Answers to Exercises

Stochastic Calculus for Finance I: The Binomial Asset Pricing Model by *Steven E. Shreve*

Chapter 1 The Binomial No-Arbitrage Pricing Model

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Please refer to the book for the exercises themselves. The text in front of each answer serves only as a summary of the question.

Exercise 1.1 Show that $0 < d < 1 + r < u$ precludes arbitrage.

Answer: Arbitrage is defined as a trading strategy that begins with zero wealth and ends with nonnegative wealth in all cases, and positive wealth in some cases. That is, a trading strategy such that initial wealth $X_0 = 0$, and terminal wealth $X_1(\omega) \geq 0$ for all $\omega \in \Omega$, and $X_1(\omega) > 0$ for some $\omega \in \Omega$.

Now with $X_0 = 0$, and $X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0)$ for two states H and T of the coin toss, we have

$$\begin{cases} X_1(H) = \Delta_0 S_1(H) + (1+r)(X_0 - \Delta_0 S_0) = \Delta_0 u S_0 + (1+r)(X_0 - \Delta_0 S_0) \\ X_1(T) = \Delta_0 S_1(T) + (1+r)(X_0 - \Delta_0 S_0) = \Delta_0 d S_0 + (1+r)(X_0 - \Delta_0 S_0) \end{cases}$$

Eliminating Δ_0, we arrive at

$$0 = X_0 = \frac{1}{1+r}\left[\frac{1+r-d}{u-d} X_1(H) + \frac{u-1-r}{u-d} X_1(T)\right]$$

which reduces to

$$0 = (1+r-d)\, X_1(H) + (u-1-r)\, X_1(T)$$

Given assumption $0 < d < 1 + r < u$, if $X_1(H) > 0$, then $X_1(T) < 0$. Vice versa, if $X_1(T) > 0$, then $X_1(H) < 0$.

Thus, we have shown that given the assumption $0 < d < 1 + r < u$, we cannot observe nonnegative terminal wealth in all cases with the existence of positive terminal wealth in some cases. In other words, arbitrage cannot exist in the model.

Exercise 1.2 Show that arbitrage does not exist in a situation involving buying Δ_0 shares of stock and Γ_0 options.

Answer: The terminal wealth at time 1 is

$$X_1 = \Delta_0 S_1 + \Gamma_0 (S_1 - 5)^+ - \frac{5}{4}(4\Delta_0 + 1.20\Gamma_0)$$

Plugging in the two states of the time-1 stock price $S_1(H) = 8$ and $S_1(T) = 2$, we have

$$\begin{cases} X_1(H) = 8\Delta_0 + 3\Gamma_0 - 5\Delta_0 - 1.5\Delta_0 = 3\Delta_0 + 1.5\Gamma_0 \\ X_1(T) = 2\Delta_0 + 0\Gamma_0 - 5\Delta_0 - 1.5\Delta_0 = -3\Delta_0 - 1.5\Gamma_0 \end{cases}$$

while implies $X_1(H) + X_1(T) = 0$, or that one being positive will lead to the other being negative, so that arbitrage cannot exist in the model.

Exercise 1.3 Use risk-neutral pricing formula to find the time-zero price V_0 for a derivative security that pays off the stock price at time one (This can be regarded as an European call option with $K = 0$).

Answer: $V_0 = \dfrac{1}{1+r}\Big[\tilde{p}V_1(H) + \tilde{q}V_1(T)\Big] = \dfrac{1}{1+r}\Big(\tilde{p}S_1 + \tilde{q}S_1\Big) = \dfrac{1}{1+r}\Big[\big(\tilde{p}+\tilde{q}\big)S_1\Big] = \dfrac{S_1}{1+r}.$

Exercise 1.4 In the proof of Theorem 1.1.2 (replication in the multiperiod binomial model), show under the induction hypothesis that

$$X_{n+1}(\omega_1\omega_2\cdots\omega_n T) = V_{n+1}(\omega_1\omega_2\cdots\omega_n T)$$

Answer: Fix the first n coin tosses $\omega_1\omega_2\cdots\omega_n$ and suppress it. We have

$$\begin{aligned}
X_{n+1}(T) &= \Delta_n S_{n+1}(T) + (1+r)(X_n - \Delta_n S_n) \\
&= (1+r)X_n + \Delta_n\big[S_{n+1}(T) - (1+r)S_n\big] \\
&= (1+r)V_n + \frac{V_{n+1}(H) - V_{n+1}(T)}{S_{n+1}(H) - S_{n+1}(T)}\Big[S_{n+1}(T) - (1+r)S_n\Big] & \cdots \text{ by hypothesis, } X_n = V_n \\
&= (1+r)V_n + \frac{V_{n+1}(H) - V_{n+1}(T)}{u-d}\Big[d - (1+r)\Big] \\
&= (1+r)\Big(\frac{1}{1+r}\Big)\Big[\tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T)\Big] + \Big[-\tilde{p}V_{n+1}(H) + \tilde{p}V_{n+1}(T)\Big] \\
&= \Big(\tilde{q} + \tilde{p}\Big)V_{n+1}(T) \\
&= V_{n+1}(T)
\end{aligned}$$

Exercise 1.5 Walk through the hedging strategy along a particular path (HT) in a three-period binomial model.

Answer: Start at time one with the portfolio valued at $X_1(H) = V_1(H) = \$2.24$. The agent should take a position of

$$\Delta_1(H) = \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)} = \frac{3.2 - 2.4}{16 - 4} = 0.0\dot{6}$$

share in the stock. This will leave her $X_1(H) - \Delta_1(H)S_1(H) = \$2.24 - 0.0\dot{6} \times \$8 = \$1.70\dot{6}$ to invest in the money market.

Next, at time two, if the stock goes up again, she will have a portfolio valued at

$$X_2(HH) = \Delta_1(H)S_2(HH) + (1+r)\big[X_1(H) - \Delta_1(H)S_1(H)\big] = 0.0\dot{6} \times \$16 + (1+25\%) \times \$1.70\dot{6} = \$3.2$$

If, at time two, the stock goes down, her portfolio will be worth

$$X_2(HT) = \Delta_1(H)S_2(HT) + (1+r)\big[X_1(H) - \Delta_1(H)S_1(H)\big] = 0.0\dot{6} \times \$4 + (1+25\%) \times \$1.70\dot{6} = \$2.4$$

We have shown that $X_2(HH) = V_2(HH) = \$3.2$ and $X_2(HT) = V_2(HT) = \$2.4$.

Assuming that the stock goes up in the first period and down in second period, the agent should take a position of

$$\Delta_2(HT) = \frac{V_3(HTH) - V_3(HTT)}{S_3(HTH) - S_3(HTT)} = \frac{0-6}{8-2} = -1$$

share in the stock. That is, she should short sell 1 share of stock. This will leave her $X_2(HT) - \Delta_2(HT)S_2(HT) = \$2.4 - (-1) \times \$4 = \$6.4$ to invest in the money market.

Finally, at time three, if the stock goes up, she will have a portfolio valued at

$$X_3(HTH) = \Delta_2(HT)S_3(HTH) + (1+r)\big[X_2(HT) - \Delta_2(HT)S_2(HT)\big] = (-1) \times \$8 + (1+25\%) \times \$6.4 = \$0$$

If, at time three, the stock goes down, her portfolio will be worth

$X_3(HTT) = \Delta_2(HT)S_3(HTT) + (1+r)\big[X_2(HT) - \Delta_2(HT)S_2(HT)\big] = (-1) \times \$2 + (1+25\%) \times \$6.4 = \$6$

We have shown that $X_3(HTH) = V_3(HTH) = \$0$ and $X_3(HTT) = V_3(HTT) = \$6$. In other words, she has hedged her short position in the option.

Exercise 1.6 (Hedging a Long Position: One Period) In a given example, specify how a bank's trader should invest in the stock and money markets to earn the interest rate on the capital tied up by a long position in a European call.

Answer: To hedge a long position in a European call with strike $K = \$5$, valued at $V_0 = \$1.20$, the trader should short sell

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = \frac{(8-5)^+ - (2-5)^+}{8-2} = \frac{3-0}{8-2} = \frac{1}{2} = 0.5$$

share of stock. This will generate $\Delta_0 S_0 = 0.5 \cdot \$4 = \$2$ in cash, which should be invested in the money market.

Next, at time one, if the stock goes up to $S_1(H) = \$8$, the trader will collect \$3 from option payoff, and get $(1+r)\Delta_0 S_0 = (1+25\%) \times \$2 = \$2.5$ from the money market, but have to spend $\Delta_0 S_1(H) = 0.5 \times \$8 = \$4$ to buy back the stock to close out the short selling position. In total, the trader will have $\$3 + \$2.5 - \$4 = \$1.5$.

If, at time one, the stock goes down to $S_1(T) = \$2$, there will be no option payoff to collect. The trader will still get $\$2.5$ from the money market, and need to spend $\Delta_0 S_1(T) = 0.5 \times \$2 = \$1$ to buy back the stock to close out the short selling position. In total, the trader will, again, have $\$2.5 - \$1 = \$1.5$.

We have shown a hedging strategy in which, given $\$1.2$ of capital tied up in a long call option, the trader can always earn 25% interest rate and get $(1+25\%) \times \$1.2 = \$1.5$ after one period.

Exercise 1.7 (Hedging a Long Position: Multiple Periods) Specify how a bank's trader should invest in the stock and money markets to earn the interest rate on the capital tied up by a long position in a lookback option in a three-period binomial model.

Answer: At each period n, the trader should use delta-hedging formula

$$\Delta_n(\omega_1 \cdots \omega_n) = \frac{V_{n+1}(\omega_1 \cdots \omega_n H) - V_{n+1}(\omega_1 \cdots \omega_n T)}{S_{n+1}(\omega_1 \cdots \omega_n H) - S_{n+1}(\omega_1 \cdots \omega_n T)}$$

to work out the number of shares Δ_n to short sell or to hold in the hedging portfolio. Since the aim is to hedge a long option position in the option, $\Delta_n > 0$ implies short selling, while $\Delta_n < 0$ implies holding the stock. Then the trader should work out the money to borrow from or invest in the money market with $V_n(\omega_1 \cdots \omega_n) - \Delta_n S_n(\omega_1 \cdots \omega_n)$. Again, since the aim is to hedge a long option position, a positive number implies borrowing, while a negative number implies investing. Note that the hedging strategy is *self-financing* in that any change in the stock position is funded by (or will lead to) a change in the money market position.

In what follows, we walk through the hedging portfolio from the perspective of a balance sheet, assuming that the coin tosses are THH or THT. Note that regardless of the coin toss result the total value is always $(1+25\%)^3 \times \$1.376 = \$2.6875$ at time three.

Time	Assets	Liabilities	Portfolio Value	Option Value	Total Value
0 (after hedging)	cash \$0.6933	stock \$0.6933	$\$0$	$\$1.376$	$\$1.376$
1 (T : before hedging)	cash $\$0.8667$	stock $\$0.3467$	$\$0.52$	$\$1.2$	$\$1.72$
1 (T : after hedging)	stock $\$0.9333$	cash $\$0.4133$	$\$0.52$	$\$1.2$	$\$1.72$
2 (TH : before hedging)	stock $\$1.8667$	cash $\$0.5167$	$\$1.35$	$\$0.8$	$\$2.15$
2 (TH : after hedging)	stock $\$1.3333$ + cash $\$0.0167$		$\$1.35$	$\$0.8$	$\$2.15$
3 (THH)	stock $\$2.6667$ + cash $\$0.0280$		$\$2.6875$	$\$0$	$\$2.6875$
3 (THT)	stock $\$0.6667$ + cash $\$0.0280$		$\$0.6875$	$\$2$	$\$2.6875$

Exercise 1.8 (Asian Option) Consider the three-period model with $S_0 = 4, u = 2, d = \frac{1}{2}, r = \frac{1}{4}$ so that $\tilde{p} = \tilde{q} = \frac{1}{2}$. Consider an Asian call option that expires at time three and has strike $K = 4$. Define $Y_n = \sum_{k=0}^{n} S_k$. Let $v_n(s, y)$ denote the price of this option at time n if $S_n = s$ and $Y_n = y$.

(i) Develop an algorithm for computing v_n recursively.

(ii) Compute $v_0(4, 4)$, the price of the Asian at time zero.

(iii) Provide a formula for $\delta_n(s, y)$, the number of shares of stock that should be held by the replicating portfolio.

Answer: (i) For $n = 3$

$$v_n(s,y) = \left(\frac{y}{n+1} - K\right)^+ = \left(\frac{y}{4} - 4\right)^+$$

For $n = 0, 1, 2$

$$v_n(s,y) = \frac{1}{1+r}\Big[\tilde{p}v_{n+1}(us, y+us) + \tilde{q}v_{n+1}(ds, y+ds)\Big] = \frac{2}{5}\Big[v_{n+1}(2s, y+2s) + v_{n+1}(\frac{s}{2}, y+\frac{s}{2})\Big]$$

(ii) Applying the above algorithm, for $n = 3$

$$\begin{aligned}
v_3(32,60) &= \left(\frac{1}{4}\cdot 60 - 4\right)^+ = 11, & v_3(8,36) &= \left(\frac{1}{4}\cdot 36 - 4\right)^+ = 5,\\
v_3(8,24) &= \left(\frac{1}{4}\cdot 24 - 4\right)^+ = 2, & v_3(8,18) &= \left(\frac{1}{4}\cdot 18 - 4\right)^+ = 0.5,\\
v_3(2,18) &= \left(\frac{1}{4}\cdot 18 - 4\right)^+ = 0.5, & v_3(2,12) &= \left(\frac{1}{4}\cdot 12 - 4\right)^+ = 0,\\
v_3(2,9) &= \left(\frac{1}{4}\cdot 9 - 4\right)^+ = 0, & v_3(0.5,7.5) &= \left(\frac{1}{4}\cdot 0.5 - 4\right)^+ = 0.
\end{aligned}$$

For $n = 0, 1, 2$

$$\begin{aligned}
v_2(16,28) &= \frac{2}{5}\Big[v_3(32,60) + v_3(8,36)\Big] = \frac{2}{5}\cdot(11+5) = 6.4\\
v_2(4,16) &= \frac{2}{5}\Big[v_3(8,24) + v_3(2,18)\Big] = \frac{2}{5}\cdot(2+0.5) = 1\\
v_2(4,10) &= \frac{2}{5}\Big[v_3(8,18) + v_3(2,12)\Big] = \frac{2}{5}\cdot(0.5+0) = 0.2\\
v_2(1,7) &= \frac{2}{5}\Big[v_3(2,9) + v_3(0.5,7.5)\Big] = \frac{2}{5}\cdot(0+0) = 0\\
v_1(8,12) &= \frac{2}{5}\Big[v_2(16,28) + v_2(4,16)\Big] = \frac{2}{5}\cdot(6.4+1) = 2.96\\
v_1(2,6) &= \frac{2}{5}\Big[v_2(4,10) + v_2(1,7)\Big] = \frac{2}{5}\cdot(0.2+0) = 0.08\\
v_0(4,4) &= \frac{2}{5}\Big[v_1(8,12) + v_1(2,6)\Big] = \frac{2}{5}\cdot(2.96+0.08) = 1.216
\end{aligned}$$

(iii) The number of shares of stock that should be held by the replicating portfolio at time n if $S_n = s$ and $Y_n = y$ is

$$\delta_n(s,y) = \frac{v_{n+1}(us, y+us) - v_{n+1}(ds, y+ds)}{us - ds} = \frac{v_{n+1}(2s, y+2s) - v_{n+1}(\frac{s}{2}, y+\frac{s}{2})}{\frac{3}{2}s}$$

Exercise 1.9 (Stochastic Volatility, Random Interest Rate) In a binomial model whose up and down factors and interest rate depend on coin tosses,

(i) Provide an algorithm for determining the time-zero price for a derivative security.

(ii) Provide a formula the number of shares of stock that should be held by the replicating portfolio.

(iii) Suppose $S_0 = 80$, with each head the stock price goes up by 10, each tail down by 10. Assume zero interest rate. Price a European call with strike price $K = 80$ which expires at time five.

Answer: (i) We construct a replicating portfolio whose terminal payoff is identical with that of the derivative security in any case. That is,

$$X_N(\omega_1 \cdots \omega_N) = V_N(\omega_1 \cdots \omega_N)$$

Let ω denote an arbitrary sequence of $\omega_1 \cdots \omega_n$. The wealth process of the replicating portfolio is governed by

$$\begin{cases} X_{n+1}(\omega H) = \Delta_n(\omega)S_n(\omega T) + [1 + r_n(\omega)][X_n(\omega) - \Delta_n(\omega)S_n(\omega)] \\ X_{n+1}(\omega T) = \Delta_n(\omega)S_n(\omega T) + [1 + r_n(\omega)][X_n(\omega) - \Delta_n(\omega)S_n(\omega)] \end{cases}$$

Let V_n denote X_n as we solve recursively and let

$$\begin{cases} \tilde{p}_n(\omega) = \dfrac{[1 + r_n(\omega)]S_n(\omega) - S_{n+1}(\omega T)}{S_{n+1}(\omega H) - S_{n+1}(\omega T)} \\ \tilde{q}_n(\omega) = 1 - \tilde{p}_n(\omega) = \dfrac{S_{n+1}(\omega H) - [1 + r_n(\omega)]S_n(\omega)}{S_{n+1}(\omega H) - S_{n+1}(\omega T)} \end{cases}$$

we have

$$V_n(\omega) = \frac{1}{1 + r_n(\omega)} \left[\tilde{p}_n(\omega)V_{n+1}(\omega H) + \tilde{q}_n(\omega)V_{n+1}(\omega T) \right]$$

for $n = 0, 1, \cdots, N - 1$, where V_N is known for the corresponding coin-toss result. Note that, given coin-toss results, we can determine $\{S_n\}_{n=0}^{N}$ and $\{r_n\}_{n=0}^{N-1}$ and thus can work out $\tilde{p}_n$ and $\tilde{q}_n$. Working backwards in time from V_N, the above equation will determine V_0, the time-zero price for the derivative security.

(ii) From the wealth process of the replicating portfolio, we have

$$\Delta_n(\omega) = \frac{V_{n+1}(\omega H) - V_{n+1}(\omega T)}{S_{n+1}(\omega H) - S_{n+1}(\omega T)}$$

which tells the number of shares of stock that should be held by the replicating portfolio.

(iii) Given this special setting, we have

$$\tilde{p}_n(\omega) = \frac{[1 + r_n(\omega)]S_n(\omega) - S_{n+1}(\omega T)}{S_{n+1}(\omega H) - S_{n+1}(\omega T)} = \frac{(1 + 0) \cdot S_n(\omega) - [S_n(\omega) - 10]}{[S_n(\omega) + 10] - [S_n(\omega) - 10]} = \frac{10}{20} = 0.5$$

That is, the value of $\tilde{p}_n(\omega)$ is independent of time n and coin-toss result ω. So we write $\tilde{p}_n(\omega)$ as $\tilde{p}$. The formula in the recursive algorithm now reduces to

$$V_n(\omega) = \frac{V_{n+1}(\omega H) + V_{n+1}(\omega T)}{2}$$

For computational considerations, we work on stock price s, rather than coin-toss result ω, and have

$$v_n(s) = \frac{v_{n+1}(s + 10) + v_{n+1}(s - 10)}{2}$$

for $n = 0, 1, \cdots, N - 1$, where $v_N(s) = (s - K)^+$.

Thus, we have

$$v_5(130) = (130-80)^+ = 50, \qquad v_5(110) = (110-80)^+ = 30, \qquad v_5(90) = (90-80)^+ = 10,$$
$$v_5(70) = (70-80)^+ = 0, \qquad v_5(50) = (50-80)^+ = 0, \qquad v_5(30) = (30-80)^+ = 0$$

and

$$\begin{aligned}
v_4(120) &= \frac{1}{2}\cdot[v_5(130)+v_5(110)] = \frac{1}{2}\cdot(50+30) = 40\\
v_4(100) &= \frac{1}{2}\cdot[v_5(110)+v_5(90)] = \frac{1}{2}\cdot(30+10) = 20\\
v_4(80) &= \frac{1}{2}\cdot[v_5(90)+v_5(70)] = \frac{1}{2}\cdot(10+0) = 5\\
v_4(60) &= \frac{1}{2}\cdot[v_5(70)+v_5(50)] = \frac{1}{2}\cdot(0+0) = 0\\
v_4(40) &= \frac{1}{2}\cdot[v_5(50)+v_5(30)] = \frac{1}{2}\cdot(0+0) = 0\\
v_3(110) &= \frac{1}{2}\cdot[v_4(120)+v_4(100)] = \frac{1}{2}\cdot(40+20) = 30\\
v_3(90) &= \frac{1}{2}\cdot[v_4(100)+v_4(80)] = \frac{1}{2}\cdot(20+5) = 12.5\\
v_3(70) &= \frac{1}{2}\cdot[v_4(80)+v_4(60)] = \frac{1}{2}\cdot(5+0) = 2.5\\
v_3(50) &= \frac{1}{2}\cdot[v_4(60)+v_4(40)] = \frac{1}{2}\cdot(0+0) = 0\\
v_2(100) &= \frac{1}{2}\cdot[v_3(110)+v_3(90)] = \frac{1}{2}\cdot(30+12.5) = 21.25\\
v_2(80) &= \frac{1}{2}\cdot[v_3(90)+v_3(70)] = \frac{1}{2}\cdot(12.5+2.5) = 7.5\\
v_2(60) &= \frac{1}{2}\cdot[v_3(70)+v_3(50)] = \frac{1}{2}\cdot(2.5+0) = 1.25\\
v_1(90) &= \frac{1}{2}\cdot[v_2(100)+v_2(80)] = \frac{1}{2}\cdot(21.25+7.5) = 14.375\\
v_1(70) &= \frac{1}{2}\cdot[v_2(80)+v_2(60)] = \frac{1}{2}\cdot(7.5+1.25) = 4.375\\
v_0(80) &= \frac{1}{2}\cdot[v_1(90)+v_2(70)] = \frac{1}{2}\cdot(14.375+4.375) = 9.375
\end{aligned}$$

The price of the European call is $v_0(80) = 9.375$.

In fact, for this special case, we may take advantage of the fact that $\tilde{p} = \tilde{q} = \frac{1}{2}$. Observe that in this binomial model, for each coin-toss result, each terminal payoff should be discounted by $(\frac{1}{2})^5$, regardless of its path, to arrive at the initial option value. Thus, we have

$$\begin{aligned}
v_0(80) =& \left(\frac{1}{2}\right)^5 \left[\binom{5}{0} v_5(130) + \binom{5}{1} v_5(110) + \binom{5}{2} v_5(90) + \binom{5}{3} v_5(70) + \binom{5}{4} v_5(50) + \binom{5}{5} v_5(30)\right]\\
&= \frac{1}{32}\big(1\cdot 50 + 5\cdot 30 + 10\cdot 10 + 10\cdot 0 + 5\cdot 0 + 1\cdot 0\big)\\
&= 9.375
\end{aligned}$$

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