Market Microstucture
Notes based on textbook "Algorithmic and High-Frequency Trading" Cartea, Jaimungal & Penalva (2015)

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March 4, 2016

1 A Primer on the Microstructure of Financial Markets

These notes cover several aspects of market making and informed trading. Section 2.1 considers a basic market making model that focuses on inventory and inventory risk, as well as the trade-off between execution frequency and profit per trade. It also looks at the conceptual basis for some measures of liquidity. The last two sections look at trading when there are informational differences between traders: Section 2.2 from the point of view of the better informed trader, and Section 2.3 from that of the less informed market maker.

A key dimension of the trading and price setting process is that of information. Who has what information, how does that information affect trading strategies, and

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how do those trading strategies affect trading outcomes in general, and asset prices in particular. Forty years ago finance theory introduced the tools to explicitly incorporate and evaluate the notion of price efficiency, the idea that "market prices are an efficient way of transmitting the information required to arrive at a Pareto optimal allocation of resources" (Grossman & Stiglitz (1976)). This dimension naturally appears in microstructure studies which look into the details of how different trading rules and trading conditions incorporate or hinder price efficiency. What differentiates microstructure studies from more general asset pricing ones is that they focus on two aspects that are key to trading: liquidity and price discovery.

Trading can take place in a number of possible ways: via personal deals settled over a handshake in a club, via decentralised chat rooms where traders engage each other in bilateral personal transactions, via broker-intermediated over-the-counter (OTC) deals, via specialised broker-dealer networks, on open electronic markets, etc. Our focus is on trading and trading algorithms that take place in large electronic markets, whether they be open exchanges, such as the NASDAQ stock market, or in electronic private exchanges (run by a broker-dealer, a bank, or a consortium of buy-side investors).

2 Market Making

An important type of market participant is the 'passive' market maker (MM), who facilitates trade and profits from making the spread and from her execution skills, and must be quick to adapt to changing market conditions. Another type is the 'active' trader, who exploits her ability to anticipate price movements and must identify the optimal timing for her market intervention. We start with the first group, the 'passive' traders.

Because we are focusing on trading in active exchanges, it is natural to assume that there are many market makers (MMs) in competition. Naturally, trading in a market dominated by a few MMs would need to additionally incorporate how the MMs exercise their market power and how it affects the market as a whole.

MMs play a crucial role in markets where they are responsible for providing liquidity to market participants by quoting prices to buy and sell the assets being traded, whether they be equities, financial derivatives, commodities, currencies, or others. A key dimension of liquidity as provided by MMs is immediacy: the ability of investors to buy (or sell) an asset at a particular point in time without having to wait to find a counterparty with an offsetting position to sell (or buy). By quoting buy and sell prices (or posting limit orders (LOs) on both sides of the book), the MM provides liquidity to the market, but to make this a sustainable business the MM quotes a buy price lower than her quoted sell price. For example an MM is willing to purchase shares of company XYZ at $99 and willing to sell at $101 per share. Note that by posting LOs, the MM is providing liquidity to other traders who may be looking to execute a trade quickly, e.g. by entering a market order (MO). Hence, we have the usual dichotomy that separates MMs as liquidity providers from other traders, considered as liquidity takers.

If our MM is the one offering the best prices, so that the ask is $101 and the bid $99, then the quoted spread is $2. There are a number of theories that explain what determines the spread in a competitive market. Before delving into some of these theories, we consider the issues faced by someone willing to provide liquidity.

2.1 Grossman–Miller Market Making Model

The first issue faced by an MM when providing liquidity is that by accepting one side of a trade (say buying from someone who wants to sell), the MM will hold an asset for an uncertain period of time; the time it takes for another person to come to the market with a matching demand for liquidity (wanting to buy the asset the MM bought in the previous trade). During that time, the MM is exposed to the risk that the price moves against her (in our example, as she bought the asset, she is exposed to a price decline and hence having to sell the asset at a loss in the next trade).

Recall that the MM has no intrinsic need or desire to hold any inventory, so she will only buy (sell) in anticipation of a subsequent sale (purchase). Grossman & Miller (1988) provide a market making model of how MMs obtain a liquidity premium from liquidity traders to compensate them for the price risk of holding an inventory of the asset until the MMs unload it later to another liquidity trader.

Let us consider a simplified version of their model. There are a finite number n of identical MMs who make markets in the same asset and trading takes place at three dates $t \in \{1, 2, 3\}$. To simplify the model further, there is no uncertainty about the arrival of matching orders: if at date $t = 1$ a liquidity trader, denoted by LT1, comes to the market to sell i units of the asset, there is (for sure) another liquidity trader (LT2) who arrives at the market to purchase i units (or more generally, to trade $-i$ units, so that LT1's trade (of i units) could be negative or positive (LT1 could be buying or selling). However, LT2 does not arrive to the market until $t = 2$. Moreover, all agents start with an initial cash amount equal to W_0, MMs hold no assets, LT1 holds i units and LT2 $-i$ units

There are no trading costs or direct costs for holding inventory. The focus is on price changes: the asset will have a cash value at $t = 3$ of $S_3 = \mu + \epsilon_2 + \epsilon_3$, where μ is constant, ϵ_2 and ϵ_3 are random shocks to prices and assumed independent, normally distributed random variables with mean zero and variance σ^2. These shocks to prices are publicly announced between dates $t - 1$ and t: ϵ_3 is announced between $t = 2$ and $t = 3$, and ϵ_2 is announced between $t = 1$ and $t = 2$. Hence, the realised cash value of the asset can increase or decrease (ignore the fact that there are realisations of ϵ_2 and ϵ_3 that could make the asset value negative – the model serves to illustrate a point). All traders, MMs and liquidity traders, are risk-averse. In particular, all traders' utility function is $U(X) = -\exp(-\gamma X)$, where X is cash and $\gamma > 0$ is a parameter capturing the utility penalty for taking risks (the risk aversion parameter).

To obtain the equilibrium quantities that MMs are willing to hold and the liquidity premium they command as compensation for bearing the risk of price changes, we solve the model backwards. At $t = 3$ the cash value of the asset is realised, $S_3 = \mu + \epsilon_2 + \epsilon_3$. At $t = 2$, the n MMs and LT1 come into the period with asset holdings q_1^{MM} and q_1^{LT1} respectively. LT2 comes in with $-i$ and they all exit with asset holdings q_2^j, where $j \in \{MM, LT1, LT2\}$. Note that if, for example, $q_t^j = 2$ this denotes that agent j is holding 2 units when exiting date t, so that the agent will be long (that is, has an inventory of) two units of the asset. Given the problem as described so far, at

$t = 2$ agent j chooses q_2^j to maximise their expected utility knowing the realisation of ϵ_2, which was made public before $t = 2$:

$$\max_{q_2^j} \mathbb{E}\left[U\left(X_3^j\right) \mid \epsilon_2\right]$$

subject to

$$X_3^j = X_2^j + q_2^j\, S_3\,, \qquad X_2^j + q_2^j\, S_2 = X_1^j + q_1^j\, S_2\,.$$

For each agent j these two constraints capture:

1. the cash value X_3 of assets at $t = 3$ is equal to cash X_2 plus the cash value of the agent's asset inventory q_2^j at the market value S_3, and
2. the cash value of assets when exiting date $t = 2$ (X_2, and the inventory q_2^j) is equal to the cash value of the agent's assets when entering date $t = 2$ (X_1, and the inventory q_1^j).

Given the normality assumption and the properties of the expected utility function it is straightforward to show that

$$\mathbb{E}\left[U\left(X_3^j\right) \mid \epsilon_2\right] \quad = \quad -\exp\left\{-\gamma\left(X_2^j + q_2^j\,\mathbb{E}[S_3 \mid \epsilon_2]\right) + \frac{1}{2}\gamma^2\left(q_2^j\right)^2\sigma^2\right\}\,.$$

Thus, the objective function in the maximisation problem is concave and the optimal inventory holding is

$$q_2^{j,*} = \frac{\mathbb{E}\left[S_3 \mid \epsilon_2\right] - S_2}{\gamma\,\sigma^2}\,, \tag{1}$$

for all agents: the n MMs, LT1, and LT2.

As at date $t = 2$ demand and supply for the asset have to be equal to each other, we can solve for the equilibrium price S_2:

$$n\,q_1^{MM} + q_1^{LT1} + q_1^{LT2} = n\,q_2^{MM} + q_2^{LT1} + q_2^{LT2}\,. \tag{2}$$

Here we use the convention that q_1^{LT2}, the assets LT2 exits period 1 and comes into period 2 with, is equal to his desired trade of $-i$ units. As shown in (1) all $q_2^{j,*}$ are equal, so the right-hand side of the balancing equation (2) is equal to

$$n\,q_2^{MM} + q_2^{LT1} + q_2^{LT2} = (n+2)\,\frac{\mathbb{E}[S_3 \mid \epsilon_2] - S_2}{\gamma\,\sigma^2}\,. \tag{3}$$

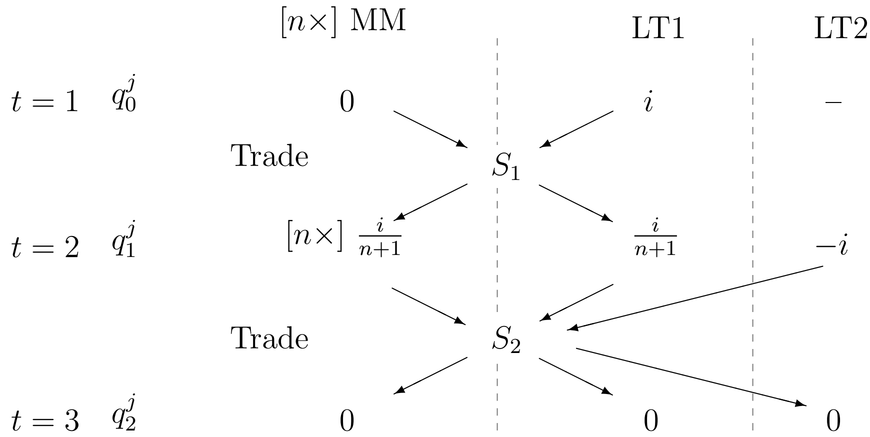

Figure 1: Trading and price setting in the Grossman–Miller model.

Similarly, we know that at date 1 the total quantity of the asset available is equal to the quantity of assets LT1 brought to the market, so the left-hand side of (2) is

$$n\, q_1^{MM} + q_1^{LT1} + q_1^{LT2} = i + q_1^{LT2} = i - i = 0\,.$$

Hence, substituting into (3) and solving, we obtain that in equilibrium, at date $t = 2$, $S_2 = \mathbb{E}[S_3] = \mu + \epsilon_2 + \mathbb{E}[\epsilon_3] = \mu + \epsilon_2$, and therefore, $q_2^j = 0$. This makes sense, as at $t = 2$ there are no asset imbalances, the price of the asset reflects its 'fundamental value' (efficient price) and no one will want to hold a non-zero amount of the risky asset. This analysis is captured in the bottom half of Figure 1, where we see the asset holdings of the three types of participants as they enter $t = 2$, q_1^j, $j \in \{MM, LT1, LT2\}$, and how after trading at a price equal to S_2 they end up with holdings q_2^j equal to zero.

Consider now what happens at date $t = 1$. Participating agents (the n MMs and LT1 – recall that LT2 will not appear until $t = 2$) anticipate that whatever they do, the future market price will be efficient and they will end up exiting date $t = 2$ with no inventories, so that $X_3 = X_2$. Thus, their portfolio decision is given by

$$\max_{q_1^j} \mathbb{E}\left[U\left(X_2^j\right)\right]\,,$$

subject to

$$X_2^j = X_1^j + q_1^j\, S_1, \qquad X_1^j + q_1^j\, S_1 = X_0^j + q_0^j\, S_1\,.$$

Repeating the analysis of $t = 2$ at $t = 1$, the optimal inventory holding is

$$q_1^{j,*} = \frac{\mathbb{E}[S_2] - S_1}{\gamma\,\sigma^2}\,,$$

for all agents j who are present: the n MMs, and LT1. Also, at date $t = 1$ demand and supply for the asset have to be equal to each other, so that

$$n\,q_0^{MM} + q_0^{LT1} = n\,q_1^{MM} + q_1^{LT1},$$

where $q_0^{LT1} = i$ (recall that if $i > 0$, LT1 is holding i shares he wants to sell), and $q_0^{MM} = 0$. This gives us the following equation:

$$i = (n+1)\,\frac{\mu - S_1}{\gamma\,\sigma^2} \iff S_1 = \mu - \frac{i}{n+1}\,\gamma\,\sigma^2\,.$$

The top half of Figure 1 reflects how the MMs and LT1 enter the market with asset holdings q_0^j and after trading at S_1 they exit period 1 and enter period 2 with q_1^j.

With this expression we can interpret how the market reaches a solution for LT1's liquidity needs: LT1, a trader who wants to sell a total of $i > 0$ units at $t = 1$, finds that there is no one currently in the market with a balancing liquidity need. There are MMs in the market, but they will not accept trading at the efficient price of μ because if they do, they will be taking on risky shares (they are exposed to the price risk from the realisation of ϵ_2) without compensation. But, if they receive adequate compensation (which we call a liquidity discount, as for $i > 0$, $S_1 < \mathbb{E}[S_2] = \mu$), then they will accept the LT1's shares. On the other hand, LT1 is price-sensitive, so if he has to accept a discount on the shares, he will not sell all the i shares at once. In equilibrium: each MM holds $q_1^{j,*} = \frac{i}{n+1}$ units of the asset each, LT1 sells $\frac{n}{n+1}i$ units and holds on to $\frac{i}{n+1}$ units to be sold later. Trading occurs at a price below the efficient price, $S_1 = \mu - \frac{i}{n+1}\,\gamma\,\sigma^2$. The difference between the trading price and the efficient price, namely $|S_1 - \mu| = \left|\frac{i}{n+1}\,\gamma\,\sigma^2\right|$, represents the (liquidity) discount the MMs receive in order to hold LT1's shares. This size of the discount is influenced by the variables in the model: the size of the liquidity demand ($|i|$), the amount of competition amongst MMs (captured by n), the market's risk aversion (γ), and the risk/volatility of the underlying asset (σ^2). These variables all affect the discount in an intuitive way: the size of the liquidity shock, risk aversion, and volatility all increase the discount, while competition reduces it. This occurs when LT1 wants to sell, i.e. $i > 0$. If LT1 wanted to buy, $i < 0$, then the solution would be the same except that instead of a discount, the MMs would receive a premium equal to $|S_1 - \mu|$ per share when selling to LT1.

From this analysis we also see that as competition increases (i.e. as n becomes larger), the liquidity discount goes to zero, the price converges to the efficient level, $S_1 = \mu$, and LT1's optimal initial is able to trade the i units of the asset in period 1.

3 Trading on an Informational Advantage

So far we have side-stepped one of the main issues in trading: informational differences. Many trades originate not because someone needs cash and sells an asset, or has extra cash and wants to invest, but because one party has (or believes she has) better information about what the price is going to do than is reflected in current prices. So, having seen the basic market making models in the context of public information we turn to the next fundamental issue: how to exploit an informational advantage while taking into account one's price impact. The primary reference in this case is Kyle (1985).

Kyle (1985) looks at the decision problem of a trader who has a strong informational advantage (the case of several competing informed traders is studied in Kyle (1989)) in a context where the price is 'efficient'. The model in Kyle (1985) tells us how the informed trader optimally adjusts his trading strategy to take into account the market reaction, and in particular, the price impact that his trades generate in equilibrium.

To get into the details of the model we first need to define what we mean by 'a strong informational advantage' and price efficiency in this context. To keep things simple we only consider the investor's static decision problem. The same basic idea extends to a dynamic setting. The formal static model is as follows: there is a market for an asset that opens at one point in time. The asset is traded at price S, and after trading, the asset has a cash value equal to v. The future cash value of the asset, v, is uncertain. In particular, v is assumed to be normally distributed with mean μ and variance σ^2. In the market, there are three types of traders: an informed trader, an anonymous mass of price-insensitive liquidity traders (traders who need to execute trades whatever the cost), and a large number of MMs who observe and compete for the order flow – that is, the MMs observe and compete for the flow of incoming buy and sell orders from the informed and the liquidity traders.

In contrast to the Grossman & Miller (1988) setting, MMs are risk-neutral, so they do not need a liquidity premium to compensate for the price risk from holding inventory. Therefore, any liquidity premium that arises will come from the need to compensate MMs for their informational disadvantage – and which will be borne by the price-insensitive liquidity traders. These liquidity traders have, in aggregate, a net demand represented by the random quantity u. If on aggregate $u > 0$, liquidity traders want to buy u units, while if $u < 0$, these traders want to sell $|u|$ units of the asset. Assume that u is normally distributed with mean zero, variance σ_u^2, and is independent of v. In principle, as liquidity traders are not sensitive to the price (u does not depend on S) MMs could charge very large liquidity premia, but competition for order flow between MMs drives the liquidity premium to zero, so that (when there are only MMs and liquidity traders) $S = \mathbb{E}[v]$.

Now consider the possibility that a new trader enters the market, and that this trader (the "insider") knows the exact value of v. The insider is the only one who knows v and chooses how much to trade. Let $x(v)$ denote the number of shares traded by the insider. MMs, on the other hand, know that there is an informed trader in the market, but do not know who this trader is.

To make the analysis formal, the model is structured as follows: (i) the insider observes v, (ii) on observing v the insider chooses $x(v)$, (iii) u is realised, (iv) the MMs observe the net order flow, $x(v) + u$, (v) based on the net order flow MMs compete to set the asset price, S.

To solve the model we use the solution concept of (Bayesian) Nash equilibrium; without going into all the details, this means that all agents optimise given the decisions of all other players, according to their beliefs (which are updated according to Bayes' rule whenever possible). Thus, we require that in equilibrium the insider chooses $x(v)$ to maximise his expected profit, taking into account the dynamics of the game (i.e. that his order will be mixed in with those of the liquidity traders), and anticipating that MMs will set their prices on the basis of what they learn from observing the order flow and what they know about the informed trader's decision problem. Also, we require that MMs choose their prices taking into account the strategy of the insider (in particular, they anticipate the functional form of $x(v)$) and the properties of the uninformed order flow that comes from liquidity traders. In particular, MMs set the market price as a function of net order flow, $S(x+u)$. This is important, as the model tells us that prices are affected by order flow, so that trading automatically generates a price impact – the average price per unit traded, S, moves with net order flow, $x+u$. We need to look at the equilibrium of the model to see what that price impact function looks like. Nevertheless, in equilibrium, the insider anticipates the functional form of $S(x+u)$, that is, he incorporates price impact when choosing $x(v)$.[1] The equilibrium is a fixed point in the optimisation of x given the functional form of S, and of S given the functional form of x.

Consider what the insider should do. The most natural response is: sell if $v < \mathbb{E}[v] = \mu$ and buy if $v > \mu$, and whether selling or buying, do so as much as possible to leverage his informational advantage. This seems natural, but we must take into account that MMs adjust their prices to the order flow they observe. Hence, even if $v < \mu$, the insider cannot expect $S = \mu$. In the extreme case where there are no liquidity traders everyone knows that any trade comes from the insider and so the MMs, anticipating the demand as a function of the realisation of v, behave optimally and set prices that incorporate all information on v in $x(v)$. Fortunately for the insider, there are liquidity traders that add noise into order flow and allow the insider to camouflage his trade to gain positive expected profits.

So, how do MMs set their prices? The first thing to note is that as MMs compete for order flow, any profits they could extract are competed away. Thus, whatever the

[1] Formally, liquidity traders are substituted by a "nature" player that executes the random demand u.

price strategy, it will lead to zero expected profits for our (risk-neutral) MMs – though never negative profits as they can always choose not to trade. The zero (expected) profit condition forces prices to have a very specific property: $S = \mathbb{E}[v \mid \mathcal{F}]$, where $\mathcal{F}$ represents all information available to MMs. This property is known as **semi-strong efficiency**: prices reflect all publicly available information (which in our case is order flow and is all the information MMs have).[2] This is why we can readily identify a fundamental property of the MMs' equilibrium strategy:

$$S(x+u) = \mathbb{E}[v \mid x+u]\,.$$

To solve the model we need to find an $x(v)$ that is optimal, i.e. it maximises the insider's expected trading profits, conditional on this pricing rule. Because of the normality of v and u, we hypothesise that $S(x+u)$ is linear in net order flow. In particular, let

$$S(x+u) = \mu + \lambda\,(x+u),$$

where λ is an unknown parameter representing the linear sensitivity of the price S to order flow.

Taking this particular functional form as given, consider the insider's problem:

$$\max_x \mathbb{E}\left[x\,v - x\,S(x+u)\right]$$

where the first term in the brackets are the revenues from liquidating the x units of the asset at the value v and the second term is the cost of purchasing x units at price $S(x+u)$.

Substituting for $S(x+u) = \mu + \lambda\,(x+u)$ and taking expectations with respect to u, we obtain that the objective function is concave and the first-order condition yields

$$x^*(v) = \beta\,(v-\mu),$$

where $\beta = (2\lambda)^{-1}$.

[2]The notion of price efficiency was introduced by the recent Nobel Laureate, Eugene Fama, see Fama (1970).

Because we have hypothesised the functional form of the price function, we must now confirm that the functional form is consistent with the optimal $x(v)$ and at the same time we can characterise λ. We know that $S = \mathbb{E}\left[v \,|\, x+u\right]$. From the optimal x, we know that

$$x + u = \beta(v - \mu) + u\,.$$

As v and u are independent and normal, $x+u$ is normal with mean 0 and variance $\beta^2\sigma^2 + \sigma_u^2$. We can now compute the joint distribution of v and $x+u$, and from it we can derive $S = \mathbb{E}\left[v \,|\, x+u\right]$, which (using the projection theorem for normal random variables and simplifying) is given by

$$S = \mu + (x+u)\,\frac{\sigma}{2\,\sigma_u},$$

so that the linear sensitivity parameter is $\lambda = \sigma/(2\,\sigma_u)$. This confirms that the hypothesised equilibrium is indeed an equilibrium (for a formal proof, see Kyle (1985)).

Even within the simple, static version of the Kyle model we can clearly see the issues that arise when facing informed trading (also referred to as “toxic order flow”). While in the previous models MMs just needed a liquidity premium (discount) to cover the expected cost from future price uncertainty, the presence of informed traders implies that MMs are adversely selected, buying when informed traders know it would be better to sell and selling when it would be better to buy. This adverse selection requires a higher premium borne by other (more impatient liquidity) traders. In this model, the additional premium takes the form of price adjustment to order flow (price impact) as described by Kyle’s lambda (the λ parameter we have just derived). This premium accounts not for the risk that future price movements are random, as described in Section 2.1, but for the adverse selection faced by MMs, as prices will on average move *against* the MMs’ position because they trade with better informed traders in the market. The sign of λ is the same as in Grossman & Miller (1988): prices move with the order flow, increasing as buy MOs hit the market and falling as traders sell aggressively.

4 Market Making with an Informational Disadvantage

The Kyle model focuses on the informed trader's problem, while using competition to characterise the MM's decisions. As we are very interested in the MM's problem, we now turn to Glosten & Milgrom (1985) for a model that puts the MM at the centre of the problem of trading with counterparties who have superior information.

Again, we look at a simplified and (essentially) static version of the model that allows us to capture the nature of the MM's decision problem. The situation is as before: there are liquidity traders, informed traders, and a competitive group of MMs. The MM is risk-neutral and has no explicit costs from carrying inventory.

Our simple model (described in Figure 2) has a future cash value of the asset equal to v which we limit to two possible values: $V_H > V_L$, that is a High, and a Low value. The unconditional probability of $v = V_H$ is p. All orders are of one unit, MMs post an LO to sell one unit at price a, and a buy LO for one unit at price b. We start by assuming that liquidity traders are price insensitive and want to buy with probability 1/2 and want to sell with probability 1/2. There are many informed traders, all of whom know v but are limited to trade a single unit, which simplifies their decision: when $v = V_H$ they buy one unit if $a < V_H$, and do nothing otherwise, while when $v = V_L$ they sell one unit if $b > V_L$ and do nothing otherwise. The total population of liquidity and informed traders is normalised to one, and of these, a proportion α are informed and a proportion $(1 - \alpha)$ are uninformed liquidity traders.

Figure 2 captures the probabilistic structure of the model: Nature randomly determines whether the underlying state is V_H or V_L. Independently of the state, a trader is picked at random from the population, so that with probability α she is informed, and with probability $1 - \alpha$ she is uninformed. An informed trader will always buy at the ask price (a) when the asset's value is V_H and sell at the bid (b) when the asset's value is V_L, while an uninformed trader will buy or sell with equal probability, independent of the true (unknown) value of the asset.

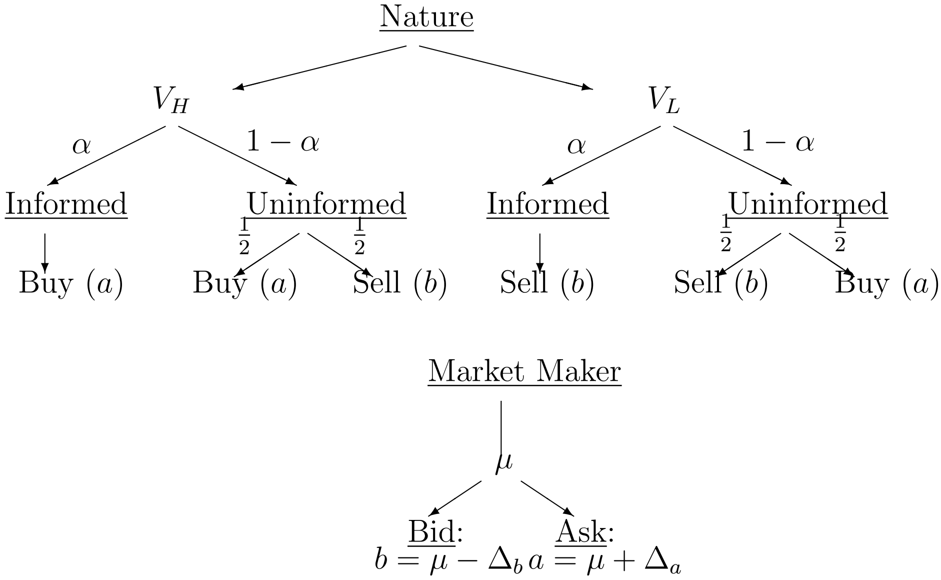

Figure 2: The Glosten-Milgrom model.

The MM's problem is to choose a and b in this setting. Because liquidity traders are price-insensitive, the optimal solution is trivial: set $a = V_H$ and $b = V_L$, but since MMs compete for business, prices are set to their (semi-strong) efficient levels – again, this happens because MMs use only public information, *which includes order flow*. Were the MMs to have private information, in addition to the order flow, competition for order flow would incorporate some of that information into prices.

Competition between MMs drives their expected profits to zero. Hence, a and b are determined by the no-profit condition. Rather than solve for a and b directly, define the ask- and bid-halfspreads, Δ_a and Δ_b respectively. The sum of the two, $\Delta_a + \Delta_b$, represents the (quoted) spread. Let the expected value of the asset $\mu = \mathbb{E}[v \,|\, \mathcal{F}]$ where $\mathcal{F}$ represents all public information prior to trading. Then, as described at the bottom of Figure 2, MMs chooses $a = \mu + \Delta_a$ and $b = \mu - \Delta_b$ optimally. To determine the effect of choosing a and b on the expected profit and loss, consider what happens when a buy order comes in:

- if it comes from an uninformed liquidity trader she makes an expected profit of $a - \mu = \Delta_a$,

- if it comes from an informed trader she makes an expected loss of $a - V_H = \Delta_a - (V_H - \mu)$.

From the point of view of the MM, the probability that a liquidity trader wants to buy is 1/2, while the probability that an informed trader wants to buy is p (as all informed traders will buy if the state is $v = V_H$ which occurs with probability p). As there are $1-\alpha$ liquidity traders and α informed ones, the expected profit from posting a price $a = \mu + \Delta_a$ is

$$\frac{(1-\alpha)/2}{\alpha\, p + (1-\alpha)/2}\Delta_a + \frac{\alpha\, p}{\alpha\, p + (1-\alpha)/2}\left(\Delta_a - (V_H - \mu)\right) .$$

Setting this expected profit to zero we obtain

$$\Delta_a = \frac{1}{1 + \frac{1-\alpha}{\alpha}\frac{1/2}{p}}\left(V_H - \mu\right) ,$$

and following similar reasoning,

$$\Delta_b = \frac{1}{1 + \frac{1-\alpha}{\alpha}\frac{1/2}{1-p}}\left(\mu - V_L\right) .$$

To interpret these equations let us label the variables. If we think of asymmetric information as 'toxicity' then we can think of α as the prevalence of toxicity, $1-p$ and $V_H - \mu$ as the magnitude of buy-toxicity and $1-p$ and $\mu - V_L$ that of sell-toxicity. Then, the equations above describe how MMs adjust the ask-half-spread and the bid-half-spread, and increase it with the prevalence and magnitude of buy- and sell-toxicity.

5 Exercises

1. In the Grossman-Miller model developed in Subsection 2.1 assume that every time there is a trade there is a fee $\eta > 0$ per unit of the asset. For example, if in period 1 LT1 sells one unit of the asset, he receives $S_1 - \eta$. Show that

   - the equilibrium holdings in period 2 are:

$$q_2^{j,*} = \frac{\mathbb{E}[S_3 \,|\, \epsilon_2] - S_2}{\gamma\, \sigma^2} , \quad j \in \{MM, LT1, LT2\} ,$$

- the equilibrium price in period 1 is

$$S_1 = \mu - \frac{i}{n+1}\,\gamma\,\sigma^2\,,$$

- and LT1 holdings at the end of period 1 are

$$q_1^{LT1,*} = \frac{i}{n+1} + 2\,\frac{n}{n+1}\,\frac{\eta}{\gamma\,\sigma^2}\,.$$

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