

# Nonlinear physics, dynamical systems and chaos theory

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# Duffing oscillator

A non-harmonic oscillator

- ▶ Let us consider as an example the Duffing oscillator given by

$$\ddot{x} + \delta \dot{x} + \alpha x + \beta x^3 = 0.$$

with  $\alpha = -1$ ,  $\beta = 1$  and  $\delta = 1/2$ .

- ▶ It describes the motion of a damped oscillator with a more complex potential than simple harmonic motion.
  - ↪ Example: a spring pendulum whose spring's stiffness does not exactly obey Hooke's law.

# Duffing oscillator

A non-harmonic oscillator

- ▶ Introducing  $y = \dot{x}$ , this second-order nonlinear ODE can be recast as a set of two first-order ODE with a nonlinear coupling term

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -\frac{1}{2}y + x - x^3.\end{aligned}$$

- ▶ Let us first discuss the different fixed points of the system.

# Duffing oscillator

## Fixed points

- ▶ The fixed points of the system are given by

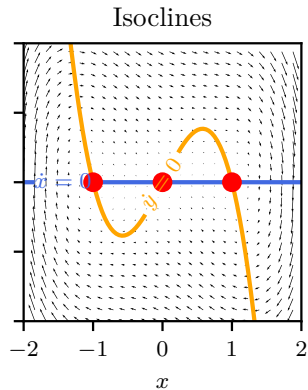
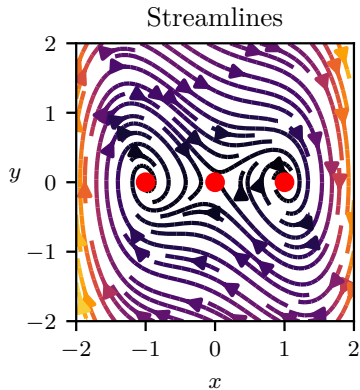
$$\begin{aligned}y &= 0 \\ x(1 - x^2) &= 0.\end{aligned}$$

- ▶ The system thus admits three different fixed points given by

$$(x_1, y_1) = (0, 0) \quad (x_2, y_2) = (1, 0) \quad (x_3, y_3) = (-1, 0)$$

# Duffing oscillator

Fixed points



Phase plane and isoclines for the Duffing oscillator.

# Duffing oscillator

## Linear stability

- ▶ The Jacobian matrix of the system reads

$$\mathbf{A}(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ 1 - 3x^2 & -0.5 \end{bmatrix}$$

- ▶ As seen in the previous course, the linear stability of each fixed point  $\mathbf{x}^*$  is governed by the eigenspectrum of  $\mathbf{A}(\mathbf{x}^*)$ .

# Duffing oscillator

## Linear stability

$$\mathbf{x}_1^* = (0, 0)$$

- ▶ Eigenvalues of  $\mathbf{A}$  are

$$\lambda_1 = 0.78$$

and

$$\lambda_2 = -1.28$$

- ▶ This fixed point is a **saddle**.

$$\mathbf{x}_2^* = (1, 0)$$

- ▶ Eigenvalues of  $\mathbf{A}$  are

$$\lambda_1 = -0.25 + 1.39i$$

and

$$\lambda_2 = -0.25 - 1.39i$$

- ▶ This fixed point is a **stable focus**.

$$\mathbf{x}_3^* = (-1, 0)$$

- ▶ Eigenvalues of  $\mathbf{A}$  are

$$\lambda_1 = -0.25 + 1.39i$$

and

$$\lambda_2 = -0.25 - 1.39i$$

- ▶ This fixed point is a **stable focus**.

# Duffing oscillator

A dissipative dynamical system

- Let us derive an equation for the total energy of the system

$$\dot{x} (\ddot{x} - x + x^3) = -\frac{1}{2}\dot{x}$$

- After some simplification, this equation can be re-written as

$$\frac{d}{dt} \underbrace{\left[ \frac{1}{2} (\dot{x})^2 - \frac{1}{2} x^2 + \frac{1}{4} x^4 \right]}_{\mathcal{H}} = -\frac{1}{2} (\dot{x})^2,$$

where  $\mathcal{H}$  is the total energy of the system.



# Duffing oscillator

A dissipative dynamical system

- ▶ The governing equation for the total energy  $\mathcal{H}$  finally reads

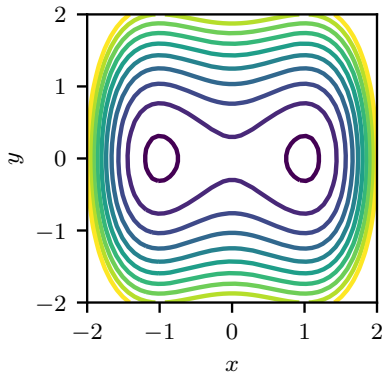
$$\frac{d}{dt}\mathcal{H} = -\frac{1}{2}(\dot{x})^2.$$

- ▶ Clearly, as  $t \rightarrow +\infty$ , the total energy  $\mathcal{H}$  tends to a constant value.

This is a **dissipative** dynamical system.

# Duffing oscillator

A dissipative dynamical system



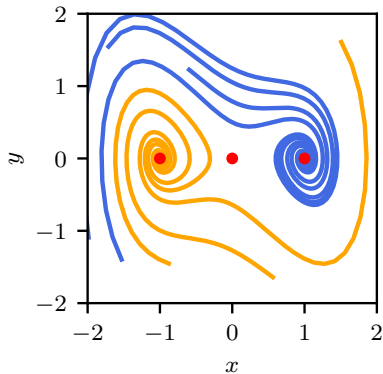
Isocontours of the total energy  $\mathcal{H}$  of the Duffing oscillator.

## Question

Given an initial condition  $x_0$ , toward which fixed point will it evolve as  $t \rightarrow +\infty$ ?

# Duffing oscillator

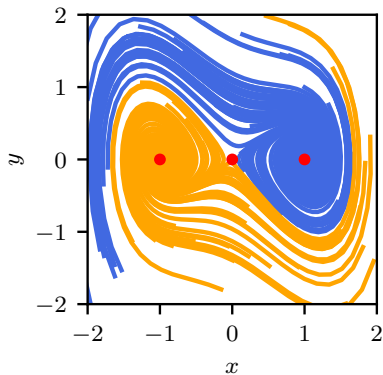
Basin of attraction



Trajectories of 8 randomly distributed initial conditions.

# Duffing oscillator

Basin of attraction



Trajectories of 100 randomly distributed initial conditions.

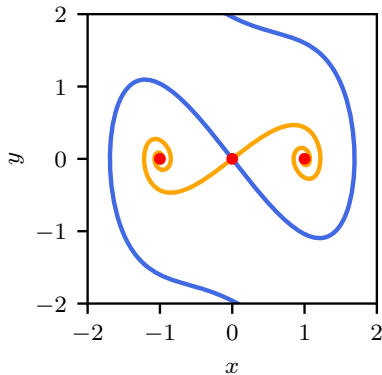
# Duffing oscillator

## Basin of attraction

- ▶ Not all initial conditions  $x_0$  end up to the same fixed point.
- ▶ Two different regions appear well separated.
  - ↪ These two regions are called the **basins of attraction** of each fixed point.
- ▶ For the present dynamical system, a sharp frontier delimits these two regions. The saddle node  $x_1^* = (0, 0)$  moreover appears to "sit" on this frontier.

# Duffing oscillator

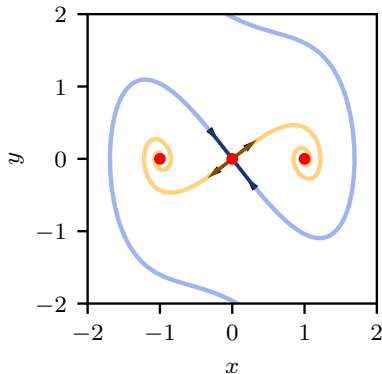
Stable and unstable manifolds



Delimitation (blue) of the two basins of attraction.

# Duffing oscillator

Stable and unstable manifolds



Arrows depict the linearly stable and unstable eigendirections of the saddle  $\mathbf{x}_1^* = (0, 0)$ .



# Duffing oscillator

## Stable and unstable manifolds

- ▶ On the previous plot, the blue line is called the **stable** manifold  $W^s$  of  $x_1^*$ , while the orange line is its **unstable** manifold  $W^u$ .
- ▶  $W^s$  and  $W^u$  are invariant sets, that is

*An invariant set is a subset  $W$  of the phase space such that for any  $x \in W$  and  $t \in \mathbb{R}$ , we have  $\phi_t(x) \in W$ .*

- ▶ Note that, if  $\|x\|$  is small enough, the stable (resp. unstable) manifold is tangent to the stable (resp. unstable) eigendirection of the fixed point.

### Stable Manifold Theorem

Suppose the origin is a fixed point of  $\dot{x} = f(x)$ . Let  $E^s$  and  $E^u$  be the stable and unstable subspaces of the linearization  $\dot{x} = Ax$ , where  $A$  is the Jacobian matrix of  $f$  at the origin. If  $\|f(x) - Ax\| = \mathcal{O}(\|x\|^2)$ , then  $\exists$  **local stable and unstable manifolds**  $W_{loc}^s(0)$  and  $W_{loc}^u(0)$  which have the same dimension as  $E^s$  and  $E^u$  and are tangent to them at 0.

# Duffing oscillator

How to compute these manifolds?

## Brute force

- ▶ Integrate numerically forward (resp. backward) in time an initial condition lying in the unstable (resp. stable) linear subspace of the fixed point.

## Maths

- ▶ Use Taylor expansion to compute an analytical expression of the stable and unstable manifolds.

# Duffing oscillator

How to compute these manifolds?

- ▶ Let us assume that

$$\begin{aligned}y &= h(x) \\&= a_1x + a_2x^2 + a_3x^3 + \dots \\&= \sum_{k=1}^n a_k x^k.\end{aligned}$$

- ▶ Note moreover that

$$\dot{y} = \frac{dx}{dt} \frac{dy}{dx}$$

# Duffing oscillator

How to compute these manifolds?

- ▶ We then obtain that

$$\begin{aligned}\dot{y} &= h'(x)\dot{x} \\ &= \dot{x} \sum_{k=1}^n a_k k x^{k-1}\end{aligned}$$

- ▶ By equating both sides, one obtains  $n$  algebraic equations allowing us to determine the coefficients  $a_k$  ( $k = 1 \cdots n$ ).

# Duffing oscillator

Application to our example

- ▶ Let us compute cubic approximations of the stable and unstable manifolds of the saddle  $x_1^* = (0, 0)$ . We thus assume that

$$\begin{aligned} y &= h(x) \\ &= ax + bx^2 + cx^3. \end{aligned}$$

- ▶ Differentiating  $h(x)$  with respect to  $x$  gives

$$h'(x) = a + 2bx + 3cx^2.$$

# Duffing oscillator

Application to our example

► Finally, we can write

$$h'(x)\dot{x} - \dot{y} = 0$$

$$h'(x)y + \frac{1}{2}y - x + x^3 = 0$$

$$(a + \frac{1}{2} + 2bx + 3cx^2)(ax + bx^2 + cx^3) - x + x^3 = 0$$

# Duffing oscillator

Application to our example

- After some calculations, we finally obtain that the coefficients in front of  $x^k$  are solution to

$$\begin{aligned} x &: a^2 + \frac{1}{2}a - 1 = 0 \\ x^2 &: \left(3a + \frac{1}{2}\right)b = 0 \\ x^3 &: \left(4a + \frac{1}{2}\right)c + 1 = 0 \end{aligned}$$

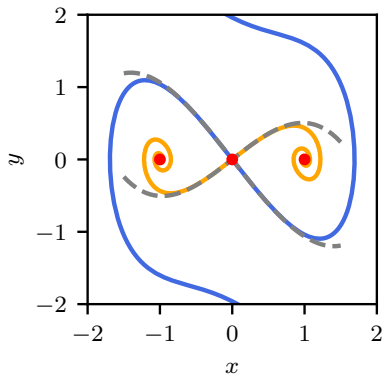
- Finally, we obtain that the stable and unstable manifolds can be approximated by

$$h_{\pm}(x) = a_{\pm}x + c_{\pm}x^3.$$



# Duffing oscillator

Stable and unstable manifolds



Dashed gray lines depict the polynomial approximations of  $W^s$  and  $W^u$ .

# Why do we bother with manifolds?

A toy-model for subcritical transition to turbulence

# Subcritical transition to turbulence

A toy-model

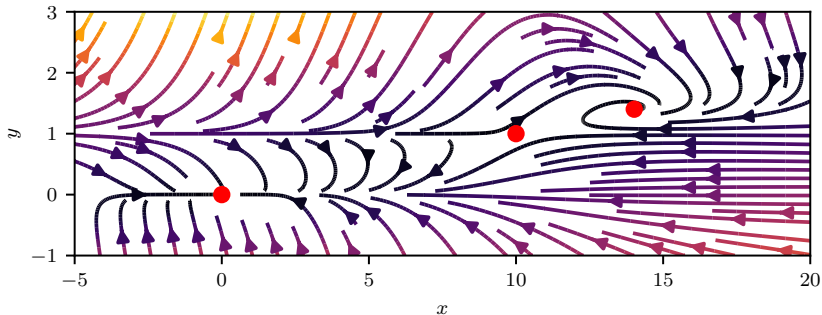
- ▶ In 2014, Kerswell *et al.* have proposed a simple toy-model to illustrate some aspect of subcritical transition to turbulence and so-called *nonlinear optimal perturbations*.
- ▶ This model reads

$$\dot{x} = -x + 10y$$

$$\dot{y} = y(10e^{-x^2/100} - y)(y - 1).$$

# Subcritical transition to turbulence

A toy-model



Phase plane and fixed points of the toy-model considered.

# Subcritical transition to turbulence

A toy-model

The model admits three fixed points.

## Laminar solution

- ▶  $\mathbf{x}_1^* = (0, 0)$
- ▶ Linearly stable sink.

## The Edge

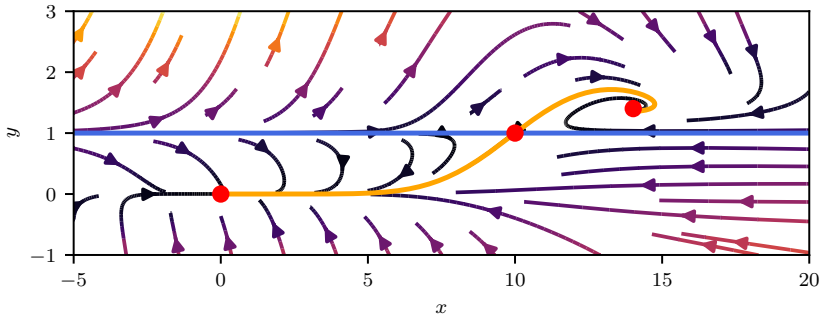
- ▶  $\mathbf{x}_2^* = (10, 1)$
- ▶ Saddle point.

## Turbulent solution

- ▶  $\mathbf{x}^* = (14.017, 1.4017)$
- ▶ Linearly stable focus.

# Subcritical transition to turbulence

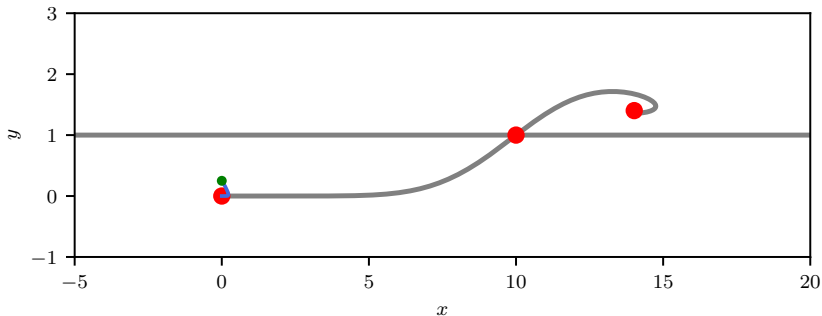
A toy-model



Stable (blue) and unstable (orange) manifolds of the Edge.

# Subcritical transition to turbulence

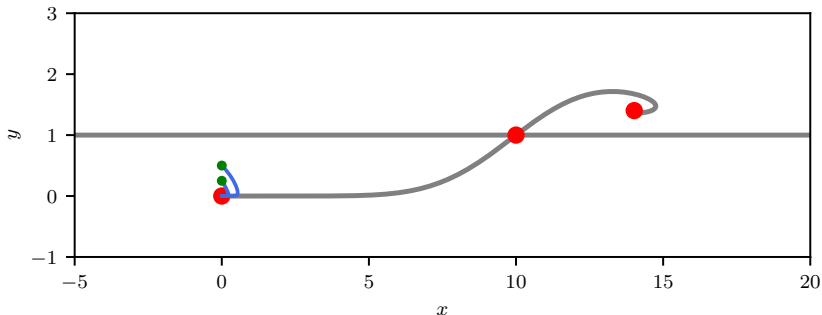
A toy-model



Trajectories for perturbations of different initial amplitude ( $y_0 = 0.25, 0.5, 0.75, 0.999, 1.00001$ ).

# Subcritical transition to turbulence

A toy-model

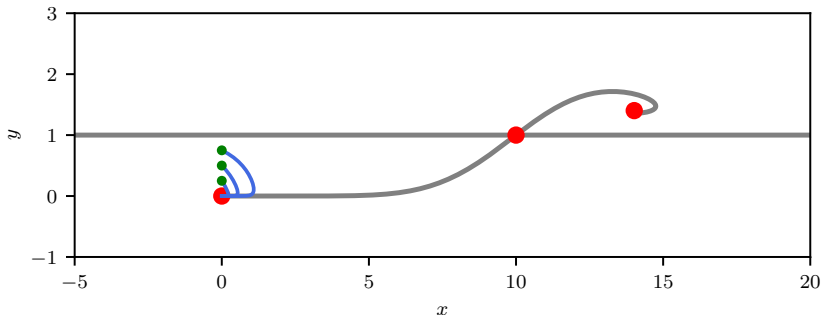


Trajectories for perturbations of different initial amplitude ( $y_0 = 0.25, 0.5, 0.75, 0.999, 1.00001$ ).



# Subcritical transition to turbulence

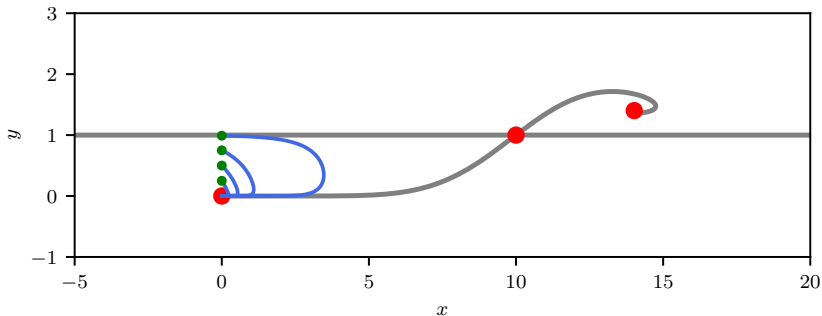
A toy-model



Trajectories for perturbations of different initial amplitude ( $y_0 = 0.25, 0.5, 0.75, 0.999, 1.00001$ ).

# Subcritical transition to turbulence

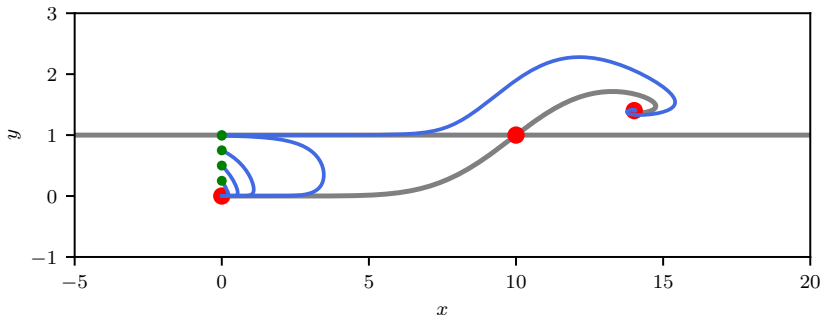
A toy-model



Trajectories for perturbations of different initial amplitude ( $y_0 = 0.25, 0.5, 0.75, 0.999, 1.00001$ ).

# Subcritical transition to turbulence

A toy-model



Trajectories for perturbations of different initial amplitude ( $y_0 = 0.25, 0.5, 0.75, 0.999, 1.00001$ ).

# Exercises

Do It Yourself

# Exercise

A fairly simple example

- ▶ Let us consider the following dynamical system

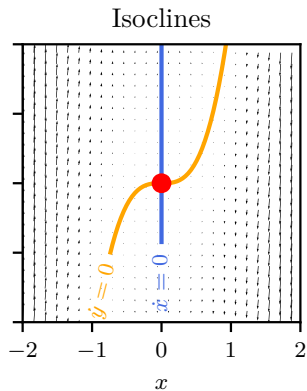
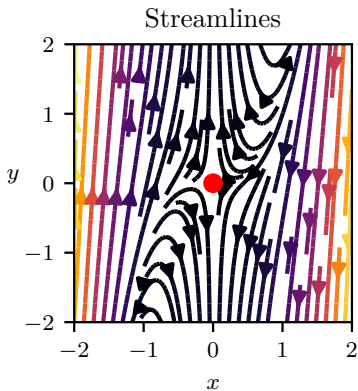
$$\dot{x} = -x$$

$$\dot{y} = 2y - 5x^3.$$

- ▶ You need to
  1. Determine the fixed point of the system.
  2. Compute its linear stable and unstable eigenspaces.
  3. Compute an approximation of its stable and unstable manifolds.

# Exercise

A fairly simple example



Phase plane and isoclines of the dynamical system considered.

# Another exercise

A slightly more complex one

- ▶ Let us now consider the following dynamical system

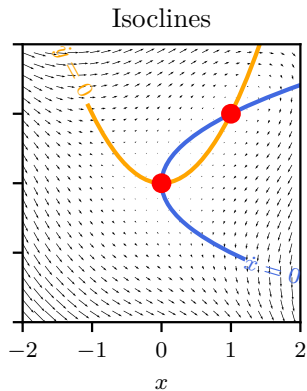
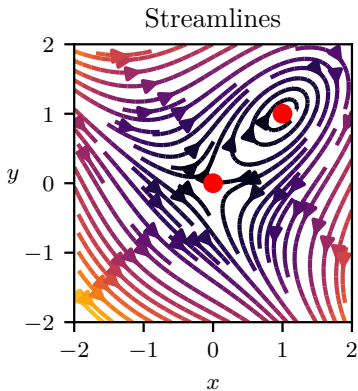
$$\dot{x} = -x + y^2$$

$$\dot{y} = y - x^2.$$

- ▶ Do the same as before.

# Exercise

A slightly more complex one



Phase plane and isoclines of the dynamical system considered.