{x}} = \boldsymbol{\mathcal{A}}\mathbf{x}.$$

↪ Characterize the linear stability of the fixed point $\mathbf{X}^*$ based on the eigenspectrum of $\boldsymbol{\mathcal{A}}$.

Question

Let us now consider a parametrized dynamical system

$$\dot{\mathbf{x}} = \boldsymbol{\mathcal{F}}\left(\mathbf{x}, \mu\right).$$

How do its fixed points evolve when varying the parameter μ? Can we characterize this evolution and make predictions?

Bifurcations of first-order systems

Flows on the line (again)

Bifurcations of first-order systems

- Let us consider a first-order dynamical system

$$\dot{x} = f(x, \mu),$$

where μ is our **control parameter**.

- We have seen that such systems have relatively simple dynamics dictated by fixed points.

- These fixed points may however change as a function of μ.
  - $\hookrightarrow$ Qualitative variations of the dynamics are called **bifurcations**.
  - $\hookrightarrow$ The values of μ at which these changes occurs are called **bifurcation points**.

Bifurcations of first-order systems

- To facilitate discussions to come, the Taylor expansion of $f(x)$ (for a constant μ) is given by

$$f(x) \simeq a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

- Depending on the coefficients a_k, different behaviors will be observed.

Saddle-node bifurcation

First-order dynamical system

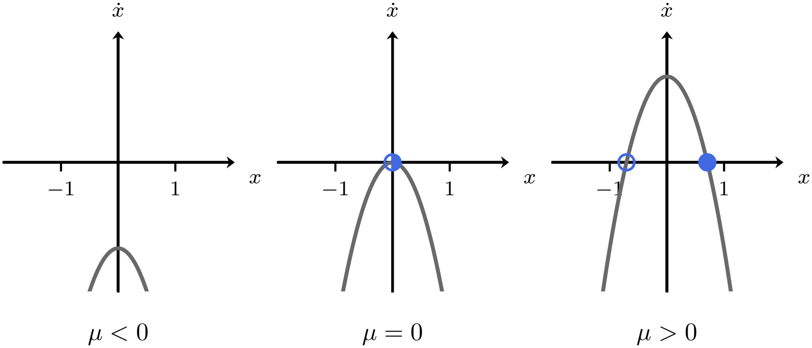
- As a starting point, let us look at the system

$$\dot{x} = \mu - x^2$$

and plot its phase line for different values of μ.

Saddle-node bifurcation

Phase line



Evolution of the phase line of the system for $\mu = {}^{-1}/_{2}, 0$ and ${}^{1}/_{2}$.

Saddle-node bifurcation

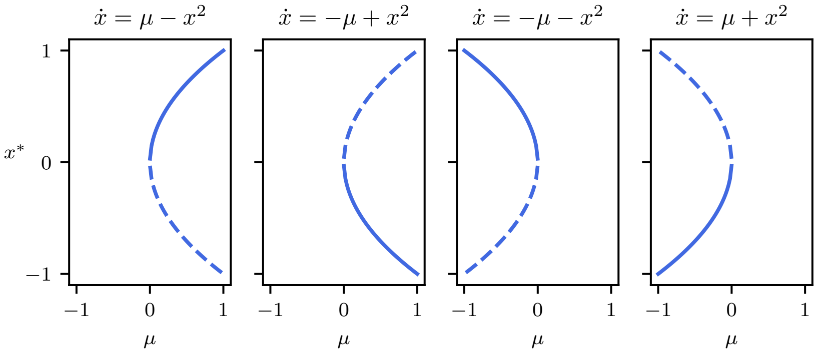
Fixed points and stability

- Depending on the value of μ, different behaviors are possible.
  - ↪ For $\mu < 0$, the system admits no fixed points and $\lim\limits_{t \to \infty} x(t) = -\infty$.
  - ↪ For $\mu = 0$, the system admits a single **meta-stable** fixed point $x^* = 0$. For $x(0) > 0$, $\lim\limits_{t \to \infty} x(t) = 0$, otherwise, for $x(0) < 0$, $\lim\limits_{t \to \infty} x(t) = -\infty$.
  - ↪ For $\mu > 0$, the system admits to fixed points $x^* = \pm\sqrt{\mu}$. One is linearly stable, while the other one is linearly unstable.

- As μ becomes positive, we observe a transition from the absence of fixed points to the creation of two of them, one stable and the other unstable. This is known as the **saddle node bifurcation**.

Saddle-node bifurcation

Bifurcation diagram



Bifurcation diagrams for the different combinations of saddle-node bifurcations.

Saddle-node bifurcation

Example from real life

- Let us consider a damped pendulum driven by a constant torque

$$mL^2\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + b\frac{\mathrm{d}\theta}{\mathrm{d}t} + mgL\sin(\theta) = \Gamma.$$

- Introducing the time scale $t = T\tau$, one can write

$$\frac{L}{gT^2}\ddot{\theta} + \frac{b}{mgLT}\dot{\theta} + \sin(\theta) = \frac{\Gamma}{mgL}.$$

Saddle-node bifurcation

Example from real life

- If ${}^{b}\!/_{mgT} \gg {}^{L}\!/_{gT^2}$, we can neglect $\ddot{\theta}$ and our equation becomes

$$\dot{\theta} = \gamma - \sin(\theta),$$

with $T = {}^{b}\!/_{mgL}$ and $\gamma = {}^{\Gamma}\!/_{mgL}$.

- You can now easily show that the system experiences a saddle-node bifurcation at $\gamma = 1$.
- Interpret your results from physical point of view!

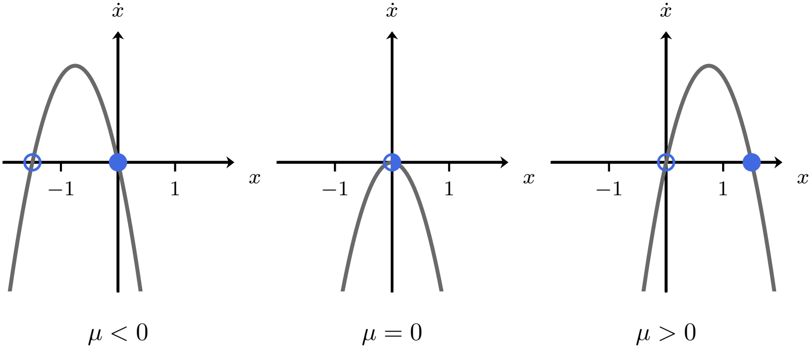
- Let us now consider the following first-order dynamical system

$$\dot{x} = \mu x - x^2$$

and plot its phase line for different values of μ.

Transcritical bifurcation

Phase line



Evolution of the phase line of the system for $\mu = {}^{-3}\!/_{2}, 0$ and ${}^{3}\!/_{2}$.

Transcritical bifurcation

Fixed points and linear stability

- The system admits two fixed points

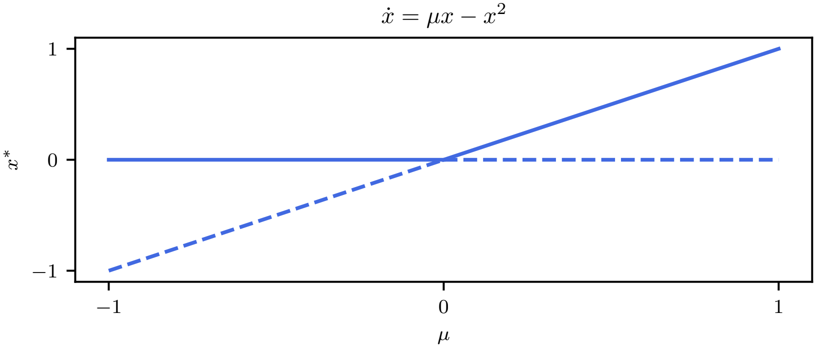
$$x_1^* = 0 \text{ and } x_2^* = \mu.$$

- Depending on the sign of μ, we have
  - $\hookrightarrow$ For $\mu < 0$, x_1^* is linearly stable while x_2^* is linearly unstable.
  - $\hookrightarrow$ For $\mu = 0$, $x_1^* = x_2^*$ is meta-stable.
  - $\hookrightarrow$ For $\mu > 0$, x_1^* is now linearly unstable, while x_2^* has become linearly stable.

- As μ becomes positive, the two fixed points have exchanged their stability. This is known as the **transcritical bifurcation**.

Transcritical bifurcation

Bifurcation diagram



Bifurcation diagram of the transcritical bifurcation.

Supercritical pitchfork bifurcation

First-order dynamical system

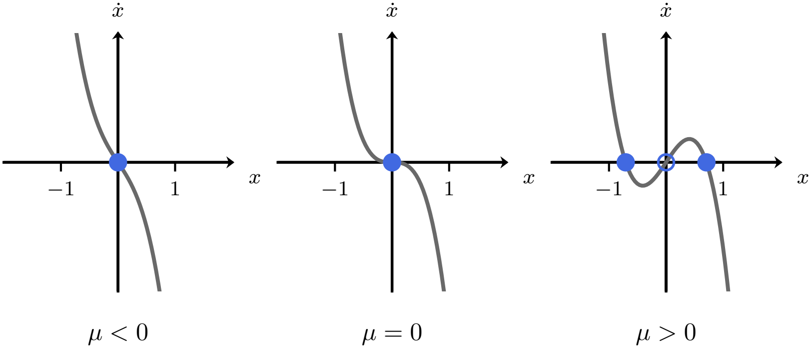
- Let us consider the following system

$$\dot{x} = \mu x - x^3$$

and plot its phase line for different values of μ.

Supercritical pitchfork bifurcation

Phase line



Evolution of the phase line of the system for $\mu = -1/2, 0$ and $1/2$.

Supercritical pitchfork bifurcation

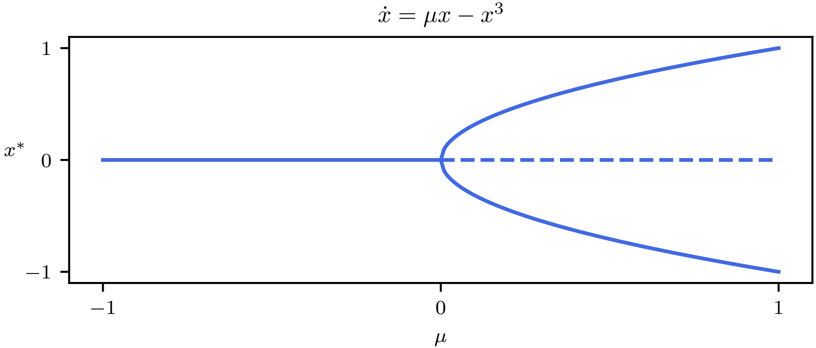
Fixed points and linear stability

- Depending on the value of μ, different behaviors are possible.
  - ↪ For $\mu < 0$, the system admits a single linearly stable fixed point $x^* = 0$.
  - ↪ For $\mu = 0$, the fixed point $x^* = 0$ is marginal from a linear point of view, yet still nonlinearly stable.
  - ↪ For $\mu > 0$, the system now admits three fixed points. $x_1^* = 0$ is now linearly unstable, while $x_{2,3}^* = \pm\sqrt{\mu}$ are linearly stable.

- As μ becomes positive, we observe that the origin becomes linearly unstable and two additional stable fixed points are created. This is known as the **supercritical pitchfork bifurcation**.

Supercritical pitchfork bifurcation

Bifurcation diagram



Bifurcation of the supercritical pitchfork.

Subcritical pitchfork bifurcation

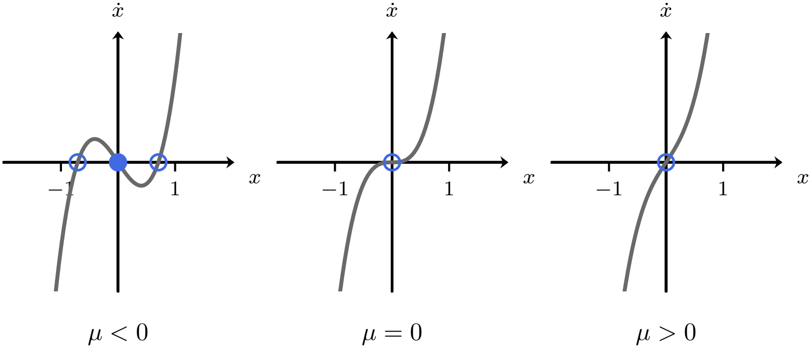
- Let us consider the following system

$$\dot{x} = \mu x + x^3$$

and plot its phase line for different values of μ.

Subcritical pitchfork bifurcation

Phase line



Evolution of the phase line of the system for $\mu = {}^{-1}/_{2}, 0$ and ${}^{1}/_{2}$.

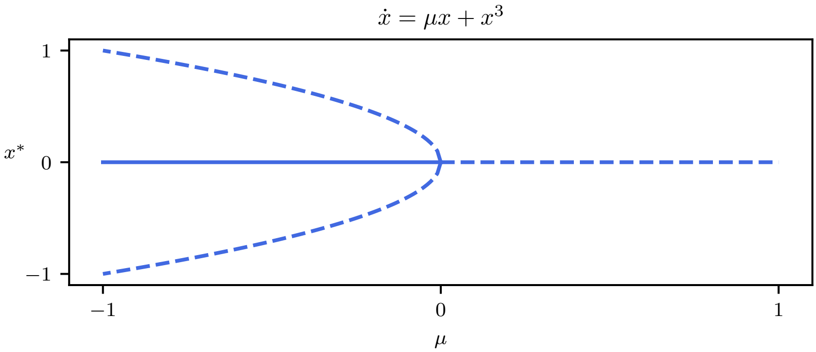
Subcritical pitchfork bifurcation

Fixed points and linear stability

- Depending on the value of μ, different behaviors are possible.
  - ↪ For $\mu < 0$, the system admits three fixed points. $x_1^* = 0$ is linearly stable, while $x_{2,3}^* = \pm\sqrt{-\mu}$ are linearly unstable.
  - ↪ For $\mu = 0$, the fixed point $x^* = 0$ is marginal from a linear point of view, but nonlinearly unstable.
  - ↪ For $\mu < 0$, the system now admits a single linearly unstable fixed point $x^* = 0$.
- As μ becomes positive, we observe that the origin becomes linearly unstable and the other two unstable fixed points are destroyed. This is known as the **subcritical pitchfork bifurcation**.

Subcritical pitchfork bifurcation

Bifurcation diagram



Bifurcation of the subcritical pitchfork.

Bifurcations of first-order systems

Summary

	f	f_x	f_μ	f_{xx}	$f_{x\mu}$	f_{xxx}
Fixed point	0					
Bifurcation	0	0	$\neq 0$			
Saddle-node	0	0	$\neq 0$	$\neq 0$		
Transcritical	0	0	0	$\neq 0$	$\neq 0$	
Pitchfork	0	0	0	0	$\neq 0$	$\neq 0$

Exercise

- Consider the following dynamical system

$$\dot{x} = \mu x + x^3 - 0.25x^5$$

and study its different fixed points and bifurcations.

Bifurcations of second-order systems

Creation of limit cycles

Bifurcations of second-order systems

- Let us now consider a second-order dynamical system given by

$$\dot{x} = f(x, y, \mu)$$
$$\dot{y} = g(x, y, \mu).$$

- As seen in the previous lectures, such systems have dynamics much richer that those of first-order systems.

- How do they evolve as the control parameter μ changes?
  $\hookrightarrow$ Note that all bifurcations seen so far also apply to fixed points of second-order dynamical systems.

Saddle-node bifurcation revisited

The reason it is called saddle-node

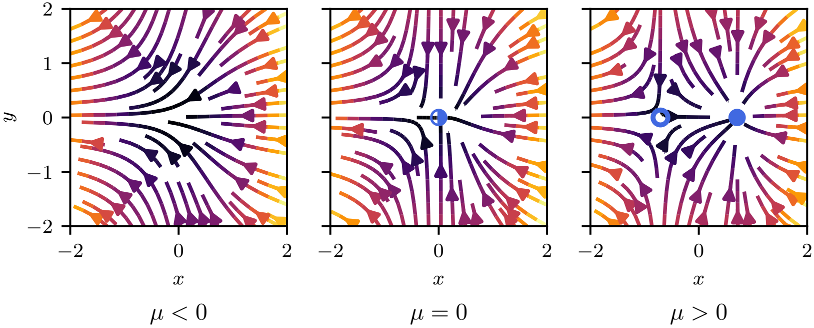
- Consider the following system

$$
\begin{aligned}
\dot{x} &= \mu - x^2 \\
\dot{y} &= -y
\end{aligned}
$$

and draw qualitatively its phase space for $\mu < 0$, $\mu = 0$ and $\mu > 0$.

Saddle-node bifurcation revisited

The reason it is called saddle-node



Phase plane of the system considered for varying μ.

Hopf bifurcation

Creation of limit cycles

- Let us consider the following system

$$\dot{x} = \mu x - \omega y - (x^2 + y^2)x$$
$$\dot{y} = \omega x + \mu y - (x^2 + y^2)y.$$

- It admits a single fixed point given by

$$(x^*, y^*) = (0, 0).$$

Hopf bifurcation

Exercise

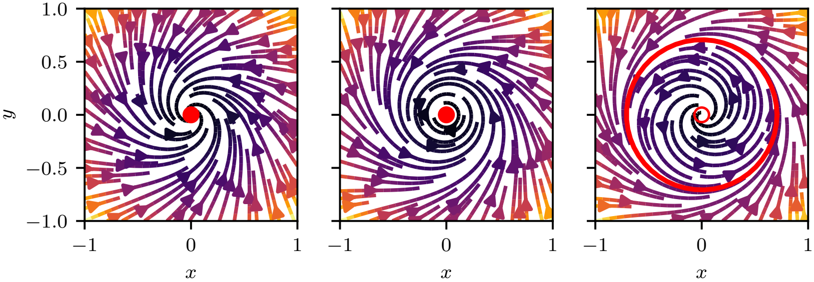
1. Study the linear stability of the fixed point as μ varies.
2. Introducing the complex variable $z = x + iy$, show that the equation for z reads

$$\dot{z} = (\mu + i\omega)z - |z|^2 z.$$

3. From this complex equation, determine the first-order system that governs the amplitude of oscillation $r = \sqrt{x^2 + y^2}$.
4. Study the properties of this equation an determine what type of bifurcation does the first-order system $\dot{r} = f(r, \mu)$ experiences.
5. Sketch the evolution of the phase plane of our original system as μ varies and conclude.

Hopf bifurcation

Phase plane

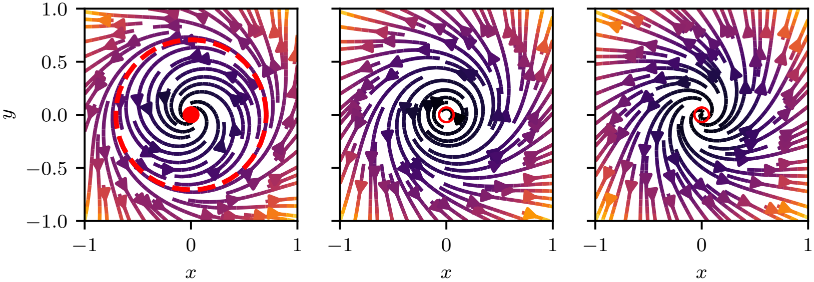


$\mu < 0$ $\mu = 0$ $\mu > 0$

Evolution of the phase plane of the system as a function of μ for the **supercritical Hopf bifurcation**.

Hopf bifurcation

Phase plane



$\mu < 0$ $\mu = 0$ $\mu > 0$

Evolution of the phase plane of the system as a function of μ for the **subcritical Hopf bifurcation**.

Hopf bifurcation

Normal form

- The **normal form** of the Hopf bifurcation (in polar coordinates) reads

$$\begin{aligned}\dot{r} &= \mu r \pm r^3 \\ \dot{\theta} &= \omega,\end{aligned}$$

where r is the amplitude of oscillator and θ its phase.

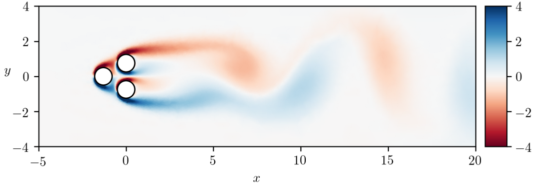
Hopf bifurcation

Example from real life

- The dynamics of the flow are governed by the Navier-Stokes equations

$$
\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \frac{1}{Re}\nabla^2 \boldsymbol{u}
$$
$$
\nabla \cdot \boldsymbol{u} = 0.
$$

- These are partial differential equations having only a quadratic nonlinearity.



Evolution of the vorticity field for the fluidic pinball at $Re = 60$.

Hopf bifurcation

Example from real life

Question

How come a system with quadratic nonlinearities exhibit a Hopf bifurcation whose normal form involve cubic ones?

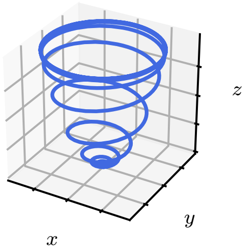
Hopf bifurcation

Example from real life

These dynamics can be modeled by the following generalized mean-field model

$$\begin{aligned}\dot{x} &= \sigma x - y - xz\\ \dot{y} &= x + \sigma y - yz\\ \dot{z} &= -\lambda(z - x^2 - y^2).\end{aligned}$$

where x and y capture the vortex shedding and z describes the *mean flow distortion*.



Trajectory given by the generalized mean field model.

Hopf bifurcation

Example from real life

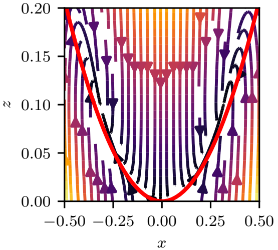
- Let us compute the two-dimensional unstable manifold. For that purpose, assume

$$
\begin{aligned}
z &= h(x, y) \\
&= ax^2 + bxy + cy^2.
\end{aligned}
$$

- After some calculations, we finally get

$$
z = \frac{1}{2\sigma + 1} \left(x^2 + y^2 \right).
$$

↪ If $\lambda \gg \sigma$, the system rapidly evolves onto this two-dimensional paraboloid manifold.



Slice in the $y = 0$ plane of the phase space.

Hopf bifurcation

Example from real life

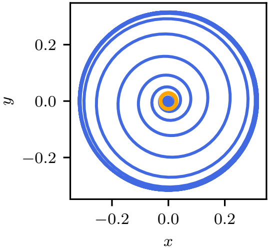
- Our dynamical system finally reduces to

$$\dot{x} = \sigma x - y - \alpha(x^2 + y^2)x$$
$$\dot{y} = x + \sigma y - \alpha(x^2 - y^2)y.$$

- Introducing $A = x + iy$, we finally arrive to the normal form of the supercritical Hopf bifurcation

$$\dot{A} = (\sigma + i)A - \alpha|A|^2 A.$$

- Though our system has only quadratic nonlinearities, dynamics on the manifold mimic cubic ones.



Trajectory of the system on the 2D manifold.

Stability of bifurcations

Perturbing the very structure of the model

Unperturbed normal forms

What we have seen so far.

- So far, we have seen the following normal forms:
  - ↪ Saddle-node :
$$\dot{x} = \mu - x^2$$
  - ↪ Transcritic :
$$\dot{x} = \mu x - x^2$$
  - ↪ Supercritical pitchfork :
$$\dot{x} = \mu x - x^3$$
  - ↪ Subcritical pitchfork :
$$\dot{x} = \mu x + x^3$$
  - ↪ Supercritical Hopf :
$$\begin{aligned}\dot{x} &= \mu x - \omega y + \left(x^2 + y^2\right)(\alpha x - \beta y)\\ \dot{y} &= \omega x + \mu y + \left(x^2 + y^2\right)(\beta x + \alpha y)\,.\end{aligned}$$

Question

What if we now perturb the very structure of the model (i.e. add an ϵ term)?

Answer

This is your very first homework! It is due for early January.

- For each normal form, study the cases

$\hookrightarrow$ $\epsilon < 0,$ $\hookrightarrow$ $\epsilon = 0,$ $\hookrightarrow$ $\epsilon > 0.$

- Which type of bifurcations are *structurally* stable and which are not?