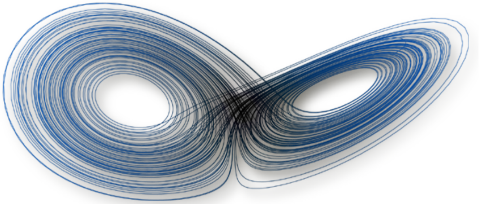

Nonlinear physics, dynamical systems and chaos theory

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Rayleigh-Bénard convection

Context and governing equations

Rayleigh-Bénard convection

Overview

- Buoyancy-driven flow of a fluid heated from below and cooled from above.
- Applications in geophysics, astrophysics, meteorology, oceanography and engineering.
- Well-known model for nonlinear and chaotic dynamics, pattern formation and fully developed turbulence.

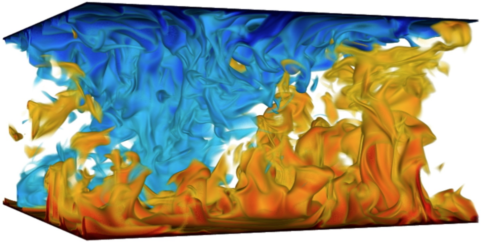


Illustration of turbulent Rayleigh-Bénard convection.

Rayleigh-Bénard convection

Driving force: Buoyancy

Archimedes' principle (c. 250 BC)

Any object, wholly or partially immersed in a stationary fluid, is buoyed up by a force equal to the weight of the fluid displaced by the object.

- This force is modeled as

$$\overrightarrow{\pi} = -\rho_f V_f \overrightarrow{g}$$

where ρ_f is the fluid's density, V_f is the volume of fluid displaced by the object and $\overrightarrow{g}$ is gravitational acceleration.

Rayleigh-Bénard convection

Density and temperature

- Assuming an incompressible flow, the state equation can be approximated as

$$\rho = \rho_0 \left(1 - \alpha(T - T_0)\right)$$

where α is the coefficient of thermal expansion.

- This approximation is known as **Boussinesq approximation**.

Rayleigh-Bénard convection

Governing equations

- The flow is governed by

$$\begin{aligned} \nabla \cdot \boldsymbol{u} &= 0 \\ \frac{\partial \boldsymbol{u}}{\partial t} + \left(\boldsymbol{u} \cdot \nabla \right) \boldsymbol{u} &= -\frac{1}{\rho} \nabla p + \nu \nabla^2 \boldsymbol{u} + \delta \rho \boldsymbol{g} \\ \frac{\partial T}{\partial t} + \left(\boldsymbol{u} \cdot \nabla \right) T &= \kappa \nabla^2 T, \end{aligned}$$

where $\boldsymbol{u}$ is the fluid's velocity, ρ its density and T is the temperature. ν and κ are the kinematic viscosity and the thermal diffusivity, respectively.

Rayleigh-Bénard convection

Nondimensional form

- Under appropriate nondimensionalization, the governing equations read

$$
\begin{aligned}
\nabla \cdot \boldsymbol{u} &= 0 \\
\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u} &= -\nabla p + Pr \nabla^2 \boldsymbol{u} + (Ra\ Pr)\, \theta \boldsymbol{e}_y \\
\frac{\partial \theta}{\partial t} + (\boldsymbol{u} \cdot \nabla)\, \theta &= \nabla^2 \theta,
\end{aligned}
$$

where Pr is the Prandtl number and Ra the Rayleight number. θ is the nondimensional temperature given by $\theta = {}^{T-T_c}/_{T_h-T_c}$.

Rayleigh-Bénard convection

Base state and linear stability analysis

Rayleigh-Bénard convection

Fixed point: the conducting state

- The fixed point is solution to

$$\frac{\mathrm{d}^2\Theta}{\mathrm{d}y^2} = 0$$

with appropriate boundary conditions.

- The nondimensional temperature profile $\Theta(y)$ is solution to the heat equation. It is given by

$$\Theta(y) = 1 - y.$$

- It corresponds to a pure conduction state (i.e. $\boldsymbol{u} = 0$).

Rayleigh-Bénard convection

Linear stability analysis

- Let us consider the linear stability of this conducting state towards two-dimensional perturbations (see Squire theorem).

- The linearized equations read

$$
\begin{aligned}
\frac{\partial}{\partial t}\nabla^2\psi &= -Ra\ Pr\frac{\partial\theta}{\partial x} + Pr\nabla^2\psi \\
\frac{\partial\theta}{\partial t} &= -\frac{\partial\psi}{\partial x} + \nabla^2\theta,
\end{aligned}
$$

where ψ is the streamfunction of the perturbation.

Rayleigh-Bénard convection

Linear stability analysis

- Solutions are sought in the form of *normal modes*, i.e.

$$\boldsymbol{q}(x, y, t) = \hat{\boldsymbol{q}}(y) e^{ikx + \lambda t} + \text{c.c},$$

where λ is the growth rate and k the perturbation's wavenumber.

- We obtain the following generalized eigenvalue problem

$$\lambda \begin{bmatrix} D^2 - k^2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\psi} \\ \hat{\theta} \end{bmatrix} = \begin{bmatrix} Pr(D^2 - k^2) & -ikRa\ Pr \\ -ik & D^2 - k^2 \end{bmatrix} \begin{bmatrix} \hat{\psi} \\ \hat{\theta} \end{bmatrix}$$

where $D = {}^{\mathrm{d}}/_{\mathrm{d}y}$.

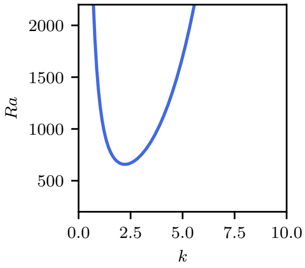
Rayleigh-Bénard convection

Linear stability analysis

- This problem has been solved analytically in 1916, assuming free-slip boundary conditions, i.e.

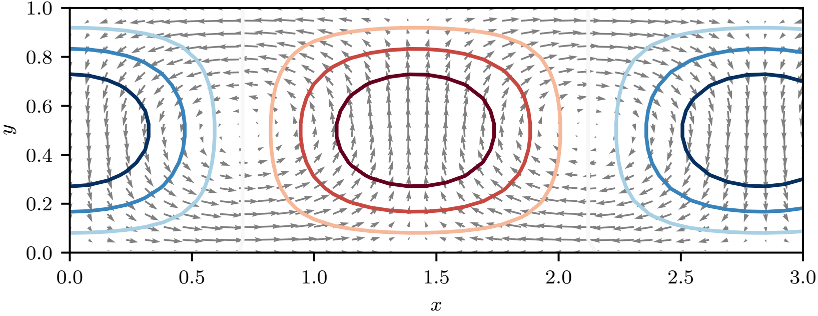
$$\psi(x, y, t) = \hat{\psi}(t) \sin(n\pi y) \sin(kx)$$
$$\theta(x, y, t) = \hat{\theta}(t) \sin(n\pi y) \cos(kx).$$

- The dispersion relation reduces to a quadratic equation. One then obtains $Ra_c = {}^{27\pi^4}/_4$ and $k_c = {}^{\pi}/_{\sqrt{2}}$.



Rayleigh-Bénard convection

Linear stability analysis



Isocontours of temperature and velocity field of the instability mode.

From Navier-Stokes to Lorenz

Investigating the nonlinearities

From Navier-Stokes to Lorenz

Nonlinear equations

- The governing equations read

$$
\begin{aligned}
\frac{\partial}{\partial t}\nabla^2\psi - Pr\nabla^2\psi + (Ra\ Pr)\frac{\partial\theta}{\partial x} &= \mathcal{J}\left(\nabla^2\psi, \psi\right) \\
\frac{\partial\theta}{\partial t} - \nabla^2\theta + \frac{\partial\psi}{\partial x} &= \mathcal{J}\left(\theta, \psi\right),
\end{aligned}
$$

where the nonlinear terms are expressed as a Jacobian operator $\mathcal{J}$ given by

$$
\mathcal{J}(f,g) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y}.
$$

- We futhermore assume free-slip boundary conditions as before.

From Navier-Stokes to Lorenz

Truncated Galerkin expansion

- It has been shown by Saltzmann (1962) that the general solution can be expressed as an infinite Fourier series.

- Let us however consider a *truncated Galerkin expansion* such that

$$
\begin{aligned}
\psi(x, y, t) &= a(t) \sin(\pi y) \sin(k \pi x) + \cdots \\
\theta(x, y, t) &= b(t) \sin(\pi y) \cos(k \pi x) + c(t) \sin(2 \pi y) + \cdots
\end{aligned}
$$

- $a(t)$ and $b(t)$ correspond to the convection rolls with wavenumber k in the x-direction. The term $c(t)$ describes the modification of the mean temperature profile due to convection.

From Navier-Stokes to Lorenz

Derivation of the low-order model

It is now up to you to derive the low-order model :)

From Navier-Stokes to Lorenz

Low-order model

- Finally, we obtain the following low-order model

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\mathrm{Pr}\ \pi^2(1+k^2)a - \frac{k\pi}{\pi^2(1+k^2)}\mathrm{Pr}\ \mathrm{Ra}\ b$$
$$\frac{\mathrm{d}b}{\mathrm{d}t} = -k\pi a - \pi^2(1+k^2)b - k\pi^2 ac$$
$$\frac{\mathrm{d}c}{\mathrm{d}t} = \frac{1}{2}k\pi^2 ab - 4\pi^2 c.$$

- This low-dimensional model of thermal convection is a rescaled version of the one originally introduced by Lorenz in 1963.

Lorenz system

An example of chaotic dynamics

Lorenz system

1963 model

- Lorenz-1963 model reads

$$
\begin{aligned}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= x(\rho - z) - y \\
\dot{z} &= xy - \beta z,
\end{aligned}
$$

where $\sigma = Pr$, $\rho = {}^{Ra}/_{Ra_c}$ and $\beta = {}^{2\pi^2}/_{\pi^2 + k^2}$ is the aspect ratio of the convection cells.

- We consider the same parameters as Lorenz, i.e. $\sigma = 10$ (water) and $\beta = {}^{8}/_{3}$.

Lorenz system

Properties

- **Symmetry**: Invariant to the transformation $(x, y, z) \to (-x, -y, z)$.
- **Invariant z-axis**: If $x(0) = y(0) = 0$, then $x(t) = y(t) = 0 \ \forall t$. Then $\dot{z} = -\beta z$ and hence $z(t) = e^{-\beta z}$. The z-axis is thus always part of the stable manifold for the equilibrium at the origin.
- **Dissipative**: We have $\nabla \cdot \boldsymbol{\mathcal{F}}(\mathbf{x}) = -\sigma - 1 - \beta < 0$. Any given volume V of phase points will eventually tend to 0 as $t \to \infty$.

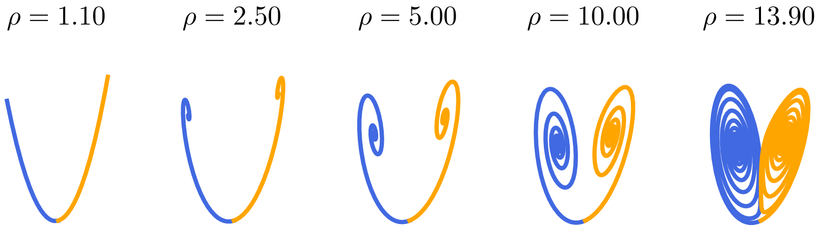
Lorenz system

Primary bifurcation

1. Compute the fixed points of the system as a function of ρ.
2. When does the conduction state (i.e. $\boldsymbol{x}^* = 0$) loose its stability?
3. Are the resulting fixed points linearly stable or not? Conclude about the type of bifurcation encountered.

Lorenz system

Dynamics for $1 \leq \rho \leq 14$

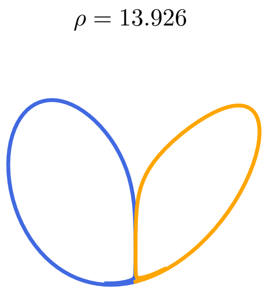


Lorenz system

Homoclinic connection ($\rho \simeq 13.926$)

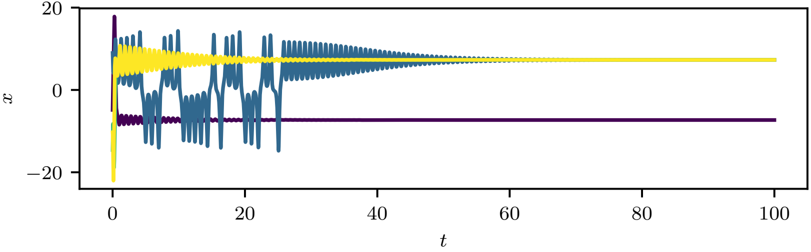
- Existence of a homoclinic connection for $\rho \simeq 13.926$.
  - ↪ Perturbation leaves the conducting state along its unstable manifold and returns to it along its stable one.

- For $\rho > 14$, the system exhibits *pre-chaotic* transients.



Lorenz system

Pre-chaotic transients $(14 \leq \rho \leq 24)$



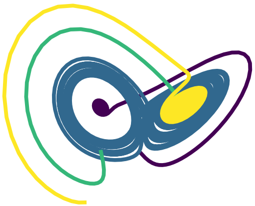
Pre-chaotic transient time series of $x(t)$ for $\rho = 21$.

Lorenz system

Pre-chaotic transients ($14 \leq \rho \leq 24$)

For $14 \leq \rho \leq 24$, the system exhibits pre-chaotic transients.

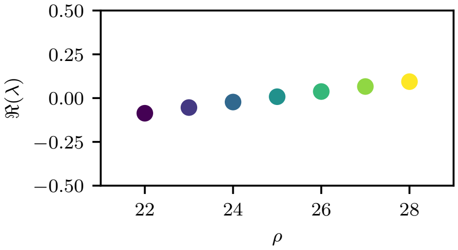
- As $t \to \infty$, the system settles onto one of its stable equilibria.
- It nonetheless exhibits some kind of *sensitivity to initial conditions*.

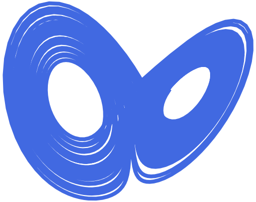


Lorenz system

Subcritical Hopf bifucation at $\rho \simeq 24.74$

- A (subcritical) Hopf bifurcation occurs at $\rho \simeq 24.74$.
- Trajectories escape from the symmetric fixed points and are repelled toward a distant attractor.

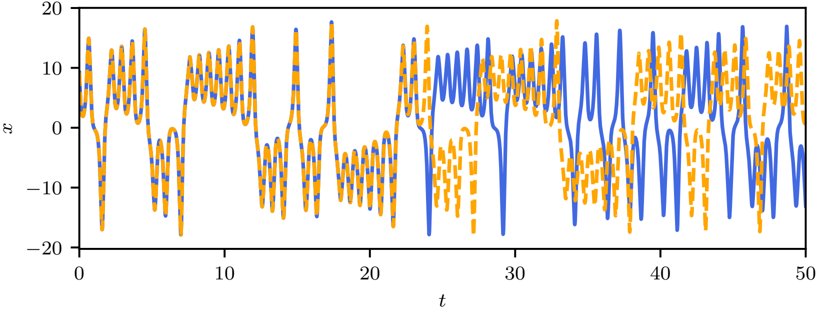




Chaos is *aperiodic long-term* behavior in a *deterministic* system that exhibits *sensitive dependence on initial conditions*.

Lorenz system

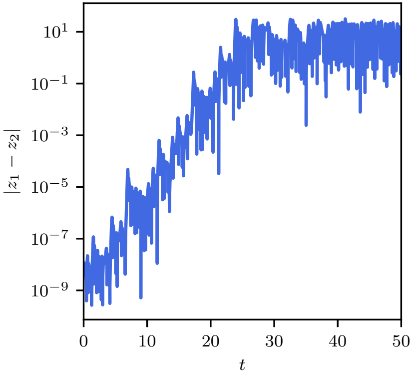
Sensitivity to initial conditions



Lorenz system

Sensitivity to initial conditions

- Two nearby trajectories diverge exponentially rapidly from one another.
- Prediction horizon hardly depends on how well the initial condition is characterized.
- From a statistical point of view, the two trajectories nonetheless have the same properties.



Lorenz system

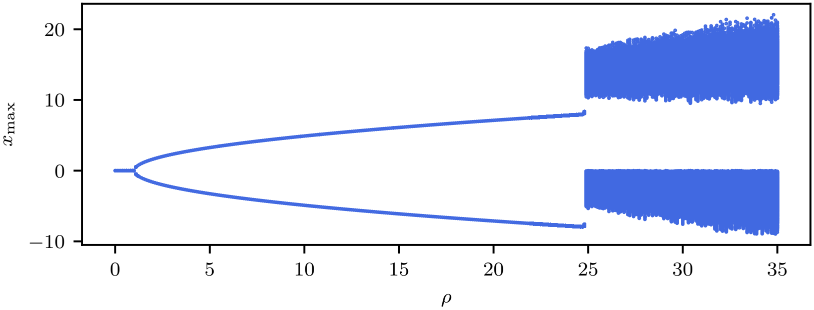
Butterfly effect

If a single flap of a butterfly's wing can be instrumental in generating a tornado, so all the previous and subsequent flaps of its wings, as can the flaps of the wings of millions of other butterflies, no to mention activities of innumerable more powerful creatures, including our own species. If a flap of a butterfly's wing can be instrumental in generating a tornado, it can equally well be instrumental in preventing a tornado.

Conference by E. Lorenz, *Predictability: does the flap of a butterfly's wing in Brazil set off a tornado in Texas?*, 1972.

Lorenz system

Bifurcation diagram



Bifurcation diagram of the Lorenz system for $0 \leq \rho \leq 35$.

On the importance of chaos in science

A (very) brief history

On the importance of chaos in science

A (very) brief history

- **1814**: Laplacian determinism.
- **1820 – 1868**: Cauchy-Lipschitz theorem on the existence and uniqueness of solutions to ordinary differential equations.
- **1888**: Poincaré and the *three bodies problem*. First example of sensitivity to initial conditions.

On the importance of chaos in science

Laplacian determinism

Nous devons [...] envisager l'état présent de l'univers comme l'effet de son état antérieur, et comme la cause de celui qui va suivre. Une intelligence qui pour un instant donné connaîtrait toutes les forces dont la nature est animée et la situation respective des êtres qui la composent [...] embrasserait dans la même formule les mouvements des plus grands corps de l'univers et ceux du plus léger atome : rien ne serait incertain pour elle, et l'avenir comme le passé serait présent à ses yeux.

Essai philosophique sur les probabilités,
Pierre SImon de Laplace, 1814.

On the importance of chaos in science

Poincaré and the sensitivity to initial conditions

Une cause très petite, qui nous échappe, détermine un effet considérable que nous ne pouvons pas ne pas voir, et alors nous disons que cet effet est dû au hasard. Si nous connaissions exactement les lois de la nature et la situation de l'univers à l'instant initial, nous pourrions prédire exactement la situation de ce même univers à un instant ultérieur. Mais, lors même que les lois naturelles n'auraient plus de secret pour nous, nous ne pourrions connaître la situation qu'approximativement. Si cela nous permet de prévoir la situation ultérieure avec la même approximation, c'est tout ce qu'il nous faut, nous disons que le phénomène a été prévu, qu'il est régi par des lois ; mais il n'en est pas toujours ainsi, il peut arriver que de petites différences dans les conditions initiales en engendrent de très grandes dans les phénomènes finaux ; une petite erreur sur les premières produirait une erreur énorme sur les derniers. La prédiction devient impossible et nous avons le phénomène fortuit.

Calcul des probabilités,
Henri Poincaré, 1912.

On the importance of chaos in science

A (very) brief history

- **1963**: Edward Lorenz introduced his simplified model of thermal convection.
- **1975**: Tien-Yien Li and James A. Yorke coined the term *deterministic chaos*.
- **1976**: Introduction of the Rössler model.
- **1971**: Ruelle & Takens introduced the concept of *strange attractors*.