Work in Progress

Answers to Exercises

Stochastic Calculus for Finance II: Continuous-Time Models by *Steven E. Shreve*

Chapter 3 Brownian Motion

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19 April, 2019

Please refer to the book for the exercises themselves. The text in front of each answer serves only as a summary of the question.

Exercise 3.1 For $0 \leq t < u_1 < u_2$, show that $W(u_2) - W(u_1)$ is independent of $\mathcal{F}(t)$.

Answer:

1. Definition of Brownian motion's filtration (independence of future increment) $\implies W(u_2) - W(u_1)$ is independent of $\mathcal{F}(u_1) \implies \sigma\big(W(u_2) - W(u_1)\big)$ is independent of $\mathcal{F}(u_1)$
2. Definition of Brownian motion's filtration (information accumulates), $t < u_1 \implies \mathcal{F}(t) \subset \mathcal{F}(u_1)$

By (1) and (2), for any events $A \in \sigma\big(W(u_2) - W(u_1)\big), B \in \mathcal{F}(t) \subset \mathcal{F}(u_1)$, we have $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ $\implies \sigma\big(W(u_2) - W(u_1)\big)$ and $\mathcal{F}(t)$ are independent $\implies W(u_2) - W(u_1)$ is independent of $\mathcal{F}(t)$.

Exercise 3.2 Show that $W^2(t) - t$ is a martingale.

Answer: For $0 \leq s < t \leq T$, we have

$$
\begin{aligned}
&\mathbb{E}\Big[W^2(t) - t \mid \mathbb{F}(s)\Big] \\
=&\mathbb{E}\Big[\big(W(t) - W(s) + W(s)\big)^2 - t \mid \mathbb{F}(s)\Big] \\
=&\mathbb{E}\Big[\big(W(t) - W(s)\big)^2 + 2\big(W(t) - W(s)\big)W(s) + W^2(s) - t \mid \mathbb{F}(s)\Big] \\
=&\mathbb{E}\Big[\big(W(t) - W(s)\big)^2\Big] + 2\mathbb{E}\Big[W(t) - W(s)\Big]W(s) + W^2(s) - t && \cdots \text{independence, take out what is known} \\
=&(t - s) + 2 \cdot 0 \cdot W(s) + W^2(s) - t && \cdots \text{Brownian motion's definition} \\
=&W^2(s) - s
\end{aligned}
$$

Thus, $W^2(t) - t$ is a martingale.

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for discussion, please write to aaron.fu@alumni.ust.hk