

## Answers to Exercises

### Stochastic Calculus for Finance I: The Binomial Asset Pricing Model by Steven E. Shreve

#### Chapter 1 The Binomial No-Arbitrage Pricing Model

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*Please refer to the book for the exercises themselves. The text in front of each answer serves only as a summary of the question.*

**Exercise 1.1** Show that  $0 < d < 1 + r < u$  precludes arbitrage.

**Answer:** Arbitrage is defined as a trading strategy that begins with zero wealth and ends with nonnegative wealth in all cases, and positive wealth in some cases. That is, a trading strategy such that initial wealth  $X_0 = 0$ , and terminal wealth  $X_1(\omega) \geq 0$  for all  $\omega \in \Omega$ , and  $X_1(\omega) > 0$  for some  $\omega \in \Omega$ .

Now with  $X_0 = 0$ , and  $X_1 = \Delta_0 S_1 + (1 + r)(X_0 - \Delta_0 S_0)$  for two states  $H$  and  $T$  of the coin toss, we have

$$\begin{cases} X_1(H) = \Delta_0 S_1(H) + (1 + r)(X_0 - \Delta_0 S_0) = \Delta_0 u S_0 + (1 + r)(X_0 - \Delta_0 S_0) \\ X_1(T) = \Delta_0 S_1(T) + (1 + r)(X_0 - \Delta_0 S_0) = \Delta_0 d S_0 + (1 + r)(X_0 - \Delta_0 S_0) \end{cases}$$

Eliminating  $\Delta_0$ , we arrive at

$$0 = X_0 = \frac{1}{1 + r} \left[ \frac{1 + r - d}{u - d} X_1(H) + \frac{u - 1 - r}{u - d} X_1(T) \right]$$

which reduces to

$$0 = (1 + r - d) X_1(H) + (u - 1 - r) X_1(T)$$

Given assumption  $0 < d < 1 + r < u$ , if  $X_1(H) > 0$ , then  $X_1(T) < 0$ . Vice versa, if  $X_1(T) > 0$ , then  $X_1(H) < 0$ .

Thus, we have shown that given the assumption  $0 < d < 1 + r < u$ , we cannot observe nonnegative terminal wealth in all cases with the existence of positive terminal wealth in some cases. In other words, arbitrage cannot exist in the model.

**Exercise 1.2** Show that arbitrage does not exist in a situation involving buying  $\Delta_0$  shares of stock and  $\Gamma_0$  options.

**Answer:** The terminal wealth at time 1 is

$$X_1 = \Delta_0 S_1 + \Gamma_0 (S_1 - 5)^+ - \frac{5}{4} (4\Delta_0 + 1.20\Gamma_0)$$

Plugging in the two states of the time-1 stock price  $S_1(H) = 8$  and  $S_1(T) = 2$ , we have

$$\begin{cases} X_1(H) = 8\Delta_0 + 3\Gamma_0 - 5\Delta_0 - 1.5\Delta_0 = 3\Delta_0 + 1.5\Gamma_0 \\ X_1(T) = 2\Delta_0 + 0\Gamma_0 - 5\Delta_0 - 1.5\Delta_0 = -3\Delta_0 - 1.5\Gamma_0 \end{cases}$$

while implies  $X_1(H) + X_1(T) = 0$ , or that one being positive will lead to the other being negative, so that arbitrage cannot exist in the model.

**Exercise 1.3** Use risk-neutral pricing formula to find the time-zero price  $V_0$  for the derivative security that pays off the stock price at time 1 (e.g. a call option with  $K = 0$ ).

**Answer:**  $V_0 = \frac{1}{1+r} [\tilde{p}V_1(H) + \tilde{q}V_1(T)] = \frac{1}{1+r} (\tilde{p}S_1 + \tilde{q}S_1) = \frac{1}{1+r} [(\tilde{p} + \tilde{q})S_1] = \frac{S_1}{1+r}.$

**Exercise 1.4** In the proof of Theorem 1.1.2 (replication in the multiperiod binomial model), show under the induction hypothesis that

$$X_{n+1}(\omega_1\omega_2 \cdots \omega_n T) = V_{n+1}(\omega_1\omega_2 \cdots \omega_n T)$$

**Answer:** Fix the first  $n$  coin tosses  $\omega_1\omega_2 \cdots \omega_n$  and suppress it. We have

$$\begin{aligned} X_{n+1}(T) &= \Delta_n S_{n+1}(T) + (1+r)(X_n - \Delta_n S_n) \\ &= (1+r)X_n + \Delta_n [S_{n+1}(T) - (1+r)S_n] \\ &= (1+r)V_n + \frac{V_{n+1}(H) - V_{n+1}(T)}{S_{n+1}(H) - S_{n+1}(T)} [S_{n+1}(T) - (1+r)S_n] \quad \cdots \text{by hypothesis, } X_n = V_n \\ &= (1+r)V_n + \frac{V_{n+1}(H) - V_{n+1}(T)}{u-d} [d - (1+r)] \\ &= (1+r) \left( \frac{1}{1+r} \right) [\tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T)] + [-\tilde{p}V_{n+1}(H) + \tilde{p}V_{n+1}(T)] \\ &= (\tilde{q} + \tilde{p})V_{n+1}(T) \\ &= V_{n+1}(T) \end{aligned}$$

**Exercise 1.5** Walk through the hedging strategy along a particular path in a three-period binomial model.

**Answer:** Start at time 1 with portfolio valued at  $X_1(H) = V_1(H) = \$2.24$ . The agent takes a position of

$$\Delta_1(H) = \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)} = \frac{3.2 - 2.4}{16 - 4} = \frac{1}{15}$$

share in the stock. This leaves her  $X_1(H) - \Delta_1(H)S_1(H) = \$2.24 - \frac{1}{15} \cdot \$8 = \$\frac{128}{75}$  to invest in the money market.

Then, at time 2, if the stock goes up again, she will have a portfolio valued at

$$X_2(HH) = \Delta_1(H)S_2(HH) + (1+r)[X_1(H) - \Delta_1(H)S_1(H)] = \frac{1}{15} \cdot \$16 + \frac{5}{4} \cdot \$\frac{128}{75} = \$3.2$$

If the stock goes down, her portfolio will be worth

$$X_2(HT) = \Delta_1(H)S_2(HT) + (1+r)[X_1(H) - \Delta_1(H)S_1(H)] = \frac{1}{15} \cdot \$4 + \frac{5}{4} \cdot \$\frac{128}{75} = \$2.4$$

We have shown that  $X_2(HH) = V_2(HH) = \$3.2$  and  $X_2(HT) = V_2(HT) = \$2.4$ .

Next, assuming that the stock goes up in the first period and down in second period, the agent takes a position of

$$\Delta_2(HT) = \frac{V_3(HTH) - V_3(HTT)}{S_3(HTH) - S_3(HTT)} = \frac{0 - 6}{8 - 2} = -1$$

share in the stock. That is, she short sells 1 unit of stock. This leaves her

$$X_2(HT) - \Delta_2(HT)S_2(HT) = \$2.4 - (-1) \cdot \$4 = \$6.4 \text{ to invest in the money market.}$$

Finally, at time 3, if the stock goes up, she will have a portfolio valued at

$$X_3(HTH) = \Delta_2(HT)S_3(HTH) + (1+r)[X_2(HT) - \Delta_2(HT)S_2(HT)] = (-1) \cdot \$8 + \frac{5}{4} \cdot \$6.4 = \$0$$

If the stock goes down, her portfolio will be worth

$$X_3(HTT) = \Delta_2(HT)S_3(HTT) + (1+r)[X_2(HT) - \Delta_2(HT)S_2(HT)] = (-1) \cdot \$2 + \frac{5}{4} \cdot \$6.4 = \$6$$

We have shown that  $X_3(HTH) = V_3(HTH) = \$0$  and  $X_3(HTT) = V_3(HTT) = \$6$ . In other words, she has hedged her short position in the option.

**Exercise 1.6 (Hedging a Long Position: One Period)** Specify how a bank's trader should invest in the stock and money markets to earn the interest rate on the capital tied up by a long position in a European call.

**Answer:** To hedge a long position in a European call with strike  $K = \$5$ , valued at  $V_0 = \$1.20$ , the trader should short sell

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = \frac{(8-5)^+ - (2-5)^+}{8-2} = \frac{3-0}{8-2} = \frac{1}{2} = 0.5$$

share of stock. This will generate  $\Delta_0 S_0 = 0.5 \cdot \$4 = \$2$  in cash, which should be invested in the money market.

At time 1, if the stock goes up to  $S_1(H) = \$8$ , the trader will collect \$3 from option payoff, and get  $(1+r)\Delta_0 S_0 = \frac{5}{4} \cdot \$2 = \$2.5$  from the money market, but spend  $\Delta_0 S_1(H) = 0.5 \cdot \$8 = \$4$  to buy back the stock to close out the short selling position. In total, the trader will have  $\$3 + \$2.5 - \$4 = \$1.5$ .

If the stock goes down to  $S_1(T) = \$2$ , there will be no option payoff to collect. The trader will still get \$2.5 from the money market, but spend  $\Delta_0 S_1(T) = 0.5 \cdot \$2 = \$1$  to buy back the stock to close out the short selling position. In total, the trader will, again, have  $\$2.5 - \$1 = \$1.5$ .

We have shown a hedging strategy in which, given \$1.2 of capital tied up in a long call option, the trader can always earn 25% interest rate and get  $\frac{5}{4} \cdot \$1.2 = \$1.5$  after one period.

**Exercise 1.7 (Hedging a Long Position: Multiple Periods)** Specify how a bank's trader should invest in the stock and money markets to earn the interest rate on the capital tied up by a long position in a lookback option in a three-period binomial model.

**Answer:** At each period  $n$ , the trader should use delta-hedging formula

$$\Delta_n(\omega_1 \cdots \omega_n) = \frac{V_{n+1}(\omega_1 \cdots \omega_n H) - V_{n+1}(\omega_1 \cdots \omega_n T)}{S_{n+1}(\omega_1 \cdots \omega_n H) - S_{n+1}(\omega_1 \cdots \omega_n T)}$$

to work out the number of shares  $\Delta_n$  to short sell or to hold in the hedging portfolio.

Since the aim is to hedge a long option position in the option,  $\Delta_n > 0$  implies short selling, while  $\Delta_n < 0$  implies holding the stock.

Then the trader should work out the money to borrow from or invest in the money market with  $V_n(\omega_1 \cdots \omega_n) - \Delta_n S_n(\omega_1 \cdots \omega_n)$ .

Again, since the aim is to hedge a long option position, a positive number implies borrowing, while a negative number implies investing.

Note that the hedging strategy is *self-financing* in that any change in the stock position is funded by (or will lead to) a change in the money market position.

In what follows, we walk through the hedging portfolio in the perspective of a balance sheet, assuming that the coin tosses are  $THH$  or  $THT$ .

Note that regardless of the coin toss result the total value is always  $(\frac{5}{4})^3 \cdot \$1.376 = \$2.6875$  at time three.

| Time | Assets | Liabilities | Portfolio Value | Option Value | Total Value |
|------|--------|-------------|-----------------|--------------|-------------|
|------|--------|-------------|-----------------|--------------|-------------|

| Time                       | Assets                         | Liabilities    | Portfolio Value | Option Value | Total Value |
|----------------------------|--------------------------------|----------------|-----------------|--------------|-------------|
| 0 (after hedging)          | cash \$0.6933                  | stock \$0.6933 | \$0             | \$1.376      | \$1.376     |
| 1 ( $T$ : before hedging)  | cash \$0.8667                  | stock \$0.3467 | \$0.52          | \$1.2        | \$1.72      |
| 1 ( $T$ : after hedging)   | stock \$0.9333                 | cash \$0.4133  | \$0.52          | \$1.2        | \$1.72      |
| 2 ( $TH$ : before hedging) | stock \$1.8667                 | cash \$0.5167  | \$1.35          | \$0.8        | \$2.15      |
| 2 ( $TH$ : after hedging)  | stock \$1.3333 + cash \$0.0167 |                | \$1.35          | \$0.8        | \$2.15      |
| 3 ( $THH$ )                | stock \$2.6667 + cash \$0.0280 |                | \$2.6875        | \$0          | \$2.6875    |
| 3 ( $THT$ )                | stock \$0.6667 + cash \$0.0280 |                | \$0.6875        | \$2          | \$2.6875    |

**Exercise 1.8 (Asian Option)** Consider the three-period model with  $S_0 = 4$ ,  $u = 2$ ,  $d = \frac{1}{2}$ ,  $r = \frac{1}{4}$  so that  $\tilde{p} = \tilde{q} = \frac{1}{2}$ . Consider an Asian call option that expires at time three and has strike  $K = 4$ . Define  $Y_n = \sum_{k=0}^n S_k$ . Let  $v_n(s, y)$  denote the price of this option at time  $n$  if  $S_n = s$  and  $Y_n = y$ .

(i) Develop an algorithm for computing  $v_n$  recursively.

(ii) Compute  $v_0(4, 4)$ , the price of the Asian at time zero.

(iii) Provide a formula for  $\delta_n(s, y)$ , the number of shares of stock that should be held by the replicating portfolio.

**Answer:** (i) For  $n = 3$

$$v_n(s, y) = \left( \frac{y}{n+1} - K \right)^+ = \left( \frac{y}{4} - 4 \right)^+$$

For  $n = 0, 1, 2$

$$v_n(s, y) = \frac{1}{1+r} \left[ \tilde{p} v_{n+1}(us, y+us) + \tilde{q} v_{n+1}(ds, y+ds) \right] = \frac{2}{5} \left[ v_{n+1}(2s, y+2s) + v_{n+1}\left(\frac{s}{2}, y+\frac{s}{2}\right) \right]$$

(ii) Applying the above algorithm, for  $n = 3$

$$\begin{aligned} v_3(32, 60) &= \left( \frac{1}{4} \cdot 60 - 4 \right)^+ = 11, & v_3(8, 36) &= \left( \frac{1}{4} \cdot 36 - 4 \right)^+ = 5, \\ v_3(8, 24) &= \left( \frac{1}{4} \cdot 24 - 4 \right)^+ = 2, & v_3(8, 18) &= \left( \frac{1}{4} \cdot 18 - 4 \right)^+ = 0.5, \\ v_3(2, 18) &= \left( \frac{1}{4} \cdot 18 - 4 \right)^+ = 0.5, & v_3(2, 12) &= \left( \frac{1}{4} \cdot 12 - 4 \right)^+ = 0, \\ v_3(2, 9) &= \left( \frac{1}{4} \cdot 9 - 4 \right)^+ = 0, & v_3(0.5, 7.5) &= \left( \frac{1}{4} \cdot 0.5 - 4 \right)^+ = 0. \end{aligned}$$

For  $n = 0, 1, 2$

$$\begin{aligned}
v_2(16, 28) &= \frac{2}{5} [v_3(32, 60) + v_3(8, 36)] = \frac{2}{5} \cdot (11 + 5) = 6.4 \\
v_2(4, 16) &= \frac{2}{5} [v_3(8, 24) + v_3(2, 18)] = \frac{2}{5} \cdot (2 + 0.5) = 1 \\
v_2(4, 10) &= \frac{2}{5} [v_3(8, 18) + v_3(2, 12)] = \frac{2}{5} \cdot (0.5 + 0) = 0.2 \\
v_2(1, 7) &= \frac{2}{5} [v_3(2, 9) + v_3(0.5, 7.5)] = \frac{2}{5} \cdot (0 + 0) = 0 \\
v_1(8, 12) &= \frac{2}{5} [v_2(16, 28) + v_2(4, 16)] = \frac{2}{5} \cdot (6.4 + 1) = 2.96 \\
v_1(2, 6) &= \frac{2}{5} [v_2(4, 10) + v_2(1, 7)] = \frac{2}{5} \cdot (0.2 + 0) = 0.08 \\
v_0(4, 4) &= \frac{2}{5} [v_1(8, 12) + v_1(2, 6)] = \frac{2}{5} \cdot (2.96 + 0.08) = 1.216
\end{aligned}$$

(iii) The number of shares of stock that should be held by the replicating portfolio at time  $n$  if  $S_n = s$  and  $Y_n = y$  is

$$\delta_n(s, y) = \frac{v_{n+1}(us, y + us) - v_{n+1}(ds, y + ds)}{us - ds} = \frac{v_{n+1}(2s, y + 2s) - v_{n+1}(\frac{s}{2}, y + \frac{s}{2})}{\frac{3}{2}s}$$

**Exercise 1.9 (Stochastic Volatility, Random Interest Rate)** In a binomial model whose up and down factors and interest rate depend on coin tosses,

(i) Provide an algorithm for determining the time-zero price for a derivative security.

(ii) Provide a formula the number of shares of stock that should be held by the replicating portfolio.

(iii) Suppose  $S_0 = 80$ , with each head the stock price goes up by 10, each tail down by 10. Assume zero interest rate. Price a European call with strike price  $K = 80$  which expires at time five.

**Answer:** (i) We construct a replicating portfolio whose terminal payoff is identical with that of the derivative security in any case. That is,

$$X_N(\omega_1 \cdots \omega_N) = V_N(\omega_1 \cdots \omega_N)$$

Let  $\omega$  denote an arbitrary sequence of  $\omega_1 \cdots \omega_n$ . The wealth process of the replicating portfolio is governed by

$$\begin{cases} X_{n+1}(\omega H) = \Delta_n(\omega)S_n(\omega T) + [1 + r_n(\omega)][X_n(\omega) - \Delta_n(\omega)S_n(\omega)] \\ X_{n+1}(\omega T) = \Delta_n(\omega)S_n(\omega T) + [1 + r_n(\omega)][X_n(\omega) - \Delta_n(\omega)S_n(\omega)] \end{cases}$$

Let  $V_n$  denote  $X_n$  as we solve recursively and let

$$\begin{cases} \tilde{p}_n(\omega) = \frac{[1 + r_n(\omega)]S_n(\omega) - S_{n+1}(\omega T)}{S_{n+1}(\omega H) - S_{n+1}(\omega T)} \\ \tilde{q}_n(\omega) = 1 - \tilde{p}_n(\omega) = \frac{S_{n+1}(\omega H) - [1 + r_n(\omega)]S_n(\omega)}{S_{n+1}(\omega H) - S_{n+1}(\omega T)} \end{cases}$$

we have

$$V_n(\omega) = \frac{1}{1 + r_n(\omega)} [\tilde{p}_n(\omega)V_{n+1}(\omega H) + \tilde{q}_n(\omega)V_{n+1}(\omega T)]$$

for  $n = 0, 1, \dots, N - 1$ , where  $V_N$  is known for the corresponding coin-toss result.

Note that, given coin-toss results, we can determine  $\{S_n\}_{n=0}^N$  and  $\{r_n\}_{n=0}^{N-1}$  and thus can work out  $\tilde{p}_n$  and  $\tilde{q}_n$ .

Working backwards in time from  $V_N$ , the above equation will determine  $V_0$ , the time-zero price for the derivative security.

(ii) From the wealth process of the replicating portfolio, we have

$$\Delta_n(\omega) = \frac{V_{n+1}(\omega H) - V_{n+1}(\omega T)}{S_{n+1}(\omega H) - S_{n+1}(\omega T)}$$

which tells the number of shares of stock that should be held by the replicating portfolio.

(iii) Given this special setting, we have

$$\tilde{p}_n(\omega) = \frac{[1 + r_n(\omega)]S_n(\omega) - S_{n+1}(\omega T)}{S_{n+1}(\omega H) - S_{n+1}(\omega T)} = \frac{(1 + 0) \cdot S_n(\omega) - [S_n(\omega) - 10]}{[S_n(\omega) + 10] - [S_n(\omega) - 10]} = \frac{10}{20} = 0.5$$

That is, the value of  $\tilde{p}_n(\omega)$  is independent of time  $n$  and coin-toss result  $\omega$ . So we write  $\tilde{p}_n(\omega)$  as  $\tilde{p}$ . The formula in the recursive algorithm now reduces to

$$V_n(\omega) = \frac{V_{n+1}(\omega H) + V_{n+1}(\omega T)}{2}$$

For computational considerations, we work on stock price  $s$ , rather than coin-toss result  $\omega$ , and have

$$v_n(s) = \frac{v_{n+1}(s + 10) + v_{n+1}(s - 10)}{2}$$

for  $n = 0, 1, \dots, N - 1$ , where  $v_N(s) = (s - K)^+$ .

Thus, we have

$$\begin{aligned} v_5(130) &= (130 - 80)^+ = 50, & v_5(110) &= (110 - 80)^+ = 30, & v_5(90) &= (90 - 80)^+ = 10, \\ v_5(70) &= (70 - 80)^+ = 0, & v_5(50) &= (50 - 80)^+ = 0, & v_5(30) &= (30 - 80)^+ = 0 \end{aligned}$$

and

$$\begin{aligned}
v_4(120) &= \frac{1}{2} \cdot [v_5(130) + v_5(110)] = \frac{1}{2} \cdot (50 + 30) = 40 \\
v_4(100) &= \frac{1}{2} \cdot [v_5(110) + v_5(90)] = \frac{1}{2} \cdot (30 + 10) = 20 \\
v_4(80) &= \frac{1}{2} \cdot [v_5(90) + v_5(70)] = \frac{1}{2} \cdot (10 + 0) = 5 \\
v_4(60) &= \frac{1}{2} \cdot [v_5(70) + v_5(50)] = \frac{1}{2} \cdot (0 + 0) = 0 \\
v_4(40) &= \frac{1}{2} \cdot [v_5(50) + v_5(30)] = \frac{1}{2} \cdot (0 + 0) = 0 \\
v_3(110) &= \frac{1}{2} \cdot [v_4(120) + v_4(100)] = \frac{1}{2} \cdot (40 + 20) = 30 \\
v_3(90) &= \frac{1}{2} \cdot [v_4(100) + v_4(80)] = \frac{1}{2} \cdot (20 + 5) = 12.5 \\
v_3(70) &= \frac{1}{2} \cdot [v_4(80) + v_4(60)] = \frac{1}{2} \cdot (5 + 0) = 2.5 \\
v_3(50) &= \frac{1}{2} \cdot [v_4(60) + v_4(40)] = \frac{1}{2} \cdot (0 + 0) = 0 \\
v_2(100) &= \frac{1}{2} \cdot [v_3(110) + v_3(90)] = \frac{1}{2} \cdot (30 + 12.5) = 21.25 \\
v_2(80) &= \frac{1}{2} \cdot [v_3(90) + v_3(70)] = \frac{1}{2} \cdot (12.5 + 2.5) = 7.5 \\
v_2(60) &= \frac{1}{2} \cdot [v_3(70) + v_3(50)] = \frac{1}{2} \cdot (2.5 + 0) = 1.25 \\
v_1(90) &= \frac{1}{2} \cdot [v_2(100) + v_2(80)] = \frac{1}{2} \cdot (21.25 + 7.5) = 14.375 \\
v_1(70) &= \frac{1}{2} \cdot [v_2(80) + v_2(60)] = \frac{1}{2} \cdot (7.5 + 1.25) = 4.375 \\
v_0(80) &= \frac{1}{2} \cdot [v_1(90) + v_1(70)] = \frac{1}{2} \cdot (14.375 + 4.375) = 9.375
\end{aligned}$$

The price of the European call is  $v_0(80) = 9.375$ .

In fact, for this special case, we may take advantage of the fact that  $\tilde{p} = \tilde{q} = \frac{1}{2}$ . Observe that in this binomial model, for each coin-toss result, each terminal payoff should be discounted by  $(\frac{1}{2})^5$ , regardless of its path, to arrive at the initial option value. Thus, we have

$$\begin{aligned}
v_0(80) &= \left(\frac{1}{2}\right)^5 \left[ \binom{5}{0} \cdot v_5(130) + \binom{5}{1} \cdot v_5(110) + \binom{5}{2} \cdot v_5(90) + \binom{5}{3} \cdot v_5(70) + \binom{5}{4} \cdot v_5(50) + \binom{5}{5} \cdot v_5(30) \right] \\
&= \frac{1}{32} (1 \cdot 50 + 5 \cdot 30 + 10 \cdot 10 + 10 \cdot 0 + 5 \cdot 0 + 1 \cdot 0) \\
&= 9.375
\end{aligned}$$

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