Work in Progress

Answers to Exercises

Stochastic Calculus for Finance I: The Binomial Asset Pricing Model by *Steven E. Shreve*

Chapter 4 American Derivative Securities

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Please refer to the book for the exercises themselves. The text in front of each answer serves only as a summary of the question.

Exercise 4.1 In the three-period model with $S_0 = 4$, let the interest rate be $r = \frac{1}{4}$ so the risk-neutral probabilities are $\tilde{p} = \tilde{q} = \frac{1}{2}$.

(i) Determine the time-zero price of an American put V_0^P that expires at time three and has intrinsic value $g_P(s) = (4 - s)^+$.

(ii) Determine the time-zero price of an American call V_0^C that expires at time three and has intrinsic value $g_C(s) = (s - 4)^+$.

(iii) Determine the time-zero price of an American straddle V_0^S that expires at time three and has intrinsic value $g_S(s) = g_P(s) + g_C(s)$.

(iv) Explain why $V_0^S < V_0^P + V_0^C$.

Answer: The American algorithm is, for N and $n = N-1, N-2, \cdots, 0$

$$
\begin{aligned}
v_N(s) &= \max\{g(s), 0\} \\
v_n(s) &= \max\Big\{g(s), \frac{1}{1+r}\Big[\tilde{p}v_{n+1}(us) + \tilde{q}v_{n+1}(ds)\Big]\Big\}
\end{aligned}
$$

(i) Applying the American algorithm backward in time

```
                                                                        32
                                                                        max(-28, 0) = 0
                                                16
                                                max(0, 0) = 0

                        8                                               8
                        max(0, 0.32) = 0.32                             max(-4, 0) = 0

4                                               4
max(0, 0.928) = 0.928                           max(0, 0.8) = 0.8

                        2                                               2
                        max(2, 1.52) = 2                                max(2, 0) = 2

                                                1
                                                max(3, 2.2) = 3

                                                                        0.5
                                                                        max(3.5, 0) = 3.5
```

we have the time-zero price for the American put $V_0^P = 0.928$.

(ii) Applying the American algorithm backward in time

```
                                                                       32
                                                                       max(28, 0) = 28
                                                 16
                                                 max(12, 12.8) = 12.8

                        8                                              8
                        max(4, 5.76) = 5.76                            max(4, 0) = 4

4                                                4
max(0, 2.56) = 2.56                              max(0, 1.6) = 1.6

                        2                                              2
                        max(0, 0.64) = 0.64                            max(-2, 0) = 0

                                                 1
                                                 max(0, 0) = 0

                                                                       0.5
                                                                       max(-3.5, 0) = 0
```

we have the time-zero price for the American call $V_0^C = 2.56$.

(iii) Note that the intrinsic value is $g_S(s) = (4 - s)^+ + (s - 4)^+ = |s - 4|$. Applying the American algorithm backward in time

```
                                                                       32
                                                                       |32 - 4| = 28
                                                 16
                                                 max(12, 12.8) = 12.8

                        8                                              8
                        max(4, 6.08) = 6.08                            |8 - 4| = 4

4                                                4
max(0, 3.296) = 3.296                            max(0, 2.4) = 2.4

                        2                                              2
                        max(2, 2.16) = 2.16                            |2 - 4| = 2

                                                 1
                                                 max(3, 2.2) = 3

                                                                       0.5
                                                                       |0.5 - 4| = 3.5
```

we have the time-zero price for the American straddle $V_0^S = 3.296$.

(iv) Consider two portfolios.

Portfolio A has an American put and an American call. Portfolio B has an American straddle.

While the two portfolios always have the same intrinsic value, portfolio A has more freedom in exercise. Its holder may choose to exercise the two options at different times to the best of her interest. For example, she may exercise the put at time one but exercise the call at time three.

Portfolio B, however, does not enjoy this flexibility.

Intuitively, this explains why $V_0^S < V_0^C + V_0^P$.

Mathematically, this can be expressed as

$$\begin{aligned} V_0^S &= \max_{\tau \,\in\, \mathcal{S}} \widetilde{\mathbb{E}}\left[\mathbb{I}_{\{\tau \le N\}} \frac{1}{(1+r)^\tau} G_\tau^S\right] \\ &= \max_{\tau \,\in\, \mathcal{S}} \widetilde{\mathbb{E}}\left[\mathbb{I}_{\{\tau \le N\}} \frac{1}{(1+r)^\tau} (G_\tau^P + G_\tau^C)\right] \\ &\le \max_{\tau_1 \,\in\, \mathcal{S}} \widetilde{\mathbb{E}}\left[\mathbb{I}_{\{\tau_1 \le N\}} \frac{1}{(1+r)^{\tau_1}} G_{\tau_1}^P\right] + \max_{\tau_2 \,\in\, \mathcal{S}} \widetilde{\mathbb{E}}\left[\mathbb{I}_{\{\tau_2 \le N\}} \frac{1}{(1+r)^{\tau_2}} G_{\tau_2}^C\right] \\ &= V_0^P + V_0^C \end{aligned}$$

where $\mathcal{S}$ is the set of all stopping times.

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