Work in Progress

Answers to Exercises

Stochastic Calculus for Finance I: The Binomial Asset Pricing Model by *Steven E. Shreve*

Chapter 6 Interest-Rate-Dependent Assets

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Please refer to the book for the exercises themselves. The text in front of each answer serves only as a summary of the question.

Please refer to the table below for notation conversion. I convert the symbols to make them more compact, and make the forward measure more distinguishable from the risk-neutral one.

Book	Answers	Note
$B_{n,m}$	B_n^m	time-n price of a bond that matures at time m
$\Delta_{n,m}$	Δ_n^m	number of m-maturity zero-coupon bonds held by the agent between time n and $n+1$
$\widetilde{\mathbb{P}}^m$	$\widehat{\mathbb{P}}^m$	probability under the m-forward measure
$\widetilde{\mathbb{E}}_n^m$	$\widehat{\mathbb{E}}_n^m$	time-n conditional expectation under the m-forward measure

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Exercise 6.2 Verify that the discounted value of the static hedging portfolio for a forward is a martingale under $\widetilde{\mathbb{P}}$.

Answer: To hedge a short position in a forward contract that is initiated at time n with delivery date m, the agent should, at time n, long 1 share of stock and short S_n/B_n^m unit of m-maturity zero-coupon bond. This constructs a static hedging portfolio. The time-n value of the hedging portfolio is

$$X_n = S_n - \Big(\frac{S_n}{B_n^m}\Big)B_n^m \ (=0)$$

To show its discounted value is $\widetilde{\mathbb{P}}$-martingale, note that

$$\begin{aligned}
\widetilde{\mathbb{E}}_n(D_{n+1}X_{n+1}) &= \widetilde{\mathbb{E}}_n\Big[D_{n+1}\Big(S_{n+1} - \Big(\frac{S_n}{B_n^m}\Big)B_{n+1}^m\Big)\Big] \\
&= \widetilde{\mathbb{E}}_n(D_{n+1}S_{n+1}) - \Big(\frac{S_n}{B_n^m}\Big)\widetilde{\mathbb{E}}_n(D_{n+1}B_{n+1}^m) && \cdots\text{linearity} \\
&= D_nS_n - \Big(\frac{S_n}{B_n^m}\Big)D_nB_n^m && \cdots\text{discounted stock, bond prices are }\widetilde{\mathbb{P}}\text{-martingale} \\
&= D_nX_n
\end{aligned}$$

Note: that discounted bond prices are $\widetilde{\mathbb{P}}$-martingale can be shown by, for $0 \le k \le n \le m$,

$$\widetilde{\mathbb{E}}_k(D_nB_n^m) = \widetilde{\mathbb{E}}_k\Big[D_n\widetilde{\mathbb{E}}_n\Big(\frac{D_m}{D_n}\Big)\Big] = \widetilde{\mathbb{E}}_k[\widetilde{\mathbb{E}}_n(D_m)] = \widetilde{\mathbb{E}}_k(D_m) = \widetilde{\mathbb{E}}_k\Big(D_k\cdot\frac{D_m}{D_k}\Big) = D_k\widetilde{\mathbb{E}}_k\Big(\frac{D_m}{D_k}\Big) = D_kB_k^m$$

Exercise 6.3 Use properties of conditional expectations to show that

$$\frac{1}{D_n}\widetilde{\mathbb{E}}_n[D_{m+1}R_m] = B_n^m - B_n^{m+1}$$

Answer: Note that $(1+R_m) = \frac{D_m}{D_{m+1}} \implies R_m = \frac{D_m}{D_{m+1}} - 1$. We have

$$\begin{aligned}
\frac{1}{D_n}\widetilde{\mathbb{E}}_n[D_{m+1}R_m] &= \frac{1}{D_n}\widetilde{\mathbb{E}}_n\Big[D_{m+1}\big(\frac{D_m}{D_{m+1}} - 1\big)\Big] \\
&= \frac{1}{D_n}\widetilde{\mathbb{E}}_n[D_m - D_{m+1}] \\
&= \widetilde{\mathbb{E}}_n\Big[\frac{D_m}{D_n}\Big] - \widetilde{\mathbb{E}}_n\Big[\frac{D_{m+1}}{D_n}\Big] && \cdots \text{take in what is known, linearity} \\
&= B_n^m - B_n^{m+1} && \cdots \text{definition}
\end{aligned}$$

Remark: What's the difference between

- (a) the time-n contract price to pay R_m at time $m+1$ (see *page 155*), and
- (b) the time-n forward price to deliver R_m at time $m+1$ (see *page 156*)?

Expressed in formula, the former is $B_n^{m+1}F_n^m = B_n^m - B_n^{m+1}$ while the latter is just F_n^m.

But the fundamental difference is that the former is the price to pay at time n, while the latter, as any other strike price of a forward contract, is the price to pay at time $m+1$.

An example with concrete numbers.

Say $n=3$, $m=7$. At time $n=3$, let's say $B_n^m = B_3^7 = \$0.93$ and $B_n^{m+1} = B_3^8 = \$0.87$.

It can be calculated that the time-n forward interest rate for the period between times m and $m+1$ is

$$F_n^m = \frac{B_n^m}{B_n^{m+1}} - 1 = \frac{\$0.93}{\$0.87} - 1 = 6.90\%$$

At time $n=3$, we do not know $R_m = R_7$ which is not unveiled until time $m=7$.

However, we may, at time $n=3$, sign a contract that promises to pay $R_m = R_7$ at time $m+1=8$.

For this contract we charge a no-arbitrage price at $B_n^{m+1}F_n^m = \$0.87 \times 6.90\% = \$0.06$, which the counterparty must pay immediately at time $n=3$, for the entitlement to receive R_7 at time $m+1=8$. This amount can also be worked out by $B_n^m - B_n^{m+1} = \$0.93 - \$0.87 = \$0.06$.

As another case, we may short a forward contract, which promises to deliver $R_m = R_7$ at time $m+1=8$. Out counterparty, who long the forward contract, is required to pay a strike price fixed at $F_n^m = \$0.069$ at time $m+1=8$ in exchange for the delivery of $R_m = R_7$.

Exercise 6.4 Using data provided in the book, construct a hedge for a short position in a caplet paying $(R_2 - \frac{1}{3})^+$ at time three. In particular,

(i) Determine $V_1(H)$ and $V_1(T)$.

(ii) Show how to begin with $\frac{2}{21}$ at time zero and invest in the money market and maturity-two bond in order to have a portfolio X_1 at time one that agrees with V_1. Why do we invest in the maturity-two bond rather than the maturity-three bond?

(iii) Show how to take the portfolio value X_1 at time one to a portfolio value X_2 at time two that agrees with V_2. Why do we at this step invest in the maturity-three bond rather than the maturity-two bond?

Answer: (i) According to the risk-neutral pricing formula, , $D_1V_1 = \widetilde{\mathbb{E}}_1(D_2V_2)$, or $V_1 = \widetilde{\mathbb{E}}_1(\frac{D_2}{D_1}V_2) = \widetilde{\mathbb{E}}_1(\frac{1}{1+R_1}V_2)$. In particular,

$$V_1(H) = \frac{1}{1+R_1(H)}\Big[\widetilde{\mathbb{P}}\{\omega_2 = H \mid \omega_1 = H\}\cdot V_2(HH) + \widetilde{\mathbb{P}}\{\omega_2 = T \mid \omega_1 = H\}\cdot V_2(HT)\Big] = \frac{1}{1+\frac{1}{6}}\cdot\Big(\frac{2}{3}\cdot\$\frac{1}{3} + \frac{1}{3}\cdot\$0\Big) = \$\frac{4}{21}$$

$$V_1(T) = \frac{1}{1+R_1(T)}\Big[\widetilde{\mathbb{P}}\{\omega_2 = H \mid \omega_1 = T\}\cdot V_2(TH) + \widetilde{\mathbb{P}}\{\omega_2 = T \mid \omega_1 = T\}\cdot V_2(TT)\Big] = \frac{1}{1+\frac{2}{5}}\cdot\Big(\frac{1}{2}\cdot\$0 + \frac{1}{2}\cdot\$0\Big) = \$0$$

(ii) The number of maturity-two zero-coupon bonds to hold between times zero and one is

$$\Delta_0^2 = \frac{V_1(H) - V_1(T)}{B_1^2(H) - B_1^2(T)} = \frac{\frac{4}{21} - 0}{\frac{6}{7} - \frac{5}{7}} = \frac{4}{3}$$

Since we have sold the caplet for $X_0 = V_0 = \$\frac{2}{21}$, the money-market account position is

$$X_0 - \Delta_0^2 B_0^2 = \$\frac{2}{21} - \frac{4}{3}\cdot\$\frac{11}{14} = -\$\frac{20}{21}$$

The negative number indicates we should borrow $\$\frac{20}{21}$ from the money market.

If the first coin toss is H, then at time one, the hedging portfolio has value

$$X_1(H) = \Delta_0^2 B_1^2(H) + (1+R_0)(X_0 - \Delta_0^2 B_0^2) = \frac{4}{3}\cdot\$\frac{6}{7} - (1+0)\cdot(-\$\frac{20}{21}) = \$\frac{4}{21}$$

If the first coin toss is V, then at time one, the hedging portfolio has value

$$X_1(T) = \Delta_0^2 B_1^2(T) + (1+R_0)(X_0 - \Delta_0^2 B_0^2) = \frac{4}{3}\cdot\$\frac{5}{7} - (1+0)\cdot(-\$\frac{20}{21}) = \$0$$

Note that the hedging portfolio has a time-one value X_1 that agrees with V_1, regardless of the coin-toss result.

- Why do we invest in the maturity-two bond?

Because at time zero, we want to hedge against the stochastic interest rate R_1 which is unveiled at time one, which is incorporated in the price of the maturity-two bond. In fact, the price of the maturity-two bond is fully determined by two interest rates: R_0, which is known at time zero, and R_1, which we want to hedge against.

- Why do we not invest in the maturity-three bond?

The price of the maturity-three bond additionally depends on R_2, which we shouldn't worry about at time zero. Moreover, in this case, $B_1^3(H) = B_1^3(T) = \frac{4}{7}$, which means it has no randomness between times zero and one, and thus cannot be used as a hedge. A binomial model requires $0 < d < 1 + r < u$, and this violates the assumption as $d = u$.

(iii) If the first coin toss is H, the number of maturity-three zero-coupon bonds to hold between times one and two is

$$\Delta_1^3(H) = \frac{V_2(HH) - V_2(HT)}{B_2^3(HH) - B_2^3(HT)} = \frac{\frac{1}{3} - 0}{\frac{1}{2} - 1} = -\frac{2}{3}$$

The negative number indicates that we should short $-\frac{2}{3}$ unit of the maturity-three bond. The cash generated will be invested in the money market.

With hedging portfolio value $X_1(H) = \$\frac{4}{21}$, the money market position is

$$X_1(H) - \Delta_1^3(H)B_1^3(H) = \$\frac{4}{21} - (-\frac{2}{3})\cdot\$\frac{4}{7} = \$\frac{4}{7}$$

If the first two coin tosses are HH, the hedging portfolio has value

$$X_2(HH) = \Delta_1^3(H)B_2^3(HH) + [1+R_1(H)][X_1(H) - \Delta_1^3(H)B_1^3(H)] = (-\frac{2}{3})\cdot \$\frac{1}{2} + (1+\frac{1}{6})\cdot \$\frac{4}{7} = \$\frac{1}{3}$$

If the first two coin tosses are HT, the hedging portfolio has value

$$X_2(HT) = \Delta_1^3(H)B_2^3(HT) + [1+R_1(H)][X_1(H) - \Delta_1^3(H)B_1^3(H)] = (-\frac{2}{3})\cdot \$1 + (1+\frac{1}{6})\cdot \$\frac{4}{7} = \$0$$

If, however, the first coin toss is T, the hedging strategy is trivial. We start with $X_1(T) = \$0$ and do nothing, as $V_2(TH) = V_2(TT) = \$0$.

We have shown that the hedging portfolio has time-two value X_2 that agrees with V_2. That is, $X_2(HH) = V_2(HH) = \$\frac{1}{3}$, and they are both zero in all other cases.

It is by the same token that we invest in the maturity-three bond rather than the maturity-two bond at this step, since the randomness of R_2, which we want to hedge against, is not incorporated in the price of the latter.

Exercise 6.5 (i) Use (6.4.8) and (6.2.5) to show that $F_n^m, n = 0, 1, \cdots, m$ is a martingale under the $(m+1)$-forward measure $\widehat{\mathbb{P}}^{m+1}$.

(ii) Compute F_0^2, $F_1^2(H)$, and $F_1^2(H)$ in an example in the book and verify the martingale property $\widehat{\mathbb{E}}^3[F_1^2] = F_0^2$.

Answer: (i) For $0 \le k \le n \le m$,

$$\begin{aligned}
\widehat{\mathbb{E}}_k^{m+1}(F_n^m) &= \frac{1}{D_k B_k^{m+1}} \widetilde{\mathbb{E}}_k(D_{m+1}F_n^m) && \cdots (6.4.8) \\
&= \frac{1}{D_k B_k^{m+1}} \widetilde{\mathbb{E}}_k\left[D_{m+1}\frac{B_n^m - B_n^{m+1}}{B_n^{m+1}}\right] && \cdots \text{definition} \\
&= \frac{1}{D_k B_k^{m+1}} \widetilde{\mathbb{E}}_k\left[D_{m+1}\frac{B_n^m - B_n^{m+1}}{\frac{1}{D_n}\widetilde{\mathbb{E}}_n(D_{m+1})}\right] && \cdots \text{definition, take out what is known} \\
&= \frac{1}{D_k B_k^{m+1}} \widetilde{\mathbb{E}}_k\left[(D_nB_n^m - D_nB_n^{m+1})\frac{D_{m+1}}{\widetilde{\mathbb{E}}_n(D_{m+1})}\right] && \cdots \text{rearranging} \\
&= \frac{1}{D_k B_k^{m+1}} (D_kB_k^m - D_kB_k^{m+1})\widetilde{\mathbb{E}}_k\left[\frac{D_{m+1}}{\widetilde{\mathbb{E}}_n(D_{m+1})}\right] && \cdots (6.2.5)\text{, take out what is known} \\
&= F_k^m \widetilde{\mathbb{E}}_k\left[\frac{D_{m+1}}{\widetilde{\mathbb{E}}_n(D_{m+1})}\right] && \cdots \text{definition} \\
&= F_k^m \widetilde{\mathbb{E}}_k\left[\widetilde{\mathbb{E}}_n\left[\frac{D_{m+1}}{\widetilde{\mathbb{E}}_n(D_{m+1})}\right]\right] && \cdots \text{iterated conditioning} \\
&= F_k^m \widetilde{\mathbb{E}}_k\left[\frac{\widetilde{\mathbb{E}}_n(D_{m+1})}{\widetilde{\mathbb{E}}_n(D_{m+1})}\right] && \cdots \text{take out what is known} \\
&= F_k^m
\end{aligned}$$

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