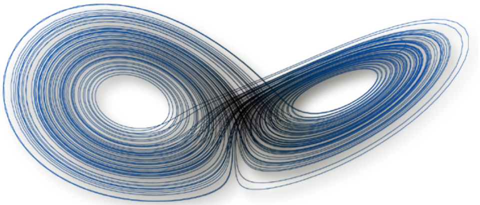

Nonlinear physics, dynamical systems and chaos theory

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Duffing oscillator

A non-harmonic oscillator

- Let us consider as an example the Duffing oscillator given by

$$\ddot{x} + \delta \dot{x} + \alpha x + \beta x^3 = 0.$$

with $\alpha = -1$, $\beta = 1$ and $\delta = 1/2$.

- It describes the motion of a damped oscillator with a more complex potential than simple harmonic motion.
  - $\hookrightarrow$ Example: a spring pendulum whose spring's stiffness does not exactly obey Hooke's law.

Duffing oscillator

A non-harmonic oscillator

- Introducing $y = \dot{x}$, this second-order nonlinear ODE can be recast as a set of two first-order ODE with a nonlinear coupling term

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\frac{1}{2}y + x - x^3. \end{aligned}$$

- Let us first discuss the different fixed points of the system.

Duffing oscillator

Fixed points

- The fixed points of the system are given by

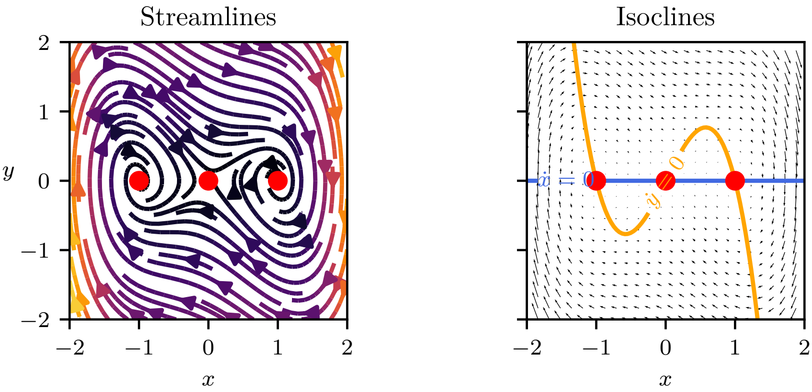
$$
\begin{aligned}
y &= 0 \\
x(1 - x^2) &= 0.
\end{aligned}
$$

- The system thus admits three different fixed points given by

$$
(x_1, y_1) = (0, 0) \quad (x_2, y_2) = (1, 0) \quad (x_3, y_3) = (-1, 0)
$$

Duffing oscillator

Fixed points



Phase plane and isoclines for the Duffing oscillator.

Duffing oscillator

Linear stability

- The Jacobian matrix of the system reads

$$\boldsymbol{A}(\boldsymbol{x}) = \begin{bmatrix} 0 & 1 \\ 1 - 3x^2 & -0.5 \end{bmatrix}$$

- As seen in the previous course, the linear stability of each fixed point $\boldsymbol{x}^*$ is governed by the eigenspectrum of $\boldsymbol{A}(\boldsymbol{x}^*)$.

Duffing oscillator

Linear stability

$$\boldsymbol{x}_1^* = (0, 0)$$

- Eigenvalues of $\boldsymbol{A}$ are

$$\lambda_1 = 0.78$$

and

$$\lambda_2 = -1.28$$

- This fixed point is a **saddle**.

$$\boldsymbol{x}_2^* = (1, 0)$$

- Eigenvalues of $\boldsymbol{A}$ are

$$\lambda_1 = -0.25 + 1.39i$$

and

$$\lambda_2 = -0.25 - 1.39i$$

- This fixed point is a **stable focus**.

$$\boldsymbol{x}_3^* = (-1, 0)$$

- Eigenvalues of $\boldsymbol{A}$ are

$$\lambda_1 = -0.25 + 1.39i$$

and

$$\lambda_2 = -0.25 - 1.39i$$

- This fixed point is a **stable focus**.

Duffing oscillator

A dissipative dynamical system

- Let us derive an equation for the total energy of the system

$$\dot{x}\left(\ddot{x} - x + x^3\right) = -\frac{1}{2}\dot{x}$$

- After some simplification, this equation can be re-written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\left[\frac{1}{2}\left(\dot{x}\right)^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4\right]}_{\mathcal{H}} = -\frac{1}{2}\left(\dot{x}\right)^2,$$

where $\mathcal{H}$ is the total energy of the system.

Duffing oscillator

A dissipative dynamical system

- The governing equation for the total energy $\mathcal{H}$ finally reads

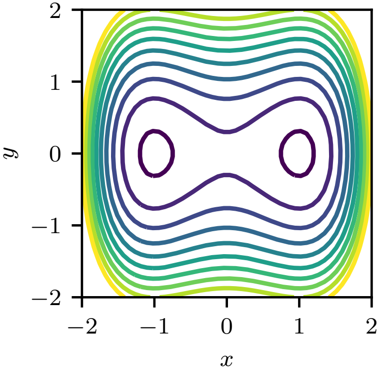
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} = -\frac{1}{2}\left(\dot{x}\right)^2 .$$

- Clearly, as $t \to +\infty$, the total energy $\mathcal{H}$ tends to a constant value.

This is a **dissipative** dynamical system.

Duffing oscillator

A dissipative dynamical system



Isocontours of the total energy $\mathcal{H}$ of the Duffing oscillator.

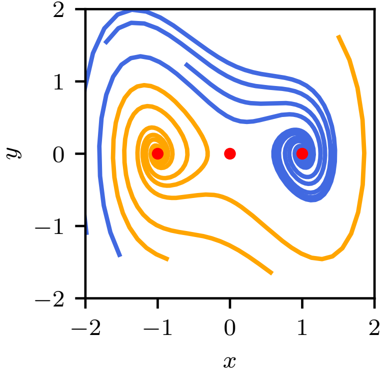
Duffing oscillator

Question

Given an initial condition x_0, toward which fixed point will it evolve as $t \to +\infty$?

Duffing oscillator

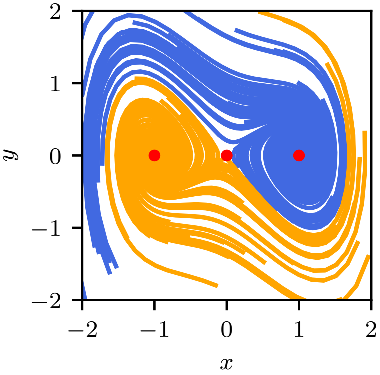
Basin of attraction



Trajectories of 8 randomly distributed initial conditions.

Duffing oscillator

Basin of attraction



Trajectories of 100 randomly distributed initial conditions.

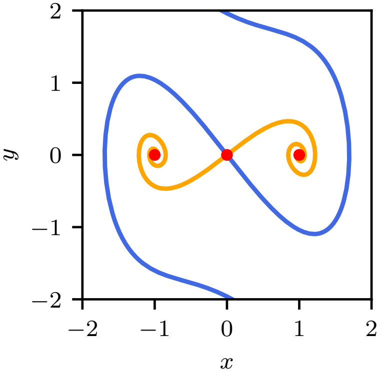
Duffing oscillator

Basin of attraction

- Not all initial conditions $\boldsymbol{x}_0$ end up to the same fixed point.
- Two different regions appear well separated.
  ↪ These two regions are called the **basins of attraction** of each fixed point.
- For the present dynamical system, a sharp frontier delimits these two regions. The saddle node $\boldsymbol{x}_1^* = (0, 0)$ moreover appears to "sit" on this frontier.

Duffing oscillator

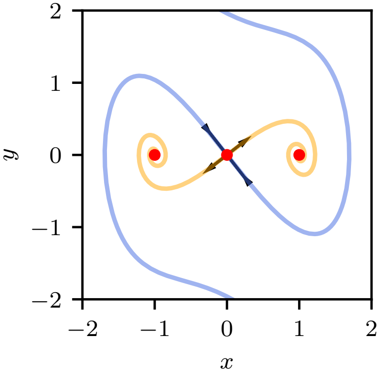
Stable and unstable manifolds



Delimitation (blue) of the two basins of attraction.

Duffing oscillator

Stable and unstable manifolds



Arrows depict the linearly stable and unstable eigendirections of the saddle $\boldsymbol{x}_1^* = (0, 0)$.

Duffing oscillator

Stable and unstable manifolds

- On the previous plot, the blue line is called the **stable** manifold $\boldsymbol{W}^s$ of $\boldsymbol{x}_1^*$, while the orange line is its **unstable** manifold $\boldsymbol{W}^u$.

- $\boldsymbol{W}^s$ and $\boldsymbol{W}^u$ are invariant sets, that is

  An invariant set is a subset $\boldsymbol{W}$ of the phase space such that for any $\boldsymbol{x} \in \boldsymbol{W}$ and $t \in \mathbb{R}$, we have $\phi_t(\boldsymbol{x}) \in \boldsymbol{W}$.

- Note that, if $\|\boldsymbol{x}\|$ is small enough, the stable (resp. unstable) manifold is tangent to the stable (resp. unstable) eigendirection of the fixed point.

Duffing oscillator

Stable Manifold Theorem

Stable Manifold Theorem

Suppose the origin is a fixed point of $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$. Let E^s and E^u be the sable and unstable subspaces of the linearization $\dot{x} = \boldsymbol{Ax}$, where $\boldsymbol{A}$ is the Jacobian matrix of f at the origin. If $\|f(\boldsymbol{x}) - \boldsymbol{Ax}\| = \mathcal{O}(\|\boldsymbol{x}\|^2)$, then $\exists$ **local stable and unstable manifolds** $\boldsymbol{W}^s_{loc}(0)$ and $\boldsymbol{W}^u_{loc}(0)$ which have the same dimension as E^s and E^u and are tangent to them at 0.

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How to compute these manifolds?

Brute force

- Integrate numerically forward (resp. backward) in time an initial condition lying in the unstable (resp. stable) linear subspace of the fixed point.

Maths

- Use Taylor expansion to compute an analytical expression of the stable and unstable manifolds.

Duffing oscillator

How to compute these manifolds?

- Let us assume that

$$
\begin{aligned}
y &= h(x) \\
&= a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\
&= \sum_{k=1}^{n} a_k x^k.
\end{aligned}
$$

- Note moreover that

$$
\dot{y} = \frac{\mathrm{d}x}{\mathrm{d}t} \frac{\mathrm{d}y}{\mathrm{d}x}
$$

Duffing oscillator

How to compute these manifolds?

- We then obtain that

$$\begin{aligned}\dot{y} &= h'(x)\dot{x}\\ &= \dot{x}\sum_{k=1}^{n} a_k k x^{k-1}\end{aligned}$$

- By equating both sides, one obtains n algebraic equations allowing us to determine the coefficients a_k $(k = 1 \cdots n)$.

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Application to our example

- Let us compute cubic approximations of the stable and unstable manifolds of the saddle $\boldsymbol{x}_1^* = (0, 0)$. We thus assume that

$$
\begin{aligned}
y &= h(x) \\
&= ax + bx^2 + cx^3.
\end{aligned}
$$

- Differentiating $h(x)$ with respect to x gives

$$
h'(x) = a + 2bx + 3cx^2.
$$

Duffing oscillator

Application to our example

- Finally, we can write

$$
\begin{aligned}
h'(x)\dot{x} - \dot{y} &= 0 \\
h'(x)y + \frac{1}{2}y - x + x^3 &= 0 \\
(a + \frac{1}{2} + 2bx + 3cx^2)(ax + bx^2 + cx^3) - x + x^3 &= 0
\end{aligned}
$$

Duffing oscillator

Application to our example

- After some calculations, we finally obtain that the coefficients in front of x^k are solution to

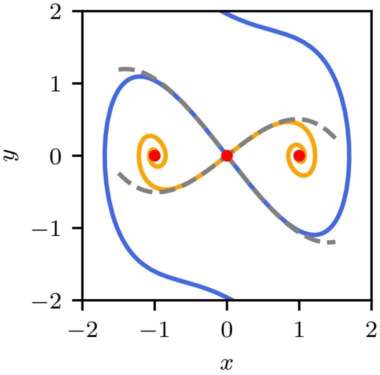
$$\begin{aligned} x \quad &: \quad a^2 + \frac{1}{2}a - 1 = 0 \\ x^2 \quad &: \quad \left(3a + \frac{1}{2}\right) b = 0 \\ x^3 \quad &: \quad \left(4a + \frac{1}{2}\right) c + 1 = 0 \end{aligned}$$

- Finally, we obtain that the stable and unstable manifolds can be approximated by

$$h_\pm(x) = a_\pm x + c_\pm x^3.$$

Duffing oscillator

Stable and unstable manifolds



Dashed gray lines depict the poylnomial approximations of $\boldsymbol{W}^s$ and $\boldsymbol{W}^u$.

Why do we bother with manifolds?

A toy-model for subcritical transition to turbulence

Subcritical transition to turbulence

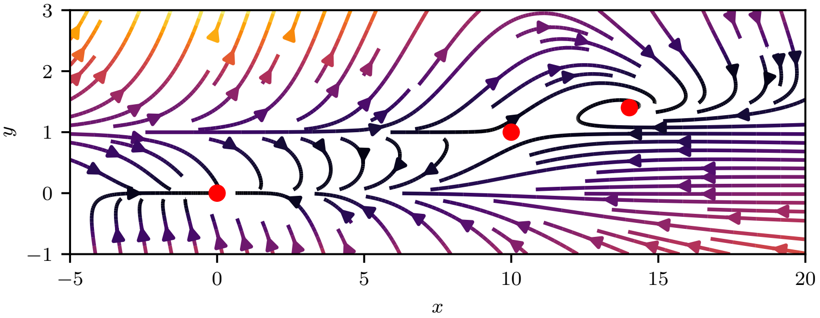
A toy-model

- In 2014, Kerswell *et al.* have proposed a simple toy-model to illustrate some aspect of subcritical transition to turbulence and so-called *nonlinear optimal perturbations*.
- This model reads

$$
\begin{aligned}
\dot{x} &= -x + 10y \\
\dot{y} &= y(10e^{-x^2/100} - y)(y - 1).
\end{aligned}
$$

Subcritical transition to turbulence

A toy-model



Phase plane and fixed points of the toy-model considered.

Subcritical transition to turbulence

A toy-model

The model admits three fixed points.

Laminar solution

- $\boldsymbol{x}_1^* = (0, 0)$
- Linearly stable sink.

The Edge

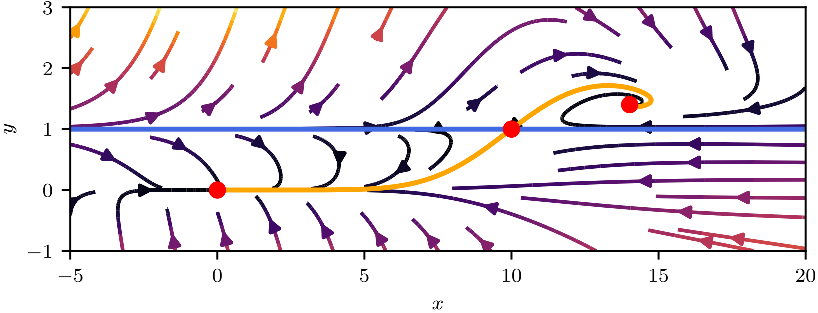
- $\boldsymbol{x}_2^* = (10, 1)$
- Saddle point.

Turbulent solution

- $\boldsymbol{x}^* = (14.017, 1.4017)$
- Linearly stable focus.

Subcritical transition to turbulence

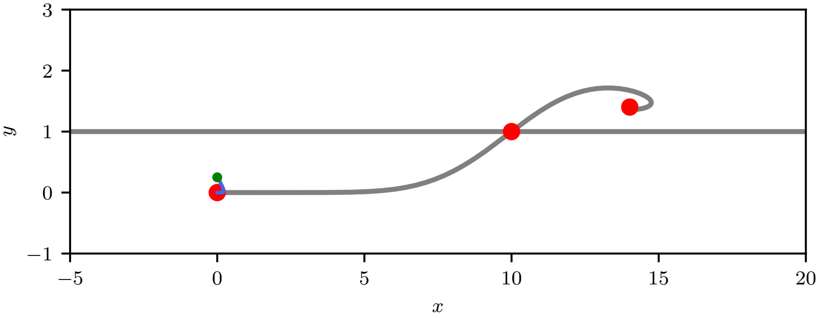
A toy-model



Stable (blue) and unstable (orange) manifolds of the Edge.

Subcritical transition to turbulence

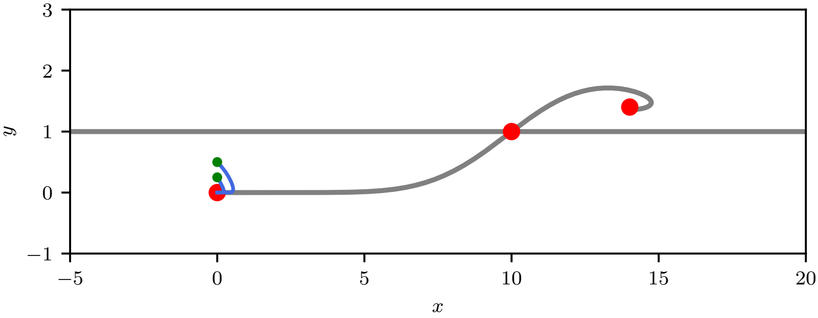
A toy-model



Trajectories for perturbations of different initial amplitude ($y_0 = 0.25, 0.5, 0.75, 0.999, 1.00001$).

Subcritical transition to turbulence

A toy-model



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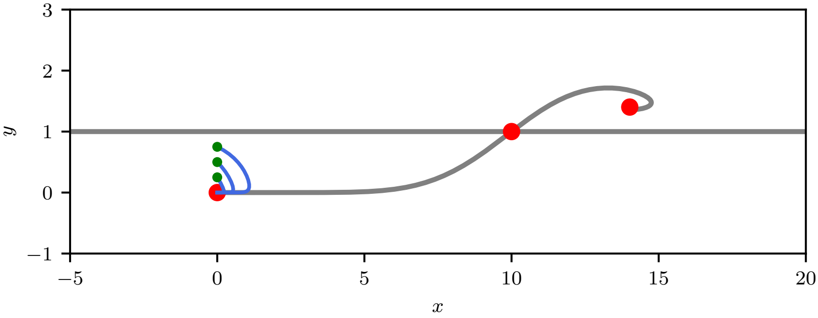
Subcritical transition to turbulence

A toy-model



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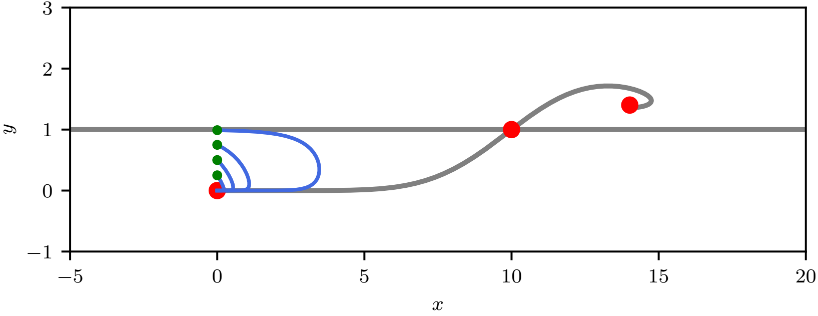
Subcritical transition to turbulence

A toy-model



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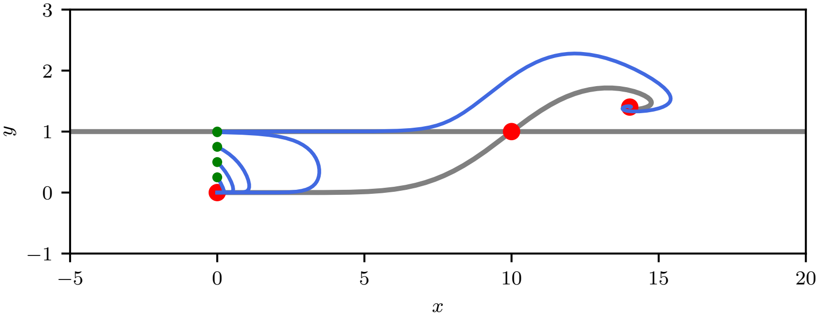
Subcritical transition to turbulence

A toy-model



Trajectories for perturbations of different initial amplitude ($y_0 = 0.25, 0.5, 0.75, 0.999, 1.00001$).

Exercises

Do It Yourself

Exercise

A fairly simple example

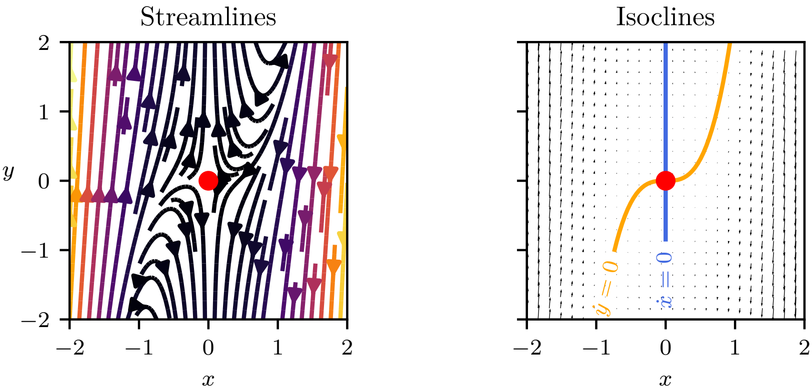
- Let us consider the following dynamical system

$$\dot{x} = -x$$
$$\dot{y} = 2y - 5x^3.$$

- You need to
  1. Determine the fixed point of the system.
  2. Compute its linear stable and unstable eigenspaces.
  3. Compute an approximation of its stable and unstable manifolds.

Exercise

A fairly simple example



Phase plane and isoclines of the dynamical system considered.

Another exercise

A slightly more complex one

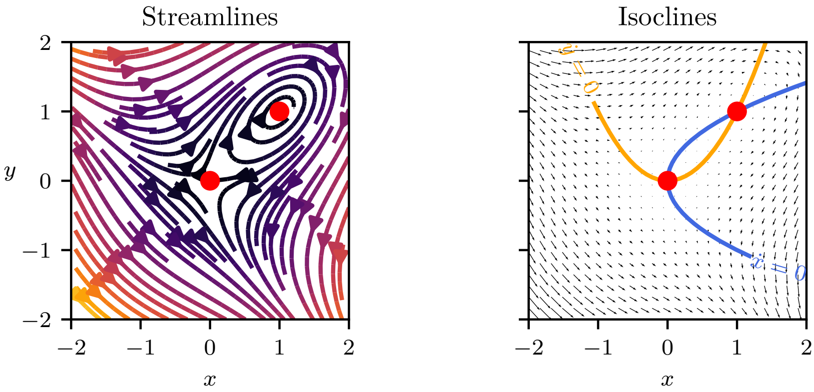
- Let us now consider the following dynamical system

$$\begin{aligned} \dot{x} &= -x + y^2 \\ \dot{y} &= y - x^2. \end{aligned}$$

- Do the same as before.

Exercise

A slightly more complex one



Phase plane and isoclines of the dynamical system considered.