

Answers to Exercises

Stochastic Calculus for Finance II: Continuous-Time Models by Steven E. Shreve

Chapter 3 Brownian Motion

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Please refer to the book for the exercises themselves. The text in front of each answer serves only as a summary of the question.

Exercise 3.1 For $0 \leq t < u_1 < u_2$, show that $W(u_2) - W(u_1)$ is independent of $\mathcal{F}(t)$.

Answer:

1. Definition of Brownian motion's filtration (independence of future increment) $\implies W(u_2) - W(u_1)$ is independent of $\mathcal{F}(u_1) \implies \sigma(W(u_2) - W(u_1))$ is independent of $\mathcal{F}(u_1)$
2. Definition of Brownian motion's filtration (information accumulates), $t < u_1 \implies \mathcal{F}(t) \subset \mathcal{F}(u_1)$

By (1) and (2), for any event $A \in \sigma(W(u_2) - W(u_1))$, $B \in \mathcal{F}(t) \subset \mathcal{F}(u_1)$, we have $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) \implies \sigma(W(u_2) - W(u_1))$ and $\mathcal{F}(t)$ are independent $\implies W(u_2) - W(u_1)$ is independent of $\mathcal{F}(t)$.

Exercise 3.2 Show $W^2(t) - t$ is a martingale.

Answer: For $0 \leq s < t \leq T$, we have

$$\begin{aligned} & \mathbb{E}[W^2(t) - t \mid \mathbb{F}(s)] \\ &= \mathbb{E}[(W(t) - W(s) + W(s))^2 - t \mid \mathbb{F}(s)] \\ &= \mathbb{E}[(W(t) - W(s))^2 + 2(W(t) - W(s))W(s) + W^2(s) - t \mid \mathbb{F}(s)] \\ &= \mathbb{E}[(W(t) - W(s))^2] + 2\mathbb{E}[W(t) - W(s)]W(s) + W^2(s) - t && \dots \text{independence, take out what is known} \\ &= (t - s) + 2 \cdot 0 \cdot W(s) + W^2(s) - t && \dots \text{Brownian motion's definition} \\ &= W^2(s) - s \end{aligned}$$

Thus, $W^2(t) - t$ is a martingale.

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