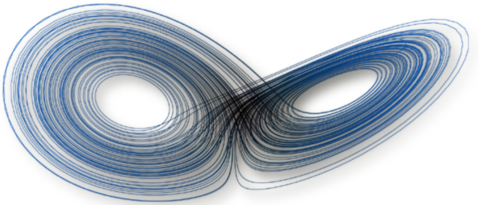

Nonlinear physics, dynamical systems and chaos theory

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Today's menu

The different routes to chaos

- So far, we have seen at least two sequences of bifurcations that cause a system to exhibit chaotic dynamics.
  - ↪ **Logistic map**: Sequence of period doubling bifurcations.
  - ↪ **Lorenz system**: For $0 \leq \rho \leq 25$, a supercritical pitchfork and a subcritical Hopf bifurcations occur before the creation of a stange attractor.
- These are two different routes to chaos. They are not the only ones though...

The routes to chaos

Subharmonic cascade

Subharmonic cascade

Logistic map

- Let us consider once again the logistic map given by

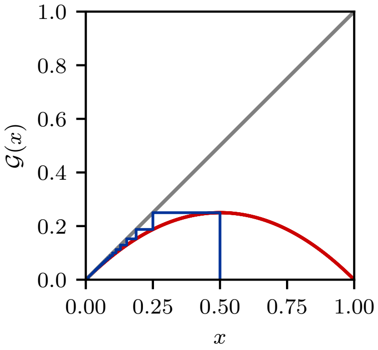
$$x_{k+1} = \mu x_k(1 - x_k),$$

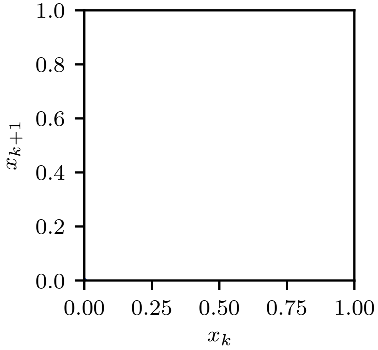
where μ is our control parameters.

- We have studied this discrete-time system during Lecture 5.

Logistic map

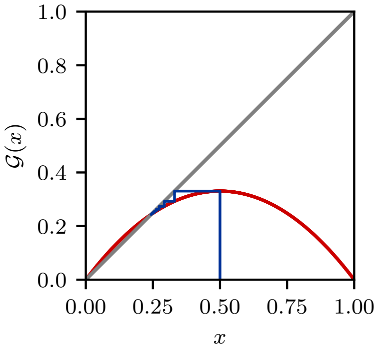
Cobweb and phase plane for $\mu = 1$

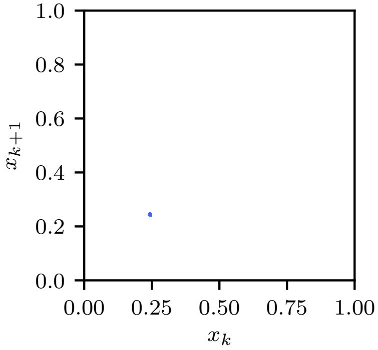




Logistic map

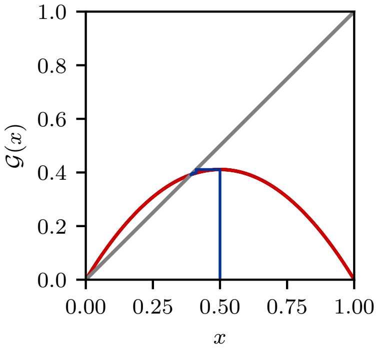
Cobweb and phase plane for $\mu = 1.32$

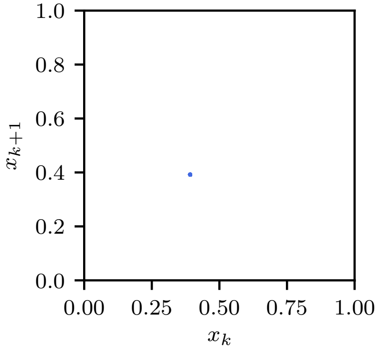




Logistic map

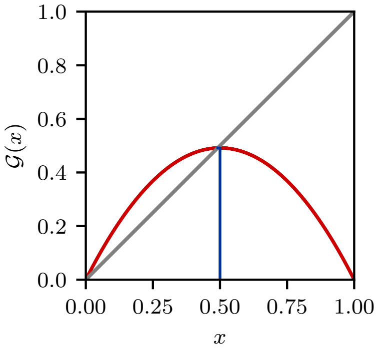
Cobweb and phase plane for $\mu = 1.64$

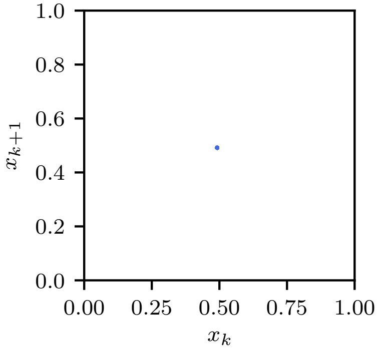




Logistic map

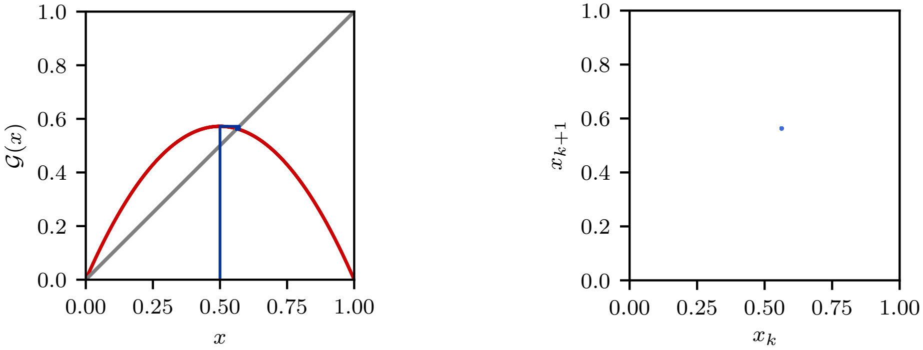
Cobweb and phase plane for $\mu = 1.96$





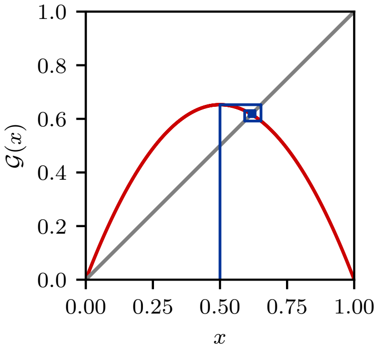
Logistic map

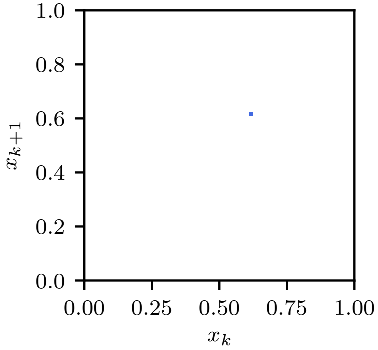
Cobweb and phase plane for $\mu = 2.28$



Logistic map

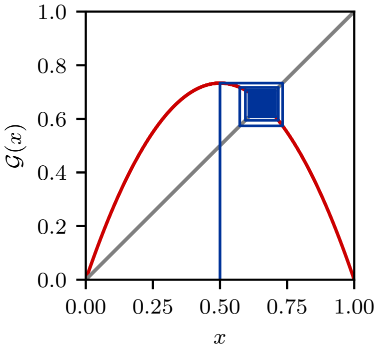
Cobweb and phase plane for $\mu = 2.61$

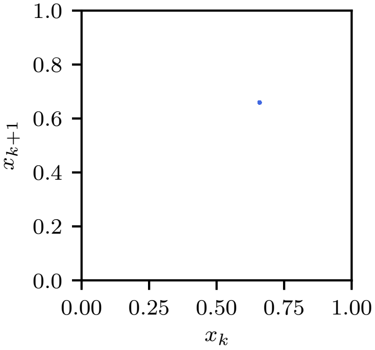




Logistic map

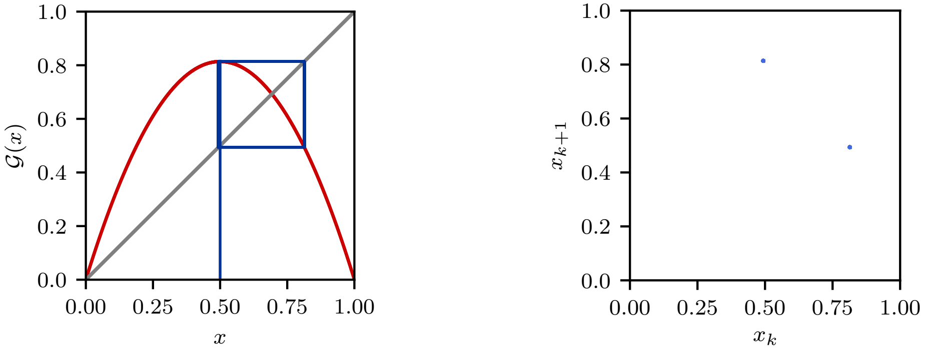
Cobweb and phase plane for $\mu = 2.93$





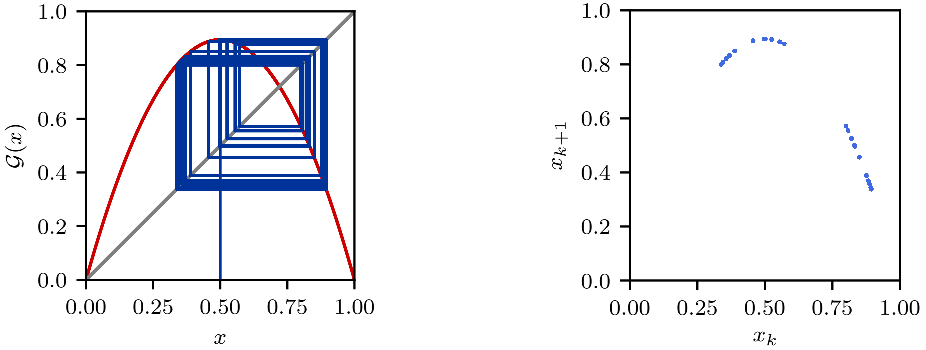
Logistic map

Cobweb and phase plane for $\mu = 3.25$



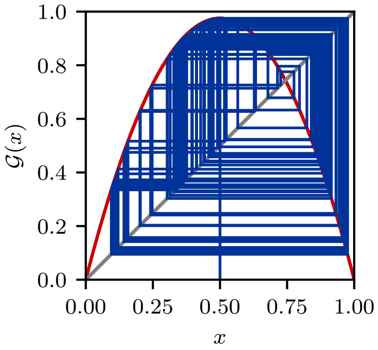
Logistic map

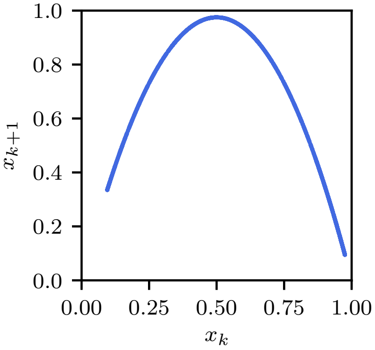
Cobweb and phase plane for $\mu = 3.57$



Logistic map

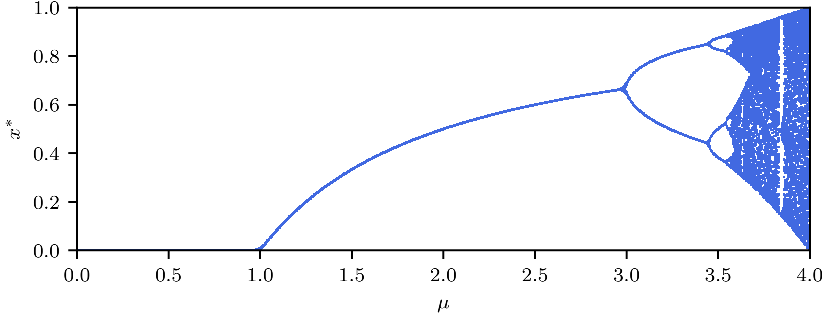
Cobweb and phase plane for $\mu = 3.9$





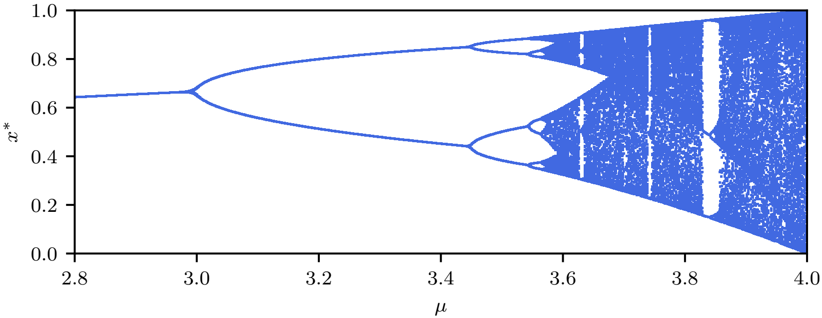
Logistic map

Bifurcation diagram



Logistic map

Bifurcation diagram



Logistic map

Subharmonic cascade

- Let us verify "experimentally" that the sequence of bifurcations taking place indeed corresponds to a subharmonic cascade.
- For that purpose, we will analyze time-series of x_k for different values of μ using the *discrete Fourier transform*.

Logistic map

Subharmonic cascade

Logistic map

Subharmonic cascade

- At each bifurcation point, the period of the limit cycle doubles.
  - $\hookrightarrow$ $\mu_1 = ?? \rightarrow$ period 1.
  - $\hookrightarrow$ $\mu_2 = ?? \rightarrow$ period 2.
  - $\hookrightarrow$ $\mu_3 = ?? \rightarrow$ period 4.
  - $\hookrightarrow$ $\mu_4 = ?? \rightarrow$ period 8.
  - $\hookrightarrow$ ...

- This scenario is not limited to discrete-time dynamical systems, but can also occurs in continuous-time systems.

Rössler system

A "simplified" Lorenz model

- In 1976, Otto Rössler introduced this "simplified" version of Lorenz model

$$\begin{aligned}\dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c).\end{aligned}$$

where we will assume $a = b = 0.2$, while c will be our control parameter.

- Although its original goal was purely theoretical, this model has since then proven able to describe a certain class of chemical reactions.

Rössler system

Subharmonic cascade

Rössler system

Bifurcation diagram

Rössler system

Subharmonic cascade

- As before, the period of the limit cycle doubles at each bifurcation point.
  - ↪ $c_1 = ?? \rightarrow$ period 1.
  - ↪ $c_2 = ?? \rightarrow$ period 2.
  - ↪ $c_3 = ?? \rightarrow$ period 4.
  - ↪ $c_4 = ?? \rightarrow$ period 8.
  - ↪ ...

- Deep down, the transition to chaos for the Rössler system obeys the same law as the transition to chaos for the logistic map.

Lorenz system

Low-order model for thermal convection

- Let us once more consider the Lorenz system

$$\begin{aligned}\dot{x} &= \sigma(y-x)\\ \dot{y} &= x(\rho-z)-y\\ \dot{z} &= xy-\beta z,\end{aligned}$$

where we set $\sigma = 10$ and $\beta = {}^8/_3$.

- The transition to chaos when ρ varies from 0 to 28 has been studied in Lecture 6.
- For the moment, we will now draw our attention when ρ varies from 313 down to 215.

Lorenz system

Bifurcation diagram

Lorenz system

Subharmonic cascade

Lorenz system

Subharmonic cascade

- Once more, the period of the limit cycle doubles at each bifurcation point.
  - ↪ $\rho_1 =?? \rightarrow$ period 1.
  - ↪ $\rho_2 =?? \rightarrow$ period 2.
  - ↪ $\rho_3 =?? \rightarrow$ period 4.
  - ↪ $\rho_4 =?? \rightarrow$ period 8.
  - ↪ ...

- Again, the transition to chaos for $215 \leq \rho \leq 313$ is very similar to that of the logistic map and of the Rössler system.

Subharmonic cascade

Question

How similar are these different systems?

Subharmonic cascade

Subharmonic cascade

	$\mu_3-\mu_2 / \mu_2-\mu_1$	$\mu_4-\mu_3 / \mu_3-\mu_2$	$\mu_5-\mu_4 / \mu_4-\mu_3$	$\mu_6-\mu_5 / \mu_5-\mu_4$	$\mu_7-\mu_6 / \mu_6-\mu_5$
Logistic map	??	??	??	??	??
Rössler	??	??	??	??	??
Lorenz	5.019923	4.710117	4.671428	4.669227	4.669203

Subharmonic cascade

- In all cases, the sequence

$$\delta_i = \frac{\mu_i - \mu_{i-1}}{\mu_{i-1} - \mu_{i-2}}$$

  converges to ??.

- This number is known as ?? and is the signature of the subharmonic cascade.

The routes to chaos

??