Mathematics and Statistics of Algorithmic Trading[1]

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[1]Notes based on textbook "Algorithmic and High-Frequency Trading" with Sebastian Jaimungal and Jose Penalva.

Optimal Execution

Assume the agent can also trade in **dark pools**, i.e. venues which, in contrast to traditional (or lit) exchanges, do not display bid and ask quotes to their clients. Trading may occur continuously, as soon as orders are matched, or consolidated and cleared periodically (sometimes referred to as throttling). We focus on a particular kind of dark pool known as a *crossing network* defined by the SEC as

> *"...systems that allow participants to enter unpriced orders to buy and sell securities, these orders are crossed at a specified time at a price derived from another market..."*

- Typically, the price at which transactions are crossed is the midprice in a corresponding lit trading venue.
- When a trader places an order in a dark pool, she may have to wait for some time until a matching order arrives so that her order is executed.
- Thus, on the one hand the trader who sends orders to the dark pool is exposed to execution risk, but on the other hand does not receive the additional temporary price impact of walking the LOB.

Model Setup

- Matching orders in the dark pool have no price impact and not necessarily the whole amount y_t she sends is executed.
- Other participants send matching orders to the dark pool which arrive at Poisson times.
- Let N_t denote a Poisson process with intensity λ, and
- Let $\{\xi_j : j = 1, 2, \dots\}$ be a collection of iid corresponding to the volume of the various matching orders which are sent into the dark pool.
- The total volume of buy orders placed in the dark pool up to time t is the compound Poisson process

$$V_t = \sum_{n=1}^{N_t} \xi_n \,.$$

- When a matching order arrives, the agent's inventory (accounting also for the continuous trading in the lit market) satisfies

$$dQ_t^{\nu,y} = -\nu_t \, dt - \min\left(y_t, \, \xi_{1+N_{t-}}\right) \, dN_t \,,$$

Model Setup

- Hence, the agent's cash process $X_t^{\nu,y}$ satisfies the SDE

$$dX_t^{\nu,y} = (S_t - k\,\nu_t)\,\nu_t\,dt + S_t\,\min\left(y_t,\,\xi_{1+N_{t^-}}\right)\,dN_t\,.$$

- Her performance criteria is

$$H^{\nu,y}(t,x,S,q) = \mathbb{E}_{t,x,S,q}\left[X_\tau + Q_\tau^{\nu,y}\left(S_\tau - \alpha\,Q_\tau^{\nu,y}\right) - \phi\int_t^\tau \left(Q_u^{\nu,y}\right)^2\,du\right],$$

  where $\mathbb{E}_{t,x,S,q}\left[\cdot\right]$ denotes expectation conditional on $X_{t^-} = x$, $S_t = S$, $Q_{t^-} = q$, and the stopping time

$$\tau = T \wedge \inf\{t\ :\ Q_t = 0\}\,,$$

  represents the time until the agent's inventory is completely liquidated, or the terminal time has arrived.

- The value function is

$$H(t,x,S,q) = \sup_{\nu,y\in\mathcal{A}} H^{\nu,y}(t,x,S,q)\,.$$

DPE

Applying the DPP shows that the value function should satisfy the DPE

$$\begin{aligned}\partial_t H + \tfrac{1}{2}\sigma^2 \partial_{SS} H - \phi\, q^2 \\ + \sup_{\nu} \{(S - k\,\nu)\,\nu\,\partial_x H - \nu\,\partial_q H\} \\ + \sup_{y \le q} \{\lambda\,\mathbb{E}\,[H\,(t, x + S\,\min(y,\xi),\, S,\, q - \min(y,\xi)) - H]\} = 0\,,\end{aligned}$$

subject to the terminal condition

$$H(T, x, S, q) = x + q\,(S - \alpha\, q)\,.$$

- The term ∂_{SS} represents the diffusion of the midprice,
- The $-\phi\, q^2$ term represents the running penalty,
- The $\sup_\nu\{\cdot\}$ term represents optimising over continuous trading in the lit market,
- The $\sup_{y \le q}$ term represents optimising over the volume posted in the dark pool and the expectation is there to account for the fact that buy volume coming into the dark pool from other traders is random.

Ansatz

Make ansatz $H(t, x, S, q) = x + q\,S + h(t, q)$. Thus

$$\begin{aligned}\partial_t h - \phi\, q^2 &+ \sup_{\nu} \left\{-k\,\nu^2 - \nu\,\partial_q h\right\} \\ &+ \lambda \sup_{y \le q} \mathbb{E}\left[h\left(t, q - \min(y, \xi)\right) - h(t, q)\right] = 0\,,\end{aligned} \tag{1}$$

subject to the terminal condition $h(T, q) = -\alpha\, q^2$. Next, the first order condition for ν implies that the optimal speed to trade in feedback control form is

$$\nu^* = -\frac{1}{2k}\partial_q h \tag{2}$$

so

$$\sup_{\nu} \left\{-k\,\nu^2 - \nu\,\partial_q h\right\} = \frac{1}{4k}\left(\partial_q h\right)^2\ .$$

Full Execution in Dark Pool

Full execution in Dark Pool

- Assume that the agent's desired execution is small relative to the volume coming into the dark pool,
- $\xi_i \geq \mathfrak{N}$ (for all $i = 1, 2, \dots$).

We hypothesise that the ansatz is a polynomial in q. Before proposing the ansatz note that the DPE contains an explicit q^2 penalty, the optimum over ν is quadratic in $\partial_q h$, and the terminal condition is $-\alpha\, q^2$. Thus this suggests the following ansatz for $h(t, q)$:

$$h(t, q) = h_0(t) + h_1(t)\, q + h_2(t)\, q^2 \,,$$

with terminal conditions $h_0(T) = h_1(T) = 0$ and $h_2(T) = -\alpha$.

The supremum over y becomes

$$
\begin{aligned}
&\sup_{y\le q}\mathbb{E}\left[h\left(t, q-\min(y,\xi)\right)-h(t,q)\right]\\
&\qquad=\sup_{y\le q}\left[h\left(t,q-y\right)-h(t,q)\right]\\
&\qquad=\sup_{y\le q}\left[-y\,h_1+(y^2-2\,q\,y)\,h_2\right]\\
&\qquad=-\frac{1}{4\,h_2}\left(h_1-2\,q\,h_2\right)^2\ ,
\end{aligned}
$$

and the optimal dark pool volume in feedback form is

$$
y^* = q+\frac{1}{2}\frac{h_1}{h_2}\,.
$$

From the terminal condition, $h_2(t)<0$. It remains to be seen that $h_1(t)\ge 0$ so that indeed $y^*\le q$ and the admissibility criteria are satisfied.
Furthermore, the optimal speed of trading, in feedback form, simplifies to

$$
\nu^* = -\frac{1}{2k}\left(h_1+2q\,h_2\right)\,.
$$

Inserting the above feedback controls into the DPE (1) leads to the coupled system of ODEs

$$\partial_t h_2 - \phi - \lambda\, h_2 + \tfrac{1}{k} h_2^2 = 0\,, \tag{3a}$$

$$\partial_t h_1 + \left(\lambda + \tfrac{1}{k}\, h_2\right) h_1 = 0\,, \tag{3b}$$

$$\partial_t h_0 + \tfrac{1}{k}\, h_1^2 - \ \tfrac{\lambda}{4}\, \frac{h_1^2}{h_2} = 0\,. \tag{3c}$$

- Since h_1 vanishes at T and its ODE in (3b) is linear in h_1, the solution is $h_1(t) = 0$ and it is also trivial to show that $h_0(t) = 0$.
- Clearly, we see that if there is no dark pool, that is $\lambda = 0$, the problem reduces to that of optimal liquidation already discussed.

Solving Riccati ODE

The equation for h_2 is of Riccati type and can be solved explicitly. Let $\zeta^\pm$ denote the roots of the polynomial $\phi + \lambda\, p - \frac{1}{k}\, p^2 = 0$, then write (3a) as

$$\partial_t h_2 = -\tfrac{1}{k}\,(h_2 - \zeta^+)(h_2 - \zeta^-)\,, \quad \zeta^\pm = \tfrac{1}{2} k\,\lambda \pm \sqrt{\tfrac{1}{4} k^2\,\lambda^2 + k\phi}\,.$$

Cross multiplying and writing as partial fractions, we have

$$\partial_t h_2 \left(\frac{1}{h_2 - \zeta^+} - \frac{1}{h_2 - \zeta^-} \right) = -\tfrac{1}{k}\,(\zeta^+ - \zeta^-)\,,$$

and integrating from t to T leads to

$$\log\left(\frac{h_2 - \zeta^-}{h_2 - \zeta^+}\right) - \log\left(\frac{\alpha + \zeta^-}{\alpha + \zeta^+}\right) = -\tfrac{1}{k}\,(\zeta^+ - \zeta^-)(T - t)\,,$$

where we have used $h_2(T) = -\alpha$. Rearranging,

$$h_2(t) = \frac{\zeta^- - \zeta^+\,\beta\,e^{-\gamma(T-t)}}{1 - \beta\,e^{-\gamma(T-t)}}\,,$$

where the constants are

$$\beta = \frac{\alpha + \zeta^-}{\alpha + \zeta^+} \quad \text{and} \quad \gamma = \tfrac{1}{k}(\zeta^+ - \zeta^-)\,.$$

Optimal liquidation and inventory

Therefore, the optimal trading strategy is

$$\nu_t^* = -\frac{1}{k}\, h_2(t)\, Q_t^{\nu^*,y^*} \qquad \text{and} \qquad y_t^* = Q_t^{\nu^*,y^*}\,. \tag{4}$$

We can obtain the optimal inventory to hold, up to the arrival of matching order in the dark pool, by solving

$$dQ_t^{\nu^*,y^*} = -\nu_t^*\, dt = \frac{1}{k}\, h_2(t)\, Q_t^{\nu^*,y^*}\, dt\,,$$

so that

$$Q_t^{\nu^*,y^*} = Q_0 \exp\left\{ \tfrac{1}{k} \int_t^T h_2(u)\, du \right\}\,,$$

and therefore by direct integration

$$Q_t^{\nu^*,\,y^*} = e^{(\zeta^-/k)\,t}\, \left(\frac{1-\beta\, e^{-\gamma(T-t)}}{1-\beta\, e^{-\gamma\, T}} \right) \mathfrak{N}\,. \tag{5}$$

Optimal liquidation and inventory II

In the limit $\alpha \to \infty$,

$$Q_t^{\nu^*,y^*} \xrightarrow{\alpha\to\infty} e^{\left(\frac{\zeta^-}{k}+\frac{\gamma}{2}\right)t}\,\frac{\sinh\left(\frac{\gamma}{2}(T-t)\right)}{\sinh\left(\frac{\gamma}{2}T\right)}\,\mathfrak{N}\,.$$

Furthermore, in the limit $\lambda \to 0$, $\zeta^- \to -\sqrt{k\phi}$ and $\gamma \to 2\sqrt{\phi/k}$ and thus

$$Q_t^{\nu^*,y^*} \xrightarrow{(\alpha,\lambda)\to(\infty,0)} \frac{\sinh\left(\sqrt{\frac{\phi}{k}}\,(T-t)\right)}{\sinh\left(\sqrt{\frac{\phi}{k}}\;T\right)}\,\mathfrak{N}\,,$$

which recovers the results from the AC case without the dark pool.

Simulations

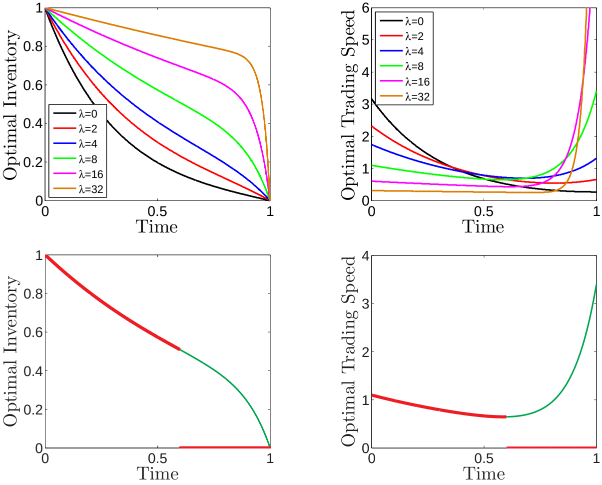

Figure: Top panels show optimal inventory path and speed of trading prior to a matching order in the dark pool. The bottom panels show the optimal inventory and trading speed where we assume that the dark pool matching order arrives at $t = 0.6$ right after which inventory drops to zero.