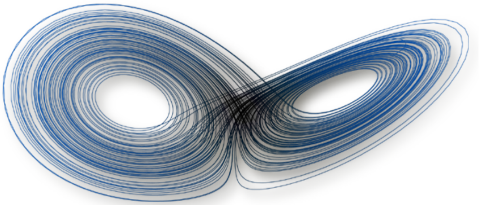

Nonlinear physics, dynamical systems and chaos theory

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First-order systems

Flows on the line

First-order systems

An apparently simple system

- Let us consider the following first-order dynamical system

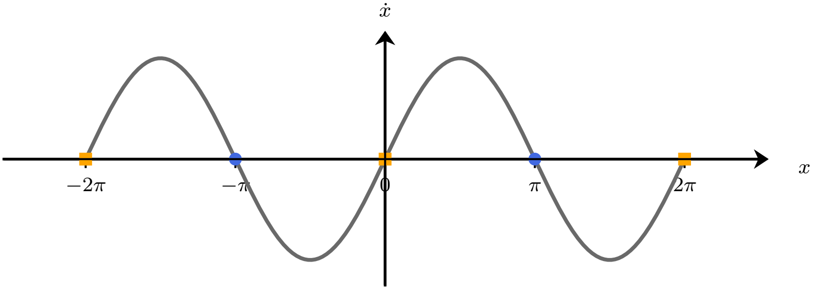
$$\dot{x} = \sin(x).$$

- Its analytical solution is given by

$$t = \ln \left| \frac{\csc(x_0) + \cot(x_0)}{\csc(x) + \cot(x)} \right|.$$

First-order systems

Phase line



Phase line of the first-order dynamical system considered.

First-order systems

Fixed points

- Fixed points x^* are equilibrium solutions characterized by

$$f(x^*) = 0.$$

- In the present case, these are given by

$$x^* = n\pi \text{ for } n \in \mathbb{N}.$$

First-order systems

Linear stability

- The dynamics of a perturbation $\eta(t) = x(t) - x^*$ is given by

$$\dot{\eta} = f(x^* + \eta).$$

- If η is small enough, $f(x^* + \eta)$ can be approximated by its first-order Taylor expansion around x^*

$$f(x^* + \eta) = f(x^*) + f'(x^*)\eta + \mathcal{O}(\eta^2).$$

First-order systems

Linear stability

- Given that $f(x^*) = 0$, the dynamics of η are governed by

$$\dot{\eta} = f'(x^*)\eta.$$

- Its analytical solution is given by

$$\eta(t) = \exp\left(f'(x^*)t\right)\eta_0.$$

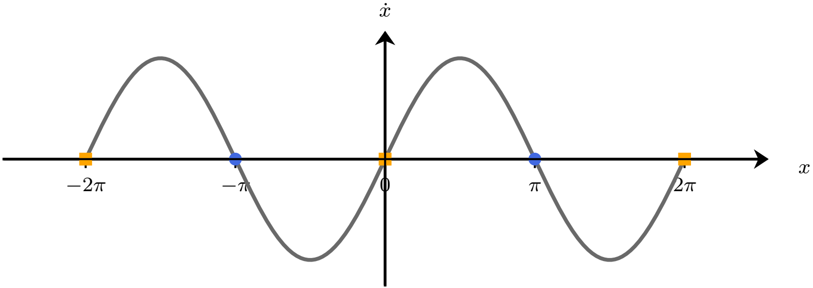
First-order systems

Linear stability

- The linear stability of a fixed point x^* is determined by the sign of $f'(x^*)$:
  - $\hookrightarrow$ if $f'(x^*) > 0$, $\eta(t)$ growths exponentially fast. The fixed point is said to be **linearly unstable**.
  - $\hookrightarrow$ if $f'(x^*) < 0$, $\eta(t)$ decays exponentially fast. The fixed point is said to be **linearly stable**.
  - $\hookrightarrow$ if $f'(x^*) = 0$, one can not conclude and nonlinear analyses are required.
- Let us now re-analyze our original system and sketch the evolution of $x(t)$.

First-order systems

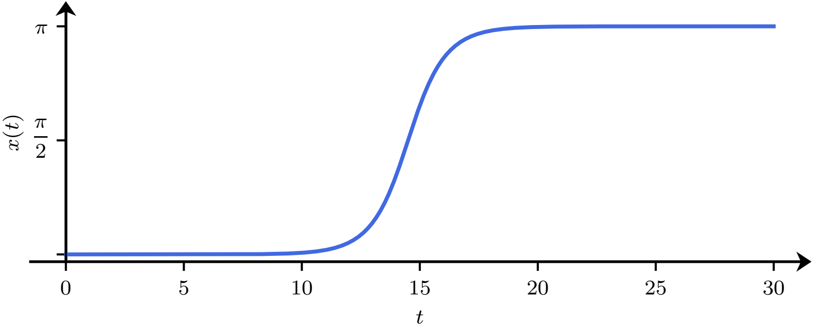
Phase line



Phase line of the first-order dynamical system considered.

First-order systems

Evolution of $x(t)$



Evolution of $x(t)$ for the initial condition $x_0 = 10^{-6}$.

First-order systems

Warning!

For a first-order system, the trajectories can only vary monotonically: either they end up on a stable fixed point, or they diverge to $\pm\infty$.

Second-order systems

Oscillators, but not only...

Second-order systems

- Second-order systems are dynamical systems which can be described by

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y).$$

- Having two degrees of freedom, they can exhibit a dynamics much richer than simple first-order systems.

Second-order system

Our working example

- For the rest of this section, let us consider the following system

$$
\begin{aligned}
\dot{x} &= x - y^2 + 1.28 + 1.4xy \\
\dot{y} &= 0.2y - x + x^3.
\end{aligned}
$$

- Note that this system is considered only for illustration purposes. To the best of my knowledge, it does not model any particular physics.

Interlude

How to compute fixed points?

How to compute fixed points?

Different techniques

- Fixed points are structuring the phase space of the dynamical system under scrutiny. Unfortunately, it may not be easy (nor possible) to compute them analytically.

- A number of different numerical techniques exist for that purpose. The following list is by no means exhaustive:
  - $\hookrightarrow$ Newton-Raphson method,
  - $\hookrightarrow$ Selective Frequency Damping,
  - $\hookrightarrow$ BoostConv,
  - $\hookrightarrow$ ...

Newton-Raphson method

$f : \mathbb{R} \to \mathbb{R}$

- Originally proposed by Isaac Newton (1645–1727) and Joseph Raphson (1648 – 1715) to solve

$$f(x) = 0.$$

- Given an initial guess x_0, the idea is to approximate $f(x)$ by its first-order Taylor expansion around x_0, i.e.

$$f(x) \simeq f(x_0) + f'(x_0)(x - x_0).$$

- A better estimate x_1 of the root of f can then be obtained by solving

$$0 = f(x_0) + f'(x_0)(x_1 - x_0).$$

Newton-Raphson method

$f : \mathbb{R} \to \mathbb{R}$

- After k iterations, the basic iteration scheme can be written as

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

- The iterative procedure stops when a user-defined criterion is fulfilled, usually

$$\|f(x_k)\| \leq \epsilon \text{ or } \|x_{k+1} - x_k\| \leq \epsilon,$$

with $\epsilon \simeq 10^{-10}$.

Newton-Raphson method

$f : \mathbb{R} \rightarrow \mathbb{R}$

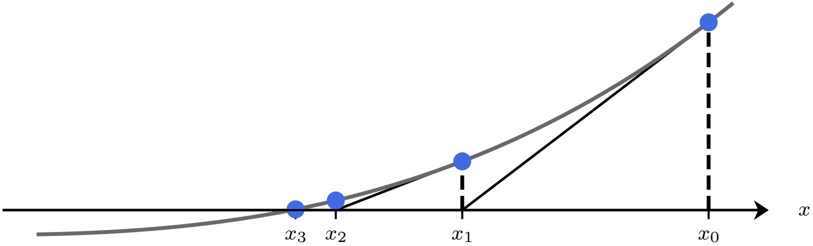


Illustration of the Newton-Raphson for $f(x) = x^3 - 2x - 5$ and $x_0 = 3.8$.

Newton-Raphson method

- Generalization of the Newton-Raphson method to the case $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ is quite straightforward.
- Given an estimate $\boldsymbol{x}_k$, the basic iteration reads

$$\boldsymbol{J}\delta\boldsymbol{x} = -\boldsymbol{f}(\boldsymbol{x}_k)$$
$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \delta\boldsymbol{x},$$

where $\boldsymbol{J}$ is the Jacobian matrix of $\boldsymbol{f}$ evaluated at $\boldsymbol{x}_k$.

Newton-Raphson method

Limitations

- Although efficient, Newton-Raphson method suffers from a number of limitations:
  - $\hookrightarrow$ The fixed points computed may depend on the initial guess $\boldsymbol{x}_0$.
  - $\hookrightarrow$ Evaluating $\boldsymbol{f}(\boldsymbol{x})$ might be computationally expensive.
  - $\hookrightarrow$ At each iteration, the Jacobian matrix $\boldsymbol{J}$ needs to be evaluated and inverted ($\mathcal{O}(n^3)$ operations).

- A number of variants of the Newton-Raphson method exist as to address these different limitations. This however is algorithmic refinement beyond the scope of the present course.

Second-order systems

Back to our example