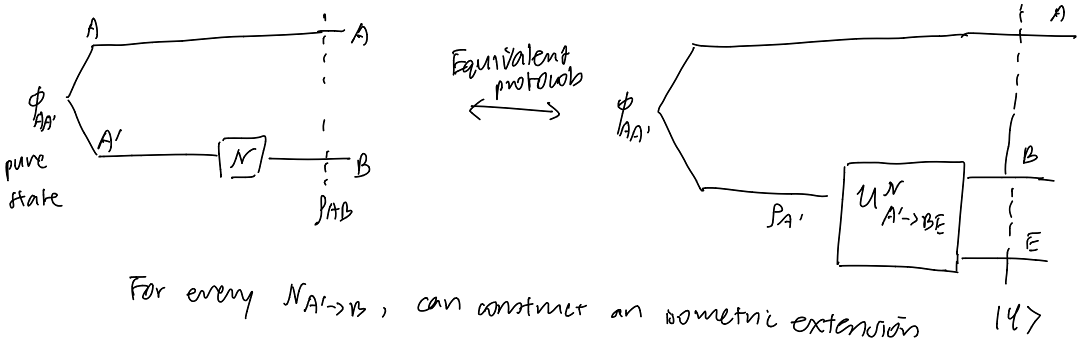
EX 13.5.1



For every $\mathcal{N}_{A'\to B}$, can construct an isometric extension $U^{\mathcal{N}}_{A'\to BE}$ s.t. $\mathcal{N}_{A'\to B}(\rho_{A'}) = \mathrm{Tr}_E(U\rho_{A'}U^\dagger) = \mathrm{Tr}_E(U^{\mathcal{N}}_{A'\to BE}|\phi\rangle_{AA'})$.

Note that on the RHS, $\phi_{AA'}$ can be seen as a purification of $\rho_{A'}$, i.e.

$\mathrm{Tr}_A(|\phi_{AA'}\rangle\langle\phi_{AA'}|) = \rho_{A'}$. ——— (*)

And b/c U is isometry, the output of the RHS protocol is a pure state $|\psi\rangle_{ABE}$ s.t. $\mathrm{Tr}_E(|\psi\rangle\langle\psi|_{ABE}) = \rho_{AB}$ ——— (**)

By defn,

$$\mathcal{N}^c(\rho_{A'}) \triangleq \mathrm{Tr}_B(U^{\mathcal{N}}_{A'\to BE}\,\rho_{A'}\,U^{\mathcal{N}\dagger}_{A'\to BE})$$

$$\mathcal{N}(\rho_{A'}) \triangleq \mathrm{Tr}_E(U^{\mathcal{N}}_{A'\to BE}\,\rho_{A'}\,U^{\mathcal{N}\dagger}_{A'\to BE})$$

$$I_c(\rho_{A'}, \mathcal{N}) = H(\mathcal{N}(\rho_{A'})) - H(\mathcal{N}^c(\rho_{A'}))$$

$$= H(\underbrace{\mathrm{Tr}_E(U^{\mathcal{N}}_{A'\to BE}\,\rho_{A'}\,U^{\mathcal{N}\dagger}_{A'\to BE})}) - H(\underbrace{\mathrm{Tr}_B(U^{\mathcal{N}}_{A'\to BE}\,\rho_{A'}\,U^{\mathcal{N}\dagger}_{A'\to BE})})$$

From (*):

$$\mathrm{Tr}_E(U^{\mathcal{N}}_{A'\to BE}\,\rho_{A'}\,U^{\mathcal{N}\dagger}_{A'\to BE}) = \mathrm{Tr}_{E,A}(U^{\mathcal{N}}_{A'\to BE}|\phi\rangle_{AA'}) = \mathrm{Tr}_{E,A}(|\psi\rangle_{ABE}) \overset{(**)}{=} \mathrm{Tr}_A(\rho_{AB}) = \rho_B$$

$$\mathrm{Tr}_B(U^{\mathcal{N}}_{A'\to BE}\,\rho_{A'}\,U^{\mathcal{N}\dagger}_{A'\to BE}) = \mathrm{Tr}_{A,B}(U^{\mathcal{N}}_{A'\to BE}|\phi\rangle_{AA'}) = \mathrm{Tr}_{A,B}(|\psi\rangle_{ABE})$$

Note I: For pure state $|\psi\rangle_{ABE}$, $H(E)_\psi = H(AB)_\psi \overset{(**)}{=} H(AB)_\rho$

$$= H(\rho_B) - H(\rho_{AB}) = I(A\rangle B)_\rho,$$ as desired.