Chap 9

Monday, February 3, 2020 10:50 AM

9.1.6

claim: $\|\omega\|_1 = \max_{-I \le \Lambda \le I} \mathrm{Tr}\{\Lambda\omega\}$

Pf

All eigenvals of a Hermitian operator are positive. Write $\omega = \sum_i \lambda_i |i\rangle\langle i|$, $\lambda_i \ge 0$ $\forall i$.

Then for $\Pi := \sum_i |i\rangle\langle i|$, $\|\omega\|_1 = \mathrm{Tr}(\Pi\omega)$. Indeed observe that Π is just the identity here, over the Hilbert space that ω lives in.

Analogously to section 9.1.3, we just need to show that $\forall \Lambda$ s.t. $-I \le \Lambda \le I$,

$$\mathrm{Tr}(\Lambda\omega) \le \mathrm{Tr}(\Pi\omega) = \mathrm{Tr}(\omega)$$

Indeed, $\mathrm{Tr}(\Lambda\omega) \le \mathrm{Tr}(\omega)$ because $\Lambda \le I$. #

9.1.7

Lemma 9.1.1 gives $\frac{1}{2}\|\rho_0 - \rho_1\|_1 = \max_{0 \le \Lambda \le I} \mathrm{Tr}\{\Lambda(\rho_0 - \rho_1)\}$

Rewriting 9.44:

$$p_{succ}(\Lambda) = \mathrm{Tr}\{\Lambda_0\rho_0\}p_0 + \mathrm{Tr}\{\underbrace{\Lambda_1}_{=(I-\Lambda_0)}\rho_1\}p_1 \qquad p_0 + p_1 = 1 \qquad \Lambda_0 + \Lambda_1 = I$$

$$= \underbrace{p_1}_{= p_1\mathrm{Tr}((\Lambda_0+\Lambda_1)\rho_1)} + p_0\mathrm{Tr}(\Lambda_0\rho_0) - p_1\mathrm{Tr}(\Lambda_0\rho_1) = p_1 + \mathrm{Tr}(\Lambda_0(p_0\rho_0 - p_1\rho_1)) \quad \text{——} \quad (1)$$

This suggests that we need to go back and redo section 9.1.3 to get an expression for $\frac{1}{2}\|p_0\rho_0 - p_1\rho_1\|_1$ in terms of $\mathrm{Tr}(\Lambda_0(p_0\rho_0 - p_1\rho_1))$.

Let P = positive part of $p_0\rho_0 - p_1\rho_1$

Q = |negative " " " |

} Clearly $|p_0\rho_0 - p_1\rho_1| = |P - Q| = P + Q$

Analogous to 9.35 – 9.36, $\mathrm{Tr}(P - Q) = \mathrm{Tr}(p_0\rho_0 - p_1\rho_1) = p_0 - p_1 = \mathrm{Tr}(P) - \mathrm{Tr}(Q)$

Analogous to 9.37,

$$\|p_0\rho_0 - p_1\rho_1\|_1 = \mathrm{Tr}(P+Q) = \mathrm{Tr}(P) + \underbrace{\mathrm{Tr}(Q)}_{\mathrm{Tr}(P) - (p_0 - p_1)} = 2\,\mathrm{Tr}(P) - (p_0 - p_1)$$

Analogous to 9.38,

$$\therefore \mathrm{Tr}(\Pi_P(p_0\rho_0 - p_1\rho_1)) = \mathrm{Tr}(\Pi_P(P - Q)) = \mathrm{Tr}(P) = \frac{1}{2}\left[\|p_0\rho_0 - p_1\rho_1\|_1 + (p_0 - p_1)\right]$$

$\forall \Lambda$, $0 \le \Lambda \le I$,

$$\mathrm{Tr}(\Lambda(p_0\rho_0 - p_1\rho_1)) = \mathrm{Tr}(\Lambda(P - Q)) \le \mathrm{Tr}(P) = \frac{1}{2}\left[\|p_0\rho_0 - p_1\rho_1\|_1 + (p_0 - p_1)\right]$$

Hence

$$\boxed{\max_{\Lambda} \mathrm{Tr}(\Lambda(p_0\rho_0 - p_1\rho_1)) = \frac{1}{2}\left[\|p_0\rho_0 - p_1\rho_1\|_1 + (p_0 - p_1)\right]} \quad \text{——} \quad (2)$$

Substitute (2) into $\max_{\Lambda}$ (1),

$$\max_{\Lambda} p_{succ}(\Lambda) = p_1 + \frac{1}{2}\left[\|p_0\rho_0 - p_1\rho_1\|_1 + (p_0 - p_1)\right]$$

$$= \frac{1}{2}\underbrace{(p_0 + p_1)}_{1} + \frac{1}{2}\|p_0\rho_0 - p_1\rho_1\|_1 = \frac{1}{2}\left(1 + \|p_0\rho_0 - p_1\rho_1\|_1\right)$$

as desired.