

# pyMPC Documentation

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# Chapter 1

## Mathematical formulation

The MPC problem solved by pyMPC is:

$$\begin{aligned} \arg \min_{\mathbf{x}, \mathbf{u}} & \underbrace{(x_N - x_{ref})^\top Q_{x_N} (x_N - x_{ref})}_{=J_{x_N}} + \underbrace{\sum_{k=0}^{N_p-1} (x_k - x_{ref})^\top Q_x (x_k - x_{ref})}_{J_x} + \\ & + \underbrace{\sum_{k=0}^{N_p-1} (u_k - u_{ref})^\top Q_u (u_k - u_{ref})}_{J_u} + \underbrace{\sum_{k=0}^{N_p-1} \Delta u_k^\top Q_{\Delta u} \Delta u_k}_{J_{\Delta u}} \end{aligned} \quad (1.1a)$$

subject to

$$x_{k+1} = Ax_k + Bu_k \quad (1.1b)$$

$$u_{min} \leq u_k \leq u_{max} \quad (1.1c)$$

$$x_{min} \leq x_k \leq x_{max} \quad (1.1d)$$

$$\Delta u_{min} \leq \Delta u_k \leq \Delta u_{max} \quad (1.1e)$$

$$x_0 = \bar{x} \quad (1.1f)$$

$$u_{-1} = \bar{u} \quad (1.1g)$$

where  $\Delta u_k = u_k - u_{k-1}$ .

The optimization variables are

$$\mathbf{x} = [x_0 \quad x_1 \quad \dots \quad x_{N_p}] , \quad (1.2)$$

$$\mathbf{u} = [u_0 \quad u_1 \quad \dots \quad u_{N_p-1}] , \quad (1.3)$$

$$(1.4)$$

In a typical implementation, the MPC input is applied in *receding horizon*. At each time step  $i$ , the problem (1.1) is solved with  $x_0 = x[i]$ ,  $u_{-1} = u[i-1]$

and an optimal input sequence  $u_0, \dots, u_{N_p}$  is obtained. The first element of this sequence  $u_0$  is the control input that is actually applied at time instant  $i$ . At time instant  $i + 1$ , a new state  $x[i + 1]$  is measured (or estimated), and the process is iterated.

Thus, formally, the MPC control law is a (static) function of the current state and the previous input:

$$u_{MPC} = K(x[i], u[i - 1]). \quad (1.5)$$

Note that this function also depends on the references  $x_{ref}$  and  $u_{ref}$  and on the system matrices  $A$  and  $B$ .

## 1.1 Quadratic Programming Formulation

The QP solver expects a problem with form:

$$\min \frac{1}{2} x^\top P x + q^\top x \quad (1.6a)$$

subject to

$$l \leq A x \leq u \quad (1.6b)$$

The challenge here is to rewrite the MPC optimization problem (1.1) in form (1.6) to use the standard QP solver.

### 1.1.1 Cost function

By direct inspection, the non-constant terms of the cost function in  $Q_x$  are:

$$\begin{aligned} J_{Q_x} = \frac{1}{2} \begin{bmatrix} x_0^\top & x_1^\top & \dots & x_{N_p-1}^\top \end{bmatrix}^\top \text{blkdiag}(Q_x, Q_x, \dots, Q_x) \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N_p-1} \end{bmatrix} + \\ + \begin{bmatrix} -x_{ref}^\top Q_x & -x_{ref}^\top Q_x & \dots & -x_{ref}^\top Q_x \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N_p-1} \end{bmatrix}^\top \end{aligned} \quad (1.7)$$

and similarly for the term  $J_{Q_{x_{N_p}}}$  and  $J_{Q_u}$ :

$$J_{Q_u} = \frac{1}{2} \begin{bmatrix} u_0^\top & u_1^\top & \dots & u_{N_p-1}^\top \end{bmatrix} \text{blkdiag}(Q_u, Q_u, \dots, Q_u) \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N_p-1} \end{bmatrix} +$$

$$+ \begin{bmatrix} -u_{ref}^\top Q_u & -u_{ref}^\top Q_u & \dots & -u_{ref}^\top Q_u \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N_p-1} \end{bmatrix} \quad (1.8)$$

For the terms in  $Q_\Delta u$  we have instead

$$J_{\Delta u} = \frac{1}{2} \begin{bmatrix} u_0 & u_1 & \dots & u_{N_p-1} \end{bmatrix}^\top \begin{bmatrix} 2Q_{\Delta u} & -Q_{\Delta u} & 0 & \dots & \dots & 0 \\ -Q_{\Delta u} & 2Q_{\Delta u} & -Q_{\Delta u} & 0 & \dots & 0 \\ 0 & -Q_{\Delta u} & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & -Q_{\Delta u} & 2Q_{\Delta u} & -Q_{\Delta u} \\ 0 & 0 & 0 & 0 & -Q_{\Delta u} & Q_{\Delta u} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N_p-1} \end{bmatrix}^\top +$$

$$+ \begin{bmatrix} -\bar{u}^\top Q_{\Delta u} & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N_p-1} \end{bmatrix}^\top \quad (1.9)$$

### 1.1.2 Constraints

#### Linear dynamics

Let us consider the linear equality constraints (1.1b) representing the system dynamics. These can be written in matrix form as

$$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N_p-1} \\ x_{N_p} \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 0 & \dots & 0 \\ A_d & 0 & \dots & 0 \\ 0 & A_d & \dots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \dots & A_d \end{bmatrix}}^{=A} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N_p-1} \\ x_{N_p} \end{bmatrix} + \overbrace{\begin{bmatrix} 0 & 0 & \dots & 0 \\ B_d & 0 & \dots & 0 \\ 0 & B_d & \dots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \dots & B_d \end{bmatrix}}^{=B} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N_p-2} \\ u_{N_p-1} \end{bmatrix} + \overbrace{\begin{bmatrix} \bar{x} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}}^{=C}$$

$$(1.10)$$

we get a set of linear equality constraints representing the system dynamics (1.1b). These constraints can be written as

$$[(\mathcal{A} - I) \quad \mathcal{B}] \begin{bmatrix} x \\ u \end{bmatrix} = \mathcal{C}. \quad (1.11)$$

**Variable bounds:  $x$  and  $u$**

Bounds on  $x$  and  $u$  are readily implemented as

$$\begin{bmatrix} x_{min} \\ u_{min} \end{bmatrix} \leq \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq \begin{bmatrix} x_{max} \\ u_{max} \end{bmatrix}. \quad (1.12)$$

**Variable bounds:  $\Delta u$**

$$\begin{bmatrix} u_{-1} + \Delta u_{min} \\ \Delta u_{min} \\ \vdots \\ \Delta u_{min} \end{bmatrix} \leq \begin{bmatrix} I & 0 & \dots & \dots & 0 & 0 \\ -I & I & 0 & \dots & 0 & 0 \\ 0 & -I & I & \dots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & \dots & 0 & -I & I \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N_c-1} \end{bmatrix} \leq \begin{bmatrix} u_{-1} + \Delta u_{max} \\ \Delta u_{max} \\ \vdots \\ \Delta u_{max} \end{bmatrix} \quad (1.13)$$

**Slack variables**

Bounds on  $x$  may result in an problem unfeasible! A common solution is to transform the hard constraints in  $x$  into soft constraints by means of *slack variables*  $\epsilon$ .

$$\begin{bmatrix} x_{min} \\ u_{min} \end{bmatrix} \leq \begin{bmatrix} I & 0 & I \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} x \\ u \\ \epsilon \end{bmatrix} \leq \begin{bmatrix} x_{max} \\ u_{max} \end{bmatrix} \quad (1.14)$$

## 1.2 Control Horizon

Sometimes, we may want to use a control horizon  $N_c < N_p$  instead of the standard  $N_c = N_p$ . The input is constant for  $N_c \geq N_p$ .

$$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N_p-1} \\ x_{N_p} \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 0 & \dots & 0 \\ A_d & 0 & \dots & 0 \\ 0 & A_d & \dots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \dots & A_d \end{bmatrix}}^{=A} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N_p-1} \\ x_{N_p} \end{bmatrix} + \overbrace{\begin{bmatrix} 0 & 0 & \dots & 0 \\ B_d & 0 & \dots & 0 \\ 0 & B_d & \dots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \dots & B_d \\ 0 & 0 & \dots & \vdots \\ 0 & 0 & \dots & B_d \end{bmatrix}}^{=B} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N_c-1} \\ \vdots \\ u_{N_c-1} \end{bmatrix} + \overbrace{\begin{bmatrix} \bar{x} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}}^{=C} \quad (1.15)$$

The contributions  $J_{Q_u}$  of the cost function also changes:

$$\begin{aligned}
J_{Q_u} &= \frac{1}{2} \begin{bmatrix} u_0^\top & u_1^\top & \dots & u_{N_p-1}^\top \end{bmatrix} \text{blkdiag} \left( Q_u, Q_u, \dots, (N_p - N_c + 1) Q_u \right) \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N_p-1} \end{bmatrix} + \\
&\quad + \begin{bmatrix} -u_{ref}^\top Q_u & -u_{ref}^\top Q_u & \dots & -(N_p - N_c + 1) u_{ref}^\top Q_u \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N_p-1} \end{bmatrix} \quad (1.16)
\end{aligned}$$

Instead,  $J_\Delta u$  does not change (because the input is constant for  $k \geq N_c$ !)





## Chapter 2

### Alternative Implementation

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N_p-1} \\ x_{N_p} \end{bmatrix} = \overbrace{\begin{bmatrix} A_d \\ A_d^2 \\ \vdots \\ A_d^{N_p-1} \\ A_d^{N_p} \end{bmatrix}}^{=A} x_0 + \overbrace{\begin{bmatrix} B_d & 0 & 0 & \dots & 0 & 0 \\ A_d B_d & B_d & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & 0 & 0 \\ A_d^{N_p-2} B_d & A_d^{N_p-3} B_d & A_d^{N_p-4} B_d & \dots & B_d & 0 \\ A_d^{N_p-1} B_d & A_d^{N_p-2} B_d & A_d^{N_p-3} B_d & \dots & A_d B_d & B_d \end{bmatrix}}^{=B} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N_p-1} \end{bmatrix} \quad (2.1)$$

$$J_{Q_x} = \frac{1}{2} (\mathcal{A}x_0 + \mathcal{B}U - X_{ref})^\top \mathcal{Q}_x (\mathcal{A}x_0 + \mathcal{B}U - X_{ref}) \quad (2.2)$$

$$J_{Q_u} = \frac{1}{2} (U - U_{ref})^\top \mathcal{Q}_u (U - U_{ref}) \quad (2.3)$$

$$J_{Q_{\Delta u}} = \frac{1}{2} U^\top \mathcal{Q}_{\Delta u} U + [-\bar{u}^\top \mathcal{Q}_{\Delta u} \quad 0 \quad \dots \quad 0] U \quad (2.4)$$

Summing up and expanding terms:

$$\begin{aligned} J = & \frac{1}{2} U^\top \mathcal{B}^\top \mathcal{Q}_x \mathcal{B} U + \frac{1}{2} (\mathcal{A}x_0 - X_{ref})^\top \mathcal{Q}_x (\mathcal{A}x_0 - X_{ref}) + (\mathcal{A}x_0 - X_{ref})^\top \mathcal{Q}_x \mathcal{B} U + \\ & + \frac{1}{2} U^\top \mathcal{Q}_u U + \frac{1}{2} U_{ref}^\top \mathcal{Q}_u U_{ref} - U_{ref}^\top \mathcal{Q}_u U + \\ & + \frac{1}{2} U^\top \mathcal{Q}_{\Delta u} U + [-\bar{u}^\top \mathcal{Q}_{\Delta u} \quad 0 \quad \dots \quad 0] U \quad (2.5) \end{aligned}$$

Neglecting constant terms and collecting:

$$J = C + \frac{1}{2} U^\top \overbrace{(\mathcal{B}^\top \mathcal{Q}_x \mathcal{B})}^{=\mathcal{P}} U + \overbrace{\left[ (\mathcal{A}x_o - X_{ref})^\top \mathcal{Q}_x \mathcal{B} - U_{ref}^\top \mathcal{Q}_u - [\bar{u}^\top Q_{\Delta u} \quad 0 \quad \dots \quad 0] \right]}^{=b^\top} U \quad (2.6)$$

Thus, we have

$$b = \mathcal{B}^\top \mathcal{Q}_x (\mathcal{A}x_o - X_{ref}) - \mathcal{Q}_u U_{ref} - \begin{bmatrix} -Q_{\Delta u} \bar{u} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (2.7)$$

For the unconstrained problem the minimum is in

$$U^{opt} = -P^{-1}b \quad (2.8)$$