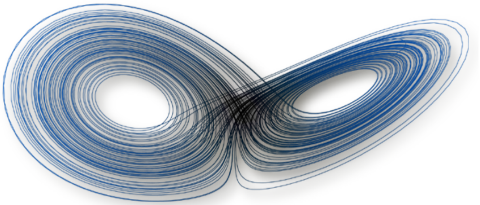


Nonlinear physics, dynamical systems and chaos theory

Jean-Christophe Loiseau

jean-christophe.loiseau@ensam.eu
DynFluid,
Arts et Métiers ParisTech, France

First-order systems

Flows on the line

First-order systems

An apparently simple system

- Let us consider the following first-order dynamical system

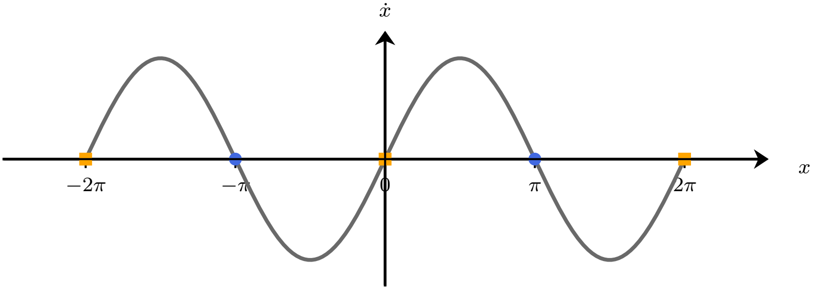
$$\dot{x} = \sin(x).$$

- Its analytical solution is given by

$$t = \ln \left| \frac{\csc(x_0) + \cot(x_0)}{\csc(x) + \cot(x)} \right|.$$

First-order systems

Phase line



Phase line of the first-order dynamical system considered.

First-order systems

Fixed points

- Fixed points x^* are equilibrium solutions characterized by

$$f(x^*) = 0.$$

- In the present case, these are given by

$$x^* = n\pi \text{ for } n \in \mathbb{N}.$$

First-order systems

Linear stability

- The dynamics of a perturbation $\eta(t) = x(t) - x^*$ is given by

$$\dot{\eta} = f(x^* + \eta).$$

- If η is small enough, $f(x^* + \eta)$ can be approximated by its first-order Taylor expansion around x^*

$$f(x^* + \eta) = f(x^*) + f'(x^*)\eta + \mathcal{O}(\eta^2).$$

First-order systems

Linear stability

- Given that $f(x^*) = 0$, the dynamics of η are governed by

$$\dot{\eta} = f'(x^*)\eta.$$

- Its analytical solution is given by

$$\eta(t) = \exp\left(f'(x^*)t\right)\eta_0.$$

First-order systems

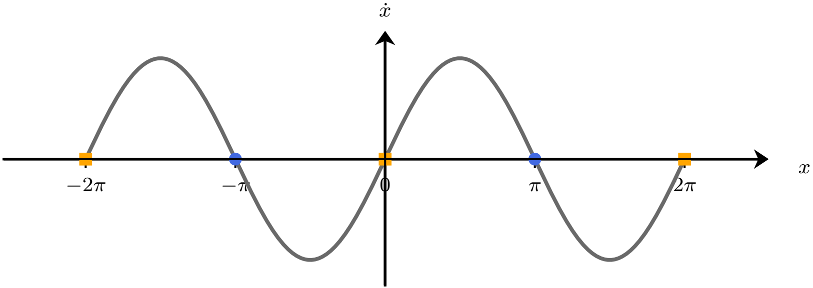
Linear stability

- The linear stability of a fixed point x^* is determined by the sign of $f'(x^*)$:
  - $\hookrightarrow$ if $f'(x^*) > 0$, $\eta(t)$ growths exponentially fast. The fixed point is said to be **linearly unstable**.
  - $\hookrightarrow$ if $f'(x^*) < 0$, $\eta(t)$ decays exponentially fast. The fixed point is said to be **linearly stable**.
  - $\hookrightarrow$ if $f'(x^*) = 0$, one can not conclude and nonlinear analyses are required.

- Let us now re-analyze our original system and sketch the evolution of $x(t)$.

First-order systems

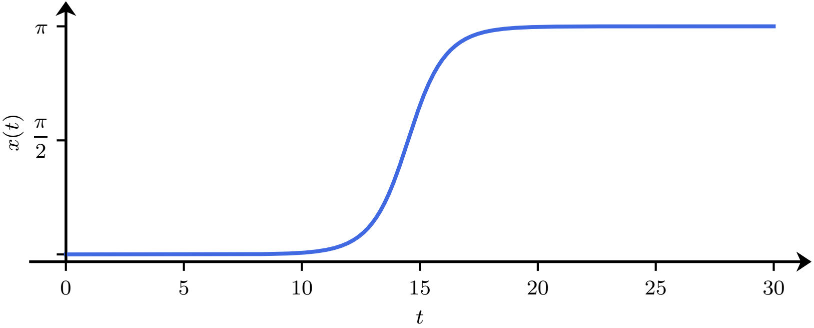
Phase line



Phase line of the first-order dynamical system considered.

First-order systems

Evolution of $x(t)$



Evolution of $x(t)$ for the initial condition $x_0 = 10^{-6}$.

First-order systems

Warning!

For a first-order system, the trajectories can only vary monotonically: either they end up on a stable fixed point, or they diverge to $\pm\infty$.

Second-order systems

Oscillators, but not only...

Second-order systems

- Second-order systems are dynamical systems which can be described by

$$\begin{aligned}\dot{x} &= f(x, y)\\ \dot{y} &= g(x, y).\end{aligned}$$

- Having two degrees of freedom, they can exhibit dynamics much richer than simple first-order systems.

Second-order system

Our working example

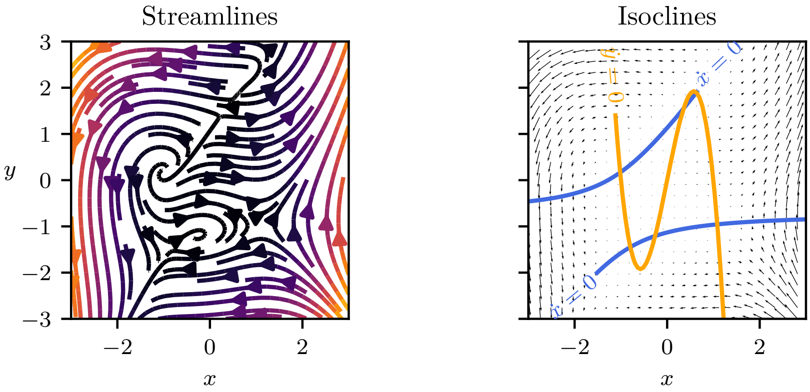
- For the rest of this section, let us consider the following system

$$\dot{x} = x - y^2 + 1.28 + 1.4xy$$
$$\dot{y} = 0.2y - x + x^3.$$

- Note that this system is considered only for illustration purposes. To the best of my knowledge, it does not model any particular physics.

Second-order systems

Phase plane and isoclines



Phase plane and isoclines (i.e. $\dot{x} = 0$ and $\dot{y} = 0$) of the system considered.

Second-order systems

Phase plane and isoclines

- For a second-order system, phase plane and isoclines are fairly easy to plot and provide valuable insights into the dynamics of the system.
  - $\hookrightarrow$ Phase plane : general overview of the dynamics of the system.
  - $\hookrightarrow$ Direct visualization of the stable and unstable fixed point of the system.

Question

How to compute these fixed points?

Interlude

How to compute fixed points?

How to compute fixed points?

Different techniques

- Fixed points are structuring the phase space of the dynamical system under scrutiny. Unfortunately, it may not be easy (nor possible) to compute them analytically.

- A number of different numerical techniques exist for that purpose. The following list is by no means exhaustive:
  - $\hookrightarrow$ Newton-Raphson method,
  - $\hookrightarrow$ Selective Frequency Damping,
  - $\hookrightarrow$ BoostConv,
  - $\hookrightarrow$ ...

Newton-Raphson method

$f : \mathbb{R} \to \mathbb{R}$

- Originally proposed by Isaac Newton (1645–1727) and Joseph Raphson (1648 – 1715) to solve

$$f(x) = 0.$$

- Given an initial guess x_0, the idea is to approximate $f(x)$ by its first-order Taylor expansion around x_0, i.e.

$$f(x) \simeq f(x_0) + f'(x_0)(x - x_0).$$

- A better estimate x_1 of the root of f can then be obtained by solving

$$0 = f(x_0) + f'(x_0)(x_1 - x_0).$$

Newton-Raphson method

$f : \mathbb{R} \to \mathbb{R}$

- After k iterations, the basic iteration scheme can be written as

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

- The iterative procedure stops when a user-defined criterion is fulfilled, usually

$$\|f(x_k)\| \leq \epsilon \text{ or } \|x_{k+1} - x_k\| \leq \epsilon,$$

with $\epsilon \simeq 10^{-10}$.

Newton-Raphson method

$f : \mathbb{R} \to \mathbb{R}$

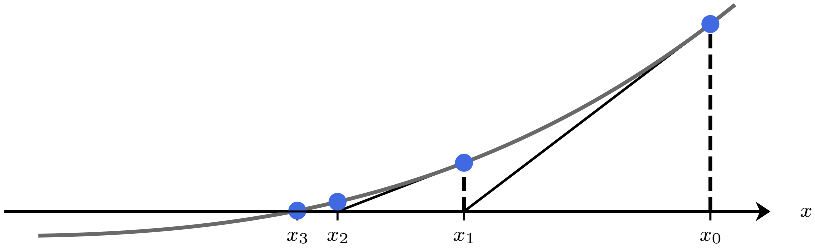


Illustration of the Newton-Raphson for $f(x) = x^3 - 2x - 5$ and $x_0 = 3.8$.

Newton-Raphson method

$\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$

- Generalization of the Newton-Raphson method to the case $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ is quite straightforward.
- Given an estimate $\boldsymbol{x}_k$, the basic iteration reads

$$
\begin{aligned}
\boldsymbol{J}\delta\boldsymbol{x} &= -\boldsymbol{f}(\boldsymbol{x}_k) \\
\boldsymbol{x}_{k+1} &= \boldsymbol{x}_k + \delta\boldsymbol{x},
\end{aligned}
$$

where $\boldsymbol{J}$ is the Jacobian matrix of $\boldsymbol{f}$ evaluated at $\boldsymbol{x}_k$.

- Although efficient, Newton-Raphson method suffers from a number of limitations:
  - ↪ The fixed points computed may depend on the initial guess $\boldsymbol{x}_0$.
  - ↪ Evaluating $\boldsymbol{f}(\boldsymbol{x})$ might be computationally expensive.
  - ↪ At each iteration, the Jacobian matrix $\boldsymbol{J}$ needs to be evaluated and inverted ($\mathcal{O}(n^3)$ operations).

- A number of variants of the Newton-Raphson method exist as to address these different limitations. This however is algorithmic refinement beyond the scope of the present course.

Second-order systems

Back to our example

Second-order systems

Fixed point computation

- Visual inspection of the isoclines has revealed that the system exhibits six different fixed points.
- In the rest, we will use the Newton-Raphson method to compute these different fixed points.
- The state-dependent Jacobian matrix required for Newton method reads

$$\boldsymbol{J} = \begin{bmatrix} 1 + 1.4y & -2y + 1.4x \\ -1 + 3x^2 & 0.2 \end{bmatrix}$$

Second-order systems

Infinitesimal perturbations

- The dynamics of an infinitesimal perturbation $\boldsymbol{q}$ evolving in the vicinity of a given fixed point $\boldsymbol{x}^*$ is given by

$$\dot{\boldsymbol{q}} = \boldsymbol{L}\boldsymbol{q},$$

where $\boldsymbol{L}$ is the Jacobian matrix of the system evaluated at $\boldsymbol{x} = \boldsymbol{x}^*$.

- The analytical solution to this **linear time-invariant dynamical system** is given by

$$\boldsymbol{q}(t) = e^{\boldsymbol{L}t}\boldsymbol{q}_0,$$

where $\boldsymbol{q}_0$ is the initial condition.

Interlude

Some elements of linear algebra

Some elements of linear algebra

Matrix decompositions

Stewart (*Comput. Sci. Engrg.*, **2**:50–59, 2000) has listed the *big six matrix decompositions*:

1. Cholesky decomposition,
2. Pivoted LU decomposition,
3. QR decomposition,
4. Spectral decomposition, also known as **Eigendecomposition**,
5. Schur decomposition,
6. Singular Value decomposition.

Some elements of linear algebra

Eigendecomposition of a matrix

- Eigendecomposition is the factorization of a matrix into its canonical form.
  - ↪ The matrix is represented in terms of its **eigenvalues** and **eigenvectors**.

- Given a $n \times n$ (i.e. square) matrix $\boldsymbol{L}$, eigendecomposition aims to rewrite it as

$$\boldsymbol{L} = \boldsymbol{V} \boldsymbol{\Lambda} \boldsymbol{V}^{-1},$$

where $\boldsymbol{\Lambda}$ is the diagonal matrix of **eigenvalues** and $\boldsymbol{V}$ is a $n \times n$ matrix whose i$^{\text{th}}$ column if the **eigenvector** $\boldsymbol{v}_i$ associated to $\boldsymbol{\Lambda}_{ii} = \lambda_i$.

Warning, unless the matrix is diagonal,

$$\left(e^{\boldsymbol{L}}\right)_{ij} \neq e^{l_{ij}}$$

Some elements of linear algebra

Matrix exponential

- The infinitie series Taylor expansion of the classical exponential function $f(x) = e^x$ is given by

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

- The same definition carries over for matrices.

$$e^{\boldsymbol{L}} = \boldsymbol{I} + \boldsymbol{L} + \frac{\boldsymbol{L}^2}{2!} + \frac{\boldsymbol{L}^3}{3!} + \cdots ,$$

where $\boldsymbol{I}$ is the identity matrix.

Some elements of linear algebra

Matrix exponential

- Evaluating $e^{\boldsymbol{L}}$ by brute force is quite impractical and extremely expansive from a computational point of view.
- Using the eigendecomposition of $\boldsymbol{L}$, the matrix exponential can be rewritten as

$$e^{\boldsymbol{L}} = \boldsymbol{V} e^{\boldsymbol{\Lambda}} \boldsymbol{V}^{-1},$$

where $e^{\boldsymbol{\Lambda}}$ can easily be evaluated as it is a diagonal matrix.

Some elements of matrix algebra

Matrix exponential

$$
\begin{aligned}
e^{\boldsymbol{L}} &= \boldsymbol{I} + \boldsymbol{L} + \frac{\boldsymbol{L}^2}{2!} + \frac{\boldsymbol{L}^3}{3!} + \cdots , \\
&= \boldsymbol{I} + \boldsymbol{V\Lambda V}^{-1} + \frac{\boldsymbol{V\Lambda V}^{-1}\boldsymbol{V\Lambda V}^{-1}}{2!} + \frac{\boldsymbol{V\Lambda V}^{-1}\boldsymbol{V\Lambda V}^{-1}\boldsymbol{V\Lambda V}^{-1}}{3!} + \cdots , \\
&= \boldsymbol{VIV}^{-1} + \boldsymbol{V\Lambda V}^{-1} + \frac{\boldsymbol{V\Lambda}^2\boldsymbol{V}^{-1}}{2!} + \frac{\boldsymbol{V\Lambda}^3\boldsymbol{V}^{-1}}{3!} + \cdots , \\
&= \boldsymbol{V}\left(\boldsymbol{I} + \boldsymbol{\Lambda} + \frac{\boldsymbol{\Lambda}^2}{2!} + \frac{\boldsymbol{\Lambda}^3}{3!} + \cdots\right)\boldsymbol{V}^{-1}, \\
&= \boldsymbol{V}e^{\boldsymbol{\Lambda}}\boldsymbol{V}^{-1}.
\end{aligned}
$$

Second-order systems

Back to our example

Second-order systems

Infinitesimal perturbations

- The dynamics of an infinitesimal perturbation $\boldsymbol{q}$ evolving in the vicinity of a given fixed point $\boldsymbol{x}^*$ is given by

$$\dot{\boldsymbol{q}} = \boldsymbol{L}\boldsymbol{q},$$

where $\boldsymbol{L}$ is the Jacobian matrix of the system evaluated at $\boldsymbol{x} = \boldsymbol{x}^*$.

- The analytical solution to this **linear time-invariant dynamical system** is given by

$$\boldsymbol{q}(t) = e^{\boldsymbol{L}t}\boldsymbol{q}_0,$$

where $\boldsymbol{q}_0$ is the initial condition.

Second-order systems

Linear instability

- Assuming the eigenvalues of $\boldsymbol{L}$ have been sorted by decreasing real parts, it is clear that if
  - ↪ $\Re(\lambda_1) > 0$, then $\lim_{t\to+\infty} e^{\boldsymbol{L}t} = +\infty$, i.e. there exist a perturbation $\boldsymbol{v}_1$ that growths exponentially fast in time. The fixed point considered is said to be **linearly unstable**.
  - ↪ $\Re(\lambda_1) < 0$, then $\lim_{t\to+\infty} e^{\boldsymbol{L}t} = 0$, i.e. all perturbations decay exponentially fast in time. The fixed point considered is said to be **linearly stable**.
- The case $\Re(\lambda_1)$ is peculiar. The fixed point is said to be **marginally stable** and one needs to consider the actual nonlinear system in order to conclude.
  - ↪ Weakly nonlinear analysis is beyond the scope of this lesson and will be addressed later on.

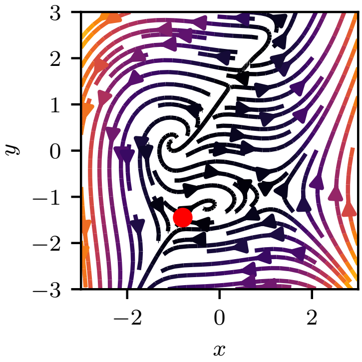
Second-order systems

Different types of fixed points

- Depending on the eigenspectrum of $\boldsymbol{L}$, the perturbation may exhibits different kind of dynamics.
- One can thus classify the type of fixed points and the typical trajectory exhibited by the system in the vicinity of said fixed point solely based on the eigenvalues of the associated Jacobian matrix $\boldsymbol{L}$.

Second-order systems

Fixed point n°1



Phase plane of the nonlinear system considered.

- The first fixed point is given by

$$\boldsymbol{x}^* = \begin{bmatrix} -0.798 & -1.449 \end{bmatrix}^T$$

- The corresponding Jacobian matrix reads

$$\boldsymbol{J} = \begin{bmatrix} -1.03 & 1.78 \\ 0.91 & 0.2 \end{bmatrix}$$

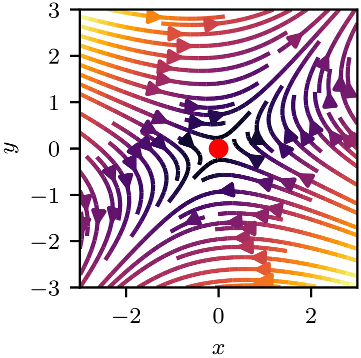
Second-order systems

Fixed point n°1

- $\boldsymbol{J}$ has two reals eigenvalues of opposite sign:

$$\lambda_1 = 0.99 \text{ and } \lambda_2 = -1.83$$

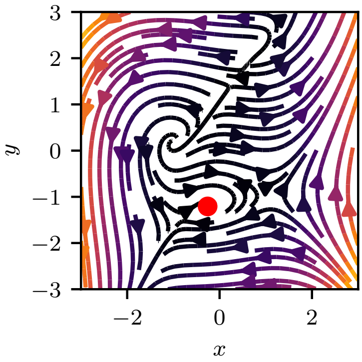
- Such a fixed point is called a **saddle**.
  - ↪ Its first eigendirection $\boldsymbol{v}_1$ is repulsive while its second, $\boldsymbol{v}_2$, is attractive.



Trajectories in the vicinity of the fixed point $\boldsymbol{x}_1^*$.

Second-order systems

Fixed point n°2



Phase plane of the nonlinear system considered.

- The first fixed point is given by

$$\boldsymbol{x}^* = \begin{bmatrix} -0.26 & -1.2 \end{bmatrix}^T$$

- The corresponding Jacobian matrix reads

$$\boldsymbol{J} = \begin{bmatrix} -0.69 & 2.05 \\ -0.79 & 0.2 \end{bmatrix}$$

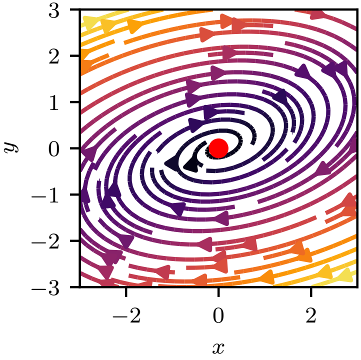
Second-order systems

Fixed point n°2

- $\boldsymbol{J}$ has complex conjugate eigenvalues:

$$\lambda_{1,2} = -0.24 \pm 1.2i$$

- Such a fixed point is called a **spiral**.
  - ↪ In this case, $\Re(\lambda) < 0$, so it is a spiral sink.



Trajectories in the vicinity of the fixed point $\boldsymbol{x}_1^*$.

Second-order systems

Classifying the fixed points

- For a 2×2 matrix, its eigenvalues are solution of

$$\lambda^2 - \mathrm{tr}(\boldsymbol{L})\lambda + \det(\boldsymbol{L}) = 0,$$

where $\mathrm{tr}(\boldsymbol{L}) = l_{11} + l_{22}$ and $\det(\boldsymbol{L}) = l_{11}l_{22} - l_{21}l_{12}$.

- All the different types of fixed points can be placed onto a $\mathrm{tr}(\boldsymbol{L})$–$\det(\boldsymbol{L})$ diagram.

Second-order systems

Classifying the fixed points