Work in Progress

Answers to Exercises

Stochastic Calculus for Finance I: The Binomial Asset Pricing Model by *Steven E. Shreve*

Chapter 4 American Derivative Securities

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Please refer to the book for the exercises themselves. The text in front of each answer serves only as a summary of the question.

Exercise 4.1 In a three-period model with $S_0 = 4, u = 2, d = \frac{1}{2}$, let the interest rate be $r = \frac{1}{4}$ so the risk-neutral probabilities are $\tilde{p} = \tilde{q} = \frac{1}{2}$. Determine the time-zero prices of

(i) an American put V_0^P that expires at time three and has intrinsic value $g_P(s) = (4-s)^+$,

(ii) an American call V_0^C that expires at time three and has intrinsic value $g_C(s) = (s-4)^+$,

(iii) an American straddle V_0^S that expires at time three and has intrinsic value $g_S(s) = g_P(s) + g_C(s)$.

(iv) Explain why $V_0^S < V_0^P + V_0^C$.

Answer: The American algorithm is, for N and $n = N-1, N-2, \cdots, 0$

$$
\begin{aligned}
v_N(s) &= \max\{g(s), 0\} \\
v_n(s) &= \max\Big\{g(s), \frac{1}{1+r}\Big[\tilde{p}v_{n+1}(us) + \tilde{q}v_{n+1}(ds)\Big]\Big\}
\end{aligned}
$$

(i) Applying the American algorithm backward in time

```
                                                                         32
                                                                         max(-28, 0) = 0
                                                16
                                                max(0, 0) = 0
                        8                                                8
                        max(0, 0.32) = 0.32                              max(-4, 0) = 0
4                                               4
max(0, 0.928) = 0.928                           max(0, 0.8) = 0.8
                        2                                                2
                        max(2, 1.52) = 2                                 max(2, 0) = 2
                                                1
                                                max(3, 2.2) = 3
                                                                         0.5
                                                                         max(3.5, 0) = 3.5
```

we have the time-zero price for the American put $V_0^P = 0.928$.

(ii) Applying the American algorithm backward in time

```
                                                                         32
                                                                         max(28, 0) = 28
                                                16
                                                max(12, 12.8) = 12.8
```

```
                                            8                                               8
                                            max(4, 5.76) = 5.76                             max(4, 0) = 4
4                                                                       4
max(0, 2.56) = 2.56                                                     max(0, 1.6) = 1.6
                                            2                                               2
                                            max(0, 0.64) = 0.64                             max(-2, 0) = 0
                                                                        1
                                                                        max(0, 0) = 0
                                                                                            0.5
                                                                                            max(-3.5, 0) = 0
```

we have the time-zero price for the American call $V_0^C = 2.56$.

(iii) Note that the intrinsic value is $g_S(s) = (4-s)^+ + (s-4)^+ = |s-4|$. Applying the American algorithm backward in time

```
                                                                                            32
                                                                                            |32 - 4| = 28
                                                                        16
                                                                        max(12, 12.8) = 12.8
                                            8                                               8
                                            max(4, 6.08) = 6.08                             |8 - 4| = 4
4                                                                       4
max(0, 3.296) = 3.296                                                   max(0, 2.4) = 2.4
                                            2                                               2
                                            max(2, 2.16) = 2.16                             |2 - 4| = 2
                                                                        1
                                                                        max(3, 2.2) = 3
                                                                                            0.5
                                                                                            |0.5 - 4| = 3.5
```

we have the time-zero price for the American straddle $V_0^S = 3.296$.

(iv) Consider two portfolios. Portfolio A has an American put and an American call. Portfolio B has an American straddle. While the two portfolios always have the same intrinsic value, portfolio A has more flexibility in exercise: its holder may choose to exercise the two options at different times to the best of her interest. For example, she may exercise the put at time one but exercise the call at time three. Portfolio B, however, does not enjoy this flexibility. Intuitively, this explains why $V_0^S < V_0^C + V_0^P$. Mathematically, this can be expressed as

$$\begin{aligned}
V_0^S &= \max_{\tau \,\in\, \mathcal{S}} \widetilde{\mathbb{E}}\left[\mathbb{I}_{\{\tau \le N\}} \frac{1}{(1+r)^\tau} G_\tau^S\right] \\
&= \max_{\tau \,\in\, \mathcal{S}} \widetilde{\mathbb{E}}\left[\mathbb{I}_{\{\tau \le N\}} \frac{1}{(1+r)^\tau} (G_\tau^P + G_\tau^C)\right] \\
&\le \max_{\tau_1 \,\in\, \mathcal{S}} \widetilde{\mathbb{E}}\left[\mathbb{I}_{\{\tau_1 \le N\}} \frac{1}{(1+r)^{\tau_1}} G_{\tau_1}^P\right] + \max_{\tau_2 \,\in\, \mathcal{S}} \widetilde{\mathbb{E}}\left[\mathbb{I}_{\{\tau_2 \le N\}} \frac{1}{(1+r)^{\tau_2}} G_{\tau_2}^C\right] \\
&= V_0^P + V_0^C
\end{aligned}$$

where $\mathcal{S}$ is the set of all stopping times.

Exercise 4.2 In a two-period model with $S_0 = 4, u = 2, d = \frac{1}{2}$, let the interest rate be $r = \frac{1}{4}$ so the risk-neutral probabilities are $\tilde{p} = \tilde{q} = \frac{1}{2}$. The time-zero value of an American put with strike price $K = 5$ can be computed to be 1.36. Consider an agent who borrows 1.36 at time zero and buys the put. Explain how this agent can generate sufficient funds to pay off his loan.

Answer: The stock and option value processes are

```
                                          16
                                          max(-11, 0) = 0
                        8
                        max(0, 0.4) = 0.4
4                                         4
max(1, 1.36) = 1.36                       max(1, 0) = 1
                        2
                        max(3, 2) = 3
                                          1
                                          max(4, 0) = 4
```

Now the agent borrows $\$1.36$ at time zero and buys the put. The number of shares he ought to hold in the hedging portfolio at time zero is

$$-\Delta_0 = -\frac{V_1(H) - V_(T)}{S_1(H) - S_1(T)} = -\frac{0.4 - 3}{8 - 2} = 0.4\dot{3}$$

Thus, he has to borrow in addition $0.4\dot{3} \times \$4 = \$1.7\dot{3}$ from the money market at time zero. In total, the loan is $\$1.36 + \$1.7\dot{3} = \$3.09\dot{3}$. This loan amount will grow to $(1 + 25\%) \times \$3.09\dot{3} = \$3.8\dot{6}$ in time one, and to $(1 + 25\%)^2 \times \$3.09\dot{3} = \$4.8\dot{3}$ in time two, if unrepaid yet.

(a) If the stock price goes down to $\$2$ at time one, then the agent should exercise the put to collect $\$5 - \$2 = \$3$ payoff. The stock will be worth $0.4\dot{3} \times \$2 = \$0.8\dot{6}$. In total, the agent will have assets worth of $\$3 + \$0.8\dot{6} = \$3.8\dot{6}$, which are sufficient to repay the loan at time one.

(b) If the stock price goes up to $\$8$ at time one, then the new number of shares to hold is

$$-\Delta_1(H) = -\frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)} = -\frac{0 - 1}{16 - 4} = 0.08\dot{3}$$

Compared with his stock position at time zero, the agent should sell $0.4\dot{3} - 0.08\dot{3} = 0.35$ share. This will generate $0.35 \times \$8 = \$2.8$ in cash, which should be invested in the money market.

- If the stock price goes up to $\$16$ at time two, the put will expire out of money. The agent will have assets worth of $0.08\dot{3} \times \$16 + (1 + 25\%) \times \$2.8 = \$4.8\dot{3}$.
- If the stock price goes down to $\$4$ at time two, the agent may collect $\$5 - \$4 = \$1$ payoff from the option. The agent will have assets worth of $\$1 + 0.08\dot{3} \times \$4 + (1 + 25\%) \times \$2.8 = \$4.8\dot{3}$.

In either case at time two, the agent will have sufficient funds to repay the loan.

Exercise 4.3 In a three-period model with $S_0 = 4, u = 2, d = \frac{1}{2}$, let the interest rate be $r = \frac{1}{4}$ so the risk-neutral probabilities are $\tilde{p} = \tilde{q} = \frac{1}{2}$. Find the time-zero price and optimal exercise policy (optimal stopping time) for the path-dependent American derivative security whose intrinsic value is $(4 - \frac{1}{n+1}\sum_{j=0}^{n} S_j)^+$, for $n = 0, 1, 2, 3$ (a put on the average stock price).

Answer: Let $v_n(s, y)$ be the time-n value of the option given stock price s and path sum s. For $n = 3$, we have

$$v_3(s, y) = (4 - \frac{y}{4})^+$$

That is,

$$\begin{aligned} V_3(HHH) &= v_3(32, 60) = 0, & V_3(HHT) &= v_3(8, 36) = 0, \\ V_3(HTH) &= v_3(8, 24) = 0, & V_3(THH) &= v_3(8, 18) = 0, \\ V_3(HTT) &= v_3(2, 18) = 0, & V_3(THT) &= v_3(2, 12) = 1, \\ V_3(TTH) &= v_3(2, 9) = \tfrac{7}{4}, & V_3(TTT) &= v_3(\tfrac{1}{2}, \tfrac{15}{2}) = \tfrac{17}{8}. \end{aligned}$$

For $n = 0, 1, 2$

$$
\begin{aligned}
v_n(s,y) &= \max\Big\{(K - \frac{1}{n+1}y)^+,\ \frac{1}{1+r}\Big[\tilde{p}v_{n+1}(us, y+us) + \tilde{q}v_{n+1}(ds, y+ds)\Big]\Big\} \\
&= \max\Big\{(4 - \frac{1}{n+1}y)^+,\ \frac{2}{5}\Big[v_{n+1}(2s, y+2s) + v_{n+1}(\frac{s}{2}, y+\frac{s}{2})\Big]\Big\}
\end{aligned}
$$

Whenever the maximum is taking the first term, it is more preferable to exercise at time n than at a later time. Working backward in time, we have

$$
\begin{aligned}
V_2(HH) &= v_2(16,28) = \max\{(4-\tfrac{28}{3})^+,\ \tfrac{2}{5}[v_3(32,60)+v_3(8,36)]\} = \max(0,0) = 0 \\
V_2(HT) &= v_2(4,16) = \max\{(4-\tfrac{16}{3})^+,\ \tfrac{2}{5}[v_3(8,24)+v_3(2,18)]\} = \max(0,0) = 0 \\
V_2(TH) &= v_2(4,10) = \max\{(4-\tfrac{10}{3})^+,\ \tfrac{2}{5}[v_3(8,18)+v_3(2,12)]\} = \max(\tfrac{2}{3},\ \tfrac{2}{5}) = \tfrac{2}{3} \quad \cdots \text{exercise} \\
V_2(TT) &= v_2(1,7) = \max\{(4-\tfrac{7}{3})^+,\ \tfrac{2}{5}[v_3(2,9)+v_3(\tfrac{1}{2},\tfrac{15}{2})]\} = \max(\tfrac{5}{3},\ \tfrac{31}{20}) = \tfrac{5}{3} \quad \cdots \text{exercise} \\
V_1(H) &= v_1(8,12) = \max\{(4-6)^+,\ \tfrac{2}{5}[v_2(16,28)+v_2(4,16)]\} = \max(0,0) = 0 \\
V_1(T) &= v_1(2,6) = \max\{(4-3)^+,\ \tfrac{2}{5}[v_2(4,10)+v_2(1,7)]\} = \max(1,\tfrac{15}{14}) = 1 \quad \cdots \text{exercise} \\
V_0 &= v_0(4,4) = \max\{(4-4)^+,\ \tfrac{2}{5}[v_1(8,12)+v_2(2,6)]\} = \max(0,\tfrac{2}{5}) = \tfrac{2}{5}
\end{aligned}
$$

Thus, time-zero price of the option is $V_0 = \frac{2}{5} = 0.4$.

The optimal exercise policy is to exercise if the first coin toss is T, and never to exercise if otherwise. The optimal stopping time is

$$
\begin{aligned}
\tau(THH) = \tau(THT) = \tau(TTH) = \tau(TTT) = 1, \\
\tau(HHH) = \tau(HHT) = \tau(HTH) = \tau(HTT) = \infty.
\end{aligned}
$$

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