Ex 11.6.9

$$\sigma_{XB} = \sum_x p_X(x) |x\rangle\langle x|_X \otimes \rho_B^x$$ represents $\mathcal{E} = \{p_X(x), \rho_B^x\}$

$$\chi(\mathcal{E}) \triangleq H(\rho_B) - \sum_x p_X(x) H(\rho_B^x)$$

$$= H\left(\sum_x p_X(x) \rho_B^x\right) - \sum_x p_X(x) H(\rho_B^x)$$

$$I(X:B) = H(B)_\sigma - H(B|X)_\sigma$$

$$= H\left(\sum_x p_X(x) \rho_B^x\right) - \sum_x p_X(x) H(\rho_B^x)$$

From defn of σ_B ($= \rho_B$) and 11.54

Ex 11.7.1

Apply chain rule to $I(A:BC)$ & $I(AC:B)$

$$I(A:BC) = I(A:C) + I(A:B|C)$$

$$\parallel$$

$$I(AC:B) = I(B:C) + I(B:A|C)$$

The desired statement follows.

Ex 11.7.5

Claim: $\sum_x p_X(x)\, H(A|B)_{\rho^x} \leq H(A|B)_\rho$ where $\rho_{AB} \triangleq \sum_x p_X(x)\, \rho^x_{AB}$

Proof:

Strong subadditivity: $I(A;B|C)_\rho \geq 0$ (Thm 11.7.1)

Consider applying $I(\tilde{A};\tilde{B}|\tilde{C}) \geq 0$ to $\sigma_{XAB} \triangleq \sum_x p_X(x)\, |x\rangle\langle x| \otimes \rho^x_{AB}$.

with $\tilde{A} = A$, $\tilde{B} = X$, $\tilde{C} = B$ (Note, $\mathrm{Tr}_X(\sigma_{XAB}) = \rho_{AB}$)

$\rightarrow I(A;X|B)_{\sigma_{XAB}} \geq 0$

X: r.v. distributed according to $p_X(x)$.

$$\leftrightarrow H(A|B)_\sigma \geq H(A|XB)_\sigma = \underbrace{H(ABX)_\sigma}_{\sum_x p_X(x) H(\rho^x_{AB}) + H(X)} - \underbrace{H(XB)_\sigma}_{\sum_x p_X(x) H(\rho^x_B) + H(X)}$$

(Entropy of a c-q state, eqn 11.33)

$$= \sum_x p_X(x) \left[H(\rho^x_{AB}) - H(\rho^x_B) \right] = \sum_x p_X(x)\, H(A|B)_{\rho^x_{AB}}$$

The claim follows from the fact that $H(A|B)_\sigma = H(A|B)_\rho$, since $\mathrm{Tr}_X(\sigma) = \rho$.

Implication: for ρ^x_{AB} a c-q state of the form $\rho^x_{AB} = \rho^x_A \otimes |x\rangle\langle x|_B$, this inequality is tight (by 11.57).

Ex 11.7.6

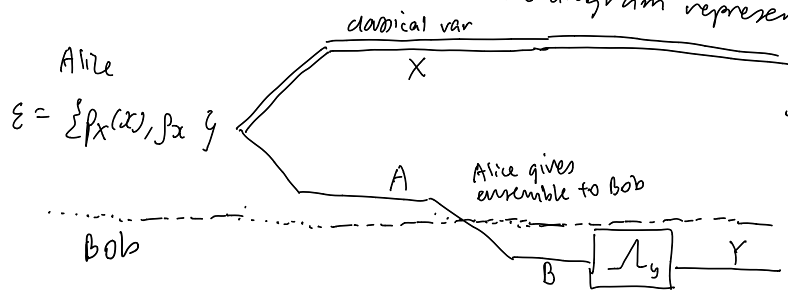
Follows straightforwardly from $I(A\rangle B)_\rho = -H(A|B)_\rho$

Ex 11.9.2

Claim: $I_{acc}(\mathcal{E}) \leq \chi(\mathcal{E})$ for $\mathcal{E} = \{p_X(x), \rho_A^x\}$

where $I_{acc}(\mathcal{E}) = \max_{\{\Lambda_y\}} I(X:Y)$ for $p_{Y|X}(y|x) = \mathrm{Tr}\{\Lambda_y \rho_x\}$

and from Ex 11.6.9, $\chi(\mathcal{E}) = I(X:A)_\sigma$ where $\sigma_{XA} = \sum_x p_X(x) |x\rangle\langle x|_X \otimes \rho_A^x$ is the ensemble prepared by Alice. The diagram representing this is



By DPI for mutual info,

$I(X:A) = I(X:B) = \chi(\mathcal{E})$

$\leq I(Y:X) = I_{acc}(\mathcal{E})$

as desired