

APPLIED MATHEMATICS

Data-driven discovery of partial differential equations

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We propose a sparse regression method capable of discovering the governing partial differential equation for a given system by time series measurements in the spatial domain. The regression framework relies on sparsity-promoting techniques to select the nonlinear and partial derivative terms of the governing equations that accurately represent the data, bypassing a combinatorially large search through all possible candidates. The method balances model complexity and regression accuracy by selecting a parsimonious model via cross-validation. Time series measurements can be made in an Eulerian framework, where the sensors are stationary, or in a Lagrangian framework, where the sensors move with the dynamics. The method is computationally efficient, robust, and demonstrated to work on a variety of canonical problems spanning a number of domains including Navier-Stokes, the quantum harmonic oscillator, and the diffusion equation. The method is capable of disambiguating between potentially nonunique dynamical terms by using multiple time series taken with different initial data. Thus, for a traveling wave, the method can distinguish between the wave equation and the Korteweg-de Vries equation, for instance. The method provides a principled technique for discovering governing equations and physical laws in parameterized spatiotemporal systems where first-principles derivations are intractable.

INTRODUCTION

Data-driven discovery methods, which have been enabled in the past decade by the plummeting cost of sensors, data storage, and computational resources, have a transformative impact on the sciences, facilitating a variety of innovations for characterizing high-dimensional data generated from experiments. Less well understood is how to uncover underlying physical laws and/or governing equations from time series data that exhibit spatiotemporal activity. Traditional theoretical methods for deriving the underlying partial differential equations (PDEs) are rooted in conservation laws, physical principles, and/or phenomenological behaviors. These first-principles derivations lead to many of the canonical models ubiquitous in physics, engineering,

recently, sparsity (15) for efficient computation (16, 17). Both the existing methods and the new method we propose balance model complexity and regression accuracy by selecting a parsimonious model via cross-validation. Time series measurements can be made in an Eulerian framework, where the sensors are stationary, or in a Lagrangian framework, where the sensors move with the dynamics. The method is computationally efficient, robust, and demonstrated to work on a variety of canonical problems spanning a number of domains including Navier-Stokes, the quantum harmonic oscillator, and the diffusion equation. The method is capable of disambiguating between potentially nonunique dynamical terms by using multiple time series taken with different initial data. Thus, for a traveling wave, the method can distinguish between the wave equation and the Korteweg-de Vries equation, for instance. The method provides a principled technique for discovering governing equations and physical laws in parameterized spatiotemporal systems where first-principles derivations are intractable.

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Table 1. Summary of regression results for PDE-FIND. The spatial domain is $x \in [0, 1]$ and the time domain is $t \in [0, 1]$. The parameters of the model for both the spatial and temporal domains are identified using the percent of data used in subsampling.

PDE	Identified PDE	Percent of data used in subsampling
KdV	$u_t + u u_x + u_{xxx} = 0$	100%
Burgers	$u_t + u u_x = \nu u_{xx}$	100%
Schrödinger	$i u_t + \frac{1}{2} u_{xx} = 0$	100%
NLS	$i u_t + \frac{1}{2} u_{xx} + u ^2 u = 0$	100%
KS	$u_t + u u_x + u_{xx} + u ^2 u = 0$	100%
Reaction Diffusion	$u_t = 0.1 \nabla^2 u + \lambda(A) u$ $v_t = 0.1 \nabla^2 v + \omega(A) v$ $A^2 = u^2 + v^2$	100%
Navier-Stokes	$u_t = 0.1 \nabla^2 u + \lambda(A) u$ $v_t = 0.1 \nabla^2 v + \omega(A) v$ $A^2 = u^2 + v^2$	100%