Related Rates vs Optimization

Read each of the word problems that follow and decide whether it is an optimization problem or a related rates problem.

1. A particle moves along the parabola $y = x^2$. At what point is the particle when it is closest to the point (3,0)?	☐ Optimization ☐ Related rates
2. A particle moves along the parabola $y = x^2$ with its *x*-coordinate increasing at a rate of 0.5 cm/sec. At what rate is the particles distance from the point $(3,0)$ changing when the particle is at $(2,4)$?	☐ Optimization ☐ Related rates
3. What are the dimensions of the cone of greatest volume that can be inscribed inside a sphere of radius 8 cm?	☐ Optimization ☐ Related rates
4. Air is pumped into a sphere at a rate of 6 cm^3/sec. At what rate is the radius of the sphere increasing when the radius is 8 cm?	☐ Optimization ☐ Related rates
5. Water flows into a cylindrical tank with base radius 3 meters and height 12 meters at a rate of $(3t^2 - 8t)$ m^3/hr. At what rate is the height of the water changing when 2 hours have elapsed?	☐ Optimization ☐ Related rates
6. Water flows into a cylindrical tank with base radius 3 meters and height 12 meters at a rate of $(3t^2 - 8t)$ m^3/hr. At what time is the rate of flow the least?	☐ Optimization ☐ Related rates

Unpacking a Related Rates Problem

For each of the following scenarios, first answer each part on your own. Then compare your answer with those of your group members and discuss any differences.

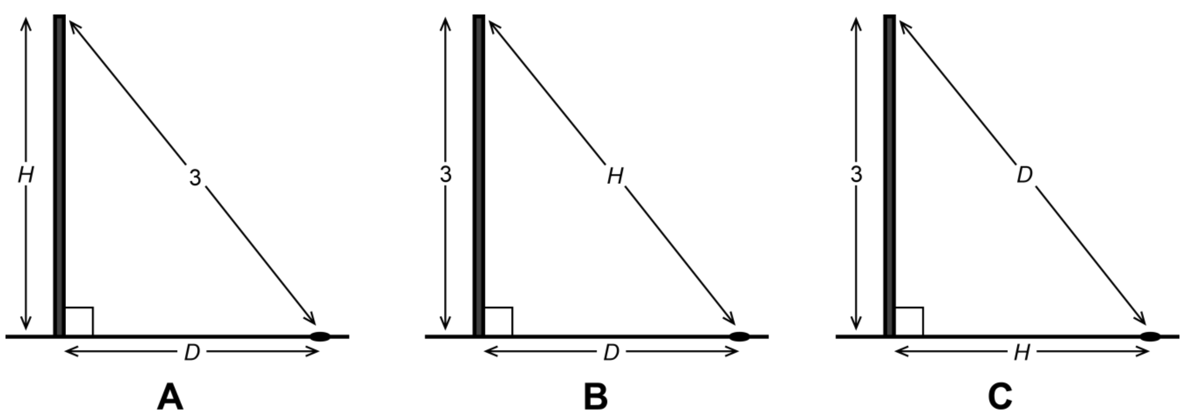
Scenario 1: An ant crawls directly away from the base of a vertical pole that is 3 feet tall. The ant is crawling at a rate of 1 foot per minute. Assume D is the distance from the ant to the base of the pole (measured in feet), and H is the distance from the ant to the top of the pole (measured in feet).

1. What is the value of the *rate of change* in the scenario (including units)?

2. How tall is the pole in the scenario (including units)?

3. How do you know that this scenario can be represented with a diagram of a **right triangle**?

4. Select which diagram appropriately represents the ant, the pole, and the quantities D and H.



5. Select which equation appropriately represents the relationship between the quantities in the problem.

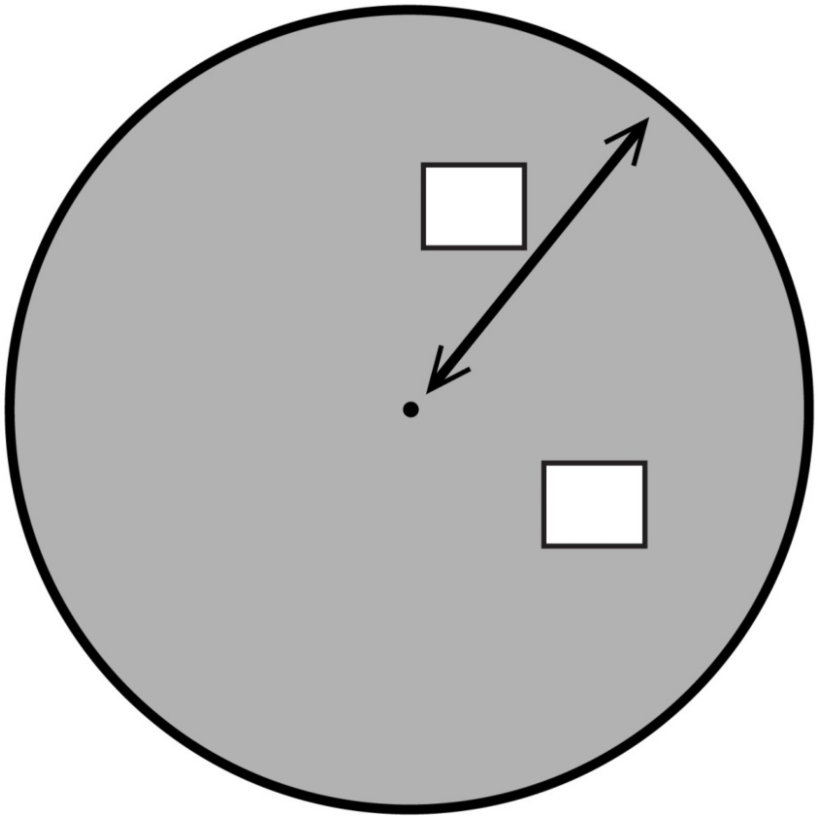
Equation A: $D^2 = 3^2 + H^2$ **Equation B:** $A = \frac{1}{2}HD$ **Equation C:** $H^2 = 3^2 + D^2$

6. At a certain time, the ant is 4 feet away from the base of the pole. What is *H* at that time? Show your work.

7. Find *D* and *H* two minutes after the time mentioned in question 6. Show your work.

Scenario 2: An oil spill in the shape of a circle is expanding on the surface of a pond. The radius of the oil spill is *r* (measured in cm) and the area of the oil spill is *A* (measured in cm^2). Suppose that at a certain time, the area of the spill is $30\ \text{cm}^2$, and the radius of the spill is increasing at 0.2 cm per second.

8. Label the accompanying diagram so that it shows the spill, labeling the area *A* and the radius of the spill *r*.



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9. Select which equation appropriately represents the relationship between the quantities in the problem.

Equation A: $A = 2\pi r$

Equation B: $A = \pi r^2$

Equation C: $A = \frac{4}{3}\pi r^3$

10. What is the rate of change given in the scenario and what variable is changing?

11. What is *r* at the “certain time” given in the scenario?

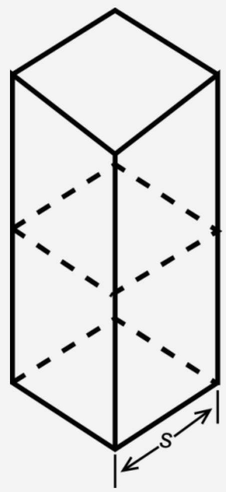
12. If the rate of change of *r* continues to be 0.2 cm per second, what is *A* after 3 more seconds?

13. Suppose the spill is expanding so that its area is increasing at a constant rate of $0.4\ \text{cm}^2$ per second. At a certain time, the area *A* is $50\ \text{cm}^2$. What is *r* at that time? What are *A* and *r* 10 seconds later?

Check your understanding

A student was given each of the following scenarios and asked to provide a response. Check the student's responses. If the student's response is incorrect, explain why.

☐ A solid object has a square base with side S inches long. It is twice as high as it is wide.
Student response:



☐ The volume of the solid is V. Write an equation relating V and S.
Student response: $V = 2S^2$

☐ What is the length of S (including units) when the volume V is 16 cubic inches?
Student response: Since $V = 2S^2$, we know that $16 = 2S^2$, so S is $2\sqrt{2}$ square inches.

Is the student's answer correct? If not, explain why.

Part I: Understanding the Derivative as a Rate of Change

Scenario: An ant crawls directly away from the base of a vertical pole which is 3 feet tall. The ant is crawling at a rate of 1 foot per minute. Suppose that *D* is the distance from the ant to the base of the pole, and *H* is the distance from the ant to the top of the pole. Differentiating both sides of the original equation $H^2 = 3^3 + D^2$ yields a "rate of change" equation of $2H\frac{dH}{dt} = 0 + 2D\frac{dD}{dt}$. Assume that at a certain time, the ant is 4 feet away from the base of the pole. What are *H* and $\frac{dh}{dt}$ at that time, including units?

1. Which of the following represents the rate at which the ant is crawling away from the pole with respect to time (1 foot per minute)?

a. D

b. H

c. $\frac{dD}{dt}$

d. $\frac{dh}{dt}$

2. Which of the following variables can we now replace with the value "4"?

a. D

b. H

c. $\frac{dD}{dt}$

d. $\frac{dH}{dt}$

3. For the scenario and conditions described in questions 1 and 2, determine which of the following quantities are constants, and which are variables.

Quantity	Constant (a set quantity)	Variable (a varying quantity)
Height of the pole		
Rate the ant is crawling		
Distance from the ant to the base of the pole		
Distance from the ant to the top of the pole		
Rate at which *H* is changing with respect to time		

4. Which of the following equations appropriately represents the scenario when the ant is 4 feet away from the base of the pole? (Recall that in Scenario 1 of the previous activity, at the "certain time," the right triangle being considered is a 3-4-5 right triangle.)

a. $2(5)\dfrac{dh}{dt} = 2(4)\dfrac{dD}{dt}$

b. $2(3)\dfrac{dh}{dt} = 2(5)\dfrac{dD}{dt}$

c. $2(5)\dfrac{dh}{dt} = 2(3)\dfrac{dD}{dt}$

5. Now solve: When the ant is 4 feet from the base of the pole, what are the values of *H* and $\dfrac{dh}{dt}$?

Part II: Solving Related Rates Problems

1. Two cars leave Central Square. The first car drives north at 35 miles per hour (mph), and its distance to Central Square is *A* (here *A* is actually a function of time). The second car drives east at 50 mph and its distance from Central Square is *B* (here *B* is also a function of time). The distance between the two cars is *D*. At a certain time, *A* = 20 miles and *B* = 40 miles.

a. Sketch the general picture of what the cars and Central Square look like. Be sure to label the diagram with *A, B,* and *D,* representing the distances mentioned in the stem of the problem.

b. Give a general formula for *D* in terms of *A* and *B*.

c. Find a formula for *D′*, the derivative of *D*, with respect to time in terms of *A, A′, B*, and *B′*.

d. Are the two cars getting closer or farther away from each other at the certain time? How fast is the distance between them changing at that time?

2. An *annulus* is a region in the plane bounded by two circles with the same center (concentric circles). Suppose S is an annulus with an inner circle of radius r and an outer circle of radius R. The radii are changing so that r is increasing at 7 cm/sec, and R is increasing at 3 cm/sec. At 9 AM, $r = 30 \text{ cm}$ and $R = 80 \text{ cm}$.

a. Sketch a picture of the annulus. Be sure to label the annulus with r and R representing the two radii mentioned in the stem of the problem.

b. Write a formula for P, the total perimeter of the region S. The formula should involve r and R. What is P at 9 AM? Include appropriate units with your answer.

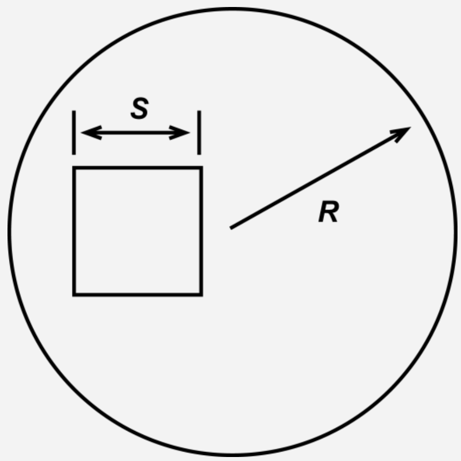
c. Write a formula for P', the rate of change of the total perimeter of the region S. The formula should involve r' and R'. What is P' at 9 AM? Include appropriate units with your answer. Is the total perimeter increasing or decreasing at that time?

d. Write a formula for A, the area of the region S. The formula should involve r and R. What is A at 9 AM? Include appropriate units with your answer.

e. Write a formula for A', the rate of change of the area of the region S The formula should involve r, r', R, and R'. What is A' at 9 AM? Include appropriate units with your answer. Is the area increasing or decreasing at that time?

Check your understanding

A square with side length *S* is inside a circle of radius *R* as shown to the right. At a certain time, *S* = 5 inches and is increasing at the rate of 0.7 inches per minute. At the same time, *R* = 8 inches and is decreasing at the rate of 0.4 inches per minute.



Fill in the blanks to complete the solutions. Be sure to indicate *units* with every exact or approximate numerical answer.

☐ The area *A* inside the circle and outside the square is given by $A = \pi$______________, and the rate of change of this area is given by $A' =$ __________________ . At the certain time, $A =$ ________________ ≈ ____________ ________________ (units) and *A* is changing at the rate of ________ ≈ _________ ___________ per ____________ . *A* is ____creasing.

☐ The total perimeter *P* of the area *A* is given by $P =$ ______________ , and the rate of change of this total perimeter is given by $P' =$ ______________ . At the certain time, $P =$ _____________ _____________ (units), and *P* is changing at the rate of _______________ ≈ _______ ___________ per _____________ . *P* is ___creasing.