Mathematics and Statistics of Algorithmic Trading[1]

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[1]Notes based on textbook "Algorithmic and High-Frequency Trading" with Sebastian Jaimungal and Jose Penalva.

POV and VWAP

- Executions are delegated to brokers
- How can investor's measure the performance of brokers?
- Obtaining what the market has borne is sometimes enough
- A popular benchmark is VWAP, weighted average price,

$$\text{VWAP} = \frac{\int_0^T S_u \, \mu_u \, du}{\int_0^T \mu_u \, du} \,, \tag{1}$$

where S_t is the midprice and μ_t is the volume traded by the entire market.

Targeting VWAP via POV or POCV

- Algorithms that track VWAP and those that target a percentage of the volume traded (POV) are closely interlinked because the investor also requires to minimise market impact.
- Reducing price impact forces the algorithm to slice the parent order over a time window and doing it in a volume-based fashion will help to target VWAP.
- Our formulation is to set up a performance criteria where the investor seeks to execute a large order over a trading horizon T and the speed of trading targets POV or targets a percentage of cumulative volume (POCV).

Volume patterns

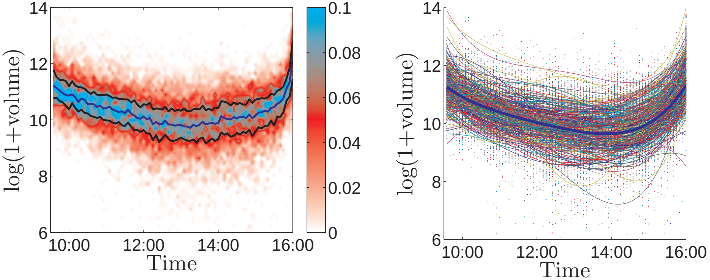

Figure: ORCL traded volume for orders sent to NASDAQ in all of 2013 using 5 minute buckets. (left) a heat-map of the data and 25%, 50% and 75% quantiles (right) functional data regression (using Legendre polynomials) on the daily curves and the estimated mean curve

Targeting VWAP via POV

The Model

- The investor searches for an optimal speed to liquidate $\mathfrak{N}$ shares over a trading horizon T.
- Inventory
$$dQ_t^\nu = -\nu_t \, dt \,, \qquad Q_0^\nu = \mathfrak{N} \,. \tag{2}$$
where ν_t is the speed of liquidation.
- The rest of the market trades at speed μ^+ for buy side and and μ^- for sell side.
- And the idea is to target, at every instant in time, a percentage $\rho \in [0, 1]$ of the overall traded volume.
- The overall traded volume, at every instant in time, is
$$\mu_t^+ + \mu_t^- + \nu_t \,.$$

The Model, price impact

- Execution price

$$\hat{S}_t^\nu = S_t^\nu - k\,\nu_t\,, \tag{3}$$

  $k > 0$ is the temporary impact parameter.

- The midprice satisfies

$$dS_t^\nu = b\left(\mu_t^+ - (\nu_t + \mu_t^-)\right)\,dt + dM_t\,, \qquad S_0^\nu = S\,, \tag{4}$$

  where $b \geq 0$ is the permanent impact parameter, and $M = \{M_t\}_{0\leq t\leq T}$ is a martingale (independent of all other processes).

- Cash

$$dX_t^\nu = \hat{S}_t^\nu\,\nu_t\,dt\,, \qquad X_0^v = X_0\,. \tag{5}$$

The Model (cont)

Performance criteria is

$$H^{\nu}(t,x,S,\boldsymbol{\mu},q) = \mathbb{E}_{t,x,S,\boldsymbol{\mu},q}\left[X^{\nu}_{\tau^{\nu}} + Q^{\nu}_{\tau^{\nu}}\,(S^{\nu}_{\tau^{\nu}} - \alpha\, Q^{\nu}_{\tau^{\nu}}) - \tilde{\varphi}\int_t^{\tau^{\nu}} (\nu_u - \chi^{\nu}_u)^2\, du\right], \tag{6}$$

where $\boldsymbol{\mu} = \{\mu^+, \mu^-\}$,

$$\chi^{\nu}_t := \tilde{\rho} \times \left(\mu^+_t + \mu^-_t + \nu_t\right)\,, \tag{7}$$

$\tau^{\nu} = T \wedge \inf\{t\,:\, Q^{\nu}_t = 0\}$, and her value function

$$H(t,x,S,\boldsymbol{\mu},q) = \sup_{\nu\in\mathcal{A}} H^{\nu}(t,x,S,\boldsymbol{\mu},q)\,, \tag{8}$$

where $\mathcal{A}$ is the set of admissible strategies consisting of $\mathcal{F}$-predictable processes such that $\int_0^T |\nu_u|\, du < +\infty$, $\mathbb{P}$-a.s.

DPE

The value function satisfies

$$0 = \left(\partial_t + \mathcal{L}^M + \mathcal{L}^{\boldsymbol{\mu}}\right) H \\ + \sup_{\nu} \left\{ (S - k\nu)\,\nu\,\partial_x H - \nu\,\partial_q H + b\,((\mu^+ - \mu^-) - \nu)\,\partial_S H - \varphi\,(\nu - \rho\,\mu)^2 \right\} \tag{9}$$

for $q > 0$, with terminal condition

$$H(t, x, S, \boldsymbol{\mu}, q) = x + q(S - \alpha q)\,, \tag{10}$$

where $\mu = \mu^+ + \mu^-$, $\mathcal{L}^{\boldsymbol{\mu}}$ represents the infinitesimal generator of $\boldsymbol{\mu}$, $\mathcal{L}^M$ the infinitesimal generator of M, and we have introduced the re-scaled parameters

$$\varphi = \tilde{\varphi}(1 - \tilde{\rho})^2 \qquad \text{and} \qquad \rho = \tilde{\rho}/(1 - \tilde{\rho})\,.$$

Solving the DPE

The DPE (9) admits the solution

$$H(t,x,S,\boldsymbol{\mu},q) = x + q\,S + h_0(t,\boldsymbol{\mu}) + h_1(t,\boldsymbol{\mu})\,q + h_2(t)\,q^2\,, \quad q>0 \quad (11)$$

where

$$h_2(t) = -\left(\frac{T-t}{k+\varphi} + \frac{1}{\alpha - \frac{1}{2}b}\right)^{-1} - \frac{1}{2}\,b\,, \tag{12a}$$

$$\begin{aligned} h_1(t,\boldsymbol{\mu}) &= \frac{\varphi\,\rho}{(T-t)+\zeta}\int_t^T \mathbb{E}_{t,\boldsymbol{\mu}}\left[\mu_s^+ + \mu_s^-\right]\,ds \\ &\quad + \frac{b}{(T-t)+\zeta}\int_t^T \left((T-s)+\zeta\right)\mathbb{E}_{t,\boldsymbol{\mu}}\left[\mu_s^+ - \mu_s^-\right]ds\,, \end{aligned} \tag{12b}$$

$$h_0(t,\boldsymbol{\mu}) = \int_t^T \mathbb{E}_{t,\boldsymbol{\mu}}\left[\frac{\left(h_1(t,\boldsymbol{\mu}_s) - 2\varphi\rho(\mu_s^+ + \mu_s^-)\right)^2}{4(k+\varphi)} - \varphi\rho^2(\mu_s^+ + \mu_s^-)^2\right]ds\,, \tag{12c}$$

and the constant

$$\zeta = \frac{k+\varphi}{\alpha - \frac{1}{2}\,b}\,.$$

Optimal liquidation speed

$$\nu_t^* = \frac{1}{(T-t)+\zeta}\, Q_t^{\nu^*} \tag{13a}$$

$$+ \frac{\varphi}{k+\varphi}\,\rho\left\{(\mu_t^+ + \mu_t^-) - \frac{1}{(T-t)+\zeta}\int_t^T \mathbb{E}[\,(\mu_s^+ + \mu_s^-)\ |\,\mathcal{F}_t^{\boldsymbol{\mu}}\,]\,ds\right\} \tag{13b}$$

$$- \frac{b}{k+\varphi}\int_t^T \frac{(T-s)+\zeta}{(T-t)+\zeta}\,\mathbb{E}\left[\,(\mu_s^+ - \mu_s^-)\ |\,\mathcal{F}_t^{\boldsymbol{\mu}}\right]\,ds\,, \qquad \forall t < \tau^{\nu^*} \tag{13c}$$

where $\mathcal{F}_t^{\boldsymbol{\mu}}$ denotes the natural filtration generated by $\boldsymbol{\mu}$, and $\nu_t^* = 0$ for $t \geq \tau$, is the admissible optimal control we seek.

Optimal liquidation speed

- The first component is TWAP-like.

- The second component consists of targeting instantaneous total order-flow plus a weighted average of future expected total order-flow. Although the POV target is $\rho\,\mu_t$, the strategy targets a lower amount since $\frac{\varphi\,\rho}{k+\varphi} \le \rho$, where equality is achieved if the costs of missing the target are $\varphi \to \infty$ and k remains finite, or there is no temporary impact $k \downarrow 0$.

- The last component acts to correct her trading based on her expectations of the net order-flow from that point in time until the end of the trading horizon. When there is a current surplus of buy trades she slows down her trading rate to allow the midprice to appreciate before liquidating the rest of her order.

Targeting VWAP

Investor must liquidate all shares. Do this by letting $\alpha \to \infty$

$$\begin{aligned}
\lim_{\alpha\to\infty} \nu_t^* \quad = \quad & \frac{1}{T-t} Q_t^{\nu^*} \\
& + \frac{\varphi}{k+\varphi} \left(\rho\, \mu_t - \frac{1}{T-t} \int_t^T \mathbb{E}[\, \mu_s^+ + \mu_s^- \mid \mathcal{F}_t^{\boldsymbol{\mu}} \,]\, ds \right) \\
& - \frac{b}{k+\varphi} \, \frac{\int_t^T (T-s)\, \mathbb{E}\,[\mu_s^+ - \mu_s^- \mid \mathcal{F}_t^{\boldsymbol{\mu}}]\; ds}{T-t} \,,
\end{aligned}$$

and follow this by taking the limit $\varphi \to \infty$, resulting in

$$\lim_{\varphi\to\infty} \lim_{\alpha\to\infty} \nu_t^* \quad = \quad \frac{1}{T-t} Q_t^{\nu^*} + \rho \Bigg(\left(\mu_t^+ + \mu_t^-\right) \tag{15}$$

$$- \frac{1}{T-t} \int_t^T \mathbb{E}\left[\left(\mu_s^+ + \mu_s^-\right) \,\middle|\, \mathcal{F}_t^{\boldsymbol{\mu}} \right] \; ds \Bigg) \,. \tag{16}$$

Targeting VWAP via POCV

Accumulated Volume

- Accumulated volume V of orders, excluding the agent's, is

$$V_t = \int_0^t \left(\mu_u^+ + \mu_u^-\right)\, du\,.$$

- The investor's performance criteria is now modified to

$$\begin{aligned} H^\nu(t,x,S,\boldsymbol{\mu},V,q) \;&=\; \mathbb{E}_{t,x,S,\boldsymbol{\mu},V,q}\Bigg[X^\nu_{\tau^\nu} + Q^\nu_{\tau^\nu}\left(S^\nu_{\tau^\nu} - \alpha Q^\nu_{\tau^\nu}\right) \\ &\qquad -\tilde{\varphi}\int_t^{\tau^\nu}\Big((\mathfrak{N} - Q^\nu_u) - \tilde{\rho}\left(V_u + (\mathfrak{N} - Q^\nu_u)\right)\Big)^2 du\Bigg] \end{aligned}$$

and $\tau^\nu = T \wedge \inf\{t \,:\, Q^\nu_t = 0\}$.

- The investor's value function is

$$H(t,x,S,\boldsymbol{\mu},V,q) = \sup_{\nu\in\mathcal{A}} H^\nu(t,x,S,\boldsymbol{\mu},V,q)\,. \tag{17}$$

DPE

The value function should satisfy the DPE

$$\begin{aligned} 0 = &\left(\partial_t + \mathcal{L}^M + \mathcal{L}^{\boldsymbol{\mu},V}\right) H - \varphi \left((\mathfrak{N} - q) - \rho\, V\right)^2 \\ &+ \sup_{\nu} \left\{ b\,((\mu^+ - \mu^-) - \nu)\,\partial_S H + (S - k\,\nu)\;\nu\,\partial_x H - \nu\,\partial_q H \right\}\,, \end{aligned} \tag{18}$$

for $q > 0$, subject to $H(T, x, S, y, q) = x + q\,(S - \alpha q)$ and

$$\varphi = \tilde{\varphi}(1 - \tilde{\rho})^2 \qquad \text{and} \qquad \rho = \tilde{\rho}/(1 - \tilde{\rho})\,.$$

Here, $\mathcal{L}^{\boldsymbol{\mu},V}$ denotes the infinitesimal generator of the joint process $(\boldsymbol{\mu}_t, V_t)_{0 \le t \le T}$.

The DPE (18) admits the solution

$$H(t,x,S,\boldsymbol{\mu},V,q) = x + q\,S + h_0(t,\boldsymbol{\mu},V) + h_1(t,\boldsymbol{\mu},V)\,q + h_2(t)\,q^2\,, \quad q > 0 \tag{19}$$

where

$$h_2(t) = -\sqrt{k\,\varphi}\,\frac{\gamma\,e^{\xi\,(T-t)} + e^{\xi\,(T-t)}}{\gamma\,e^{\xi(T-t)} - e^{-\xi\,(T-t)}} - \tfrac{1}{2}\,b\,, \tag{20a}$$

$$\begin{aligned} h_1(t,\boldsymbol{\mu},V) = 2\,\varphi \int_t^T \ell(u,t)\,\left(\mathfrak{N} - \rho\,\mathbb{E}_{t,\boldsymbol{\mu},V}\,[V_u]\right)\,du \\ + b\int_t^T \ell(u,t)\,\mathbb{E}_{t,\boldsymbol{\mu},V}\left[(\mu_u^+ - \mu_u^-)\right]\,du\,, \end{aligned} \tag{20b}$$

$$h_0(t,\boldsymbol{\mu},V) = \int_t^T \mathbb{E}_{t,\boldsymbol{\mu}}\left[\tfrac{1}{4k}\big(h_1(u,\boldsymbol{\mu}_u,V_u)\big)^2 - \varphi\big(\mathfrak{N} - \rho\,V_u\big)^2\right]\,du\,, \tag{20c}$$

the function

$$\ell(u,t) := \frac{\gamma\,e^{\xi\,(T-u)} - e^{-\xi\,(T-u)}}{\gamma\,e^{\xi\,(T-t)} - e^{-\xi\,(T-t)}}\,, \tag{20d}$$

and the constants

$$\xi = \sqrt{\frac{\varphi}{k}}\,, \quad \text{and} \quad \gamma = \frac{\alpha - \frac{1}{2}b + \sqrt{k\,\varphi}}{\alpha - \frac{1}{2}b - \sqrt{k\,\varphi}}\,.$$

Optimal control for POCV

Moreover, the trading speed, for $q > 0$, is given by

$$\nu_t^* = \xi \, \frac{1+\gamma}{\gamma \, e^{\xi(T-t)} - e^{-\xi\,(T-t)}} \, Q_t^{\nu^*} \tag{21a}$$

$$- \xi^2 \int_t^T \ell(u,t) \left\{ \left(\mathfrak{N} - Q_t^{\nu^*} \right) - \rho \, \mathbb{E}\left[V_u \,\middle|\, \mathcal{F}_t^{\boldsymbol{\mu},V} \right] \right\} du \tag{21b}$$

$$- \frac{b}{2\,k} \int_t^T \ell(u,t) \, \mathbb{E}\left[(\mu_u^+ - \mu_u^-) \,\middle|\, \mathcal{F}_t^{\boldsymbol{\mu},V} \right] du \,, \qquad \forall\, t < \tau^{\nu^*} . \tag{21c}$$

And the optimal inventory path

$$Q_t^{\nu^*} = \ell(t,0)\, \mathfrak{N} + \int_0^t \int_u^T \ell(u,s) \left\{ \tfrac{\varphi}{k} \left(\mathfrak{N} - \rho \, \mathbb{E}\left[V_s \,\middle|\, \mathcal{F}_u^{\boldsymbol{\mu},V} \right] \right) + \tfrac{b}{2k} \mathbb{E}\left[(\mu_s^+ - \mu_s^-) \,\middle|\, \mathcal{F}_u^{\boldsymbol{\mu},V} \right] \right\} ds \, du \,. \tag{22}$$

Interpreting liquidation speed

- (21a) may be interpreted as an Almgren-Chriss like strategy which approaches TWAP near maturity.
- The second term (21b) corrects for the weighted average of the difference between what the investor has liquidated so far $\left(\mathfrak{N} - Q_t^{\nu^*}\right)$ and her future targeted volume $\rho\,\mathbb{E}\left[V_u \,\middle|\, \mathcal{F}_t^{\boldsymbol{\mu},V}\right]$. This term will be positive most of the time because: i) total volume is a submartingale and so $\mathbb{E}\left[V_u \,\middle|\, \mathcal{F}_t^{\boldsymbol{\mu},V}\right] \geq V_t$, and ii) since at time t the investor targets $\left(\mathfrak{N} - Q_t^{\nu^*}\right)$ to $\rho\,V_t$, it is likely that
$$\left(\mathfrak{N} - Q_t^{\nu^*}\right) - \rho\,\mathbb{E}\left[V_u \,\middle|\, \mathcal{F}_t^{\boldsymbol{\mu},V}\right] \sim \rho\,V_t - \rho\,\mathbb{E}\left[V_u \,\middle|\, \mathcal{F}_t^{\boldsymbol{\mu},V}\right] \leq 0\,.$$
- The third term (21c) contributes a weighted average of future net order-flow and accounts for the permanent impact which trades have on the midprice. If future net order-flow is expected to be buy heavy, then investor slows down
- Finally, as the trading horizon ends, t approaching T, the second and third terms become negligible and the investor ignores order-flow entirely and instead focuses on completing her trades.

Targeting VWAP

Let $\alpha \to \infty$, so that the investor ensures that she completely liquidates her position by the terminal time, $\gamma \to 1$, therefore $\ell(u,t) \to \frac{\sinh(\xi\,(T-u))}{\sinh(\xi\,(T-t))}$ and so we have

$$\lim_{\alpha\to\infty} \nu_t^* = \xi \,\frac{Q_t^{\nu^*}}{\sinh\left(\xi\,(T-t)\right)} - \xi^2 \int_t^T \frac{\sinh\left(\xi\,(T-u)\right)}{\sinh\left(\xi\,(T-t)\right)} \left\{ \left(\mathfrak{N} - Q_t^{\nu^*}\right) - \rho\,\mathbb{E}\left[V_u \,\middle|\, \mathcal{F}_t^{\boldsymbol{\mu},V} \right] \right\} du$$
$$- \frac{b}{2\,k} \int_t^T \frac{\sinh\left(\xi\,(T-u)\right)}{\sinh\left(\xi\,(T-t)\right)}\,\mathbb{E}\left[(\mu_u^+ - \mu_u^-) \,\middle|\, \mathcal{F}_t^{\boldsymbol{\mu},V} \right] du\,, \tag{23}$$

and

$$\lim_{\alpha\to\infty} Q_t^{\nu^*} = \frac{\sinh(\xi\,(T-t))}{\sinh(\xi\,T)}\,\mathfrak{N} + \xi^2 \int_0^t \int_u^T \frac{\sinh(\xi\,(T-u))}{\sinh(\xi\,(T-s))} \left(\mathfrak{N} - \rho\,\mathbb{E}\left[V_s \,\middle|\, \mathcal{F}_u^{\boldsymbol{\mu},V} \right]\right) ds\,du$$
$$+ \frac{b}{2k} \int_0^t \int_u^T \frac{\sinh(\xi\,(T-u))}{\sinh(\xi\,(T-s))}\,\mathbb{E}\left[(\mu_s^+ - \mu_s^-) \,\middle|\, \mathcal{F}_u^{\boldsymbol{\mu},V} \right] ds\,du\,.$$

Strategy Performance

Assume that other market participants trades follow

$$d\mu_t^+ = -\kappa^+ \, \mu_t^+ \, dt + \eta^+_{1+N^+_{t^-}} \, dN_t^+ \,, \tag{24a}$$

$$d\mu_t^- = -\kappa^- \, \mu_t^- \, dt + \eta^-_{1+N^-_{t^-}} \, dN_t^- \,, \tag{24b}$$

where

- $\kappa^\pm \geq 0$ are the mean-reversion rates,
- N_t^+ and N_t^- are independent homogeneous Poisson processes with intensities λ^+ and λ^-, respectively,
- $\{\eta_1^\pm, \eta_2^\pm, \dots\}$ are non-negative i.i.d. random variables with distribution function F, with finite first moment, independent from all processes.
- In addition, we require $\kappa^\pm > \lambda^\pm \mathbb{E}[\eta_1^\pm]$ to ensure that $\mu^\pm$ remain bounded $\mathbb{P}$-a.s..

VWAP as target

Compute VWAP as

$$\text{VWAP} = \frac{\int_0^T S_u^\nu \,(\mu_u^+ + \mu_u^- + \nu_u)\; du}{\int_0^T \left(\mu_u^+ + \mu_u^- + \nu_u\right)\; du}\,, \tag{25a}$$

while the execution price is computed as

$$\text{Exec. Price} = \frac{X_T^\nu}{\mathfrak{N}} = \frac{\int_0^T \hat{S}_u^\nu \, \nu_u \, du}{\mathfrak{N}}\,. \tag{25b}$$

POV-VWAP performance

Table: The statistics of the execution price, VWAP, and relative error (computed as $(Exec.Price - VWAP)/VWAP$ for each simulation) and reported in basis points (i.e., $\times 10^4$).

		FARO		SMH		NTAP	
		VWAP	Rel.Error	VWAP	Rel.Error	VWAP	Rel.Error
	mean	$ 40.54	8.9	$ 37.90	2.98	$ 38.30	0.19
	stdev	$0.11	16.9	$0.04	6.10	$0.06	0.87
quantile	5%	$40.35	-4.2	$37.83	-1.03	$38.20	-0.78
	25%	$40.46	-0.4	$37.87	-0.16	$38.26	-0.20
	50%	$40.54	2.5	$37.90	0.53	$38.30	0.03
	75%	$40.61	12.0	$37.92	3.60	$38.34	0.36
	95%	$40.72	42.6	$37.96	15.29	$38.40	1.66
		27.3%		18.4%		0.6%	

POCV-VWAP performance

Table: The statistics of the execution price, VWAP, and relative error (computed as $(Exec.Price - VWAP)/VWAP$ for each simulation) and reported in basis points (i.e., $\times 10^4$). **For POCV we set $\varphi = 10^5 \times k$**

		FARO		SMH		NTAP	
		VWAP	Rel.Error	VWAP	Rel.Error	VWAP	Rel.Error
	mean	\$40.54	5.0	\$37.89	2.75	\$38.30	1.88
	stdev	\$0.11	14.2	\$0.04	6.02	\$0.06	2.35
quantile	5%	\$40.35	-13.6	\$37.83	-5.43	\$38.20	-1.65
quantile	25%	\$40.46	-3.0	\$37.87	-0.78	\$38.26	0.28
quantile	50%	\$40.54	1.9	\$37.89	1.79	\$38.30	1.72
quantile	75%	\$40.61	11.0	\$37.92	5.51	\$38.34	3.27
quantile	95%	\$40.73	31.9	\$37.96	13.76	\$38.40	6.00
		0. 83%		0.013%		0.55%	