TEMPLATE: MATH 3113, HOMEWORK 2

AUTHOR = YOU

(1) The following questions concern finding formula and utilizing induction.
   (a) For g and h in a group G, prove that $(gh)^n = g(hg)^{n-1}h$ for all $n > 0$.

   Proof. $\square$

   (b) For g and h in a group G, prove that $(gh)^n = e$ if and only if $(hg)^n = e$. Does this force $gh = hg$?

   Proof. $\square$

(2) What is the cycle decomposition of

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 7 & 8 & 3 & 1 & 2 & 6 & 4 \end{pmatrix}.$$

Proof. $\square$

(3) Suppose G is a cyclic group of order 144. How many elements are in the subgroup $\langle g^{26} \rangle$? How many subgroups contain this subgroup?

Proof. $\square$

(4) Which of the following subsets are subgroups of $\mathrm{GL}_2(\mathbf{R})$?
   (a) $\left\{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) : a, c \neq 0\right\}$.
   (b) $\left\{\begin{pmatrix} a & b \\ 1/b & 1/a \end{pmatrix} : a, b \neq 0\right\}$.
   (c) $\left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : ad - bc = 1\right\}$.
   (d) $\left\{\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) : a \neq 0\right\}$.

   Proof. $\square$

(5) Fix $G = (\mathbf{Z}/7\mathbf{Z})^\times$ and $g = a \bmod 7$ with $(a, 7) = 1$. Consider the function $\varphi_a \colon G \to G$ defined by $\varphi(x \bmod 7) \mapsto 2x \bmod 7$.

(a) Show that φ_a is a bijection.

Proof. □

(b) Prove that $\varphi_a \circ \varphi_b = \varphi_{ab}$.

Proof. □

(c) Show that the ordered set $\{\varphi_a(1 \bmod 7), \ldots, \varphi_a(6 \bmod 7)\}$ is a permutation of the standard representatives modulo 7.

Proof. □

(d) What is the cycle decomposition of the permutation corresponding to φ_3?

Proof. □

(e) Is every permutation in S_6 realized this way?

Proof. □