pyMPC Documentation

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Chapter 1

Mathematical formulation

The MPC problem solved by pyMPC is:

$$
\begin{aligned}
&\arg\min_{\mathbf{x},\mathbf{u}} \overbrace{\big(x_N - x_{ref}\big)^\top Q_{x_N}\big(x_N - x_{ref}\big)}^{=J_{x_N}} + \overbrace{\sum_{k=0}^{N_p-1} \big(x_k - x_{ref}\big)^\top Q_x\big(x_k - x_{ref}\big)}^{J_x} + \\
&+ \overbrace{\sum_{k=0}^{N_p-1} \big(u_k - u_{ref}\big)^\top Q_u\big(u_k - u_{ref}\big)}^{J_u} + \overbrace{\sum_{k=0}^{N_p-1} \Delta u_k^\top Q_{\Delta u} \Delta u_k}^{J_{\Delta u}} && \text{(1.1a)} \\
&\text{subject to} \\
&x_{k+1} = Ax_k + Bu_k && \text{(1.1b)} \\
&u_{min} \leq u_k \leq u_{max} && \text{(1.1c)} \\
&x_{min} \leq x_k \leq x_{max} && \text{(1.1d)} \\
&\Delta u_{min} \leq \Delta u_k \leq \Delta u_{max} && \text{(1.1e)} \\
&x_0 = \bar{x} && \text{(1.1f)} \\
&u_{-1} = \bar{u} && \text{(1.1g)}
\end{aligned}
$$

where $\Delta u_k = u_k - u_{k-1}$.

The optimization variables are

$$
\begin{aligned}
\mathbf{x} &= \begin{bmatrix} x_0 & x_1 & \dots & x_{N_p} \end{bmatrix}, && \text{(1.2)} \\
\mathbf{u} &= \begin{bmatrix} u_0 & u_1 & \dots & u_{N_p-1} \end{bmatrix}, && \text{(1.3)} \\
& && \text{(1.4)}
\end{aligned}
$$

In a typical implementation, the MPC input is applied in *receding horizon.* At each time step i, the problem (1.1) is solved with $x_0 = x[i],\ u_{-1} = u[i-1]$

and an optimal input sequence $u_0, \dots, u_{N_p}$ is obtained. The first element of this sequence u_0 is the control input that is actually applied at time instant i. At time instant $i+1$, a new state $x[i+1]$ is measured (or estimated), and the process is iterated.

Thus, formally, the MPC control law is a (static) function of the current state and the previous input:

$$u_{MPC} = K(x[i], u[i-1]). \tag{1.5}$$

Note that this function also depends on the references x_{ref} and u_{ref} and on the system matrices A and B.

1.1 Quadratic Programming Formulation

The QP solver expects a problem with form:

$$\begin{aligned}
&\min \frac{1}{2} x^\top P x + q^\top x && (1.6a)\\
&\text{subject to} &&\\
&l \leq Ax \leq u && (1.6b)
\end{aligned}$$

The challenge here is to rewrite the MPC optimization problem (1.1) in form (1.6) to use the standard QP solver.

1.1.1 Cost function

By direct inspection, the non-constant terms of the cost function in Q_x are:

$$\begin{aligned}
J_{Q_x} = \frac{1}{2} \begin{bmatrix} x_0^\top & x_1^\top & \dots & x_{N_p-1}^\top \end{bmatrix}^\top \mathrm{blkdiag}(Q_x, Q_x, \dots, Q_x) \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N_p-1} \end{bmatrix} + \\
+ \begin{bmatrix} -x_{ref}^\top Q_x & -x_{ref}^\top Q_x & \dots & -x_{ref}^\top Q_x \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N_p-1} \end{bmatrix}^\top
\end{aligned} \tag{1.7}$$

and similarly for the term $J_{Q_{x_{N_p}}}$ and J_{Q_u}:

$$J_{Q_u} = \frac{1}{2}\begin{bmatrix} u_0^\top & u_1^\top & \dots & u_{N_p-1}^\top \end{bmatrix} \operatorname{blkdiag}(Q_u, Q_u, \dots, Q_u) \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N_p-1} \end{bmatrix} + \\ + \begin{bmatrix} -u_{ref}^\top Q_u & -u_{ref}^\top Q_u & \dots & -u_{ref}^\top Q_u \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N_p-1} \end{bmatrix} \quad (1.8)$$

For the terms in $Q_\Delta u$ we have instead

$$J_{\Delta u} = \frac{1}{2}\begin{bmatrix} u_0 & u_1 & \dots & u_{N_p-1} \end{bmatrix}^\top \begin{bmatrix} 2Q_{\Delta u} & -Q_{\Delta u} & 0 & \dots & \dots & 0 \\ -Q_{\Delta u} & 2Q_{\Delta u} & -Q_{\Delta u} & 0 & \dots & 0 \\ 0 & -Q_{\Delta u} & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & -Q_{\Delta u} & 2Q_{\Delta u} & -Q_{\Delta u} \\ 0 & 0 & 0 & 0 & -Q_{\Delta u} & Q_{\Delta u} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N_p-1} \end{bmatrix}^\top + \\ + \begin{bmatrix} -\bar{u}^\top Q_{\Delta u} & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N_p-1} \end{bmatrix}^\top \quad (1.9)$$

1.1.2 Constraints

Linear dynamics

Let us consider the linear equality constraints (1.1b) representing the system dynamics. These can be written in matrix form as

$$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N_p-1} \\ x_{N_p} \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 0 & \dots & 0 \\ A_d & 0 & \dots & 0 \\ 0 & A_d & \dots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \dots & A_d \end{bmatrix}}^{=\mathcal{A}} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N_p-1} \\ x_{N_p} \end{bmatrix} + \overbrace{\begin{bmatrix} 0 & 0 & \dots & 0 \\ B_d & 0 & \dots & 0 \\ 0 & B_d & \dots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \dots & B_d \end{bmatrix}}^{=\mathcal{B}} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N_p-2} \\ u_{N_p-1} \end{bmatrix} + \overbrace{\begin{bmatrix} \bar{x} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}}^{=\mathcal{C}}$$

(1.10)

we get a set of linear equality constraints representing the system dynamics (1.1b). These constraints can be written as

$$\begin{bmatrix}(\mathcal{A}-I) & \mathcal{B}\end{bmatrix}\begin{bmatrix}x\\ u\end{bmatrix} = \mathcal{C}. \tag{1.11}$$

Variable bounds: x **and** u

Bounds on x and u are readily implemented as

$$\begin{bmatrix}x_{min}\\ u_{min}\end{bmatrix} \leq \begin{bmatrix}I & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}x\\ u\end{bmatrix} \leq \begin{bmatrix}x_{max}\\ u_{max}\end{bmatrix}. \tag{1.12}$$

Variable bounds: Δu

$$\begin{bmatrix}u_{-1}+\Delta u_{min}\\ \Delta u_{min}\\ \vdots\\ \Delta u_{min}\end{bmatrix} \leq \begin{bmatrix}I & 0 & \dots & \dots & 0 & 0\\ -I & I & 0 & \dots & 0 & 0\\ 0 & -I & I & \dots & 0 & 0\\ \vdots & & & & & \\ 0 & 0 & \dots & 0 & -I & I\end{bmatrix}\begin{bmatrix}u_0\\ u_1\\ \vdots\\ u_{N_c-1}\end{bmatrix} \leq \begin{bmatrix}u_{-1}+\Delta u_{max}\\ \Delta u_{max}\\ \vdots\\ \Delta u_{max}\end{bmatrix} \tag{1.13}$$

Slack variables

Bounds on x may result in an problem unfeasible! A common solution is to transform the hard constraints in x into soft constraints by means of *slack variables* ϵ.

$$\begin{bmatrix}x_{min}\\ u_{min}\end{bmatrix} \leq \begin{bmatrix}I & 0 & I\\ 0 & I & 0\end{bmatrix}\begin{bmatrix}x\\ u\\ \epsilon\end{bmatrix} \leq \begin{bmatrix}x_{max}\\ u_{max}\end{bmatrix} \tag{1.14}$$

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1.2 Control Horizon

Sometimes, we may want to use a control horizon $N_c < N_p$ instead of the standard $N_c = N_p$. The input is constant for $N_c \geq Np$.

$$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N_p-1} \\ x_{N_p} \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 0 & \dots & 0 \\ A_d & 0 & \dots & 0 \\ 0 & A_d & \dots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \dots & A_d \end{bmatrix}}^{=\mathcal{A}} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N_p-1} \\ x_{N_p} \end{bmatrix} + \overbrace{\begin{bmatrix} 0 & 0 & \dots & 0 \\ B_d & 0 & \dots & 0 \\ 0 & B_d & \dots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \dots & B_d \\ 0 & 0 & \dots & \vdots \\ 0 & 0 & \dots & B_d \end{bmatrix}}^{=\mathcal{B}} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N_c-1} \\ \vdots \\ u_{N_c-1} \end{bmatrix} + \overbrace{\begin{bmatrix} \bar{x} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}}^{=\mathcal{C}} \tag{1.15}$$

The contributions J_{Q_u} of the cost function also changes:

$$\begin{aligned} J_{Q_u} = {} & \frac{1}{2} \begin{bmatrix} u_0^\top & u_1^\top & \dots & u_{N_p-1}^\top \end{bmatrix} \operatorname{blkdiag}\Big(Q_u, Q_u, \dots, (N_p - N_c + 1) Q_u \Big) \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N_p-1} \end{bmatrix} + \\ & + \begin{bmatrix} -u_{ref}^\top Q_u & -u_{ref}^\top Q_u & \dots & -(N_p - N_c + 1) u_{ref}^\top Q_u \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N_p-1} \end{bmatrix} \end{aligned} \tag{1.16}$$

Instead, $J_\Delta u$ does not change (because the input is constant for $k \geq N_c$!

Chapter 2

Alternative Implementation

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N_p-1} \\ x_{N_p} \end{bmatrix} = \overbrace{\begin{bmatrix} A_d \\ A_d^2 \\ \vdots \\ A_d^{N_p-1} \\ A_d^{N_p} \end{bmatrix}}^{=\mathcal{A}} x_0 + \overbrace{\begin{bmatrix} B_d & 0 & 0 & \dots & 0 & 0 \\ A_d B_d & B_d & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & 0 & 0 \\ A_d^{N_p-2} B_d & A_d^{N_p-3} B_d & A_d^{N_p-4} B_d & \dots & B_d & 0 \\ A_d^{N_p-1} B_d & A_d^{N_p-2} B_d & A_d^{N_p-3} B_d & \dots & A_d B_d & B_d \end{bmatrix}}^{=\mathcal{B}} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N_p-1} \end{bmatrix} \tag{2.1}$$

$$J_{Q_x} = \frac{1}{2} \left(\mathcal{A}x_0 + \mathcal{B}U - X_{ref}\right)^\top \mathcal{Q}_x \left(\mathcal{A}x_0 + \mathcal{B}U - X_{ref}\right) \tag{2.2}$$

$$J_{Q_u} = \frac{1}{2}(U - U_{ref})^\top \mathcal{Q}_u (U - U_{ref}) \tag{2.3}$$

$$J_{Q_{\Delta u}} = \frac{1}{2} U^\top \mathcal{Q}_{\Delta u} U + \begin{bmatrix} -u_{-1}{}^\top Q_{\Delta u} & 0 & \dots & 0 \end{bmatrix} U \tag{2.4}$$

Summing up and expanding terms:

$$\begin{multline} J = \frac{1}{2} U^\top \mathcal{B}^\top \mathcal{Q}_x \mathcal{B} U + \frac{1}{2} (\mathcal{A}x_o - X_{ref})^\top \mathcal{Q}_x (\mathcal{A}x_0 - X_{ref}) + (\mathcal{A}x_0 - X_{ref})^\top \mathcal{Q}_x \mathcal{B} U + \\ + \frac{1}{2} U^\top \mathcal{Q}_u U + \frac{1}{2} U_{ref}^\top \mathcal{Q}_u U^{ref} + -U_{ref}^\top \mathcal{Q}_u U + \\ + \frac{1}{2} U^\top \mathcal{Q}_{\Delta u} U + \begin{bmatrix} -u_{-1}{}^\top Q_{\Delta u} & 0 & \dots & 0 \end{bmatrix} U \end{multline} \tag{2.5}$$

Neglecting constant terms and collecting:

$$J = C + \frac{1}{2}U^\top \overbrace{\left(\mathcal{B}^\top \mathcal{Q}_x \mathcal{B} + \mathcal{Q}_u + \mathcal{Q}_{\Delta u}\right)}^{=\mathcal{P}} U + \\ \overbrace{\left[(\mathcal{A}x_o - X_{ref})^\top \mathcal{Q}_x \mathcal{B} - U_{ref}^\top \mathcal{Q}_u - \begin{bmatrix} {u_{-1}}^\top Q_{\Delta u} & 0 & \dots & 0 \end{bmatrix}\right)}^{=b^\top} U \tag{2.6}$$

Thus, we have

$$p = \mathcal{B}^\top \mathcal{Q}_x(\mathcal{A}x_0 - X_{ref}) - \mathcal{Q}_u U_{ref} + \begin{bmatrix} -Q_{\Delta u}u_{-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.7}$$

expanding

$$p = \overbrace{\mathcal{B}^\top \mathcal{Q}_x \mathcal{A}}^{=p_{x_0}} x_0 \ + \ \overbrace{-\mathcal{B}^\top \mathcal{Q}_x}^{p_{X_{ref}}} X_{ref} \ + \ \overbrace{-\mathcal{Q}_u}^{=p_{U_{ref}}} U_{ref} + \overbrace{\begin{bmatrix} -Q_{\Delta u} \\ 0 \\ \vdots \\ 0 \end{bmatrix}}^{=p_{u_{-1}}} u_{-1} \tag{2.8}$$

For an unconstrained problem the minimum is in

$$U^{opt} = -P^{-1}p \tag{2.9}$$

Expanding

$$U^{opt} = k_{x_0}x_0 + k_{X_{ref}}X_{ref} + k_{U_{ref}}U_{ref}k_{u_{-1}}u_{-1} \tag{2.10}$$

with

$$k_{x_0} = -P^{-1}p_{x_0} \tag{2.11}$$
$$k_{X_{ref}} = -P^{-1}p_{X_{ref}} \tag{2.12}$$
$$k_{U_{ref}} = -P^{-1}p_{U_{ref}} \tag{2.13}$$
$$k_{u_{-1}} = -P^{-1}p_{u_{-1}} \tag{2.14}$$