Analyzing Problems Involving Related Rates

Plan

Learning Goals

Students will be able to:

- identify when a problem involves optimization versus when it involves related rates.
- select a labeled diagram that represents the problem when given a geometric related rates problem.
- select an appropriate equation linking the quantities in the problem.
- determine which quantities in the problem are constants and which are variables, and assign given values to quantities and rates of change of variables.
- calculate a differential equation, substitute initial conditions, and solve for a missing value.
- interpret the answers to related rates problems using the context of the problem (including units).

Solving related rates problems requires connecting the concepts of differentiation and rates of change in the context of a given scenario.

This lesson helps to build skill 1.D by having students first determine whether the problem scenario is an optimization or a related rates problem. The lesson also helps students break down the problem scenario, while addressing common misunderstandings or issues that may arise as students are faced with the question of "where to begin." Students will also be introduced to the concept of the derivative as a rate of change, and begin applying their understanding to solving related rates problems (which helps to develop skill 1.E as well).

Student misunderstandings

- The student may not be able to distinguish between an optimization problem and a related rates problem.
- The student may misunderstand the description of a geometric setting for a problem (e.g., confuse altitude with base in a triangle, confuse area and volume in three-dimensional problems) OR e.g., confuse sides in a right triangle altitude with base in a triangle when hypotenuse and base are changing or confuse area and volume in three-dimensional problems when several edges are changing independently.
- The student may not understand how to write an equation linking variables and constants based on a labeled diagram (e.g., students may misrepresent variables as constants, or vice versa, when transferring information from a diagram to an equation; or use the correct values but with the wrong equation).
- The student may confuse which quantities are variables and which are constants, or confuse the rate of change of a variable with the value of the variable.
- The student may select an equation that misrepresents the given problem (e.g., selecting $A = 2\pi r$, instead of $A = \pi r^2$, for a problem about the area of a circle; or selecting $A = \frac{1}{2}bh$ instead of $a^2 + b^2 = c^2$ for a problem relating the three sides of a right triangle).

Materials

The following supplies are needed:

- student activity sheets (1 per student)

AP CALCULUS STUDENT HANDOUT

Related Rates vs Optimization

Read each of the word problems that follow and decide whether it is an optimization problem or a related rates problem.

1. A particle moves along the parabola $y = x^2$. At what point is the particle when it is closest to the point (3,0)?	Answer: ☑ Optimization ☐ Related rates
2. A particle moves along the parabola $y = x^2$ with its x-coordinate increasing at a rate of 0.5 cm/sec. At what rate is the particles distance from the point $(3,0)$ changing when the particle is at $(2,4)$?	☐ Optimization Answer: ☑ Related rates
3. What are the dimensions of the cone of greatest volume that can be inscribed inside a sphere of radius 8 cm?	Answer: ☑ Optimization ☐ Related rates
4. Air is pumped into a sphere at a rate of 6 cm^3/sec. At what rate is the radius of the sphere increasing when the radius is 8 cm?	☐ Optimization Answer: ☑ Related rates
5. Water flows into a cylindrical tank with base radius 3 meters and height 12 meters at a rate of $(3t^2 - 8t)$ m^3/hr. At what rate is the height of the water changing when 2 hours have elapsed?	☐ Optimization Answer: ☑ Related rates
6. Water flows into a cylindrical tank with base radius 3 meters and height 12 meters at a rate of $(3t^2 - 8t)$ m^3/hr. At what time is the rate of flow the least?	Answer: ☑ Optimization ☐ Related rates

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Teach

Engage

Some students struggle with related rates problems at the very beginning because they have trouble discerning when a contextual problem calls for an optimization procedure and when it calls for a related rates procedure. Certainly related rates problems involve rates, but sometimes optimization problems do also. The key is in the question: *Are we being asked to find a rate or are we being asked to optimize some quantity?*

AP CALCULUS STUDENT HANDOUT

Unpacking a Related Rates Problem

For each of the following scenarios, first answer each part on your own. Then compare your answer with those of your group members and discuss any differences.

Scenario 1: An ant crawls directly away from the base of a vertical pole that is 3 feet tall. The ant is crawling at a rate of 1 foot per minute. Assume D is the distance from the ant to the base of the pole (measured in feet), and H is the distance from the ant to the top of the pole (measured in feet).

1. What is the value of the *rate of change* in the scenario (including units)?

Answer: 1 foot per minute

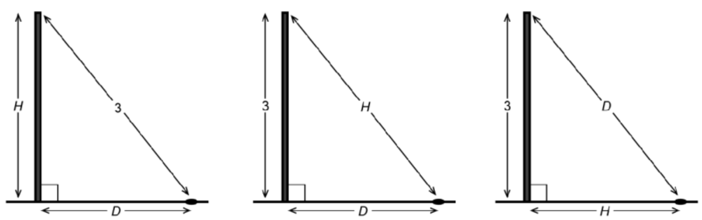
2. How tall is the pole in the scenario (including units)?

Answer: 3 feet

3. How do you know that this scenario can be represented with a diagram of a **right triangle**?

Answer: A vertical pole would make a right angle with the ground, and the ant is crawling away from it in a straight line.

4. Select which diagram appropriately represents the ant, the pole, and the quantities D and H.



A **B** **C**

Answer: Diagram B

5. Select which equation appropriately represents the relationship between the quantities in the problem.

Equation A: $D^2 = 3^2 + H^2$ **Equation B:** $A = \frac{1}{2}HD$ **Equation C:** $H^2 = 3^2 + D^2$

Answer: Equation C

Guided Practice

Remind students that the equations relating varying quantities can be differentiated, resulting in relating the original quantities and also their rates of change. The information in the scenario can then be used to predict an otherwise unknown rate of change from known values of the quantities and other rates of change.

Have students try to answer each set of questions on their own first, then compare their answers in small groups. Go over any questions at the end of the exercise.

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6. At a certain time, the ant is 4 feet away from the base of the pole. What is *H* at that time? Show your work.

Answer: 5 feet

$$H^2 = 3^2 + D^2$$
$$H^2 = 9 + (4)^2$$
$$H^2 = 25$$
$$H = \sqrt{25} = 5$$

Note that because the scenario is a right triangle, the values presented here would make it a 3-4-5 triangle.

7. Find *D* and *H* two minutes after the time mentioned in question 6. Show your work.

Answer: Approximately 7.616 feet. The ant is crawling at a rate of 1 foot per minute, so 2 minutes later the ant will have crawled 2 more feet, meaning that $D = 5 + 2 = 7$ feet.

$$H^2 = 3^2 + D^2$$
$$H^2 = 9 + (7)^2$$
$$H^2 = 58$$
$$H = \sqrt{58} \approx 7.616$$

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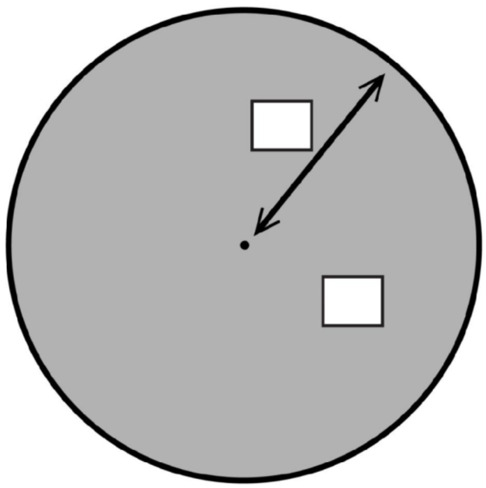
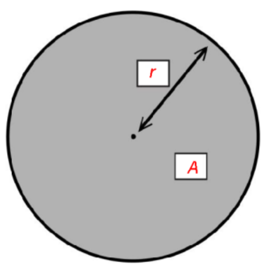
Scenario 2: An oil spill in the shape of a circle is expanding on the surface of a pond. The radius of the oil spill is r (measured in cm) and the area of the oil spill is A (measured in cm^2). Suppose that at a certain time, the area of the spill is 30 cm^2, and the radius of the spill is increasing at 0.2 cm per second.

8. Label the accompanying diagram so that it shows the spill, labeling the area A and the radius of the spill r.



Answer:



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9. Select which equation appropriately represents the relationship between the quantities in the problem.

Equation A: $A = 2\pi r$

Equation B: $A = \pi r^2$

Equation C: $A = \frac{4}{3}\pi r^3$

Answer: Equation B

10. What is the rate of change given in the scenario and what variable is changing?

Answer: The given rate of change is 0.2 cm per second, and the variable r is changing.

11. What is r at the “certain time” given in the scenario?

Answer: If $A = 30 \text{ cm}^2$, then $r = \sqrt{\frac{30}{\pi}} \approx 3.0902$ cm.

12. If the rate of change of r continues to be 0.2 cm per second, what is A after 3 more seconds?

Answer: r increases to $3.989 + (3)(0.2) = 4.589$ cm, so that $A = \pi(4.589)^2 \approx 21.05\pi \text{ cm}^2$.

13. Suppose the spill is expanding so that its area is increasing at a constant rate of 0.4 cm^2 per second. At a certain time, the area A is 50 cm^2. What is r at that time? What are A and r 10 seconds later?

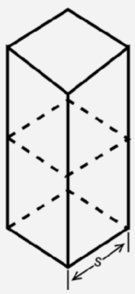
Answer: If $A = 50 \text{ cm}^2$, then $r = \sqrt{\frac{50}{\pi}} \approx 3.9894$ cm. A increases 10 seconds later to $50 + 10(0.4) = 54 \text{ cm}^2$, so then $r = \sqrt{\frac{54}{\pi}} \approx 4.146$ cm.

AP CALCULUS STUDENT HANDOUT

Check your understanding

A student was given each of the following scenarios and asked to provide a response. Check the student's responses. If the student's response is incorrect, explain why.

- ☐ A solid object has a square base with side S inches long. It is twice as high as it is wide.
  Student response:



Answer: The student is correct.

- ☐ The volume of the solid is V. Write an equation relating V and S.
  Student response: $V = 2S^2$

Answer: The student is incorrect. The student may have confused area and volume. The correct equation is $V = 2S^3$.

- ☐ What is the length of S (including units) when the volume V is 16 cubic inches?
  Student response: Since $V = 2S^2$, we know that $16 = 2S^2$, so S is $2\sqrt{2}$ square inches.

Is the student's answer correct? If not, explain why.

Answer: The student is using an incorrect equation and gets a numerically wrong answer. The unit used is also wrong, and should be inches. The correct equation $V = 2S^3$ gives $S = 2$ inches.

AP CALCULUS STUDENT HANDOUT

Part I: Understanding the Derivative as a Rate of Change

Scenario: An ant crawls directly away from the base of a vertical pole which is 3 feet tall. The ant is crawling at a rate of 1 foot per minute. Suppose that *D* is the distance from the ant to the base of the pole, and *H* is the distance from the ant to the top of the pole. Differentiating both sides of the original equation $H^2 = 3^3 + D^2$ yields a "rate of change" equation of $2H\frac{dH}{dt} = 0 + 2D\frac{dD}{dt}$. Assume that at a certain time, the ant is 4 feet away from the base of the pole. What are *H* and $\frac{dh}{dt}$ at that time, including units?

1. Which of the following represents the rate at which the ant is crawling away from the pole with respect to time (1 foot per minute)?

a. D **b.** H **c.** $\frac{dD}{dt}$ **d.** $\frac{dh}{dt}$

Answer: C

2. Which of the following variables can we now replace with the value "4"?

a. D **b.** H **c.** $\frac{dD}{dt}$ **d.** $\frac{dH}{dt}$

Answer: A

3. For the scenario and conditions described in questions 1 and 2, determine which of the following quantities are constants, and which are variables.

Quantity	Constant (a set quantity)	Variable (a varying quantity)
Height of the pole	Answer: ✓	
Rate the ant is crawling	Answer: ✓ (because $\frac{dD}{dt} = 1$ foot per minute)	
Distance from the ant to the base of the pole		Answer: ✓ (because D is increasing)
Distance from the ant to the top of the pole		Answer: ✓ (variable H)
Rate at which *H* is changing with respect to time		Answer: ✓ (variable $\frac{dH}{dt}$)

Independent Practice

Have students work individually on completing the table. If time does not permit for them to complete the assignment in class, this could be given as homework.

For the table in problem 3:
Many students may think that any time a quantity refers to a "rate," that that quantity is a variable (i.e., that it is "varying at a particular rate"). Clarify this misunderstanding by explaining that a rate can be a constant value because it can be the same rate at all times – that is, unvarying. For any values that have not been given a designated value (e.g., *D*, the distance from the ant to the base of the pole), those would be considered variables.

AP CALCULUS STUDENT HANDOUT

4. Which of the following equations appropriately represents the scenario when the ant is 4 feet away from the base of the pole? (Recall that in Scenario 1 of the previous activity, at the "certain time," the right triangle being considered is a 3-4-5 right triangle.)

a. $2(5)\frac{dh}{dt} = 2(4)\frac{dD}{dt}$

b. $2(3)\frac{dh}{dt} = 2(5)\frac{dD}{dt}$

c. $2(5)\frac{dh}{dt} = 2(3)\frac{dD}{dt}$

Answer: A

5. Now solve: When the ant is 4 feet from the base of the pole, what are the values of H and $\frac{dh}{dt}$?

Answer: **H = 5 feet,** and $\frac{dH}{dt} = 0.8$ feet per minute.

We know from the Guided Practice activity that H = 5 feet when D = 4 feet. Substituting those values into the differentiated equation gives us the following solution:

$$2H\frac{dh}{dt} = 2D\frac{dD}{dt}$$

$$2(5)\frac{dh}{dt} = 2(4)(1)$$

$$10\frac{dh}{dt} = 8$$

$$\frac{dh}{dt} = \frac{8}{10} = 0.8 \text{ feet per minute}$$

Independent Practice (cont.)

For problem 5:
A solution based on "solving" for H is also possible here (that is, to use "explicit differentiation" instead of implicit differentiation). It would begin with the equation $H = \sqrt{3^2 + D^2}$ and then continue by calculating the derivative; it would produce the same answer. Note, however, that this approach tends to be more problematic with more complex equations, so students should be encouraged to practice using implicit differentiation throughout this performance task.

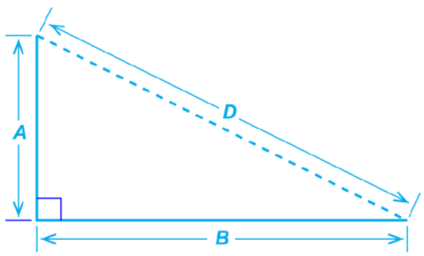
AP CALCULUS STUDENT HANDOUT

Part II: Solving Related Rates Problems

1. Two cars leave Central Square. The first car drives north at 35 miles per hour (mph), and its distance to Central Square is *A* (here *A* is actually a function of time). The second car drives east at 50 mph and its distance from Central Square is *B* (here *B* is also a function of time). The distance between the two cars is *D*. At a certain time, *A* = 20 miles and *B* = 40 miles.

   a. Sketch the general picture of what the cars and Central Square look like. Be sure to label the diagram with *A*, *B*, and *D*, representing the distances mentioned in the stem of the problem.

   Answer:



   b. Give a general formula for *D* in terms of *A* and *B*.

   Answer: $D = \sqrt{A^2 + B^2}$

   c. Find a formula for *D*′, the derivative of *D*, with respect to time in terms of *A*, *A*′, *B*, and *B*′.

   Answer: $D' = \frac{1}{2}(A^2 + B^2)^{-\frac{1}{2}}(2AA' + 2BB')$

   d. Are the two cars getting closer or farther away from each other at the certain time? How fast is the distance between them changing at that time?

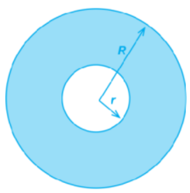
   Answer: $D' = \frac{1}{2}(20^2 + 40^2)^{-\frac{1}{2}}(2(20)(35) + 2(40)(50)) \approx 60.374$ miles per hour. Since this answer is positive, the cars are getting farther away from each other at that certain time.

AP CALCULUS STUDENT HANDOUT

2. An *annulus* is a region in the plane bounded by two circles with the same center (concentric circles). Suppose S is an annulus with an inner circle of radius r and an outer circle of radius R. The radii are changing so that r is increasing at 7 cm/sec, and R is increasing at 3 cm/sec. At 9 AM, $r = 30$ cm and $R = 80$ cm.

a. Sketch a picture of the annulus. Be sure to label the annulus with r and R representing the two radii mentioned in the stem of the problem.

Answer:



b. Write a formula for P, the total perimeter of the region S. The formula should involve r and R. What is P at 9 AM? Include appropriate units with your answer.

Answer: $P = 2\pi r + 2\pi R$. At 9 AM, $P = 2\pi(30) + 2\pi(80) \approx 691.150$ cm.

c. Write a formula for P', the rate of change of the total perimeter of the region S. The formula should involve r' and R'. What is P' at 9 AM? Include appropriate units with your answer. Is the total perimeter increasing or decreasing at that time?

Answer: $P' = 2\pi r' + 2\pi R'$. At 9 AM, $P' = 2\pi(7) + 2\pi(3) \approx 62.832$ cm/sec. Since this is positive, the total perimeter is increasing at that time.

d. Write a formula for A, the area of the region S. The formula should involve r and R. What is A at 9 AM? Include appropriate units with your answer.

Answer: $A = \pi R^2 - \pi r^2$. At 9 AM, $A = \pi(80)^2 - \pi(30)^2 \approx 17278.760\text{ cm}^2$.

e. Write a formula for A', the rate of change of the area of the region S The formula should involve r, r', R, and R'. What is A' at 9 AM? Include appropriate units with your answer. Is the area increasing or decreasing at that time?

Answer: $A' = 2\pi RR' - 2\pi rr'$. At 9 AM, $A' = 2\pi(80)(3) - 2\pi(30)(7) \approx 188.496\text{ cm}^2/\text{sec}$. Since this is positive, the area is increasing at that time.

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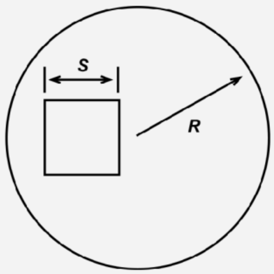
AP CALCULUS STUDENT HANDOUT

Check your understanding

A square with side length S is inside a circle of radius R as shown to the right. At a certain time, S = 5 inches and is increasing at the rate of 0.7 inches per minute. At the same time, R = 8 inches and is decreasing at the rate of 0.4 inches per minute.



Fill in the blanks to complete the solutions. Be sure to indicate *units* with every exact or approximate numerical answer.

- ☐ The area A inside the circle and outside the square is given by $A = \pi$______________, and the rate of change of this area is given by $A' =$ __________________. At the certain time, $A =$ ________________ ≈ ____________ ________________ (units) and A is changing at the rate of ________ ≈ _________ ___________ per ____________. A is ____creasing.

  Answer: The area A inside the circle and outside the square is given by $A = \pi R^2 - S^2$, and the rate of change of this area is given by $A' = 2\pi RR' - 2SS'$. At the certain time, $A = 64\pi - 25 \approx 176.062$ square inches, and A is changing at the rate of $-6.4\pi - 7 \approx -27.106$ square inches per minute. A is decreasing.

- ☐ The total perimeter P of the area A is given by $P =$ ______________, and the rate of change of this total perimeter is given by $P' =$ ______________. At the certain time, $P =$ _____________ _____________ (units), and P is changing at the rate of _______________ ≈ _______ ___________ per _____________. P is ___creasing.

  Answer: The total perimeter P of the area A is given by $P = 2\pi R + 4S$, and the rate of change of this total perimeter is given by $P' = 2\pi R' + 4S'$. At the certain time, $P = 16\pi + 20$ inches, and P is changing at the rate of $-0.8\pi + 2.8 \approx 0.287$ inches per minute. P is increasing.

Assess

Direct students to complete the Topic Questions.

Student handouts with answers:

Related Rates vs Optimization

Read each of the word problems that follow and decide whether it is an optimization problem or a related rates problem.

1. A particle moves along the parabola $y = x^2$. At what point is the particle when it is closest to the point (3,0)?	Answer: ☑ Optimization ☐ Related rates
2. A particle moves along the parabola $y = x^2$ with its *x*-coordinate increasing at a rate of 0.5 cm/sec. At what rate is the particles distance from the point $(3,0)$ changing when the particle is at $(2,4)$?	☐ Optimization Answer: ☑ Related rates
3. What are the dimensions of the cone of greatest volume that can be inscribed inside a sphere of radius 8 cm?	Answer: ☑ Optimization ☐ Related rates
4. Air is pumped into a sphere at a rate of 6 cm^3/sec. At what rate is the radius of the sphere increasing when the radius is 8 cm?	☐ Optimization Answer: ☑ Related rates
5. Water flows into a cylindrical tank with base radius 3 meters and height 12 meters at a rate of $(3t^2 - 8t)$ m^3/hr. At what rate is the height of the water changing when 2 hours have elapsed?	☐ Optimization Answer: ☑ Related rates
6. Water flows into a cylindrical tank with base radius 3 meters and height 12 meters at a rate of $(3t^2 - 8t)$ m^3/hr. At what time is the rate of flow the least?	Answer: ☑ Optimization ☐ Related rates

Unpacking a Related Rates Problem

For each of the following scenarios, first answer each part on your own. Then compare your answer with those of your group members and discuss any differences.

Scenario 1: An ant crawls directly away from the base of a vertical pole that is 3 feet tall. The ant is crawling at a rate of 1 foot per minute. Assume D is the distance from the ant to the base of the pole (measured in feet), and H is the distance from the ant to the top of the pole (measured in feet).

1. What is the value of the *rate of change* in the scenario (including units)?

Answer: 1 foot per minute

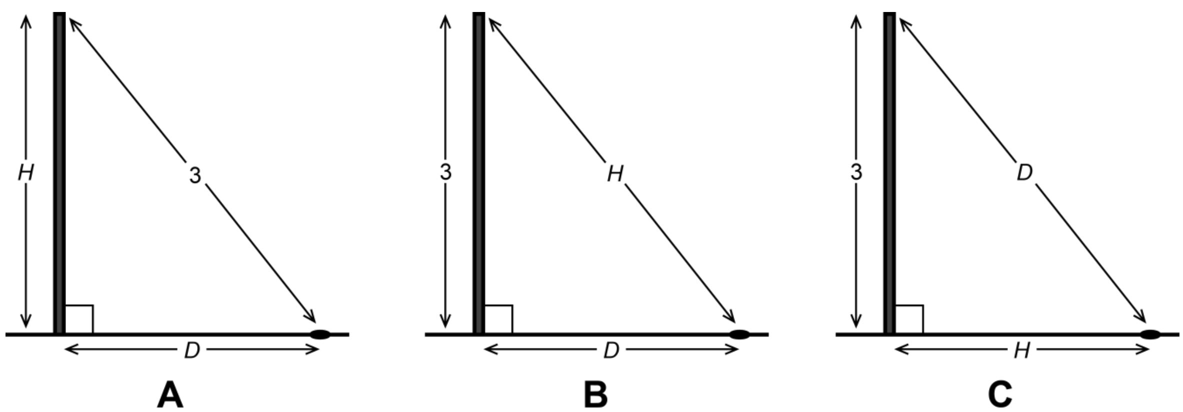
2. How tall is the pole in the scenario (including units)?

Answer: 3 feet

3. How do you know that this scenario can be represented with a diagram of a **right triangle**?

Answer: A vertical pole would make a right angle with the ground, and the ant is crawling away from it in a straight line.

4. Select which diagram appropriately represents the ant, the pole, and the quantities D and H.



Answer: Diagram B

5. Select which equation appropriately represents the relationship between the quantities in the problem.

Equation A: $D^2 = 3^2 + H^2$ **Equation B:** $A = \frac{1}{2}HD$ **Equation C:** $H^2 = 3^2 + D^2$

Answer: Equation C

6. At a certain time, the ant is 4 feet away from the base of the pole. What is *H* at that time? Show your work.

Answer: 5 feet

$$H^2 = 3^2 + D^2$$
$$H^2 = 9 + (4)^2$$
$$H^2 = 25$$
$$H = \sqrt{25} = 5$$

Note that because the scenario is a right triangle, the values presented here would make it a 3-4-5 triangle.

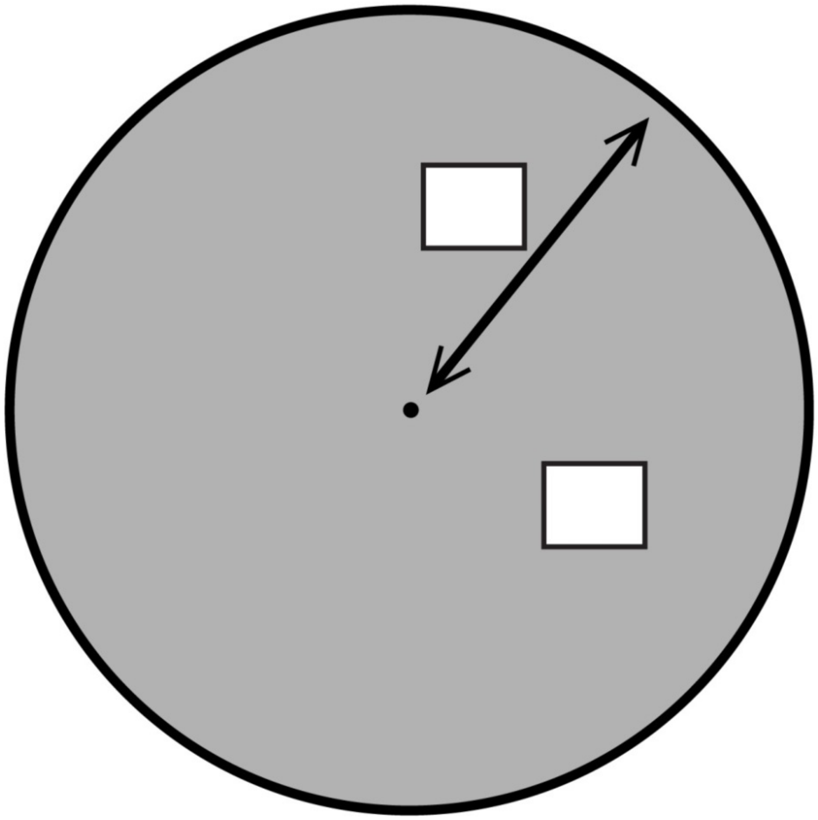
7. Find *D* and *H* two minutes after the time mentioned in question 6. Show your work.

Answer: Approximately 7.616 feet. The ant is crawling at a rate of 1 foot per minute, so 2 minutes later the ant will have crawled 2 more feet, meaning that $D = 5 + 2 = 7$ feet.

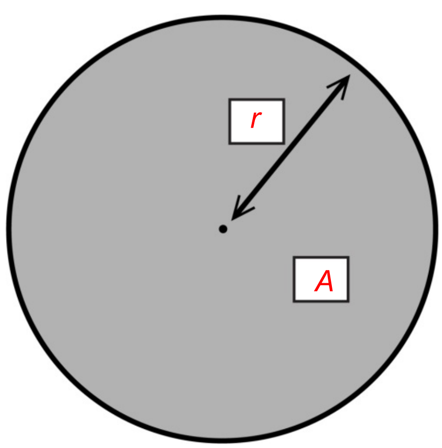
$$H^2 = 3^2 + D^2$$
$$H^2 = 9 + (7)^2$$
$$H^2 = 58$$
$$H = \sqrt{58} \approx 7.616$$

Scenario 2: An oil spill in the shape of a circle is expanding on the surface of a pond. The radius of the oil spill is r (measured in cm) and the area of the oil spill is A (measured in cm^2). Suppose that at a certain time, the area of the spill is $30\ \text{cm}^2$, and the radius of the spill is increasing at 0.2 cm per second.

8. Label the accompanying diagram so that it shows the spill, labeling the area A and the radius of the spill r.



Answer:



9. Select which equation appropriately represents the relationship between the quantities in the problem.

Equation A: $A = 2\pi r$

Equation B: $A = \pi r^2$

Equation C: $A = \frac{4}{3}\pi r^3$

Answer: Equation B

10. What is the rate of change given in the scenario and what variable is changing?

Answer: The given rate of change is 0.2 cm per second, and the variable r is changing.

11. What is r at the "certain time" given in the scenario?

Answer: If $A = 30\text{ cm}^2$, then $r = \sqrt{\frac{30}{\pi}} \approx 3.0902$ cm.

12. If the rate of change of r continues to be 0.2 cm per second, what is A after 3 more seconds?

Answer: r increases to $3.989 + (3)(0.2) = 4.589$ cm, so that $A = \pi(4.589)^2 \approx 21.065\text{ cm}^2$.

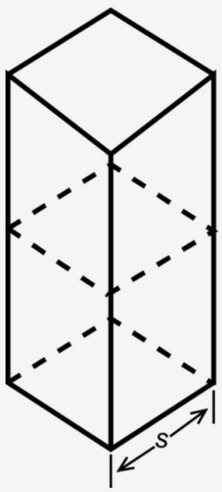
13. Suppose the spill is expanding so that its area is increasing at a constant rate of 0.4 cm^2 per second. At a certain time, the area A is 50 cm^2. What is r at that time? What are A and r 10 seconds later?

Answer: If $A = 50\text{ cm}^2$, then $r = \sqrt{\frac{50}{\pi}} \approx 3.9894$ cm. A increases 10 seconds later to $50 + 10(0.4) = 54\text{ cm}^2$, so then $r = \sqrt{\frac{54}{\pi}} \approx 4.146$ cm.

Check your understanding

A student was given each of the following scenarios and asked to provide a response. Check the student's responses. If the student's response is incorrect, explain why.

- ☐ A solid object has a square base with side S inches long. It is twice as high as it is wide.
  Student response:



Answer: The student is correct.

- ☐ The volume of the solid is V. Write an equation relating V and S.
  Student response: $V = 2S^2$

Answer: The student is incorrect. The student may have confused area and volume. The correct equation is $V = 2S^3$.

- ☐ What is the length of S (including units) when the volume V is 16 cubic inches?
  Student response: Since $V = 2S^2$, we know that $16 = 2S^2$, so S is $2\sqrt{2}$ square inches.

Is the student's answer correct? If not, explain why.

Answer: The student is using an incorrect equation and gets a numerically wrong answer. The unit used is also wrong, and should be inches. The correct equation $V = 2S^3$ gives $S = 2$ inches.

Part I: Understanding the Derivative as a Rate of Change

Scenario: An ant crawls directly away from the base of a vertical pole which is 3 feet tall. The ant is crawling at a rate of 1 foot per minute. Suppose that *D* is the distance from the ant to the base of the pole, and *H* is the distance from the ant to the top of the pole. Differentiating both sides of the original equation $H^2 = 3^3 + D^2$ yields a "rate of change" equation of $2H\frac{dH}{dt} = 0 + 2D\frac{dD}{dt}$. Assume that at a certain time, the ant is 4 feet away from the base of the pole. What are *H* and $\frac{dh}{dt}$ at that time, including units?

1. Which of the following represents the rate at which the ant is crawling away from the pole with respect to time (1 foot per minute)?

a. D

b. H

c. $\frac{dD}{dt}$

d. $\frac{dh}{dt}$

Answer: C

2. Which of the following variables can we now replace with the value "4"?

a. D

b. H

c. $\frac{dD}{dt}$

d. $\frac{dH}{dt}$

Answer: A

3. For the scenario and conditions described in questions 1 and 2, determine which of the following quantities are constants, and which are variables.

Quantity	Constant (a set quantity)	Variable (a varying quantity)
Height of the pole	Answer: ✓	
Rate the ant is crawling	Answer: ✓ (because $\frac{dD}{dt} = 1$ foot per minute)	
Distance from the ant to the base of the pole		Answer: ✓ (because D is increasing)
Distance from the ant to the top of the pole		Answer: ✓ (variable H)
Rate at which *H* is changing with respect to time		Answer: ✓ (variable $\frac{dH}{dt}$)

4. Which of the following equations appropriately represents the scenario when the ant is 4 feet away from the base of the pole? (Recall that in Scenario 1 of the previous activity, at the "certain time," the right triangle being considered is a 3-4-5 right triangle.)

a. $2(5)\frac{dh}{dt} = 2(4)\frac{dD}{dt}$

b. $2(3)\frac{dh}{dt} = 2(5)\frac{dD}{dt}$

c. $2(5)\frac{dh}{dt} = 2(3)\frac{dD}{dt}$

Answer: A

5. Now solve: When the ant is 4 feet from the base of the pole, what are the values of *H* and $\frac{dh}{dt}$?

Answer: ***H* = 5 feet,** and $\frac{dH}{dt} = 0.8$ feet per minute.

We know from the Guided Practice activity that *H* = 5 feet when *D* = 4 feet. Substituting those values into the differentiated equation gives us the following solution:

$$2H\frac{dh}{dt} = 2D\frac{dD}{dt}$$

$$2(5)\frac{dh}{dt} = 2(4)(1)$$

$$10\frac{dh}{dt} = 8$$

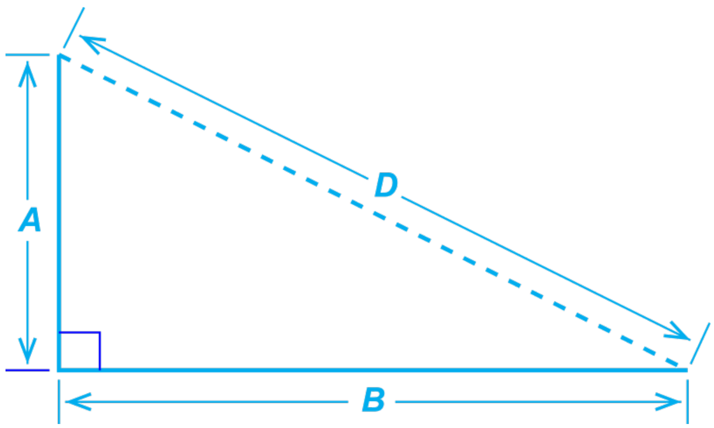
$$\frac{dh}{dt} = \frac{8}{10} = 0.8 \text{ feet per minute}$$

Part II: Solving Related Rates Problems

1. Two cars leave Central Square. The first car drives north at 35 miles per hour (mph), and its distance to Central Square is A (here A is actually a function of time). The second car drives east at 50 mph and its distance from Central Square is B (here B is also a function of time). The distance between the two cars is D. At a certain time, A = 20 miles and B = 40 miles.

a. Sketch the general picture of what the cars and Central Square look like. Be sure to label the diagram with *A, B,* and *D,* representing the distances mentioned in the stem of the problem.

Answer:



b. Give a general formula for D in terms of A and B.

Answer: $D = \sqrt{A^2 + B^2}$

c. Find a formula for D', the derivative of D, with respect to time in terms of A, A', B, and B'.

Answer: $D' = \frac{1}{2}(A^2 + B^2)^{-\frac{1}{2}}(2AA' + 2BB')$

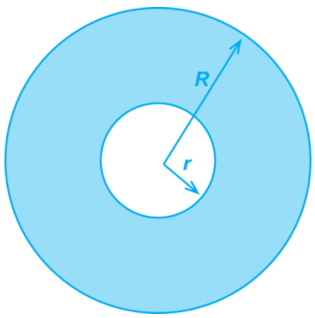
d. Are the two cars getting closer or farther away from each other at the certain time? How fast is the distance between them changing at that time?

Answer: $D' = \frac{1}{2}(20^2 + 40^2)^{-\frac{1}{2}}(2(20)(35) + 2(40)(50)) \approx 60.374$ miles per hour. Since this answer is positive, the cars are getting farther away from each other at that certain time.

2. An *annulus* is a region in the plane bounded by two circles with the same center (concentric circles). Suppose S is an annulus with an inner circle of radius r and an outer circle of radius R. The radii are changing so that r is increasing at 7 cm/sec, and R is increasing at 3 cm/sec. At 9 AM, $r = 30$ cm and $R = 80$ cm.

a. Sketch a picture of the annulus. Be sure to label the annulus with r and R representing the two radii mentioned in the stem of the problem.

Answer:



b. Write a formula for P, the total perimeter of the region S. The formula should involve r and R. What is P at 9 AM? Include appropriate units with your answer.

Answer: $P = 2\pi r + 2\pi R$. At 9 AM, $P = 2\pi(30) + 2\pi(80) \approx 691.150$ cm.

c. Write a formula for P', the rate of change of the total perimeter of the region S. The formula should involve r' and R'. What is P' at 9 AM? Include appropriate units with your answer. Is the total perimeter increasing or decreasing at that time?

Answer: $P' = 2\pi r' + 2\pi R'$. At 9 AM, $P' = 2\pi(7) + 2\pi(3) \approx 62.832$ cm/sec. Since this is positive, the total perimeter is increasing at that time.

d. Write a formula for A, the area of the region S. The formula should involve r and R. What is A at 9 AM? Include appropriate units with your answer.

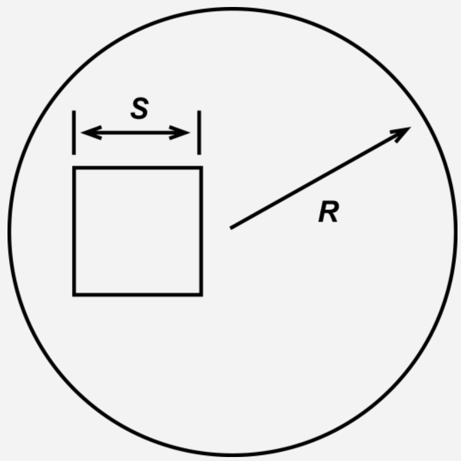
Answer: $A = \pi R^2 - \pi r^2$. At 9 AM, $A = \pi(80)^2 - \pi(30)^2 \approx 17278.760$ cm^2.

e. Write a formula for A', the rate of change of the area of the region S The formula should involve r, r', R, and R'. What is A' at 9 AM? Include appropriate units with your answer. Is the area increasing or decreasing at that time?

Answer: $A' = 2\pi RR' - 2\pi rr'$. At 9 AM, $A' = 2\pi(80)(3) - 2\pi(30)(7) \approx 188.496$ cm^2/sec. Since this is positive, the area is increasing at that time.

Check your understanding

A square with side length S is inside a circle of radius R as shown to the right. At a certain time, $S = 5$ inches and is increasing at the rate of 0.7 inches per minute. At the same time, $R = 8$ inches and is decreasing at the rate of 0.4 inches per minute.



Fill in the blanks to complete the solutions. Be sure to indicate *units* with every exact or approximate numerical answer.

- ☐ The area A inside the circle and outside the square is given by $A = \pi$________________, and the rate of change of this area is given by $A' =$ ____________________. At the certain time, $A =$ __________________ ≈ _____________ __________________ (units) and A is changing at the rate of _________ ≈ __________ ____________ per _____________. A is ____creasing.

  Answer: The area A inside the circle and outside the square is given by $A = \pi R^2 - S^2$, and the rate of change of this area is given by $A' = 2\pi RR' - 2SS'$. At the certain time, $A = 64\pi - 25 \approx 176.062$ square inches, and A is changing at the rate of $-6.4\pi - 7 \approx -27.106$ square inches per minute. A is decreasing.

- ☐ The total perimeter P of the area A is given by $P =$ ________________, and the rate of change of this total perimeter is given by $P' =$ ________________. At the certain time, $P =$ ______________ ______________ (units), and P is changing at the rate of _________________ ≈ ________ ____________ per ______________. P is ___creasing.

  Answer: The total perimeter P of the area A is given by $P = 2\pi R + 4S$, and the rate of change of this total perimeter is given by $P' = 2\pi R' + 4S'$. At the certain time, $P = 16\pi + 20$ inches, and P is changing at the rate of $-0.8\pi + 2.8 \approx 0.287$ inches per minute. P is increasing.