Stochastic Optimal Control
Lecture 3

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Overview of course[1]

- Deterministic dynamic optimisation
- Stochastic dynamic optimisation
- **Diffusions, Jumps, and infinitesimal generators**
- Dynamic programming principle
  - Diffusions
  - Jump-diffusions
- Examples:
  - Merton portfolio problems
  - Optimal execution of blocks of shares
  - others

[1]Preliminary. Contents may change as we go along.

Poisson process

- A Poisson process is a process subject to jumps of fixed size or random size.
- λ denotes the mean arrival rate of an event, during a time interval dt.
- The probability that an event will occur is given by λdt, and that it will not occur is $1 - \lambda dt$.
- The event is a jump of size u, which can itself be a random variable. The simplest is $u = 1$ so the Poisson process is a counting process.

Formally, we let N_t be the number of events that occur by time t then $N_t, t > 0$ is called a Poisson process, and it can be shown that

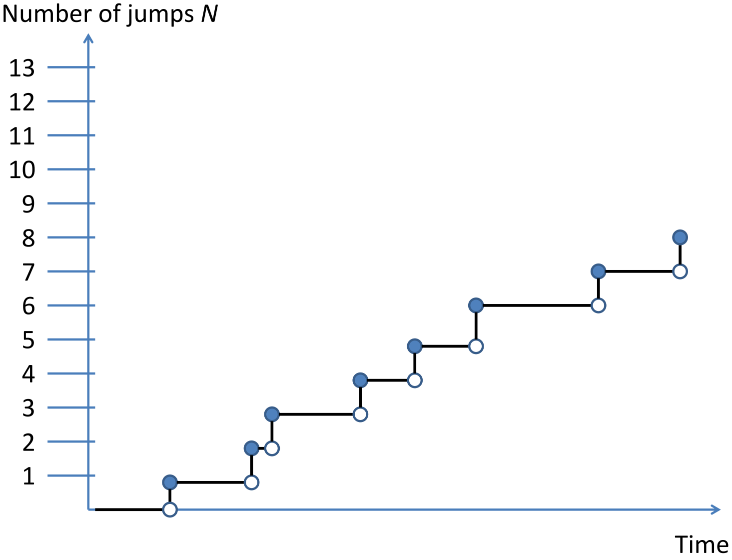
$$\mathbb{P}\{N_t = n\} = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \qquad n = 0, 1, \ldots. \tag{1}$$

1. The probability of at least one event $(ALOE)$ happening in a time period of duration Δt is

$$\mathbb{P}[ALOE] = \lambda \Delta t + o(\Delta t) \qquad \text{as } \Delta t \to 0, \lambda > 0.$$

2. The probability of two or more events happening in a period of time Δt is $o(\Delta t)$. In other words, we do not see two events happening at the same time.

Poisson Process



Interarrival times

Another way to see a Poisson counting process is by looking at the interarrival times. That is how long it takes between each Poisson event. Let T_j be the time of the jth arrival, then

$$\mathbb{P}[T_{n+1} - T_n > s \mid T_1, \cdots, T_n] = 1 - e^{-\lambda s}.$$

In other words, the interarrival times T_1, $T_2 - T_1$, $\cdots$ of a Poisson process are iid with cdf $1 - e^{-\lambda s}$. Moreover, the pdf of the interarrival times is

$$\mathbb{P}[\tau > t] = \lambda e^{-\lambda t}.$$

Exercise: Show that

$$\mathbb{P}[\tau > t + s \mid \tau > t] = \mathbb{P}[\tau > s].$$

That is, that the probability of observing an event does not depend on the past events.

A stock model: diffusion and jumps

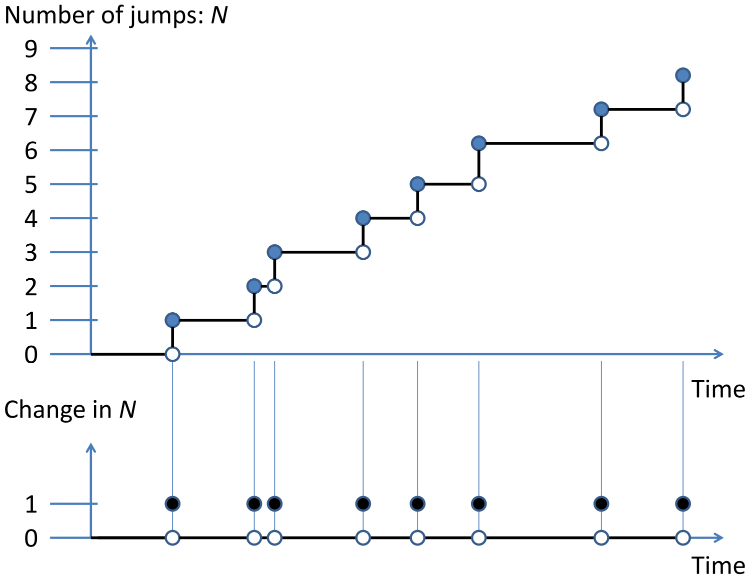
$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t - \delta dN_t, \tag{2}$$

where W_t and N_t are independent and $0 \le \delta \le 1$ is a constant.

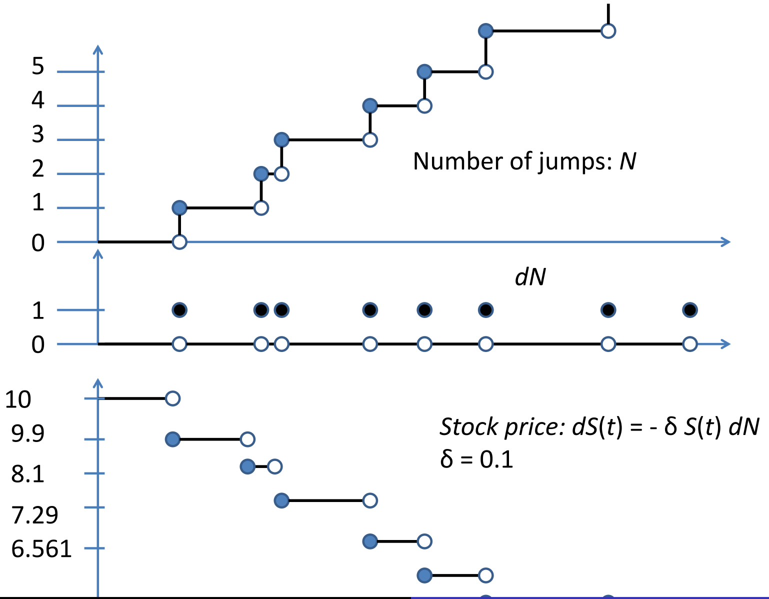
- What is the solution of this ODE?
- Assume that $\mu = 0$ and $\sigma = 0$ and solve

$$\frac{dS_t}{S_t} = -\delta dN_t\,. \tag{3}$$

Poisson Process



Stock Price with Fixed Jump Size $\delta = 0.1$



A stock model: diffusion and jumps

- Thus, the solution to

$$\frac{dS_t}{S_t} = -\delta dN_t \,,$$

is

$$S_t = S_0(1-\delta)^{N_t}.$$

- Note that all that matters is the number of jumps N_t and not when they occurred!
- And the solution to

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t - \delta dN_t, \tag{4}$$

is therefore

$$S_t = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t}(1-\delta)^{N_t}.$$

Classical Model: MJD

Merton's Jump Diffusion. Assume that the returns process follows

$$\frac{dS}{S} = (\alpha - \lambda k)dt + \sigma dW + (Y - 1)dN. \tag{5}$$

Another way to express (5) is

$$\frac{dS}{S} = (\alpha - \lambda k)dt + \sigma dW\,, \qquad \text{if no jumps occur,}$$

$$\frac{dS}{S} = (\alpha - \lambda k)dt + \sigma dW + Y - 1\,, \qquad \text{if a jump occurs.}$$

We can apply Ito's lemma to $f = \ln S$ to solve (5).

$$S_t = S_0 e^{(\alpha - \frac{1}{2}\sigma^2 - \lambda k)t + \sigma W_t} Y(n), \tag{6}$$

where $Y(n) = 1$ if $n = 0$, $Y(n) = \Pi_{j=1}^{n} Y_j$ for $n \geq 1$ where Y_j are iid and n is Poisson distributed with parameter λt.
Note that

$$d(\ln S) = \left(\alpha - \frac{1}{2}\sigma^2 - \lambda k\right) dt + \sigma dW + \ln Y dN.$$

Do we need any restriction on the support of Y?

Exercise: $\mathbb{E}[e^{-rt}S_t] =$

$$
\begin{aligned}
&= \mathbb{E}\left[S_0 e^{-rt} e^{(\alpha - \frac{1}{2}\sigma^2 - \lambda k)t + \sigma W_t} Y(n)\right] \\
&= \mathbb{E}_N\left[\mathbb{E}_W\left[S_0 e^{-rt} e^{(\alpha - \frac{1}{2}\sigma^2 - \lambda k)t + \sigma W_t} \prod_{j=1}^n Y_j | N_t = n\right]\right] \\
&= S_0 e^{-rt} e^{(\alpha - \lambda k)t} \mathbb{E}_N[\prod_{j=1}^n Y_j] \\
&= S_0 e^{-rt} e^{(\alpha - \lambda k)t} \mathbb{E}_N\left[\mathbb{E}_Y[\prod_{j=1}^n Y_j | N_t = n]\right] \\
&= S_0 e^{-rt} e^{(\alpha - \lambda k)t} \mathbb{E}_N\left[\mathbb{E}_Y[Y]^n\right] \\
&= S_0 e^{-rt} e^{(\alpha - \lambda k)t} \sum_{n=0}^{\infty} \frac{e^{-\lambda t}\left(\lambda t \mathbb{E}_Y[Y]\right)^n}{n!} \\
&= S_0 e^{-rt} e^{(\alpha - \lambda k)t} e^{-\lambda t} e^{\lambda t \mathbb{E}_Y[Y]}.
\end{aligned}
$$

Exercise

1. Let

$$dS = \mu S dt + \sigma S dW$$

where μ, σ are constants and W is a standard Brownian motion. Calculate $C^E(S, t; K, T)$ where

$$C^E(S, t; K, T) = \mathbb{E}\left[e^{-r(T-t)} \max(S_T - K, 0)\right]$$

and r is a constant (the risk-free rate).

Exercise

2. Let

$$d(\ln S) = \left(\alpha - \frac{1}{2}\sigma^2 - \lambda k\right) dt + \sigma dW + \ln Y dN.$$

where α, σ, k are constants, W is a standard Brownian motion, N is a counting process with intensity λ, and Y are iid. W, N, Y are independent.
Show that

$$\mathbb{E}\left[e^{-rT}\max(S_T - K, 0)\right] = \sum_{n=0}^{\infty}\left[\frac{e^{-\lambda T}(\lambda T)^n}{n!}\mathbb{E}_Y\left[C^E(S_0 Y^n e^{-\lambda(\mathbb{E}_Y[Y]-1)T}, 0; T, K)\right]\right].$$

$C^E(S,0;T,K) =$

$$
\begin{aligned}
&= \mathbb{E}^\star[e^{-rT}(S_T - K)^+] \\
&= \mathbb{E}^\star[e^{-rT}\max(S_0 e^{(r-\frac{1}{2}\sigma^2-\lambda(\mathbb{E}_Y[Y]-1))T+\sigma W_T} Y(n) - K, 0)] \\
&= \mathbb{E}^\star_{N,Y}\left[\mathbb{E}^\star_W[e^{-rT}(S_0 e^{(r-\frac{1}{2}\sigma^2-\lambda(\mathbb{E}_Y[Y]-1))T+\sigma W_T} Y(n) - K)^+ | N_T, Y]\right] \\
&= \mathbb{E}^\star_{N,Y}\left[\mathbb{E}^\star_W[e^{-rT}(S_0 e^{(r-\frac{1}{2}\sigma^2-\lambda(\mathbb{E}_Y[Y]-1))T+\sigma W_T} Y(n) - K)^+ | N_T, Y]\right] \\
&= \mathbb{E}^\star_{N,Y}\left[\mathbb{E}^\star_W[e^{-rT}(S_0 Y(n) e^{-\lambda(\mathbb{E}_Y[Y]-1)T} e^{(r-\frac{1}{2}\sigma^2)T+\sigma W_T} - K)^+ | N_T, Y]\right] \\
&= \mathbb{E}^\star_{N,Y)}\left[\mathbb{E}^\star_W[e^{-rT}\max(\hat{S} e^{(r-\frac{1}{2}\sigma^2)T+\sigma W_T} - K, 0) | N_T, Y]\right] \\
&= \mathbb{E}^\star_{N,Y}\left[C^E(\hat{S}, 0; T, K)\right],
\end{aligned}
$$

where $\hat{S} = S_0 Y(n) e^{-\lambda(\mathbb{E}_Y[Y]-1)T}$.

In other words

$$C^E = \sum_{n=0}^{\infty} \left[\frac{e^{-\lambda T}(\lambda T)^n}{n!} \mathbb{E}_Y \left[C^E(S_0 Y^n e^{-\lambda(\mathbb{E}_Y[Y]-1)T}, 0; T, K) \right] \right].$$

We can verify

$$\begin{aligned} \mathbb{E}^{\star}_{N,Y}\left[C^E(\hat{S}, 0; T, K) \right] &\geq C^E(\mathbb{E}^{\star}_{N,Y}[\hat{S}], 0; T, K) \\ &= C^E(S_0, 0; T, K). \end{aligned}$$

Note that we would get strict inequality if $\mathbb{P}(Y_k = 1) \neq 1$.

Ito's lemma with jumps

Consider a process X_t for $t > t_0$ of the form

$$X_t = X_0 + \int_{t_0}^{t} b(u, X_{u-})du + \int_{t_0}^{t} \sigma(u, X_{u-})dW_u + \sum_{n}^{N(t)} \Delta X_n \,, \qquad (7)$$

where $\Delta X_n = X_{\tau_n} - X_{\tau_n -}$ and τ_n denotes the jump times of the Poisson process. Here (I am being very loose with notation and conditions) the minus sign is there to denote that the variable is right before a jump occurs (if it occurs of course).
Now we want a more formal statement of Ito's lemma with Poisson jumps. Thus, assume that $f(t, X)$ is $C^{1,2}$. Then df is given by

$$\begin{aligned} df &= \left(f_t(t, X_{t-}) + f_X(t, X_{t-}) b(t, X_{t-}) + \frac{1}{2} f_{XX}(t, X_{t-}) \sigma^2(t, X_{t-}) \right) dt \\ &\quad + f_X(t, X_{t-}) \sigma(t, X_{t-}) dW_t \\ &\quad + \left(f(t, X_{t-} + \Delta X_t) - f(t, X_{t-}) \right) dN_t \end{aligned}$$

and recall that $\Delta X_t = X_t - X_{t-}$.

Exercises

The SDE in Merton's model can be written as

$$dS = \tilde{\mu} S dt + \sigma S dW + S(J-1)dq,$$

where dW is the increment of Brownian motion, q is a Poisson process with intensity λ, J is a random variable (W, q, J are independent) and $\tilde{\mu} = \mu - \lambda k$ where μ, and k are constants. Moreover, $dq = 0$ with probability $1 - \lambda dt$ and $dq = 1$ with probability λdt.

- Defining the return as $x = \frac{dS}{S}$ show that

$$\mathbb{E}[x] = \mu dt \quad \text{if and only if} \quad k = \mathbb{E}[J-1].$$

- By integrating the SDE between 0 and t and using Itô's Lemma prove that

$$S_t = S_0 e^{\hat{\mu} t + \sigma W_t} J(n),$$

where $\hat{\mu} = \mu - \lambda k - \frac{\sigma^2}{2}$ and

$$J(n) = \begin{cases} 1 & \text{if } n = 1\,, \\ \prod^{n} J_i & \text{if } n \geq 1\,, \end{cases}$$

- Choosing a portfolio in the usual way, $\Pi(S,t) = V(S,t) - \Delta S$, write the process for $d\Pi$ where V is the value of an option written on the stock S and Δ is the amount of S in the portfolio at time t. Comment on the underlying assumptions taken when choosing a Δ-hedging strategy. In what other way could we reduce the risk in the portfolio?
- In order to obtain a PDE for this problem you must assume that $\mathbb{E}[d\Pi] = r\Pi dt$. Under which specific assumptions (relevant to this problem) might one assume this?
- Based on the previous assumption and on a Δ-hedging strategy, derive the pricing PDE for Merton's problem

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r - \lambda k) S \frac{\partial V}{\partial S} - rV + \lambda \mathbb{E}[V(SJ,t) - V(S,t)] = 0.$$