Continuous Phase Modulation

Achilleas Anastasopoulos

November 11, 2008

Introduction and motivation

Given a set of signals $\mathcal{S} = \{s_1(t), s_2(t), \ldots, s_M(t)\}$ defined on $t \in [0, T]$, a memoryless modulator is defined as a device that accepts an input symbol stream $(x_n)_{n=0}^{\infty}$ (with $x_n \in \{1, 2, \ldots, M\}$) and produces at the output the signal $s(t) = \sum_{n=0}^{\infty} s_{x_n}(t-nT)$. The modulator is called memoryless because the transmitted signal $s_{x_n}(t-nT)$ during the period $[nT, nT+T)$ depends only on the input symbol x_n during the n-th symbol period.

Not all modulations are memoryless. In particular, we want to study CPM, which is a modulation with memory. The goal is to show that CPM can be considered as the serial concatenation of an FSM and a memoryless modulator. The FSM captures the modulator memory (in the same way that an FSM captures the memory of a convolutionally encoded sequence), while the memoryless modulator translates symbols to waveforms. In order to complete this decomposition we need to specify precisely the FSM and the memoryless modulator.

CPM definition

The lowpass complex equivalent CPM signal is

$$s(t) = \sqrt{2E/T} \exp(j[\phi(t; \mathbf{a}) + \phi_0]), \tag{1}$$

where

$$\phi(t; \mathbf{a}) = 2\pi h \sum_{k=0}^{\infty} a_k q(t - kT), \tag{2}$$

and $h = K/P$ with K, P relatively prime, $a_k \in \{\pm 1, \pm 3, \ldots, \pm(M-1)\}$ for M even ($a_k \in \{0, \pm 2, \ldots, \pm(M-1)\}$ for M odd), and

$$q(t) = \begin{cases} 0, & t < 0 \\ 1/2, & t > LT \end{cases} \tag{3}$$

CPM decomposition

Let's look at the phase $\phi(t;\mathbf{a})$ in the interval $t \in [nT, nT+T)$.

$$\phi(t;\mathbf{a}) = 2\pi h \sum_{k=0}^{\infty} a_k q(t-kT) \tag{4}$$

$$= 2\pi h \sum_{k=0}^{n-L} a_k q(t-kT) + 2\pi h \sum_{k=n-L+1}^{n} a_k q(t-kT) \tag{5}$$

$$= \pi h \sum_{k=0}^{n-L} a_k + 2\pi h \sum_{k=n-L+1}^{n} a_k q(t-kT). \tag{6}$$

Further, let's define $u_k = \frac{a_k+M-1}{2} \in \{0,1,\ldots,M-1\}$. Then the above equation becomes

$$\phi(t;\mathbf{a}) = \pi h \sum_{k=0}^{n-L} [2u_k - (M-1)] + 2\pi h \sum_{k=n-L+1}^{n} [2u_k - (M-1)] q(t-kT) \tag{7}$$

$$\begin{aligned} &= 2\pi h \sum_{k=0}^{n-L} u_k - \pi h \sum_{k=0}^{n-L} (M-1) + \\ &\quad + 4\pi h \sum_{k=n-L+1}^{n} u_k q(t-kT) - 2\pi h (M-1) \sum_{k=n-L+1}^{n} q(t-kT) \end{aligned} \tag{8}$$

$$\begin{aligned} &= 2\pi h \sum_{k=0}^{n-L} u_k - \pi h (n-L+1)(M-1) + \\ &\quad + 4\pi h \sum_{k=n-L+1}^{n} u_k q(t-kT) - 2\pi h (M-1) \sum_{k=n-L+1}^{n} q(t-kT), \end{aligned} \tag{9}$$

and after a change of variables $m = n-k$ we get

$$\begin{aligned} \phi(t;\mathbf{a}) &= 2\pi h \sum_{k=0}^{n-L} u_k + 4\pi h \sum_{m=0}^{L-1} u_{n-m} q(t-nT+mT) + \\ &\quad - 2\pi h (M-1) \sum_{m=0}^{L-1} q(t-nT+mT) - \pi h (n-L+1)(M-1). \end{aligned} \tag{10}$$

Observe that the first two terms (denoted as A in the sequel) depend on the input sequence $(u_k)_{k=0}^{n}$, while the last two terms (denoted as B in the sequel) are data-independent. We can rewrite the last two terms as follows.

$$\begin{aligned}
B &\stackrel{\text{def}}{=} -2\pi h(M-1)\sum_{m=0}^{L-1} q(t-nT+mT) - \pi h(n-L+1)(M-1) &(11)\\
&= -2\pi h(M-1)\sum_{m=0}^{L-1} q(t-nT+mT) - \pi h(n-L+1)(M-1)+\\
&\quad + \pi h(M-1)\frac{t}{T} - \pi h(M-1)\frac{t}{T} &(12)\\
&= -2\pi h(M-1)\sum_{m=0}^{L-1} q(t-nT+mT) - \pi hn(M-1) + \pi h(L-1)(M-1)+\\
&\quad + \pi h(M-1)\frac{t}{T} - \pi h(M-1)\frac{t}{T} &(13)\\
&= -2\pi h(M-1)\sum_{m=0}^{L-1} q(t-nT+mT) + \pi h(M-1)\frac{t-nT}{T}+\\
&\quad + \pi h(L-1)(M-1) - \pi h(M-1)\frac{t}{T}. &(14)
\end{aligned}$$

Collecting all these results we have

$$\begin{aligned}
s(t) &= \sqrt{2E/T}\exp(j[A+B+\phi_0]) &(15)\\
&= \sqrt{2E/T}\exp(j[2\pi f_0 t + \psi(t-nT;\mathbf{u}) + \phi_0]), &(16)
\end{aligned}$$

where $f_0 = -h(M-1)/(2T)$, and

$$\begin{aligned}
\psi(\tau;\mathbf{u}) &= 2\pi h\sum_{k=0}^{n-L} u_k + 4\pi h\sum_{m=0}^{L-1} u_{n-m} q(\tau+mT)+\\
&\quad + \pi h(M-1)\frac{\tau}{T} - 2\pi h(M-1)\sum_{m=0}^{L-1} q(\tau+mT) + \pi h(L-1)(M-1) &(17)\\
&= 2\pi h\sum_{k=0}^{n-L} u_k + 4\pi h\sum_{m=0}^{L-1} u_{n-m} q(\tau+mT) + w(\tau), &(18)
\end{aligned}$$

with

$$w(\tau) = \pi h(M-1)\frac{\tau}{T} - 2\pi h(M-1)\sum_{m=0}^{L-1} q(\tau+mT) + \pi h(L-1)(M-1), \qquad (19)$$

where $\tau \in [0,1)$.

Note that although the above expression for $\psi(\tau;\mathbf{u})$ (and thus for $s(t)$) involves all present and past symbols $(u_k)_{k=0}^{n}$, it can be shown that this dependence can be summarized into a finite state. In particular, the second term of $\psi(\tau;\mathbf{u})$ depends only on $(u_k)_{k=n-L+1}^{n}$. Also, since the phase is only important modulo-2π, the first term can be considered as

$$\begin{aligned} \bmod_{2\pi}(2\pi h \sum_{k=0}^{n-L} u_k) &= \bmod_{2\pi}(2\pi \frac{K}{P} \sum_{k=0}^{n-L} u_k) && (20) \\ &= \bmod_{2\pi}(2\pi \frac{K}{P} \bmod_P(\sum_{k=0}^{n-L} u_k)). && (21) \end{aligned}$$

Thus, setting $v_n \stackrel{\text{def}}{=} \bmod_P(\sum_{k=0}^{n-L} u_k)$, we get

$$\begin{aligned} \psi(\tau; \mathbf{u}) &= 2\pi h v_n + 4\pi h \sum_{m=0}^{L-1} u_{n-m} q(\tau + mT) + w(\tau) && (22) \\ &= \psi(\tau; (u_k)_{k=n-L+1}^{n}, v_n). && (23) \end{aligned}$$

If we define[1] $x_n \stackrel{\text{def}}{=} (u_n, u_{n-1}, \ldots, u_{n-L+1}, v_n) \in [M]^L \times [P]$, then, this decomposition allows us to rewrite the CPM signal as

$$s(t) = \sqrt{2E/T} \exp(j[2\pi f_0 t + \phi_0]) \sum_{n=0}^{\infty} s_{x_n}(t - nT), \tag{24}$$

where the signals $s_x(t)$ are defined as

$$s_x(\tau) = \exp(j[\psi(\tau; (u_k)_{k=n-L+1}^{n}, v_n)]), \qquad \forall\, x \in [M]^L \times [P] \tag{25}$$

If we collect all signals $s_x(\tau)$ in the alphabet $\mathcal{S} \stackrel{\text{def}}{=} \{s_x(\tau) | x \in [M]^L \times [P]\}$, we end up with the following transmission scheme:

At each symbol period n, an information symbol enters a FSM and generates an output x_n. This output is used to select the signal $s_{x_n}(\cdot) \in \mathcal{S}$. This signal is shifted by nT and transmitted in the time interval $[nT, nT + T)$.

Regarding the FSM, it is defined as follows: The state at time n is $\sigma_n \stackrel{\text{def}}{=} (u_{n-1}, \ldots, u_{n-L+1}, v_n) \in [M]^{L-1} \times [P]$. Upon reception of the current input u_n, the output $x_n = (u_n, \sigma_n)$ is produced and the FSM moves to the next state $\sigma_{n+1} \stackrel{\text{def}}{=} (u_n, \ldots, u_{n-L+2}, v_{n+1})$, where $v_{n+1} = \bmod_P(v_n + u_{n-L+1})$.

Thus we decomposed the CPM modulation scheme into a serial concatenation of a FSM and a memoryless modulator.

Example

Consider the case of $h = 1/2$, $L = 1$, and $q(t) = t/(2T)$ for $t \in [0, T)$. Then the FSM specializes as follows. The state at time n is $\sigma_n = v_n \in \{0, 1\}$. Upon reception of the current input $u_n \in \{0, 1\}$, the output $x_n = (u_n, \sigma_n)$ is produced and the FSM moves to the next state $\sigma_{n+1} = v_{n+1}$, where $v_{n+1} = \bmod_2(v_n + u_n)$. The operation of the entire transmitter is summarized in the following table

[1] We use the notation $[N] \stackrel{\text{def}}{=} \{0, 1, \ldots, N-1\}$.

Current state $\sigma_n = v_n$	Current input u_n	Next state $\sigma_{n+1} = v_{n+1}$	Current output $x_n = (u_n, v_n)$	Current signal output $s_{x_n}(\tau)$
0	0	0	(0,0)	$+1$
0	1	1	(1,0)	$\exp(j\pi\tau/T)$
1	0	1	(0,1)	-1
1	1	0	(1,1)	$-\exp(j\pi\tau/T)$