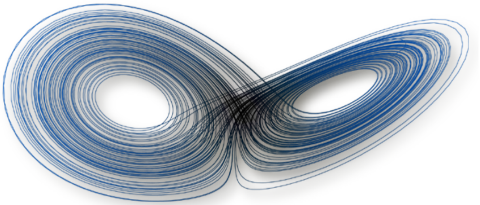

Nonlinear physics, dynamical systems and chaos theory

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Overview from last time

Given the non-linear dynamical system

$$\dot{\mathbf{X}} = \boldsymbol{\mathcal{F}}\left(\mathbf{X}\right),$$

we have seen in the previous lectures how to :

↪ Compute fixed points $\mathbf{X}^*$ of the system, i.e. solutions to

$$\boldsymbol{\mathcal{F}}\left(\mathbf{X}\right) = 0.$$

↪ Derive the linearized the equations governing the dynamics of a perturbation $\mathbf{x}$:

$$\dot{\mathbf{x}} = \boldsymbol{\mathcal{A}}\mathbf{x}.$$

↪ Characterize the linear stability of the fixed point $\mathbf{X}^*$ based on the eigenspectrum of $\boldsymbol{\mathcal{A}}$.

Question

Let us now consider a parametrized dynamical system

$$\dot{\mathbf{x}} = \boldsymbol{\mathcal{F}}\left(\mathbf{x}, \mu\right).$$

How do its fixed points evolve when varying the parameter μ? Can we characterize this evolution and make predictions?

Bifurcations of first-order systems

Flows on the line (again)

Bifurcations of first-order systems

- Let us consider a first-order dynamical system

$$\dot{x} = f(x, \mu),$$

where μ is our **control parameter**.

- We have seen that such systems have relatively simple dynamics dictated by fixed points.

- These fixed points may however change as a function of μ.
  - $\hookrightarrow$ Qualitative variations of the dynamics are called **bifurcations**.
  - $\hookrightarrow$ The values of μ at which these changes occurs are called **bifurcation points**.

Bifurcations of first-order systems

- To facilitate discussions to come, the Taylor expansion of $f(x)$ (for a constant μ) is given by

$$f(x) \simeq a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

- Depending on the coefficients a_k, different behaviors will be observed.

Saddle-node bifurcation

First-order dynamical system

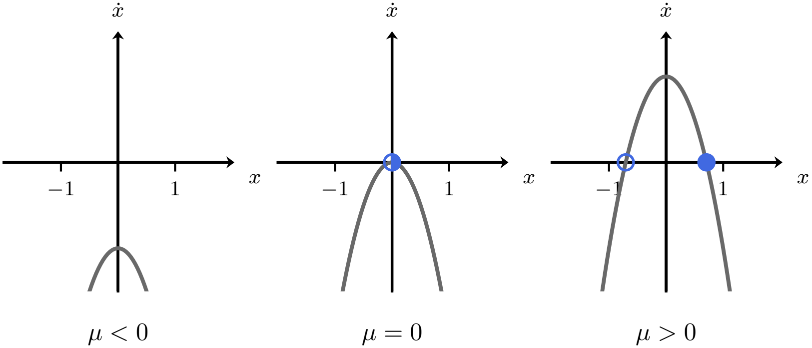
- As a starting point, let us look at the system

$$\dot{x} = \mu - x^2$$

and plot its phase line for different values of μ.

Saddle-node bifurcation

Phase line



Evolution of the phase line of the system for $\mu = -1/2, 0$ and $1/2$.

Saddle-node bifurcation

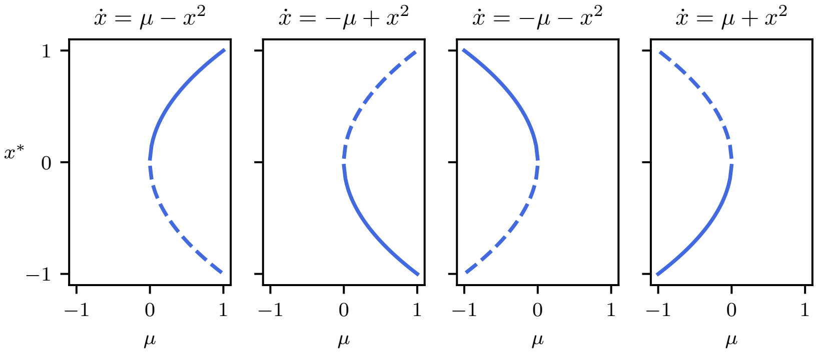
Fixed points and stability

- Depending on the value of μ, different behaviors are possible.
  - $\hookrightarrow$ For $\mu < 0$, the system admits no fixed points and $\lim_{t \to \infty} x(t) = -\infty$.
  - $\hookrightarrow$ For $\mu = 0$, the system admits a single **meta-stable** fixed point $x^* = 0$. For $x(0) > 0$, $\lim_{t \to \infty} x(t) = 0$, otherwise, for $x(0) < 0$, $\lim_{t \to \infty} x(t) = -\infty$.
  - $\hookrightarrow$ For $\mu > 0$, the system admits to fixed points $x^* = \pm\sqrt{\mu}$. One is linearly stable, while the other one is linearly unstable.

- As μ becomes positive, we observe a transition from the absence of fixed points to the creation of two of them, one stable and the other unstable. This is known as the **saddle node bifurcation**.

Saddle-node bifurcation

Bifurcation diagram



Bifurcation diagrams for the different combinations of saddle-node bifurcations.

Saddle-node bifurcation

Example from real life

- Let us consider a damped pendulum driven by a constant torque

$$mL^2 \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + b\frac{\mathrm{d}\theta}{\mathrm{d}t} + mgL\sin(\theta) = \Gamma.$$

- Introducing the time scale $t = T\tau$, one can write

$$\frac{L}{gT^2}\ddot{\theta} + \frac{b}{mgLT}\dot{\theta} + \sin(\theta) = \frac{\Gamma}{mgL}.$$

Saddle-node bifurcation

Example from real life

- If ${}^{b}\!/_{mgT} \gg {}^{L}\!/_{gT^2}$, we can neglect $\ddot{\theta}$ and our equation becomes

$$\dot{\theta} = \gamma - \sin(\theta),$$

  with $T = {}^{b}\!/_{mgL}$ and $\gamma = {}^{\Gamma}\!/_{mgL}$.

- You can now easily show that the system experiences a saddle-node bifurcation at $\gamma = 1$.

- Interpret your results from physical point of view!

Transcritical bifurcation

First-order dynamical system

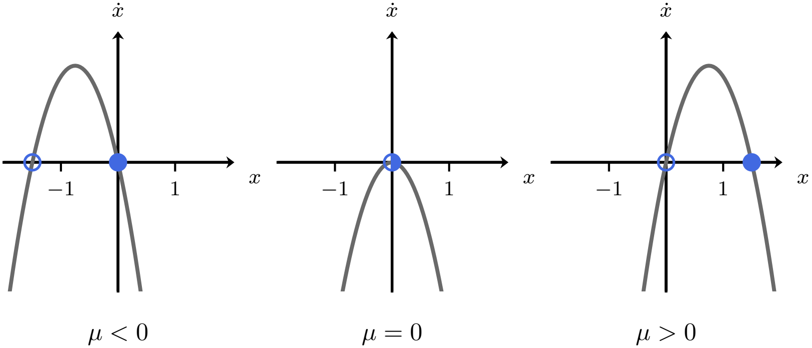
- Let us now consider the following first-order dynamical system

$$\dot{x} = \mu x - x^2$$

and plot its phase line for different values of μ.

Transcritical bifurcation

Phase line



Evolution of the phase line of the system for $\mu = -3/2, 0$ and $3/2$.

Transcritical bifurcation

Fixed points and linear stability

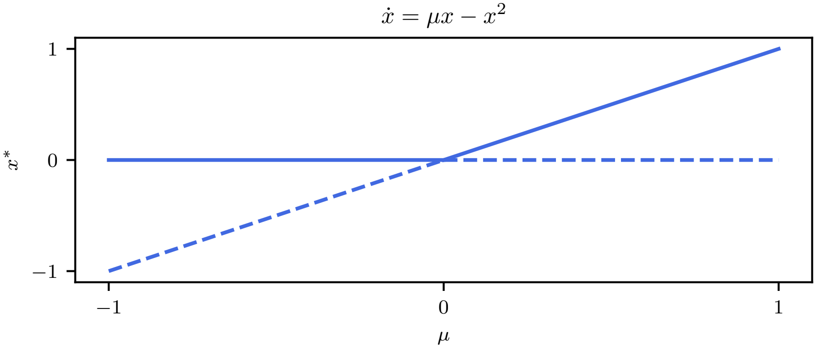
- The system admits two fixed points

$$x_1^* = 0 \text{ and } x_2^* = \mu.$$

- Depending on the sign of μ, we have
  - ↪ For $\mu < 0$, x_1^* is linearly stable while x_2^* is linearly unstable.
  - ↪ For $\mu = 0$, $x_1^* = x_2^*$ is meta-stable.
  - ↪ For $\mu > 0$, x_1^* is now linearly unstable, while x_2^* has become linearly stable.
- As μ becomes positive, the two fixed points have exchanged their stability. This is known as the **transcritical bifurcation**.

Transcritical bifurcation

Bifurcation diagram



Bifurcation diagram of the transcritical bifurcation.

Supercritical pitchfork bifurcation

First-order dynamical system

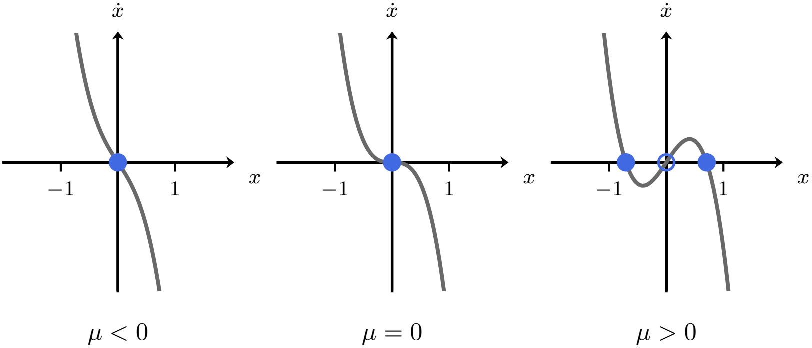
- Let us consider the following system

$$\dot{x} = \mu x - x^3$$

and plot its phase line for different values of μ.

Supercritical pitchfork bifurcation

Phase line



Evolution of the phase line of the system for $\mu = {}^{-1}/_{2}, 0$ and ${}^{1}/_{2}$.

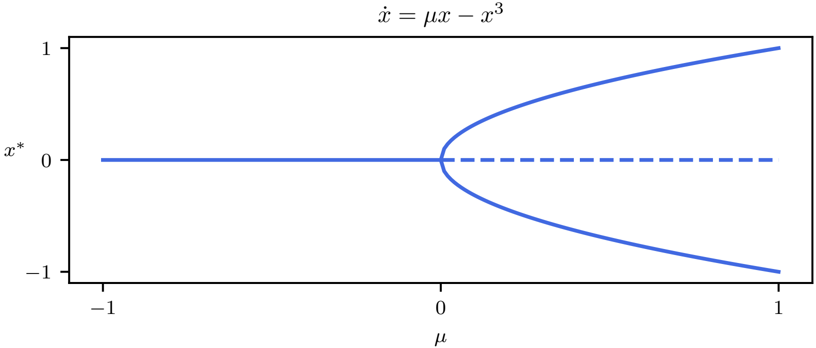
Supercritical pitchfork bifurcation

Fixed points and linear stability

- Depending on the value of μ, different behaviors are possible.
  - $\hookrightarrow$ For $\mu < 0$, the system admits a single linearly stable fixed point $x^* = 0$.
  - $\hookrightarrow$ For $\mu = 0$, the fixed point $x^* = 0$ is marginal from a linear point of view, yet still nonlinearly stable.
  - $\hookrightarrow$ For $\mu > 0$, the system now admits three fixed points. $x_1^* = 0$ is now linearly unstable, while $x_{2,3}^* = \pm\sqrt{\mu}$ are linearly stable.
- As μ becomes positive, we observe that the origin becomes linearly unstable and two additional stable fixed points are created. This is known as the **supercritical pitchfork bifurcation**.

Supercritical pitchfork bifurcation

Bifurcation diagram



Bifurcation of the supercritical pitchfork.

Subcritical pitchfork bifurcation

First-order dynamical system

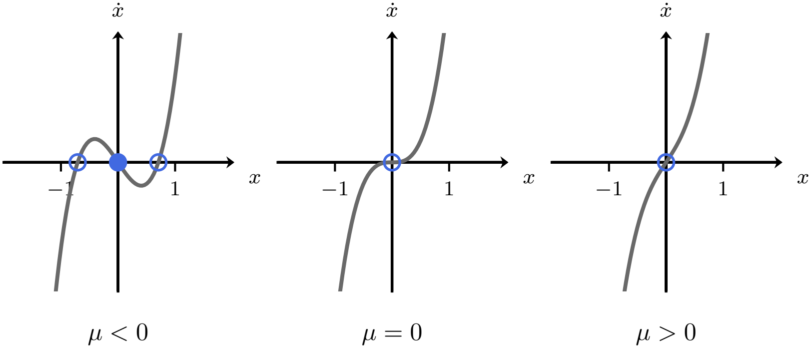
- Let us consider the following system

$$\dot{x} = \mu x + x^3$$

and plot its phase line for different values of μ.

Subcritical pitchfork bifurcation

Phase line



Evolution of the phase line of the system for $\mu = {}^{-1}/_{2}, 0$ and ${}^{1}/_{2}$.

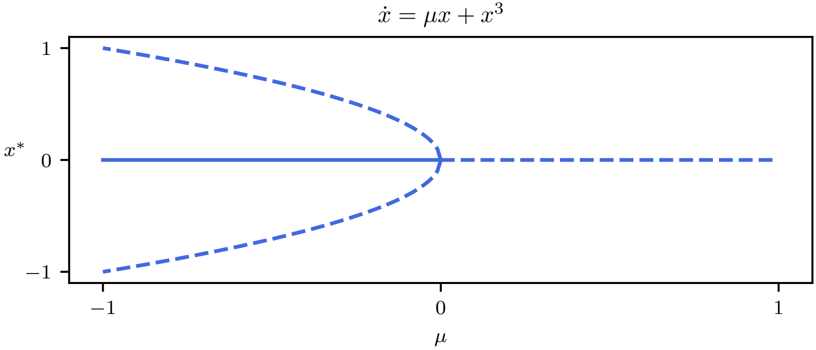
Subcritical pitchfork bifurcation

Fixed points and linear stability

- Depending on the value of μ, different behaviors are possible.
  - ↪ For $\mu < 0$, the system admits three fixed points. $x_1^* = 0$ is linearly stable, while $x_{2,3}^* = \pm\sqrt{-\mu}$ are linearly unstable.
  - ↪ For $\mu = 0$, the fixed point $x^* = 0$ is marginal from a linear point of view, but nonlinearly unstable.
  - ↪ For $\mu < 0$, the system now admits a single linearly unstable fixed point $x^* = 0$.
- As μ becomes positive, we observe that the origin becomes linearly unstable and the other two unstable fixed points are destroyed. This is known as the **subcritical pitchfork bifurcation**.

Subcritical pitchfork bifurcation

Bifurcation diagram



Bifurcation of the subcritical pitchfork.

Bifurcations of first-order systems

Summary

	f	f_x	f_μ	f_{xx}	$f_{x\mu}$	f_{xxx}
Fixed point	0					
Bifurcation	0	0	$\neq 0$			
Saddle-node	0	0	$\neq 0$	$\neq 0$		
Transcritical	0	0	0	$\neq 0$	$\neq 0$	
Pitchfork	0	0	0	0	$\neq 0$	$\neq 0$

Exercise

- Consider the following dynamical system

$$\dot{x} = \mu x + x^3 - 0.25x^5$$

and study its different fixed points and bifurcations.

Bifurcations of second-order systems

Creation of limit cycles

Bifurcations of second-order systems

- Let us now consider a second-order dynamical system given by

$$\begin{aligned} \dot{x} &= f(x, y, \mu) \\ \dot{y} &= g(x, y, \mu). \end{aligned}$$

- As seen in the previous lectures, such systems have dynamics much richer that those of first-order systems.
- How do they evolve as the control parameter μ changes?
  $\hookrightarrow$ Note that all bifurcations seen so far also apply to fixed points of second-order dynamical systems.

Saddle-node bifurcation revisited

The reason it is called saddle-node

- Consider the following system

$$\dot{x} = \mu - x^2$$
$$\dot{y} = -y$$

and draw qualitatively its phase space for $\mu < 0$, $\mu = 0$ and $\mu > 0$.

Saddle-node bifurcation revisited

The reason it is called saddle-node

Hopf bifurcation

Creation of limit cycles

- Let us consider the following system

$$\begin{aligned}\dot{x} &= \mu x - \omega y - (x^2 + y^2)x \\ \dot{y} &= \omega x + \mu y - (x^2 + y^2)y.\end{aligned}$$

- It admits a single fixed point given by

$$(x^*, y^*) = (0, 0).$$

Hopf bifurcation

Exercise

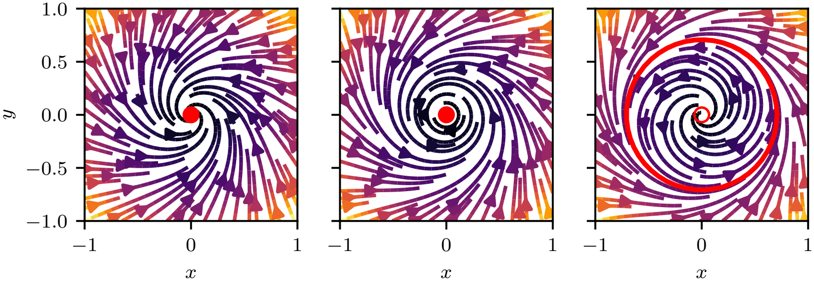
1. Study the linear stability of the fixed point as μ varies.
2. Introducing the complex variable $z = x + iy$, show that the equation for z reads

$$\dot{z} = (\mu + i\omega)z - |z|^2 z.$$

3. From this complex equation, determine the first-order system that governs the amplitude of oscillation $r = \sqrt{x^2 + y^2}$.
4. Study the properties of this equation an determine what type of bifurcation does the first-order system $\dot{r} = f(r, \mu)$ experiences.
5. Sketch the evolution of the phase plane of our original system as μ varies and conclude.

Hopf bifurcation

Phase plane

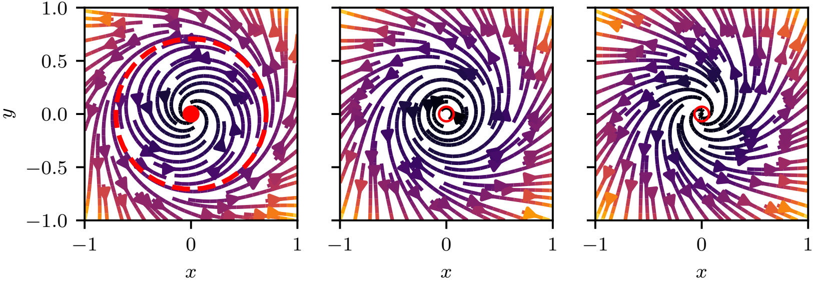


$\mu < 0$ $\mu = 0$ $\mu > 0$

Evolution of the phase plane of the system as a function of μ for the **supercritical Hopf bifurcation**.

Hopf bifurcation

Phase plane



$\mu < 0$ $\mu = 0$ $\mu > 0$

Evolution of the phase plane of the system as a function of μ for the **subcritical Hopf bifurcation**.

Hopf bifurcation

Normal form

- The **normal form** of the Hopf bifurcation reads

$$
\begin{aligned}
\dot{r} &= \mu r \pm r^3 \\
\dot{\theta} &= \omega,
\end{aligned}
$$

where r is the amplitude of oscillator and θ its phase.

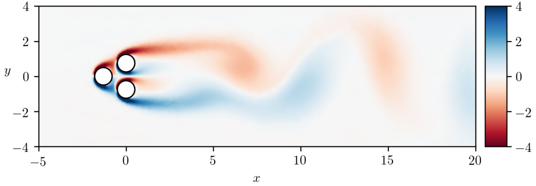
Hopf bifurcation

Example from real life

- The dynamics of the flow are governed by the Navier-Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \frac{1}{Re}\nabla^2 \boldsymbol{u}$$
$$\nabla \cdot \boldsymbol{u} = 0.$$

- These are partial differential equations having only a quadratic nonlinearity.



Evolution of the vorticity field for the fluidic pinball at $Re = 60$.

Hopf bifurcation

Example from real life

Question

How come a system with quadratic nonlinearities exhibit a Hopf bifurcation whose normal form involve cubic ones?

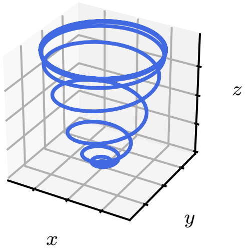
Hopf bifurcation

Example from real life

These dynamics can be modeled by the following generalized mean-field model

$$\begin{aligned}\dot{x} &= \sigma x - y - xz\\ \dot{y} &= x + \sigma y - yz\\ \dot{z} &= -\lambda(z - x^2 - y^2).\end{aligned}$$

where x and y capture the vortex shedding and z describes the *mean flow distortion*.



Trajectory given by the generalized mean field model.

Hopf bifurcation

Example from real life

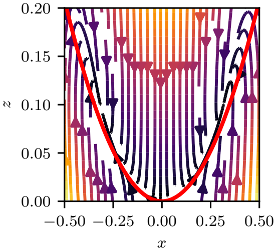
- Let us compute the two-dimensional unstable manifold. For that purpose, assume

$$
\begin{aligned}
z &= h(x, y) \\
&= ax^2 + bxy + cy^2.
\end{aligned}
$$

- After some calculations, we finally get

$$
z = \frac{1}{2\sigma + 1} \left(x^2 + y^2 \right).
$$

$\hookrightarrow$ If $\lambda \gg \sigma$, the system rapidly evolves onto this two-dimensional paraboloid manifold.



Slice in the $y = 0$ plane of the phase space.

Hopf bifurcation

Example from real life

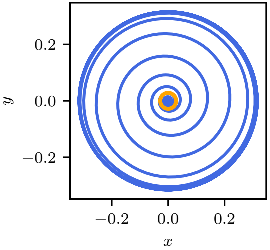
- Our dynamical system finally reduces to

$$\dot{x} = \sigma x - y - \alpha(x^2 + y^2)x$$
$$\dot{y} = x + \sigma y - \alpha(x^2 - y^2)y.$$

- Introducing $A = x + iy$, we finally arrive to the normal form of the supercritical Hopf bifurcation

$$\dot{A} = (\sigma + i)A - \alpha|A|^2 A.$$

- Though our system has only quadratic nonlinearities, dynamics on the manifold mimic cubic ones.



Trajectory of the system on the 2D manifold.