Answers to Exercises

Stochastic Calculus for Finance I: The Binomial Asset Pricing Model by *Steven E. Shreve*

Chapter 6 Interest-Rate-Dependent Assets

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22 April, 2019

Please refer to the book for the exercises themselves. The text in front of each answer serves only as a summary of the question.

Please refer to the table below for the notation conversion. Changes are made to make symbols more compact, and to make the forward measure more distinguishable from the risk-neutral one.

Book	Answers	Note
$B_{n,m}$	B_n^m	time-n price of a bond that matures at time m
$\Delta_{n,m}$	Δ_n^m	number of m-maturity zero-coupon bonds held by the agent between time n and $n+1$
$For_{n,m}$	For_n^m	time-n price for a forward contract with delivery date m
$Fut_{n,m}$	Fut_n^m	time-n price for a futures contract with delivery date m
$\widetilde{\mathbb{P}}^m$	$\widehat{\mathbb{P}}^m$	probability under the m-forward measure
$\widetilde{\mathbb{E}}_n^m$	$\widehat{\mathbb{E}}_n^m$	time-n conditional expectation under the m-forward measure

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Exercise 6.1 Prove the fundamental properties of conditional expectations using the definition based on conditional probabilities.

Answer: Recall that the conditional expectation of random variable X at time n is defined to be a random variable. Its value for a particular sequence of coin tosses $\bar{\omega}_1 \cdots \bar{\omega}_n$ is

$$\mathbb{E}_n[X](\bar{\omega}_1 \cdots \bar{\omega}_n) = \sum_{\bar{\omega}_{n+1} \cdots \bar{\omega}_N} X(\bar{\omega}_1 \cdots \bar{\omega}_n \bar{\omega}_{n+1} \cdots \bar{\omega}_N) \cdot \mathbb{P}\{\omega_{n+1} = \bar{\omega}_{n+1}, \cdots, \omega_N = \bar{\omega}_N \mid \omega_1 = \bar{\omega}_1, \cdots, \omega_n = \bar{\omega}_n\}$$

(i) ***linearity***. For all constants c_1 and c_2, we have

$$\begin{aligned}
\mathbb{E}_n[c_1 X + c_2 Y](\bar{\omega}_1 \cdots \bar{\omega}_n) &= \sum_{\bar{\omega}_{n+1} \cdots \bar{\omega}_N} (c_1 X + c_2 Y)(\bar{\omega}_1 \cdots \bar{\omega}_n \bar{\omega}_{n+1} \cdots \bar{\omega}_N) \cdot \mathbb{P}\{\omega_{n+1} = \bar{\omega}_{n+1}, \cdots, \omega_N = \bar{\omega}_N \mid \omega_1 = \bar{\omega}_1, \cdots, \omega_n = \bar{\omega}_n\} \\
&= c_1 \sum_{\bar{\omega}_{n+1} \cdots \bar{\omega}_N} X(\bar{\omega}_1 \cdots \bar{\omega}_n \bar{\omega}_{n+1} \cdots \bar{\omega}_N) \cdot \mathbb{P}\{\omega_{n+1} = \bar{\omega}_{n+1}, \cdots, \omega_N = \bar{\omega}_N \mid \omega_1 = \bar{\omega}_1, \cdots, \omega_n = \bar{\omega}_n\} + \\
&\quad c_2 \sum_{\bar{\omega}_{n+1} \cdots \bar{\omega}_N} Y(\bar{\omega}_1 \cdots \bar{\omega}_n \bar{\omega}_{n+1} \cdots \bar{\omega}_N) \cdot \mathbb{P}\{\omega_{n+1} = \bar{\omega}_{n+1}, \cdots, \omega_N = \bar{\omega}_N \mid \omega_1 = \bar{\omega}_1, \cdots, \omega_n = \bar{\omega}_n\} \\
&= c_1 \mathbb{E}_n[X](\bar{\omega}_1 \cdots \bar{\omega}_n) + c_2 \mathbb{E}_n[Y](\bar{\omega}_1 \cdots \bar{\omega}_n)
\end{aligned}$$

(ii) ***taking out what is known***. If X actually depends only on the first n coin tosses, then

$$\begin{aligned}
\mathbb{E}_n[XY](\bar{\omega}_1 \cdots \bar{\omega}_n) &= \sum_{\bar{\omega}_{n+1} \cdots \bar{\omega}_N} (XY)(\bar{\omega}_1 \cdots \bar{\omega}_n \bar{\omega}_{n+1} \cdots \bar{\omega}_N) \cdot \mathbb{P}\{\omega_{n+1} = \bar{\omega}_{n+1}, \cdots, \omega_N = \bar{\omega}_N \mid \omega_1 = \bar{\omega}_1, \cdots, \omega_n = \bar{\omega}_n\} \\
&= \sum_{\bar{\omega}_{n+1} \cdots \bar{\omega}_N} X(\bar{\omega}_1 \cdots \bar{\omega}_n) \cdot Y(\bar{\omega}_1 \cdots \bar{\omega}_n \bar{\omega}_{n+1} \cdots \bar{\omega}_N) \cdot \mathbb{P}\{\omega_{n+1} = \bar{\omega}_{n+1}, \cdots, \omega_N = \bar{\omega}_N \mid \omega_1 = \bar{\omega}_1, \cdots, \omega_n = \bar{\omega}_n\} \\
&= X(\bar{\omega}_1 \cdots \bar{\omega}_n) \sum_{\bar{\omega}_{n+1} \cdots \bar{\omega}_N} Y(\bar{\omega}_1 \cdots \bar{\omega}_n \bar{\omega}_{n+1} \cdots \bar{\omega}_N) \cdot \mathbb{P}\{\omega_{n+1} = \bar{\omega}_{n+1}, \cdots, \omega_N = \bar{\omega}_N \mid \omega_1 = \bar{\omega}_1, \cdots, \omega_n = \bar{\omega}_n\} \\
&= X(\bar{\omega}_1 \cdots \bar{\omega}_n) \cdot \mathbb{E}_n[X](\bar{\omega}_1 \cdots \bar{\omega}_n)
\end{aligned}$$

(iii) ***iterated conditioning***. Assume $0 \le n \le m \le N$. To avoid formulas becoming too long and we getting lost, let's define events

$$\begin{aligned} A &= \{\omega_1 = \bar{\omega}_1, \cdots, \omega_n = \bar{\omega}_n\}, \\ B &= \{\omega_{n+1} = \bar{\omega}_{n+1}, \cdots, \omega_m = \bar{\omega}_m\}, \\ C &= \{\omega_{m+1} = \bar{\omega}_{m+1}, \cdots, \omega_N = \bar{\omega}_N\}. \end{aligned}$$

Note that, no matter coin tosses are independent or not, we must have

$$\mathbb{P}(BC|A) = \mathbb{P}(B|A)\mathbb{P}(C|AB)$$

This can be put in a more straightforward way if we write conditional probability $\mathbb{P}(\cdot\,|A)$ as a new probability measure $\mathbb{P}_A$, so that the above formula becomes

$$\mathbb{P}_A(BC) = \mathbb{P}_A(B)\mathbb{P}_A(C|B)$$

Further, define, symbolically, summations over specific subsequence of coin tosses $\sum_{(B)} = \sum_{\bar{\omega}_n \cdots \bar{\omega}_m}$, $\sum_{(C)} = \sum_{\bar{\omega}_{m+1} \cdots \bar{\omega}_N}$, $\sum_{(BC)} = \sum_{\bar{\omega}_n \cdots \bar{\omega}_N}$.

Let $Z = \mathbb{E}_m[X]$. It is clear that Z depends on the first m coin tosses only.

$$\begin{aligned} \mathbb{E}_n\big[\mathbb{E}_m[X]\big](\bar{\omega}_1 \cdots \bar{\omega}_n) &= \mathbb{E}_n[Z](\bar{\omega}_1 \cdots \bar{\omega}_n) \\ &= \sum_{(BC)} Z(\bar{\omega}_1 \cdots \bar{\omega}_N)\mathbb{P}(BC|A) \\ &= \sum_{(BC)} Z(\bar{\omega}_1 \cdots \bar{\omega}_m)\mathbb{P}(BC|A) \\ &= \sum_{(BC)} Z(\bar{\omega}_1 \cdots \bar{\omega}_m)\mathbb{P}(B|A)\mathbb{P}(C|AB) \\ &= \sum_{(B)} Z(\bar{\omega}_1 \cdots \bar{\omega}_m)\mathbb{P}(B|A) \cdot \sum_{(C)} \mathbb{P}(C|AB) \qquad \cdots \text{see remark below} \\ &= \sum_{(B)} Z(\bar{\omega}_1 \cdots \bar{\omega}_m)\mathbb{P}(B|A) \\ &= \sum_{(B)} \big[\sum_{(C)} X(\bar{\omega}_1 \cdots \bar{\omega}_N)\mathbb{P}(C|AB)\big]\mathbb{P}(B|A) \\ &= \sum_{(BC)} X(\bar{\omega}_1 \cdots \bar{\omega}_N)\mathbb{P}(BC|A) \\ &= \mathbb{E}_n[X](\bar{\omega}_1 \cdots \bar{\omega}_n) \end{aligned}$$

Remark: Consider a simplified situation in the same spirit, where $f(x,y) = g(x)h(x)k(y)$. For $x = 0,1$ and $y = 0,1$

$$\begin{aligned} \sum_{x,y} f(x,y) &= \sum_{x,y} g(x)h(x)k(y) \\ &= g(0)h(0)k(0) + g(0)h(0)k(1) + g(1)h(1)k(0) + g(1)h(1)k(1) \\ &= \big[g(0)h(0) + g(1)h(1)\big] \cdot \big[k(0) + k(1)\big] \\ &= \sum_{x} g(x)h(x) \cdot \sum_{y} k(y) \end{aligned}$$

(iv) ***independence***. If X depends only on tosses $n+1$ through N, then

$$
\begin{aligned}
\mathbb{E}_n[X](\bar{\omega}_1\cdots\bar{\omega}_n) &= \sum_{\bar{\omega}_{n+1}\cdots\bar{\omega}_N} X(\bar{\omega}_1\cdots\bar{\omega}_n\bar{\omega}_{n+1}\cdots\bar{\omega}_N)\cdot\mathbb{P}\{\omega_{n+1}=\bar{\omega}_{n+1},\cdots,\omega_N=\bar{\omega}_N \mid \omega_1=\bar{\omega}_1,\cdots,\omega_n=\bar{\omega}_n\} \\
&= \sum_{\bar{\omega}_{n+1}\cdots\bar{\omega}_N} X(\bar{\omega}_{n+1}\cdots\bar{\omega}_N)\cdot\mathbb{P}\{\omega_{n+1}=\bar{\omega}_{n+1},\cdots,\omega_N=\bar{\omega}_N \mid \omega_1=\bar{\omega}_1,\cdots,\omega_n=\bar{\omega}_n\} \\
&= \sum_{\bar{\omega}_{n+1}\cdots\bar{\omega}_N} X(\bar{\omega}_{n+1}\cdots\bar{\omega}_N)\cdot\mathbb{P}\{\omega_{n+1}=\bar{\omega}_{n+1},\cdots,\omega_N=\bar{\omega}_N \mid \omega_1=\bar{\omega}_1,\cdots,\omega_n=\bar{\omega}_n\}\cdot\sum_{\bar{\omega}_1\cdots\bar{\omega}_n}\mathbb{P}\{\omega_1=\bar{\omega}_1,\cdots,\omega_n=\bar{\omega}_n\} \\
&= \sum_{\bar{\omega}_1\cdots\bar{\omega}_N} X(\bar{\omega}_{n+1}\cdots\bar{\omega}_N)\cdot\mathbb{P}\{\omega_{n+1}=\bar{\omega}_{n+1},\cdots,\omega_N=\bar{\omega}_N \mid \omega_1=\bar{\omega}_1,\cdots,\omega_n=\bar{\omega}_n\}\cdot\mathbb{P}\{\omega_1=\bar{\omega}_1,\cdots,\omega_n=\bar{\omega}_n\} \\
&= \sum_{\bar{\omega}_1\cdots\bar{\omega}_N} X(\bar{\omega}_{n+1}\cdots\bar{\omega}_N)\cdot\mathbb{P}\{\omega_1=\bar{\omega}_1,\cdots,\omega_n=\bar{\omega}_n,\omega_{n+1}=\bar{\omega}_{n+1},\cdots,\omega_N=\bar{\omega}_N\} \\
&= \mathbb{E}[X]
\end{aligned}
$$

(v) ***conditional Jensen's inequality***. Let φ be a convex function, and denote by $\mathcal{L}$ the set of all linear functions l that lie below φ. It can be shown as a result in analysis that for all x

$$\varphi(x) = \sup_{l\in\mathcal{L}} l(x)$$

Since, for any $l \in \mathcal{L}$, $\varphi(x) \geq l(x)$, and $l(x)$ is linear, we have, by property (i)

$$\mathbb{E}_n[\varphi(X)] \geq \mathbb{E}_n[l(X)] \geq l(\mathbb{E}_n[X])$$

Now taking the supremum over the two sides preserves the $\geq$ sign, so

$$\sup_{l\in\mathcal{L}} \mathbb{E}_n[\varphi(X)] \geq \sup_{l\in\mathcal{L}} l(\mathbb{E}_n[X])$$

That is

$$\mathbb{E}_n[\varphi(X)] \geq \varphi(\mathbb{E}[X])$$

Exercise 6.2 Verify that the discounted value of the static hedging portfolio for a forward is a martingale under $\widetilde{\mathbb{P}}$.

Answer: To hedge a short position in a forward contract that is initiated at time n with delivery date m, the agent should, at time n, long 1 share of stock and short S_n/B_n^m unit of m-maturity zero-coupon bond. This constructs a static hedging portfolio. The time-n value of the hedging portfolio is

$$X_n = S_n - \Big(\frac{S_n}{B_n^m}\Big)B_n^m \ (=0)$$

To show its discounted value is $\widetilde{\mathbb{P}}$-martingale, note that

$$
\begin{aligned}
\widetilde{\mathbb{E}}_n(D_{n+1}X_{n+1}) &= \widetilde{\mathbb{E}}_n\Big[D_{n+1}\Big(S_{n+1} - \Big(\frac{S_n}{B_n^m}\Big)B_{n+1}^m\Big)\Big] \\
&= \widetilde{\mathbb{E}}_n(D_{n+1}S_{n+1}) - \Big(\frac{S_n}{B_n^m}\Big)\widetilde{\mathbb{E}}_n(D_{n+1}B_{n+1}^m) \qquad \cdots\text{linearity} \\
&= D_nS_n - \Big(\frac{S_n}{B_n^m}\Big)D_nB_n^m \qquad \cdots\text{discounted stock, bond prices are }\widetilde{\mathbb{P}}\text{-martingale} \\
&= D_nX_n
\end{aligned}
$$

Note: that discounted bond prices are $\widetilde{\mathbb{P}}$-martingale can be shown by, for $0 \leq k \leq n \leq m$,

$$\widetilde{\mathbb{E}}_k(D_nB_n^m) = \widetilde{\mathbb{E}}_k\Big[D_n\widetilde{\mathbb{E}}_n\Big(\frac{D_m}{D_n}\Big)\Big] = \widetilde{\mathbb{E}}_k[\widetilde{\mathbb{E}}_n(D_m)] = \widetilde{\mathbb{E}}_k(D_m) = \widetilde{\mathbb{E}}_k\Big(D_k\cdot\frac{D_m}{D_k}\Big) = D_k\widetilde{\mathbb{E}}_k\Big(\frac{D_m}{D_k}\Big) = D_kB_k^m$$

Exercise 6.3 Use properties of conditional expectations to show that

$$\frac{1}{D_n}\widetilde{\mathbb{E}}_n[D_{m+1}R_m] = B_n^m - B_n^{m+1}$$

Answer: Note that $(1+R_m) = \frac{D_m}{D_{m+1}} \implies R_m = \frac{D_m}{D_{m+1}} - 1$. We have

$$\begin{aligned}
\frac{1}{D_n}\widetilde{\mathbb{E}}_n[D_{m+1}R_m] &= \frac{1}{D_n}\widetilde{\mathbb{E}}_n\left[D_{m+1}\big(\frac{D_m}{D_{m+1}} - 1\big)\right] \\
&= \frac{1}{D_n}\widetilde{\mathbb{E}}_n[D_m - D_{m+1}] \\
&= \widetilde{\mathbb{E}}_n\left[\frac{D_m}{D_n}\right] - \widetilde{\mathbb{E}}_n\left[\frac{D_{m+1}}{D_n}\right] && \cdots \text{take in what is known, linearity} \\
&= B_n^m - B_n^{m+1} && \cdots \text{definition}
\end{aligned}$$

Remark: What's the difference between

- (a) the time-n contract price to pay R_m at time $m+1$ (see *page 155*), and
- (b) the time-n forward price to deliver R_m at time $m+1$ (see *page 156*)?

Expressed in formula, the former is $B_n^{m+1}F_n^m = B_n^m - B_n^{m+1}$ while the latter is just F_n^m.

But the fundamental difference is that the former is the price to pay at time n, while the latter, as any other strike price of a forward contract, is the price to pay at time $m+1$.

An example with concrete numbers.

Say $n=3$, $m=7$. At time $n=3$, let's say $B_n^m = B_3^7 = \$0.93$ and $B_n^{m+1} = B_3^8 = \$0.87$.

It can be calculated that the time-n forward interest rate for the period between times m and $m+1$ is

$$F_n^m = \frac{B_n^m}{B_n^{m+1}} - 1 = \frac{\$0.93}{\$0.87} - 1 = 6.90\%$$

At time $n=3$, we do not know $R_m = R_7$ which is not unveiled until time $m=7$.

However, we may, at time $n=3$, sign a contract that promises to pay $R_m = R_7$ at time $m+1=8$.

For this contract we charge a no-arbitrage price at $B_n^{m+1}F_n^m = \$0.87 \times 6.90\% = \$0.06$, which the counterparty must pay immediately at time $n=3$, for the entitlement to receive R_7 at time $m+1=8$. This amount can also be worked out by $B_n^m - B_n^{m+1} = \$0.93 - \$0.87 = \$0.06$.

As another case, we may short a forward contract, which promises to deliver $R_m = R_7$ at time $m+1=8$. Out counterparty, who long the forward contract, is required to pay a strike price fixed at $F_n^m = \$0.069$ at time $m+1=8$ in exchange for the delivery of $R_m = R_7$.

Exercise 6.4 Using data provided in the book, construct a hedge for a short position in a caplet paying $(R_2 - \frac{1}{3})^+$ at time three. In particular,

(i) Determine $V_1(H)$ and $V_1(T)$.

(ii) Show how to begin with $\frac{2}{21}$ at time zero and invest in the money market and maturity-two bond in order to have a portfolio X_1 at time one that agrees with V_1. Why do we invest in the maturity-two bond rather than the maturity-three bond?

(iii) Show how to take the portfolio value X_1 at time one to a portfolio value X_2 at time two that agrees with V_2. Why do we at this step invest in the maturity-three bond rather than the maturity-two bond?

Answer: (i) According to the risk-neutral pricing formula, $D_1V_1 = \widetilde{\mathbb{E}}_1(D_2V_2)$, or $V_1 = \widetilde{\mathbb{E}}_1(\frac{D_2}{D_1}V_2) = \widetilde{\mathbb{E}}_1(\frac{1}{1+R_1}V_2)$. In particular,

$$\begin{aligned} V_1(H) &= \frac{1}{1+R_1(H)}\Big[\widetilde{\mathbb{P}}\{\omega_2 = H \mid \omega_1 = H\}\cdot V_2(HH) + \widetilde{\mathbb{P}}\{\omega_2 = T \mid \omega_1 = H\}\cdot V_2(HT)\Big] \\ &= \frac{1}{1+\frac{1}{6}}\Big(\frac{2}{3}\cdot \$\frac{1}{3} + \frac{1}{3}\cdot \$0\Big) = \$\frac{4}{21}\ , \\ V_1(T) &= \frac{1}{1+R_1(T)}\Big[\widetilde{\mathbb{P}}\{\omega_2 = H \mid \omega_1 = T\}\cdot V_2(TH) + \widetilde{\mathbb{P}}\{\omega_2 = T \mid \omega_1 = T\}\cdot V_2(TT)\Big] \\ &= \frac{1}{1+\frac{2}{5}}\Big(\frac{1}{2}\cdot \$0 + \frac{1}{2}\cdot \$0\Big) = \$0\ . \end{aligned}$$

(ii) The number of maturity-two zero-coupon bonds to hold between times zero and one is

$$\Delta_0^2 = \frac{V_1(H) - V_1(T)}{B_1^2(H) - B_1^2(T)} = \frac{\frac{4}{21} - 0}{\frac{6}{7} - \frac{5}{7}} = \frac{4}{3}$$

Since we have sold the caplet for $X_0 = V_0 = \$\frac{2}{21}$, the money-market account position is

$$X_0 - \Delta_0^2 B_0^2 = \$\frac{2}{21} - \frac{4}{3}\cdot\$\frac{11}{14} = -\$\frac{20}{21}$$

The negative number indicates we should borrow $\$\frac{20}{21}$ from the money market.

If the first coin toss is H, then at time one, the hedging portfolio has value

$$X_1(H) = \Delta_0^2 B_1^2(H) + (1+R_0)(X_0 - \Delta_0^2 B_0^2) = \frac{4}{3}\cdot\$\frac{6}{7} - (1+0)\cdot(-\$\frac{20}{21}) = \$\frac{4}{21}$$

If the first coin toss is V, then at time one, the hedging portfolio has value

$$X_1(T) = \Delta_0^2 B_1^2(T) + (1+R_0)(X_0 - \Delta_0^2 B_0^2) = \frac{4}{3}\cdot\$\frac{5}{7} - (1+0)\cdot(-\$\frac{20}{21}) = \$0$$

Note that the hedging portfolio has a time-one value X_1 that agrees with V_1, regardless of the coin-toss result.

- Why do we invest in the maturity-two bond?

Because at time zero, we want to hedge against the stochastic interest rate R_1 which is unveiled at time one, which is incorporated in the price of the maturity-two bond. In fact, the price of the maturity-two bond is fully determined by two interest rates: R_0, which is known at time zero, and R_1, which we want to hedge against.

- Why do we not invest in the maturity-three bond?

The price of the maturity-three bond additionally depends on R_2, which we shouldn't worry about at time zero. Moreover, in this case, $B_1^3(H) = B_1^3(T) = \frac{4}{7}$, which means it has no randomness between times zero and one, and thus cannot be used as a hedge. A binomial model requires $0 < d < 1 + r < u$, and this violates the assumption as $d = u$.

(iii) If the first coin toss is H, the number of maturity-three zero-coupon bonds to hold between times one and two is

$$\Delta_1^3(H) = \frac{V_2(HH) - V_2(HT)}{B_2^3(HH) - B_2^3(HT)} = \frac{\frac{1}{3} - 0}{\frac{1}{2} - 1} = -\frac{2}{3}$$

The negative number indicates that we should short $-\frac{2}{3}$ unit of the maturity-three bond. The cash generated will be invested in the money market.

With hedging portfolio value $X_1(H) = \$\frac{4}{21}$, the money market position is

$$X_1(H) - \Delta_1^3(H)B_1^3(H) = \$\frac{4}{21} - (-\frac{2}{3}) \cdot \$\frac{4}{7} = \$\frac{4}{7}$$

If the first two coin tosses are HH, the hedging portfolio has value

$$X_2(HH) = \Delta_1^3(H)B_2^3(HH) + [1 + R_1(H)][X_1(H) - \Delta_1^3(H)B_1^3(H)] = (-\frac{2}{3}) \cdot \$\frac{1}{2} + (1 + \frac{1}{6}) \cdot \$\frac{4}{7} = \$\frac{1}{3}$$

If the first two coin tosses are HT, the hedging portfolio has value

$$X_2(HT) = \Delta_1^3(H)B_2^3(HT) + [1 + R_1(H)][X_1(H) - \Delta_1^3(H)B_1^3(H)] = (-\frac{2}{3}) \cdot \$1 + (1 + \frac{1}{6}) \cdot \$\frac{4}{7} = \$0$$

If, however, the first coin toss is T, the hedging strategy is trivial. We start with $X_1(T) = \$0$ and do nothing, as $V_2(TH) = V_2(TT) = \$0$.

We have shown that the hedging portfolio has time-two value X_2 that agrees with V_2. That is, $X_2(HH) = V_2(HH) = \$\frac{1}{3}$, and they are both zero in all other cases.

It is by the same token that we invest in the maturity-three bond rather than the maturity-two bond at this step, since the randomness of R_2, which we want to hedge against, is not incorporated in the price of the latter.

Exercise 6.5 (i) Use (6.4.8) and (6.2.5) to show that $F_n^m, n = 0, 1, \cdots, m$ is a martingale under the $(m+1)$-forward measure $\widehat{\mathbb{P}}^{m+1}$.

(ii) Compute F_0^2, $F_1^2(H)$, and $F_1^2(H)$ in the *Ho-Lee model* example in the book and verify the martingale property $\widehat{\mathbb{E}}^3[F_1^2] = F_0^2$.

Answer: (i) For $0 \leq k \leq n \leq m$,

$$
\begin{aligned}
\widehat{\mathbb{E}}_k^{m+1}(F_n^m) &= \frac{1}{D_k B_k^{m+1}} \widetilde{\mathbb{E}}_k(D_{m+1} F_n^m) && \cdots (6.4.8) \\
&= \frac{1}{D_k B_k^{m+1}} \widetilde{\mathbb{E}}_k\left[D_{m+1} \frac{B_n^m - B_n^{m+1}}{B_n^{m+1}} \right] && \cdots \text{definition} \\
&= \frac{1}{D_k B_k^{m+1}} \widetilde{\mathbb{E}}_k\left[D_{m+1} \frac{B_n^m - B_n^{m+1}}{\frac{1}{D_n}\widetilde{\mathbb{E}}_n(D_{m+1})} \right] && \cdots \text{definition, take out what is known} \\
&= \frac{1}{D_k B_k^{m+1}} \widetilde{\mathbb{E}}_k\left[\widetilde{\mathbb{E}}_n\left[\frac{D_{m+1}(D_n B_n^m - D_n B_n^{m+1})}{\widetilde{\mathbb{E}}_n(D_{m+1})} \right] \right] && \cdots \text{iterated conditioning} \\
&= \frac{1}{D_k B_k^{m+1}} \widetilde{\mathbb{E}}_k\left[(D_n B_n^m - D_n B_n^{m+1}) \frac{\widetilde{\mathbb{E}}_n(D_{m+1})}{\widetilde{\mathbb{E}}_n(D_{m+1})} \right] && \cdots \text{take out what is known} \\
&= \frac{1}{D_k B_k^{m+1}} \widetilde{\mathbb{E}}_k(D_n B_n^m - D_n B_n^{m+1}) && \cdots \text{cancelling} \\
&= \frac{D_k B_k^m - D_k B_k^{m+1}}{D_k B_k^{m+1}} && \cdots (6.2.5) \\
&= \frac{B_k^m - B_k^{m+1}}{B_k^{m+1}} && \cdots \text{cancelling} \\
&= F_k^m && \cdots \text{definition}
\end{aligned}
$$

(ii) Using the numbers in the book, we have

$$
\begin{aligned}
F_0^2 &= \frac{B_0^2 - B_0^3}{B_0^3} = \frac{0.9071 - 0.8639}{0.8639} = 5.0\% \\
F_1^2(H) &= \frac{B_1^2(H) - B_1^3(H)}{B_1^3(H)} = \frac{0.9479 - 0.8955}{0.8985} = 5.5\% \\
F_1^2(T) &= \frac{B_1^2(T) - B_1^3(T)}{B_1^3(T)} = \frac{0.9569 - 0.9158}{0.9158} = 4.5\% \\
\widehat{\mathbb{E}}^3(F_1^2) &= \sum_{\omega_1 = H} \widehat{\mathbb{P}}^3(\omega_1\omega_2\omega_3) F_1^2(H) + \sum_{\omega_1 = H} \widehat{\mathbb{P}}^3(\omega_1\omega_2\omega_3) F_1^2(H) \\
&= (0.1232 + 0.1232 + 0.1244 + 0.1244) \cdot 5.5\% + (0.1256 + 0.1256 + 0.1268 + 0.1268) \cdot 4.5\% \\
&= 5.0\%
\end{aligned}
$$

Exercise 6.6 Let S_m be the time-m price of an asset in a binomial interest rate model. For $n = 0, 1, \cdots, m$, the forward price is $For_n^m = \dfrac{S_n}{B_n^m}$ and the futures price is $Fut_n^m = \widetilde{\mathbb{E}}_n[S_n]$.

(i) Suppose at each time n an agent takes a long forward position and sells this contract at time $n+1$. Show that this generates cash flow $S_{n+1} - \dfrac{S_n B_{n+1}^m}{B_n^m}$.

(ii) Show that if the interest rate is a constant r and at each time n an agent takes a long position of $(1+r)^{m+n-1}$ forward contracts, selling these contracts at time $n+1$, then the resulting cash flow is the same as the difference in the futures price $Fut_{n+1}^m - Fut_n^m$.

Answer: (i) To sell the contract at time $n+1$ is equivalent to short a forward contract at that time. Thus, at the delivery date m, the agent has two obligations: (a) to buy the assets at For_n^m and (b) to sell the assets at For_{n+1}^m. According to the risk-neutral pricing formula, the time-$(n+1)$ value of this portfolio is

$$
\begin{aligned}
V_{n+1} &= \frac{1}{D_{n+1}} \widetilde{\mathbb{E}}_{n+1}(D_m V_m) \\
&= \widetilde{\mathbb{E}}_{n+1}\left[\frac{D_{n+1}}{D_m} V_m\right] \\
&= \widetilde{\mathbb{E}}_{n+1}\left[\frac{D_{n+1}}{D_m}\left[\left(S_m - For_n^m\right) + \left(For_{n+1}^m - S_m\right)\right]\right] && \cdots \text{two obligations at time } m \\
&= \left(For_{n+1}^m - For_n^m\right)\widetilde{\mathbb{E}}_{n+1}\left[\frac{D_{n+1}}{D_m}\right] && \cdots \text{take out what is known} \\
&= \left[\frac{S_{n+1}}{B_{n+1}^m} - \frac{S_n}{B_n^m}\right] B_{n+1}^m && \cdots \text{definition} \\
&= S_{n+1} - \frac{S_{n+1} B_{n+1}^m}{B_n^m}
\end{aligned}
$$

(ii) From (i) and linearity of forward contracts, we know that the cash flow for buying and selling $(1+r)^{m-n+1}$ forward contracts is

$$
\begin{aligned}
&(1+r)^{m+n-1}\left(S_{n+1} - \frac{S_n B_{n+1}^m}{B_n^m}\right) \\
=&(1+r)^{m+n-1}\left[S_{n+1} - S_n(1+r)\right] && \cdots \text{constant interest rate} \\
=&(1+r)^{m+n-1} S_{n+1} - (1+r)^{m+n} S_n
\end{aligned}
$$

Now

$$
\begin{aligned}
&Fut_{n+1}^m - Fut_n^m \\
=&\widetilde{\mathbb{E}}_{n+1}(S_m) - \widetilde{\mathbb{E}}_n(S_m) && \cdots \text{theorem 6.5.2} \\
=&(1+r)^{m-n+1} S_{n+1} - (1+r)^{m-n} S_n && \cdots \text{constant interest rate, risk-neutral pricing}
\end{aligned}
$$

We have shown that the cash flow is equal to the futures price difference.

Exercise 6.7 Consider a binomial interest rate model in which the interest rate at time n depends on only the number of heads in the first n coin tosses. That is, for each n there is a function $r_n(k)$ such that $R_n(\omega_1 \cdots \omega_n) = r_n(\#H(\omega_1 \cdots \omega_n))$. Assume risk-neutral probabilities $\tilde{p} = \tilde{q} = \frac{1}{2}$. Consider a derivative security that pays 1 at time n if and only if there are k heads in the first n tosses, or $V_n(k) = \mathbb{I}_{\{\#H(\omega_1 \cdots \omega_n) = k\}}$. Define $\psi_0(0) = 1$ and for $n = 1, 2, \cdots$ define

$$
\psi_n(k) = \widetilde{\mathbb{E}}[D_n V_n(k)], \quad k = 0, 1, \cdots, n
$$

to be the price of this security at time zero. Show that the functions $\psi_n(k)$ can be computed by

$$
\begin{aligned}
\psi_{n+1}(0) &= \frac{\psi_n(0)}{2(1+r_n(0))}, \\
\psi_{n+1}(k) &= \frac{\psi_n(k-1)}{2(1+r_n(k-1))} + \frac{\psi_n(k)}{2(1+r_n(k))}, \quad k = 1, \cdots, n \\
\psi_{n+1}(n+1) &= \frac{\psi_n(n)}{2(1+r_n(n))}.
\end{aligned}
$$

Answer: A key observation is that, in order to have k heads in first $n+1$ coin tosses, we need to have either (a) $k-1$ heads in first n coin tosses, followed by a head, or (b) k heads in first n coin tosses, followed by a tail.

That is

$$
\begin{aligned}
V_{n+1}(k) &= \mathbb{I}_{\{\#H(\omega_1\cdots\omega_n\omega_{n+1})=k\}} \\
&= \mathbb{I}_{\{\#H(\omega_1\cdots\omega_n)=k-1,\ \omega_{n+1}=H\}\ \cup\ \{\#H(\omega_1\cdots\omega_n)=k,\ \omega_{n+1}=T\}} \\
&= \mathbb{I}_{\{\#H(\omega_1\cdots\omega_n)=k-1\}} \cdot \mathbb{I}_{\{\omega_{n+1}=H\}} + \mathbb{I}_{\{\#H(\omega_1\cdots\omega_n)=k\}} \cdot \mathbb{I}_{\{\omega_{n+1}=T\}} \\
&= V_n(k-1) \cdot \mathbb{I}_{\{\omega_{n+1}=H\}} + V_n(k) \cdot \mathbb{I}_{\{\omega_{n+1}=T\}}
\end{aligned}
$$

Thus, for $k = 1, \cdots, n$

$$
\begin{aligned}
\psi_{n+1}(k) &= \widetilde{\mathbb{E}}[D_{n+1}V_{n+1}(k)] \\
&= \widetilde{\mathbb{E}}\Big[D_{n+1}\left[V_n(k-1)\cdot\mathbb{I}_{\{\omega_{n+1}=H\}} + V_n(k)\cdot\mathbb{I}_{\{\omega_{n+1}=T\}}\right]\Big] && \cdots \text{observation} \\
&= \widetilde{\mathbb{E}}\big[D_{n+1}V_n(k-1)\cdot\mathbb{I}_{\{\omega_{n+1}=H\}}\big] + \widetilde{\mathbb{E}}\big[D_{n+1}V_n(k)\cdot\mathbb{I}_{\{\omega_{n+1}=T\}}\big] && \cdots \text{linearity} \\
&= \widetilde{\mathbb{E}}\Big[\frac{1}{1+R_n}D_nV_n(k-1)\cdot\mathbb{I}_{\{\omega_{n+1}=H\}}\Big] + \widetilde{\mathbb{E}}\Big[\frac{1}{1+R_n}D_nV_n(k)\cdot\mathbb{I}_{\{\omega_{n+1}=T\}}\Big] && \cdots \text{definition} \\
&= \widetilde{\mathbb{E}}\Big[\frac{1}{1+r_n(k-1)}D_nV_n(k-1)\cdot\mathbb{I}_{\{\omega_{n+1}=H\}}\Big] + \widetilde{\mathbb{E}}\Big[\frac{1}{1+r_n(k)}D_nV_n(k)\cdot\mathbb{I}_{\{\omega_{n+1}=T\}}\Big] && \cdots R_n \text{ fixed by } V_n \\
&= \frac{1}{1+r_n(k-1)}\widetilde{\mathbb{E}}\Big[D_nV_n(k-1)\cdot\mathbb{I}_{\{\omega_{n+1}=H\}}\Big] + \frac{1}{1+r_n(k)}\widetilde{\mathbb{E}}\Big[D_nV_n(k)\cdot\mathbb{I}_{\{\omega_{n+1}=T\}}\Big] && \cdots \text{take out functions} \\
&= \frac{1}{1+r_n(k-1)}\widetilde{\mathbb{E}}\Big[D_nV_n(k-1)\Big]\cdot\widetilde{\mathbb{E}}\Big[\mathbb{I}_{\{\omega_{n+1}=H\}}\Big] + \frac{1}{1+r_n(k)}\widetilde{\mathbb{E}}\Big[D_nV_n(k)\Big]\cdot\widetilde{\mathbb{E}}\Big[\mathbb{I}_{\{\omega_{n+1}=T\}}\Big] && \cdots \text{independence} \\
&= \frac{1}{1+r_n(k-1)}\psi_n(k-1)\cdot\widetilde{\mathbb{P}}\{\omega_{n+1}=H\} + \frac{1}{1+r_n(k)}\psi_n(k)\cdot\widetilde{\mathbb{P}}\{\omega_{n+1}=T\} && \cdots \text{definition} \\
&= \frac{\psi_n(k-1)}{2(1+r_n(k-1))} + \frac{\psi_n(k)}{2(1+r_n(k))} && \cdots \tilde{p}=\tilde{q}=\frac{1}{2}
\end{aligned}
$$

For $k = 0$ and $k = n+1$, which requires the first n coin tosses to be all tails and all heads, respectively, the formula becomes

$$
\begin{aligned}
\psi_{n+1}(0) &= \frac{\psi_n(0)}{2(1+r_n(0))}, \\
\psi_{n+1}(n+1) &= \frac{\psi_n(n)}{2(1+r_n(n))}.
\end{aligned}
$$

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for discussion, please write to aaron.fu@alumni.ust.hk