Mathematics and Statistics of Algorithmic Trading[1]

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[1] Notes based on textbook "Algorithmic and High-Frequency Trading" with Sebastian Jaimungal and Jose Penalva.

Market Making

- $S_t = S_0 + \sigma\, W_t$, $\sigma > 0$ and $W = (W_t)_{0\le t\le T}$ is a standard Brownian motion,
- $\delta^{\pm}$ depth at which the agent posts LOs. Sell LOs are posted at a price of $S_t + \delta_t^{+}$ and buy LOs at $S_t - \delta_t^{-}$,
- $M^{\pm}$ counting processes corresponding to the arrival of other participants' buy $(+)$ and sell $(-)$ market orders (MOs) which arrive at Poisson times with intensities $\lambda^{\pm}$,
- $N^{\delta,\pm}$ denote the counting processes for the agent's filled sell $(+)$ and buy $(-)$ LOs,
- Conditional on an MO arrival, the LO is filled with probability $e^{-\kappa^{\pm}\,\delta_t^{\pm}}$ with $\kappa^{\pm} \ge 0$,
- X^{δ} denotes the MM's cash process

$$dX_t^{\delta} = (S_{t-} + \delta_t^{+})\, dN_t^{\delta,+} - (S_{t-} - \delta_t^{-})\, dN_t^{\delta,-}\,. \tag{1}$$

- Q^{δ} denotes the agent's inventory process and satisfies the SDE

$$Q_t^{\delta} = N_t^{\delta,-} - N_t^{\delta,+}\,. \tag{2}$$

Market Maker's Control Problem

The MM's performance criteria is

$$H^{\delta}(t,x,S,q) = \mathbb{E}_{t,x,q,S}\left[X_T + Q_T(S_T - \alpha\, Q_T) - \phi \int_t^T (Q_u)^2\, du\right]$$

where $\alpha \geq 0$ represents the fees for taking liquidity (i.e. using an MO) as well as the impact of the MO walking the LOB, and $\phi \geq 0$ is the running inventory penalty parameter. The MM's value function is

$$H(t,x,S,q) = \sup_{\delta^{\pm}\in\mathcal{A}} H^{\delta}(t,x,S,q)\,, \tag{3}$$

and the MM caps her inventory so that it is bounded above by $\overline{q} > 0$ and below by $\underline{q} < 0$.

DPE

A DPP holds and the value function satisfies the DPE

$$
\begin{aligned}
0 = & \, \partial_t H + \tfrac{1}{2}\sigma^2 \partial_{SS} H - \phi q^2 \\
& + \lambda^+ \sup_{\delta^+} \left\{ e^{-\kappa^+ \delta^+} \left(H(t, x + (S + \delta^+), q - 1, S) - H \right) \right\} \mathbb{1}_{q > \underline{q}} \\
& + \lambda^- \sup_{\delta^-} \left\{ e^{-\kappa^- \delta^-} \left(H\left(t, x - (S - \delta^-), q + 1, S\right) - H \right) \right\} \mathbb{1}_{q < \overline{q}} \,,
\end{aligned}
\tag{4}
$$

where $\mathbb{1}$ is the indicator function, with terminal condition

$$
H(T, x, S, q) = x + q(S - \alpha q) \,. \tag{5}
$$

Recall that inventory is bounded, thus when $q_t = \overline{q}$ ($\underline{q}$) the optimal strategy is to post one-sided LOs which are obtained by solving (4) with the term proportional to λ^- (λ^+) absent as stated by the indicator function $\mathbb{1}$ in the DPE. Alternatively, one can view these boundary cases as imposing $\delta^- = +\infty$ and $\delta^+ = +\infty$ when $q = \overline{q}$ and $\underline{q}$ respectively.

Solving HJB

Make an ansatz for H. In particular, write

$$H(t, x, q, S) = x + q\,S + h(t, q)\,. \tag{6}$$

The first term is the accumulated cash, the second term is the book value of the inventory marked-to-market, and the last term is the added value from following an optimal market making strategy up to the terminal date.
Thus,

$$\begin{aligned}\phi\, q^2 = \partial_t h(t, q) &+ \lambda^+ \sup_{\delta^+} \left\{ e^{-\kappa^+ \delta^+} \left(\delta^+ + h(t, q-1) - h(t, q) \right) \right\} \mathbb{1}_{q > \underline{q}} \\ &+ \lambda^- \sup_{\delta^-} \left\{ e^{-\kappa^- \delta^-} \left(\delta^- + h(t, q+1) - h(t, q) \right) \right\} \mathbb{1}_{q < \overline{q}}\,,\end{aligned} \tag{7}$$

with terminal condition $h(T, q) = -\alpha q^2$.

Optimal Controls

Then the optimal depths in feedback form are given by

$$\delta^{+,*}(t,q) = \frac{1}{\kappa^+} - h(t,q-1) + h(t,q)\,, \quad q \neq \underline{q}\,, \tag{8a}$$

$$\delta^{-,*}(t,q) = \frac{1}{\kappa^-} - h(t,q+1) + h(t,q)\,, \quad q \neq \overline{q}\,, \tag{8b}$$

and the boundary cases are $\delta^{+,*}(t,q) = +\infty$ and $\delta^{-,*}(t,q) = +\infty$ when $q = \overline{q}$ and $\underline{q}$ respectively.

Substituting the optimal controls into the DPE we obtain

$$\begin{aligned} \phi\, q^2 = \partial_t h(t,q) &+ \tfrac{\lambda^+}{\kappa^+} e^{-1} e^{-\kappa^+(-h(t,q-1)+h(t,q)}\, \mathbb{1}_{q>\underline{q}} \\ &+ \tfrac{\lambda^-}{\kappa^-} e^{-1} e^{-\kappa^-(-h(t,q+1)+h(t,q))}\, \mathbb{1}_{q<\overline{q}}\,. \end{aligned} \tag{9}$$

Symmetric fill probability

It is possible to find an analytical solution to the DPE if the fill probabilities of LOs is the same on both sides of the LOB. In this case if $\kappa = \kappa^+ = \kappa^-$ then write

$$h(t, q) = \frac{1}{\kappa} \log \omega(t, q) \,,$$

and stack $\omega(t, q)$ into a vector

$$\boldsymbol{\omega}(t, q) = \left[\omega(t, \overline{q}), \omega(t, \overline{q} - 1), \ldots, \omega(t, \underline{q})\right]' \,.$$

Now, let $\mathbf{A}$ denote the $(\overline{q} - \underline{q} + 1)$-square matrix whose rows are labeled from $\overline{q}$ to $\underline{q}$ and whose entries are given by

$$\mathbf{A}_{i,q} = \begin{cases} -\phi\kappa\, q^2 \,, & i = q \,, \\ \lambda^+ \, e^{-1} \,, & i = q - 1 \,, \\ \lambda^- \, e^{-1} \,, & i = q + 1 \,, \\ 0 \,, & \text{otherwise.} \end{cases} \tag{10}$$

with terminal and boundary conditions $\omega(T, q) = e^{-\alpha\kappa q^2}$.

Then,

$$\boxed{\boldsymbol{\omega}(t) = e^{\mathbf{A}(T-t)}\mathbf{z}\,,} \tag{11}$$

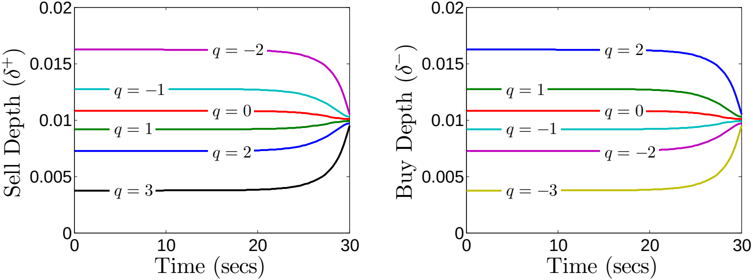
where $\mathbf{z}$ is a $(\overline{q} - \underline{q} + 1)$-dim vector where each component is $z_j = e^{-\alpha\kappa j^2}$, $j = \overline{q}, \dots, \underline{q}$. Inserting the controls (8) into the DPE equation (9) and

writing $h(t,q) = \frac{1}{\kappa}\log\omega(t,q)$, after some straightforward computations, one finds that $\omega(t,q)$ satisfy the coupled system of equations

$$\partial_t\boldsymbol{\omega}(t) + \mathbf{A}\boldsymbol{\omega}(t) = \mathbf{0}. \tag{12}$$

Optimal Postings

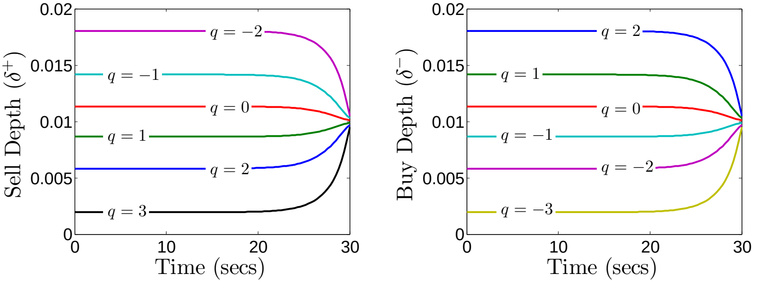
Optimal postings $\phi = 0.001$



(a) $\phi = 0.001$

Figure: The optimal depths as a function of time for various inventory levels and $T = 30$. The remaining model parameters are: $\lambda^{\pm} = 1$, $\kappa^{\pm} = 100$, $\overline{q} = -\underline{q} = 3$, $\phi = 0.001$ and $\phi = 0.02$, $\alpha = 0.0001$, $\sigma = 0.01$, $S_0 = 100$.

Optimal postings $\phi = 0.02$



(a) $\phi = 0.02$

Figure: The optimal depths as a function of time for various inventory levels and $T = 30$. The remaining model parameters are: $\lambda^{\pm} = 1$, $\kappa^{\pm} = 100$, $\overline{q} = -\underline{q} = 3$, $\phi = 0.001$ and $\phi = 0.02$, $\alpha = 0.0001$, $\sigma = 0.01$, $S_0 = 100$.

Mean reversion in inventory

Given the pair of optimal strategies $\delta^+(t,q), \delta^-(t,q)$, the expected drift in inventories q_t is given by

$$\boxed{\begin{aligned}\mu(t,q) &\triangleq \lim_{s \downarrow t} \frac{1}{s-t} \mathbb{E}\left[Q_s - Q_t \mid Q_{t^-} = q\right] \\ &= \lambda^- e^{-\kappa^- \delta^{-,*}(t,q)} - \lambda^+ e^{-\kappa^+ \delta^{+,*}(t,q)} \,.\end{aligned}} \tag{13}$$

Note that the drift $\mu(t,q)$ depends on time. For instance it is clear that for the same level of inventory the speed will be different depending on how near of far is the strategy from the terminal date because at time T the strategy tries to unwind all outstanding inventory.

Inventory

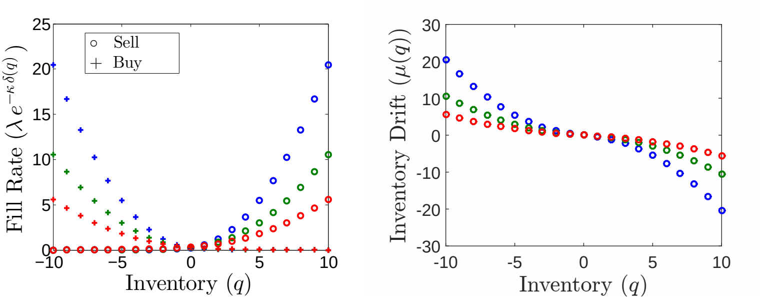


Figure: Long-term inventory level. Model parameters are: $\lambda^{\pm} = 1$, $\kappa^{\pm} = 100$, $\overline{q} = -\underline{q} = 10$, $\alpha = 0.0001$, $\sigma = 0.01$, $S_0 = 100$, and $\phi = \{0.2, 0.1, 0.05\}$.

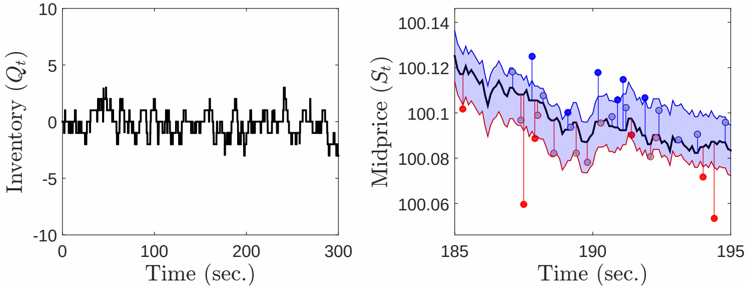


Figure: Inventory and midprice path. Model parameters are: $\lambda^{\pm} = 1$, $\kappa^{\pm} = 100$, $\overline{q} = -\underline{q} = 10$, $\phi = 0.02$, $\alpha = 0.0001$, $\sigma = 0.01$, $S_0 = 100$.

Profit and Loss

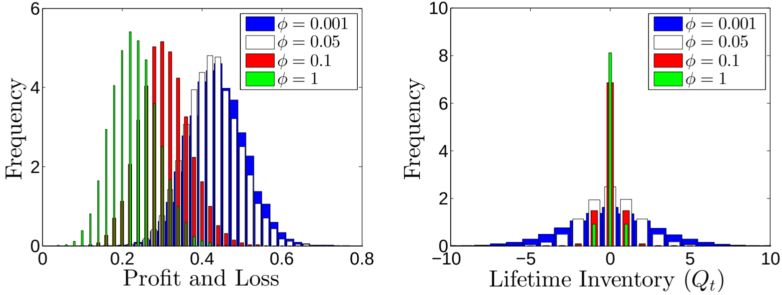


Figure: P&L and Life Inventory of the optimal strategy for 10,000 simulations. The remaining model parameters are: $\lambda^{\pm} = 1$, $\kappa^{\pm} = 100$, $\overline{q} = -\underline{q} = 10$, $\alpha = 0.0001$, $\sigma = 0.01$, and $S_0 = 100$.

Market Making at-the-touch

Market Making at-the-touch

Throughout we assume that the spread is constant and equal to Δ.
Next, let $\ell_t^{\pm} \in \{0,1\}$ denote whether the agent is posted on the sell side $(+)$ or buy side $(-)$ of the LOB. In this way, the agent may be posted on both sides of the book, only the sell side, only the buy side, or not posted at all. Her performance criteria is

$$H^{\boldsymbol{\ell}}(t,x,S,q) = \mathbb{E}_{t,x,S,q}\left[X_T^{\boldsymbol{\ell}} + Q_T^{\boldsymbol{\ell}}\,\left(S_T - \left(\tfrac{\Delta}{2} + \varphi\,Q_T^{\boldsymbol{\ell}}\right)\right) - \phi\int_t^T \left(Q_u^{\boldsymbol{\ell}}\right)^2 du\right],$$

where her cash process $X_t^{\boldsymbol{\ell}}$ now satisfies the SDE

$$dX_t^{\boldsymbol{\ell}} = \left(S_t + \tfrac{\Delta}{2}\right)\,dN_t^{+,\boldsymbol{\ell}} - \left(S_t - \tfrac{\Delta}{2}\right)\,dN_t^{-,\boldsymbol{\ell}},$$

where $N_t^{\pm,\boldsymbol{\ell}}$ denote the counting process for filled LOs. We also further assume that, if she is posted in the LOB, when a matching MO arrives her LO is filled with probability one. In this case, $N_t^{\pm,\boldsymbol{\ell}}$ are controlled doubly stochastic Poisson processes with intensity $\ell_t^{\pm}\,\lambda^{\pm}$.
The agent is not posted on the buy (sell) side if her inventory is equal to the upper (lower) inventory constraints $\overline{q}$ ($\underline{q}$) and her value function is denoted by

$$H(t,x,S,q) = \sup_{\boldsymbol{\ell}\in\mathcal{A}} H^{\boldsymbol{\ell}}(t,x,S,q)\,.$$

The Resulting DPE

Applying the DPP, we find the agent's value function H should satisfy the DPE

$$
\begin{aligned}
0 = &\left(\partial_t + \tfrac{1}{2}\sigma^2\partial_{SS}\right)H - \phi\, q^2 \\
&+ \lambda^+ \max_{\ell^+\in\{0,1\}} \left\{\left(H\left(t, x + \left(S + \tfrac{\Delta}{2}\right)\, \ell^+, S, q - \ell^+\right) - H\right)\right\}\, \mathbb{1}_{q>\underline{q}} \\
&+ \lambda^- \max_{\ell^-\in\{0,1\}} \left\{\left(H\left(t, x - \left(S - \tfrac{\Delta}{2}\right)\, \ell^-, S, q + \ell^-\right) - H\right)\right\}\, \mathbb{1}_{q<\overline{q}}\,,
\end{aligned}
$$

subject to the terminal condition

$$H(T, x, S, q) = x + q\,\left(S - \left(\tfrac{\Delta}{2} + \varphi\, q\right)\right)\,.$$

Ansatz:

$$H(t, x, S, q) = x + q\,S + h(t, q)\,,$$

and on substituting this ansatz into the above DPE we find that h satisfies

$$\begin{aligned} 0 = {} & \partial_t h - \phi\, q^2 \\ & + \lambda^+ \max_{\ell^+ \in \{0,1\}} \left\{ \left(\ell^+ \tfrac{\Delta}{2} + \left[h(t, q - \ell^+) - h(t, q) \right] \right) \right\} \mathbb{1}_{q > \underline{q}} \\ & + \lambda^- \max_{\ell^- \in \{0,1\}} \left\{ \left(\ell^- \tfrac{\Delta}{2} + \left[h(t, q + \ell^-) - h(t, q) \right] \right) \right\} \mathbb{1}_{q < \overline{q}}\,, \end{aligned}$$

subject to the terminal condition

$$h(T, q) = -q\, \left(\tfrac{\Delta}{2} + \varphi\, q \right)\,.$$

When $\ell = 0$ both terms that are being maximised are zero, hence,

$$\boxed{\begin{aligned} \ell^{+,*}(t, q) &= \mathbb{1}_{\left\{ \frac{\Delta}{2} + [h(t,q-1) - h(t,q)] > 0 \right\} \cap \{q > \underline{q}\}}\,, \\ \ell^{-,*}(t, q) &= \mathbb{1}_{\left\{ \frac{\Delta}{2} + [h(t,q+1) - h(t,q)] > 0 \right\} \cap \{q < \overline{q}\}}\,. \end{aligned}} \tag{14}$$

The interpretation of this result is that the agent posts an LO on the appropriate side of the LOB by ensuring that she only posts if the arrival of an MO, which hit/lifts her LO, produces a change in her value function larger than $-\frac{\Delta}{2}$.

Market Making with No Terminal Penalty

Solving HJB with $\alpha = 0$

Assume no penalties for liquidating inventories at time T. Thus the ansatz is

$$H(t, x, q, S) = x + qS + g(t)\,. \tag{15}$$

Note that the function $g(t)$ does not depend on q. In the problem above the Ansatz contained $h(t, q)$ because the optimal strategy had to manage inventory risk which is something that is not a problem when $\alpha = 0$ here. Thus,

$$0 = g_t(t) + \lambda^+ \sup_{\delta^+} \left\{ e^{-\kappa^+ \delta^+} \, \delta^+ \right\} + \lambda^- \sup_{\delta^-} \left\{ e^{-\kappa^- \delta^-} \, \delta^- \right\}\,, \tag{16}$$

and the optimal postings are:

$$\delta^{*,+} = \frac{1}{\kappa^+} \tag{17}$$

and

$$\delta^{*,-} = \frac{1}{\kappa^-}\,. \tag{18}$$

Solving HJB with $\alpha = 0$

Alternatively note that

- For a risk-neutral HFT, who does not penalise inventories, seeks to maximise the probability of being filled at every instant in time.
- The MM chooses $\delta^{\pm}$ to maximise the expected depth conditional on a market order hitting or lifting the appropriate side of the book: maximises $\delta^{\pm} e^{-\kappa^{\pm}\delta^{\pm}}$. The FOC

$$e^{-\kappa^{\pm}\delta^{\pm}} - \kappa^{\pm}\delta^{\pm} e^{-\kappa^{\pm}\delta^{\pm}} = 0\,. \tag{19}$$

Thus, we see that the optimal half spreads are as in (17) and (18).