Work in Progress

Answers to Exercises

Stochastic Calculus for Finance I: The Binomial Asset Pricing Model by *Steven E. Shreve*

Chapter 6 Interest-Rate-Dependent Assets

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Please refer to the book for the exercises themselves. The text in front of each answer serves only as a summary of the question.

In what follows, I use the notation B_n^m to denote the time-n price of a bond that matures at time m. This corresponds to the $B_{n,m}$ notation that Shreve uses in the book but is more compact.

Exercise 6.2 Verify that the discounted value of the static hedging portfolio for a forward is a martingale under $\tilde{\mathbb{P}}$.

Answer: To hedge a short position in a forward contract that is initiated at time n with delivery date m, the agent should, at time n, long 1 share of stock and short S_n/B_n^m unit of m-maturity zero-coupon bond. This constructs a static hedging portfolio. The time-n value of the hedging portfolio is

$$X_n = S_n - \Big(\frac{S_n}{B_n^m}\Big)B_n^m \ (=0)$$

To show its discounted value is $\tilde{\mathbb{P}}$-martingale, note that

$$\begin{aligned}
\tilde{\mathbb{E}}_n(D_{n+1}X_{n+1}) &= \tilde{\mathbb{E}}_n\Big[D_{n+1}\Big(S_{n+1} - \Big(\frac{S_n}{B_n^m}\Big)B_{n+1}^m\Big)\Big] \\
&= \tilde{\mathbb{E}}_n(D_{n+1}S_{n+1}) - \Big(\frac{S_n}{B_n^m}\Big)\tilde{\mathbb{E}}_n(D_{n+1}B_{n+1}^m) && \cdots \text{linearity} \\
&= D_nS_n - \Big(\frac{S_n}{B_n^m}\Big)D_nB_n^m && \cdots \text{discounted stock, bond prices are } \tilde{\mathbb{P}}\text{-martingale} \\
&= D_nX_n
\end{aligned}$$

Note: that discounted bond prices are $\tilde{\mathbb{P}}$-martingale can be shown by, for $0 \le k \le n \le m$,

$$\tilde{\mathbb{E}}_k(D_nB_n^m) = \tilde{\mathbb{E}}_k\Big[D_n\tilde{\mathbb{E}}_n\big(\frac{D_m}{D_n}\big)\Big] = \tilde{\mathbb{E}}_k[\tilde{\mathbb{E}}_n(D_m)] = \tilde{\mathbb{E}}_k(D_m) = \tilde{\mathbb{E}}_k\big(D_k\cdot\frac{D_m}{D_k}\big) = D_k\tilde{\mathbb{E}}_k\big(\frac{D_m}{D_k}\big) = D_kB_k^m$$

Exercise 6.3 Use properties of conditional expectations to show that

$$\frac{1}{D_n}\tilde{\mathbb{E}}_n[D_{m+1}R_m] = B_n^m - B_n^{m+1}$$

Answer: Note that $(1+R_m) = \frac{D_m}{D_{m+1}} \implies R_m = \frac{D_m}{D_{m+1}} - 1$. We have

$$
\begin{aligned}
\frac{1}{D_n}\tilde{\mathbb{E}}_n[D_{m+1}R_m] &= \frac{1}{D_n}\tilde{\mathbb{E}}_n\Big[D_{m+1}\big(\frac{D_m}{D_{m+1}}-1\big)\Big] \\
&= \frac{1}{D_n}\tilde{\mathbb{E}}_n[D_m - D_{m+1}] \\
&= \tilde{\mathbb{E}}_n\Big[\frac{D_m}{D_n}\Big] - \tilde{\mathbb{E}}_n\Big[\frac{D_{m+1}}{D_n}\Big] && \cdots \text{take in what is known, linearity} \\
&= B_n^m - B_n^{m+1} && \cdots \text{definition}
\end{aligned}
$$

Remark: What's the difference between

- (a) the time-n contract price to pay R_m at time $m+1$ (see *page 155*), and
- (b) the time-n forward price to deliver R_m at time $m+1$ (see *page 156*)?

Expressed in formula, the former is $B_n^{m+1}F_n^m = B_n^m - B_n^{m+1}$ while the latter is just F_n^m.

But the fundamental difference is that the former is the price to pay at time n, while the latter, as any other strike price of a forward contract, is the price to pay at time $m+1$.

An example with concrete numbers.

Say $n=3$, $m=7$. At time $n=3$, let's say $B_n^m = B_3^7 = \$0.93$ and $B_n^{m+1} = B_3^8 = \$0.87$.

It can be calculated that the time-n forward interest rate to govern the $[m, m+1]$ period is
$F_n^m = \dfrac{B_n^m}{B_n^{m+1}} - 1 = \dfrac{\$0.93}{\$0.87} - 1 = 6.90\%$.

At time $n=3$, we do not know $R_m = R_7$ which is not unveiled until time $m=7$.

However, we may, at time $n=3$, sign a contract that promises to pay $R_m = R_7$ at time $m+1=8$.

For this contract we charge a no-arbitrage price at $B_n^{m+1}F_n^m = \$0.87 \times 6.90\% = \$0.06$, which the counterparty must pay immediately at time $n=3$, for the entitlement to receive R_7 at time $m+1=8$. This amount can also be worked out by $B_n^m - B_n^{m+1} = \$0.93 - \$0.87 = \$0.06$.

As another case, we may short a forward contract, which promises to deliver $R_m = R_7$ at time $m+1=8$. Out counterparty, who long the forward contract, is required to pay a strike price fixed at $F_n^m = \$0.069$ at time $m+1=8$ in exchange for the delivery of $R_m = R_7$.

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