Mathematics and Statistics of Algorithmic Trading[1]

University of Oxford

Álvaro Cartea

alvaro.cartea@maths.ox.ac.uk
http://www.cartea.net

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[1] Notes based on textbook "Algorithmic and High-Frequency Trading" with Sebastian Jaimungal and Jose Penalva.

Optimal Execution

- How do we trade in or out of a large position?
- Optimal Execution with Temporary Impact and Inventory Control
  - Liquidation
  - Acquisition
- Optimal Liquidation with Temporary and Permanent Impact and Inventory Control

Model Setup

- $\nu = (\nu_t)_{\{0 \le t \le T\}}$ is the trading rate, the speed at which the agent is liquidating or acquiring shares. It is also the variable the agent controls in the optimisation problem,
- $Q^\nu = (Q_t^\nu)_{\{0 \le t \le T\}}$ is the agent's inventory, which is clearly affected by how fast she trades,
- $S^\nu = (S_t^\nu)_{\{0 \le t \le T\}}$ is the midprice process, and is also affected in principle by the speed of her trading,
- $\hat{S}^\nu = (\hat{S}_t^\nu)_{\{0 \le t \le T\}}$ corresponds to the price process at which the agent can sell or purchase the asset, i.e. the execution price, by walking the LOB , and
- $X^\nu = (X_t^\nu)_{\{0 \le t \le T\}}$ is the agent's cash process resulting from the agent's execution strategy.

Model Setup

$$dQ_t^\nu = \pm\nu_t\,dt\,, \qquad\qquad Q_0^\nu = q\,, \tag{1a}$$

while the midprice is assumed to satisfy the following SDE,

$$dS_t^\nu = \pm g(\nu_t)\,dt + \sigma\,dW_t\,, \qquad\qquad S_0^\nu = S\,, \tag{1b}$$

where

- $W = (W_t)_{\{0\le t\le T\}}$ is a standard Brownian motion,
- $g : \mathbb{R}_+ \to \mathbb{R}_+$ denotes the permanent impact that the agent's trading action incurs on the midprice.

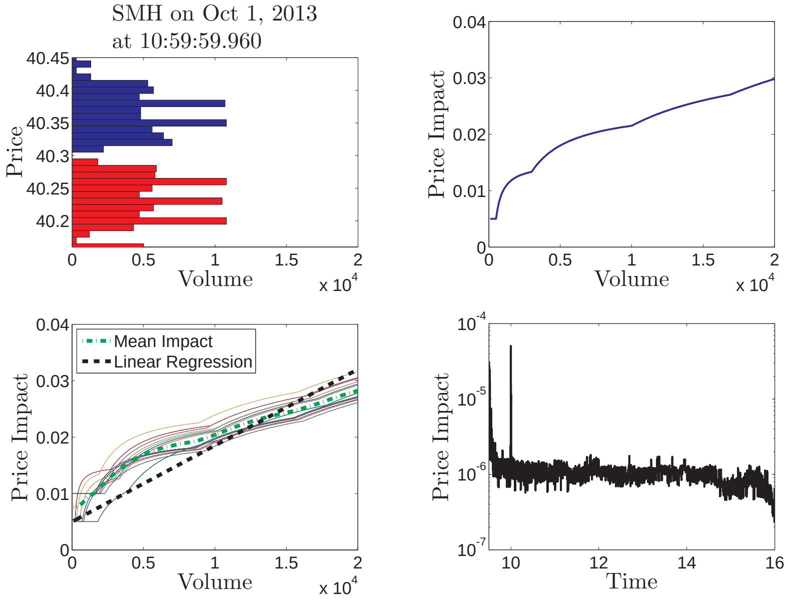
The execution price satisfies the SDE

$$\hat{S}_t^\nu = S_t^\nu \pm \left(\tfrac{1}{2}\Delta + f(\nu_t)\right)\,, \qquad\qquad \hat{S}_0^\nu = \hat{S}\,, \tag{1c}$$

where

- $f : \mathbb{R}_+ \to \mathbb{R}_+$ denotes the temporary impact that the agent's trading action has on the price they can execute the trade at.
- $\Delta \ge 0$ is the bid-ask spread, assumed here to be a constant.

SMH on Oct 1, 2013



Optimal Liquidation with Temporary Impact

Liquidation

$$H(t,S,q) = \sup_{\nu\in\mathcal{A}} \mathbb{E}_{t,S,q}\left[\int_t^T (S_u - k\,\nu_u)\,\nu_u\,du\right],$$

where $\mathbb{E}_{t,S,q}[\cdot]$ denotes expectation conditional on $S_t = S$ and $Q_t = q$, and $\mathcal{A}$ is the set of admissible strategies.

Assume $f(\nu) = k\,\nu$, and $g(\nu) = 0$, i.e. only temporary price impact (walking the LOB).

Use the DPP which suggests that H satisfies the DPE

$$\partial_t H + \tfrac{1}{2}\sigma^2\,\partial_{SS} H + \sup_{\nu}\left\{(S - k\,\nu)\,\nu - \nu\,\partial_q H\right\} = 0\,. \tag{2}$$

The agent must liquidate all the inventory by time T, hence impose

$$H(T,S,q) \xrightarrow{t\to T} -\infty, \quad \text{for} \quad q>0\,, \qquad \text{and} \qquad H(T,S,0) \xrightarrow{t\to T} 0\,.$$

Liquidation

The FOC applied to DPE (12):

$$\nu^* = \tfrac{1}{2k}\left(S - \partial_q H\right) , \tag{3}$$

which is the optimal trading speed in feedback control form. Now

$$\partial_t H + \tfrac{1}{2}\,\sigma^2\,\partial_{SS} H + \tfrac{1}{4\,k}\left(S - \partial_q H\right)^2 = 0 \tag{4}$$

for the value function. Propose ansatz (trial solution)

$$H(t,q) = q\,S + h(t,q)\,, \tag{5}$$

where $h(t,q)$ is still to be determined, though we know that it must blow up as $t \to T$.

Thus

$$\partial_t h + \tfrac{1}{4\,k}\left(\partial_q h\right)^2 = 0\,.$$

Interestingly, the volatility of the asset's midprice drops out of the problem.

Again, make ansatz $h(t,q) = q^2\, h_2(t)$ allows us to factor out q and obtain

$$\partial_t h_2 + \tfrac{1}{k}\, h_2^2 = 0\,, \tag{6}$$

which we solve by integrating between t and T to obtain

$$h_2(t) = \left(\frac{1}{h_2(T)} - \frac{1}{k}(T-t)\right)^{-1}.$$

The optimal strategy must ensure that the terminal inventory is zero and this is equivalent to requiring $h_2(T) \to -\infty$ as $t \to T$. In this way the value function heavily penalises non-zero final inventory. An alternative way to obtain this condition is to calculate the inventory path along the optimal strategy and impose that the terminal inventory be zero. To see this, use the ansatz (5) to reduce (3) to

$$\nu_t^* \quad = \quad -\tfrac{1}{k}\, h_2(t)\, Q_t^{\nu^*}\,, \tag{7}$$

then integrate $dQ_t^{\nu^*} = -\nu_t^*\,dt$ over $[0, t]$ to obtain the inventory profile along the optimal strategy:

$$\int_0^t \frac{dQ_t^{\nu^*}}{Q_t^{\nu^*}} = \int_0^t \frac{h_2(s)}{k} ds \quad \Rightarrow \quad Q_t^{\nu^*} = \frac{(T-t) - k/h_2(T)}{T - k/h_2(T)}\,\mathfrak{N}\,.$$

To satisfy the terminal inventory condition $Q_T^{\nu^*} = 0$, and also ensure that the correction $h(t, q)$ is negative, we must have

$$h_2(t) \to -\infty \quad \text{as} \quad t \to T\,. \tag{8}$$

Hence

$$h_2(t) = -k\,(T-t)^{-1}\,,$$

so that

$$Q_t^{\nu^*} = \left(1 - \frac{t}{T}\right)\,\mathfrak{N}\,, \tag{9}$$

and

$$\nu_t^* = \frac{\mathfrak{N}}{T}\,. \tag{10}$$

The result is TWAP.

Optimal Acquisition with Temporary Impact

Acquisition

Objective is to acquire $\mathfrak{N}$ shares by time T.
Thus, the agent's expected costs from strategy ν_t is

$$EC^{\nu} = \mathbb{E}\Big[\underbrace{\int_t^T \hat{S}_t\,\nu_u\,du}_{\text{Terminal Cash}} + \underbrace{(\mathfrak{N} - Q_T^{\nu})\,S_T}_{\text{Terminal execution at mid}} + \underbrace{\alpha\,(\mathfrak{N} - Q_T^{\nu})^2}_{\text{Terminal Penalty}}\Big]\,. \qquad (11)$$

To simplify notation, let $Y = (Y_t)_{0 \le t \le T}$ to denote the shares remaining to be purchased:

$$Y_t^{\nu} = \mathfrak{N} - Q_t^{\nu}\,, \qquad \text{so that} \qquad dY_t^{\nu} = -\nu_t\,dt\,,$$

and write the value function as

$$H(t,S,y) = \inf_{\nu \in \mathcal{A}} \mathbb{E}_{t,S,y}\left[\int_t^T \hat{S}_u\,\nu_u\,du + Y_T^{\nu}\,S_T + \alpha\,(Y_T^{\nu})^2\right]\,.$$

Applying the DPP, H satisfies the DPE

$$0 = \partial_t H + \tfrac{1}{2}\sigma^2 \partial_{SS} H + \inf_{\nu} \left\{ (S + k\nu)\,\nu - \nu \partial_y H \right\} , \tag{12}$$

with $H(T, S, y) = y\,S + \alpha\,y^2$. Optimal speed:

$$\nu^* = \tfrac{1}{2k}\left(\partial_y H - S\right) , \tag{13}$$

and upon substitution into the DPE above, we obtain

$$\partial_t H + \tfrac{1}{2}\sigma^2 \partial_{SS} H - \tfrac{1}{4\,k}\left(\partial_y H - S\right)^2 = 0 \,.$$

Make ansatz

$$H(t, S, y) = y\,S + h_0(t) + h_1(t)\,y + h_2(t)\,y^2 , \tag{14}$$

where $h_2(t)$, $h_1(t)$, $h_0(t)$ and

$$h_2(T) = \alpha \quad \text{and} \quad h_1(T) = h_0(T) = 0 \,.$$

Moreover, upon substituting the ansatz into the above non-linear PDE we find that

$$0 = \left\{\partial_t h_2 - \tfrac{1}{k}\, h_2^2\right\} y^2 + \left\{\partial_t h_1 - \tfrac{1}{2k}\, h_2\, h_1\right\} y + \left\{\partial_t h_0 - \tfrac{1}{4k}\, h_1^2\right\} .$$

- Equation must be valid for each y, so each term in curly braces must individually vanish
- Due to $h_1(T) = 0$, we obtain $h_1(t) = 0$.
- Similarly, since $h_0(T) = 0$ and $h_1(t) = 0$ we obtain $h_0(t) = 0$.
- Finally, since $h_2(T) = \alpha$

$$h_2(t) = \left(\tfrac{1}{k}\,(T - t) + \tfrac{1}{\alpha}\right)^{-1} .$$

Thus

$$\nu_t^* = \left((T - t) + \tfrac{k}{\alpha}\right)^{-1} \; Y_t^{\nu^*} . \tag{15}$$

Inventory path

We can solve for the optimal inventory path by solving

$$dY_t^{\nu^*} = -\left((T-t) + \tfrac{k}{\alpha}\right)^{-1} Y_t^{\nu^*}\, dt$$

for $Y_t^{\nu^*}$. Recalling that $Y_t^{\nu} = \mathfrak{N} - Q_t^{\nu}$,

$$Q_t^{\nu^*} = \frac{t}{T + \frac{k}{\alpha}}\,\mathfrak{N}\,. \tag{16}$$

- For any finite $\alpha > 0$ and finite $k > 0$,
  - it is always optimal to leave some shares to be executed at the terminal date, and
  - the fraction of shares left to execute at the end decreases with the relative price impact at the terminal date, k/α.

Substitute for $Q_t^{\nu^*}$ into the expression for ν_t^*, so that

$$\nu_t^* = \frac{\mathfrak{N}}{T + \frac{k}{\alpha}}\,. \tag{17}$$

Optimal Liquidation with Temporary and Permanent Price Impact

Liquidation with Permanent Price Impact

Then, the agent's performance criterion is

$$H^\nu(t,x,S,q) = \mathbb{E}_{t,x,S,q}\Big[\underbrace{X_T^\nu}_{\text{Terminal Cash}} + \underbrace{Q_T^\nu\,(S_T^\nu - \alpha\,Q_T^\nu)}_{\text{Terminal Execution}} - \underbrace{\phi\int_t^T (Q_u^\nu)^2\,du}_{\text{Inventory Penalty}} \Big] \tag{18}$$

and the value function

$$H(t,x,S,q) = \sup_{\nu\in\mathcal{A}} H^\nu(t,x,S,q)\,.$$

The DPP implies that the value function should satisfy the HJB equation

$$\begin{aligned} 0 = {} & \left(\partial_t + \tfrac{1}{2}\,\sigma^2\,\partial_{SS}\right) H - \phi\,q^2 \\ & + \sup_{\nu} \left\{ \left(\nu\,(S - f(\nu))\,\partial_x - g(\nu)\,\partial_S - \nu\,\partial_q\right) H \right\}\,, \end{aligned} \tag{19}$$

subject to the terminal condition $H(T,x,S,q) = x + S\,q - \alpha\,q^2$.

Assume that $f(\nu) = k\,\nu$ and $g(\nu) = b\,\nu$ for constants $k \geq 0$ and $b \geq 0$ and obtain

$$\nu^* = \tfrac{1}{2\,k}\,\frac{(S\,\partial_x - b\,\partial_S - \partial_q)H}{\partial_x H}\,. \tag{20}$$

Hence

$$0 = \left(\partial_t + \tfrac{1}{2}\sigma^2\,\partial_{SS}\right) H - \phi\,q^2 + \tfrac{1}{4\,k}\,\frac{\left[(S\,\partial_x - b\,\partial_S - \partial_q)H\right]^2}{\partial_x H}\,.$$

Make ansatz $H(t, x, S, q) = x + S\,q + h(t, S, q)$ with terminal condition $h(T, S, q) = -\alpha\,q^2$. Hence

$$0 = \left(\partial_t + \tfrac{1}{2}\sigma^2\,\partial_{SS}\right) h - \phi\,q^2 + \tfrac{1}{4k}\left[b\,(q + \partial_S h) + \partial_q h\right]^2\,.$$

Since the above PDE contains no explicit dependence on S and the terminal condition is independent of S, then $\partial_S h(t, S, q) = 0$, then

$$0 = \partial_t h(t, q) - \phi\, q^2 + \tfrac{1}{4k}\left[b\, q + \partial_q h(t, q)\right]^2 \ .$$

Use separation of variables $h(t, q) = h_2(t)\, q^2$ where $h_2(t)$ to write

$$0 = \partial_t h_2 - \phi + \tfrac{1}{k}\left[h_2 + \tfrac{1}{2} b\right]^2 \ , \tag{21}$$

subject to $h_2(T) = -\alpha$. This ODE is of Riccati type and can be integrated exactly. Let $h_2(t) = -\frac{1}{2} b + \chi(t)$, then

$$\frac{\partial_t \chi}{k\phi - \chi^2} = \frac{1}{k}\,,$$

s.t. $\chi(T) = \frac{1}{2} b - \alpha$. Next, integrating over $[t, T]$

$$\log \frac{\sqrt{k\,\phi} + \chi(T)}{\sqrt{k\,\phi} - \chi(T)} - \log \frac{\sqrt{k\,\phi} + \chi(t)}{\sqrt{k\,\phi} - \chi(t)} = 2\gamma\,(T - t)\,,$$

so that

$$\chi(t) = \sqrt{k\,\phi}\,\frac{1 + \zeta\, e^{2\gamma\,(T-t)}}{1 - \zeta\, e^{2\gamma(T-t)}}\,,$$

where

$$\gamma = \sqrt{\frac{\phi}{k}} \quad \text{and} \quad \zeta = \frac{\alpha - \frac{1}{2} b + \sqrt{k\,\phi}}{\alpha - \frac{1}{2} b - \sqrt{k\,\phi}}\ . \tag{22}$$

from (20), the optimal speed to trade at is

$$\nu_t^* = -\sqrt{\frac{\phi}{k}}\frac{1+\zeta\, e^{2\gamma\,(T-t)}}{1-\zeta\, e^{2\gamma(T-t)}}\, Q_t^{\nu^*} \ . \tag{23}$$

In addition

$$dQ_t^{\nu^*} = \frac{\chi(t)}{k}\, Q_t^{\nu^*}\, dt \quad \text{so that} \quad Q_t^{\nu^*} = \mathfrak{N} \exp\left\{\int_0^t \frac{\chi(s)}{k}\, ds\right\} \ .$$

To obtain the inventory process

$$\begin{aligned} \int_0^t \frac{\chi(s)}{k}\, ds &= \frac{1}{k}\int_0^t \sqrt{k\phi}\frac{1+\zeta e^{2\gamma(T-s)}}{1-\zeta e^{2\gamma(T-s)}}\, ds \\ &= \log\frac{\zeta\, e^{\gamma(T-t)} - e^{-\gamma(T-t)}}{\zeta\, e^{\gamma T} - e^{-\gamma T}} \ , \end{aligned}$$

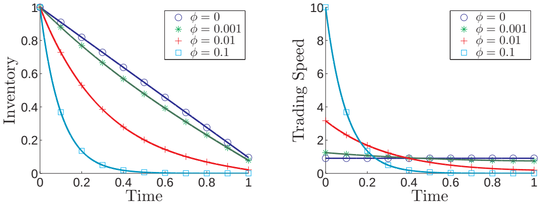
hence

$$Q_t^{\nu^*} = \frac{\zeta\, e^{\gamma(T-t)} - e^{-\gamma(T-t)}}{\zeta\, e^{\gamma T} - e^{-\gamma T}}\, \mathfrak{N} \, . \tag{24}$$

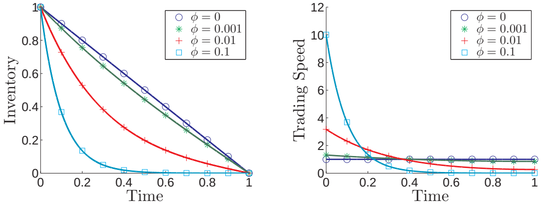
In the limit $\alpha \to +\infty$

$$Q_t^{\nu^*} \xrightarrow[\alpha\to+\infty]{} \frac{\sinh\left(\gamma(T-t)\right)}{\sinh\left(\gamma T\right)}\, \mathfrak{N} \, ,$$

which is independent of b.



(a) $\alpha = 0.01$



(b) $\alpha = +\infty$

Figure: Other model parameters are $k = 10^{-3}$, $b = 10^{-3}$.