Exercises: Optimal Execution and Market Making

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1 Exercises

1. The agent wishes to liquidate $\mathfrak{N}$ shares between t and T using MOs. The value function is

$$H(t,S,q) = \sup_{\nu\in\mathcal{A}_{t,T}} \mathbb{E}_{t,S,q}\left[\int_t^T \left(S_u - k\,\nu_u\right)\nu_u\,du + Q_T^{\nu}\left(S_T - \alpha Q_T^{\nu}\right)\right],$$

where $k>0$ is the temporary market impact, ν_t is the speed of trading, $\alpha \geq 0$ is the liquidation penalty, and $dS_t = \sigma dW_t$.

(a) Show that the value function H satisfies

$$0 \;\; = \;\; -\left(\partial_q H - S\right)^2 - 4k\partial_t H - 2k\sigma^2\partial_{SS}H\,.$$

(b) Make the ansatz

$$H(t,S,q) = h_2(t)q^2 + h_1(t)q + h_0(t) + q\,S \tag{1}$$

and show that the optimal liquidation rate is

$$\nu_t^* = \frac{Q_t^{\nu^*}}{T-t+\frac{k}{\alpha}}\,. \tag{2}$$

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(c) Let $\alpha \to \infty$ and discuss the intuition of the strategy.

2. This exercise is similar to that above but with a slightly different setup. The agent wishes to liquidate $\mathfrak{N}$ shares and her objective is to maximise expected terminal wealth which is denoted by X_T^ν (in the exercise above we wrote terminal wealth as $\int_t^T (S_u - k\,\nu_u)\,\nu_u\,du$). The value function is

$$H(t,x,S,q) = \sup_{\nu} \mathbb{E}_{t,x,S,q}\left[X_T^\nu + Q_T^\nu\,(S_T - \alpha Q_T^\nu)\right] \,, \tag{3}$$

where

$$dX_t^\nu = (S_t - k\,\nu_t)\,\nu_t dt\,. \tag{4}$$

(a) Show that the HJB satisfied by the value function $H(t,S,q,x)$ is

$$0 = \left(\partial_t + \tfrac{1}{2}\sigma^2\partial_{SS}\right) H + \sup_{\nu}\left\{\left(-\nu\partial_q + (S - k\nu)\,\nu\partial_x\right) H\right\} \,, \tag{5}$$

and the optimal liquidation rate in feedback form is

$$\nu_t^* = \frac{\partial_q H - S\partial_x H}{-2k\,\partial_x H}\,. \tag{6}$$

(b) To solve (5), use the terminal condition $H(T,x,S,q) = x + q\,S - \alpha\,q^2$ to propose the ansatz

$$H(t,S,x,q) = x + h(t)\,q^2 + q\,S\,, \tag{7}$$

where $h(t)$ is a deterministic function of time. Show that

$$h(t) = -\frac{k}{T - t + \frac{k}{\alpha}}\,, \tag{8}$$

and

$$\nu_t^* = \frac{Q_t^{\nu^*}}{T - t + \frac{k}{\alpha}}\,.$$

3. Let the stock price dynamics satisfy

$$dS_t = \mu\,dt + \sigma\,dW_t\,,$$

where $\sigma > 0$, μ is a constant and W_t is a standard Brownian motion. The agent wishes to liquidate $\mathfrak{N}$ shares and her trades create a temporary adverse move in prices so the price at which she transacts is

$$\hat{S}_t^\nu = S_t - k\,\nu_t\,,$$

with $k > 0$ and the inventory satisfies

$$dQ_t^\nu = -\nu_t \, dt \,,$$

where ν_t is the liquidation rate. Any outstanding inventory at time T is liquidated at the midprice and picks up a penalty of $\alpha \, Q_T^2$ where $\alpha \geq 0$ is a constant. The agent's value function is

$$H(t, S, q) = \sup_\nu \mathbb{E}_{t,S,q} \left[\int_t^T (S_u - k \, \nu_u) \, \nu_u \, du + Q_T^\nu \, (S_T - \alpha Q_T^\nu) \right] . \tag{9}$$

(a) Show that the optimal liquidation rate in feedback form is

$$\nu^* = \frac{\partial_q H - S}{-2k} \,. \tag{10}$$

(b) Use the ansatz $H(t, S, q) = q \, S + h(t, S, q)$ to show that the optimal liquidation rate is given by

$$\nu_t^* = \frac{Q_t^{\nu^*}}{(T-t) + \frac{k}{\alpha}} - \frac{1}{4k} \, \mu \, (T-t) \, \frac{(T-t) + 2\frac{k}{\alpha}}{(T-t) + \frac{k}{\alpha}} \,.$$

Comment on the magnitude of μ and the sign of the liquidation rate.

(c) Let $\alpha \to \infty$ and show that the inventory along the optimal strategy is given by

$$Q_t^{\nu^*} = (T-t) \left(\frac{\mathfrak{N}}{T} + \frac{\mu}{4k} t \right) \,.$$

4. Consider the framework developed in the market making lecture (see slides), where the MM posts only at-the-touch, but assume that when an MO arrives, and the agent is posted on the matching side of the LOB, her order is filled with probability $\rho < 1$. Derive the DPE and compute the optimal strategy in feedback form.