Lexicographic Product of Networked Systems

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***Abstract*—The lexicographic product graphs are discussed from the perspective of networked systems with Laplacian flow dynamics. Controllability conditions of these systems are developed depending on the controllability of their factor graphs. We show that any trajectory in these networks, with suitable initial conditions, can be decomposed into two lower dimensional trajectories, making computation of trajectories in very large product graphs less computationally expensive. We discuss practical implications of obtained results and illustrate them with numerical simulations.**

***Index Terms*—lexicographic product, networked systems, controllability, trajectory decomposition, convergence rate**

I. Introduction

Dynamical systems with multiple agents can be used to model dynamics of many real world problems, including behavior of different users in social networks, alignment patterns in animal world or collaboration of multiple mobile robots [1]. Moreover, with the increase of available data and global interconnectedness, the number of agents in such systems can become massive. One of the possible approaches to such systems is finding and utilizing an inherent structure in the network. Although composition rules can be complex and difficult for analyzing, for certain classes of composite graphs it is possible to obtain properties of the whole network based on the properties of its factors.

Graph products can be used for both analysis and synthesis of real world complex networks [2]. Also, many of the graph products produce graphs with large number of symmetries, which directly relates to the controllability of multi-agent systems [3]. Further development of those results led to conditions for determining controllability of standard graph products based on controllability of the factor graphs. Results have been mostly developed for standard graph products, including Cartesian [4], Kronecker [5] and generalized graph products [6]. Moreover, the controllability conditions were reported for corona graph product [7], and a general product called NEPSes of graphs [8]. Controllability conditions of the Cartesian graph products were further developed and improved in [9]. Distributed computation and optimization of the network properties such as $\mathcal{H}_2$ norm were obtained for series-parallel networks [10]. For the case of Cartesian products, controllability was also considered from the energy perspective [11].

The main contribution of this paper is developing control and dynamical properties of lexicographic (or composition) product graphs, known for their interesting group properties [12]. Although this product belongs to standard graph products, it differs significantly from the other members of this class. In support of the claim, we note that lexicographic product is the only standard product which cannot be regarded as generalized graph product, as introduced in [6]. However, applications of lexicographic product networks are still not widely considered, and are still very obscure from the control perspective. We hope that the following analysis might provide inspiration for using this networks in practical real-world problems. At the end, we mention that we are not aware of any previous characterization of control properties of these graphs in terms of their factor graphs.

The organization of the paper is as follows. We begin by introducing mathematical background needed for developing the main results and give short introduction to the spectral properties of lexicographic product graphs in terms of spectra of its factors. Then, in §III we discuss the setting of dynamical systems and leader-follower networks. In §IV we present our main theoretical results. We consider decomposition of trajectories into terms describing dynamics of factor graphs. Furthermore, we consider controllability of lexicographic product graphs and compare convergence rate on these graphs with the one on Cartesian product graphs. In §V we give few numerical examples to illustrate developed theoretical results, and comment their applicability.

II. Mathematical Preliminaries

In this section we introduce notation and provide a short review of the mathematical background needed in the following sections.

We use column vector notation and for vector $x \in \mathbb{R}^n$ we denote by $x^{(i)}$ its i-th component. We denote by $\mathbb{1}$ vector of all ones and by e_i standard unit vector parallel to the i-th coordinate axis. We will not explicitly denote dimension of these vectors since most of the time it will be evident from equations. Furthermore, by I_n is denoted $n \times n$ identity matrix and by J_n $n \times n$ matrix of all ones.

The Kronecker product of matrices A and B is denoted by $A \otimes B$. We recall mixed-product property of Kronecker product:

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD), \tag{1}$$

and the fact that the product is not commutative in general [13].

For a graph G we denote by $V(G)$ and $E(G)$ its vertex and edge set, respectively. We say that two nodes $u, v \in V(G)$ are adjacent in G if $(u, v) \in E(G)$. A graph is undirected if adjacency relation is symmetric and we will always assume that G is an undirected graph. For a weighted graph G with n nodes and nonnegative weights $W = [w_{i,j}]_{i,j=1}^n$ we define adjacency matrix $A \in \mathbb{R}^{n\times n}$ as:

$$A_{i,j}(G) = \begin{cases} w_{i,j}, & \text{if } (v_i, v_j) \in E(G) \\ 0, & \text{if } (v_i, v_j) \notin E(G) \end{cases} \tag{2}$$

Let $d(u)$ be degree of node u in graph G, and let $D(G) = \text{diag}(d(u_1), d(u_2), \ldots, d(u_n))$ be diagonal matrix of degrees of vertices in G. We denote by $L(G)$ Laplacian matrix of graph G defined by

$$L(G) = D(G) - A(G) \tag{3}$$

In the case of undirected graphs both matrices $A(G)$ and $L(G)$ are symmetric. Furthermore, Laplacian matrix is positive semidefinite with real eigenvalues $0 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$. Eigenvector corresponding to the smallest eigenvalue 0 is vector $\mathbb{1}$. Moreover, eigenvalue λ_2 is called algebraic connectivity of graph G.

For two graphs F and H with disjoint vertex sets $V(F)$ and $V(H)$, the graph product is a graph with the vertex set equal to $V(F) \times V(H)$, where $\times$ denotes Cartesian product of two sets.

The lexicographic product of graphs F and H, denoted by $F[H]$ is defined as following. For any node $u \in V(F)$ and $x \in V(H)$ we will denote by $[u|x]$ node from $V(F[H]) = V(F) \times V(H)$ corresponding to u and x. Then, $F[H]$ is a lexicographic product graph if for any two nodes $[u|x], [v|y] \in V(F[H])$ holds:

$$\begin{aligned}([u|x],[v|y]) \in E(F[H]) &\iff \\ &(u, v) \in E(F) \lor (u = v \land (x, y) \in E(H))\end{aligned} \tag{4}$$

For graph F with weights W_1 and graph H with weights W_2, we define weighted graph $F[H]$ with weights given by:

$$w([u|x], [v|y]) = \begin{cases} w_1(u, v), & \text{if } (u, v) \in E(F) \\ w_2(x, y), & \text{if } u = v \ \land \ (x, y) \in E(H) \end{cases} \tag{5}$$

In Figure 1 is shown an example of lexicographic product of two given graphs. Note that different colors of the nodes give intuition behind the vertex set $V(F[H])$. Also, definition

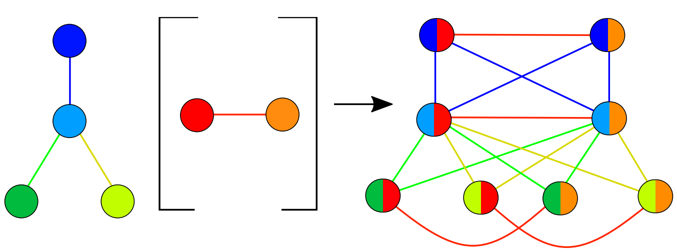

Fig. 1. Illustration of lexicographic product of two graphs.

(5) is illustrated in this figure, since different colors of the edges denote different weights of the graphs. However, note that colors of nodes and edges are not connected.

After simple but cumbersome calculations it can be shown that lexicographic product graph of graph F with m nodes and graph H with n nodes has the following adjacency and diagonal matrix:

$$A(F[H]) = I_m \otimes A(H) + A(F) \otimes J_n \tag{6}$$

$$D(F[H]) = I_m \otimes D(H) + nD(F) \otimes I_n \tag{7}$$

We recall some of the main properties of this type of graphs. Lexicographic product is one of the four standard graph products [14]. It is associative and distributive to disjoint union, but is the only noncommutative standard product [15]. Hence in the following discussion we will always use notation $F[H]$, and established results do not necessarily hold true for product $H[F]$. Furthermore, for graphs F and H with at least two vertices, $F[H]$ is connected if and only if F is connected [15].

In the following two theorems we are interested in characterizing spectrum of product graph $F[H]$ based on the spectra of its factor graphs. First, we give an explicit expression for the Laplacian of $F[H]$.

Theorem II.1. *(Proposition 26 in [14]) Let F and H be graphs with m and n vertices, respectively. Then, for Laplacian matrix of lexicographic product graph $F[H]$ holds:*

$$L(F[H]) = I_m \otimes L(H) + L(F) \otimes J_n + D(F) \otimes (nI_n - J_n) \tag{8}$$

Proof. From definition (3) and expressions (6) and (7) we obtain:

$$\begin{aligned} &L(F[H]) = D(F[H]) - A(F[H]) = I_m \otimes D(H) + \\ &+ nD(F) \otimes I_n - (I_m \otimes \underbrace{A(H)}_{D(H)-L(H)} + \underbrace{A(F)}_{D(F)-L(F)} \otimes J_n) = \\ &= I_m \otimes L(H) + L(F) \otimes J_n + D(F) \otimes (nI_n - J_n) \end{aligned}$$

□

Now we are ready to derive expressions for the spectrum of Laplacian of lexicographic product graph.

Theorem II.2. *(Theorem 27 in [14]) Let F be a connected graph with vertex set $V(F) = \{u_1, u_2, \ldots, u_m\}$ and eigenpairs $(\mu_i, \xi_i)_{i=1}^m$. Let H be an arbitrary graph with eigenpairs $(\eta_i, \zeta_i)_{i=1}^n$. Then the spectrum of $L(F[H])$ consists of the following eigenpairs:*

- *$(\mu_i n, \xi_i \otimes \mathbb{1})$, for $i = 1, \ldots, m$*
- *$(\eta_j + d(u_i)n, e_i \otimes \zeta_j)$, for $i = 1, \ldots, m,\ j = 2, \ldots, n$.*

Proof. First we prove that $(\mu_i n, \xi_i \otimes \mathbb{1})$ are eigenpairs of $L(F[H])$. We consider every term from equation (8) separately, and using mixed-product property (1) we get:

$$(I_m \otimes L(H))(\xi_i \otimes \mathbb{1}) = (I_m \xi_i) \otimes \underbrace{(L(H)\mathbb{1})}_{=0} = 0$$

$$(L(F)\otimes J_n)(\xi_i\otimes \mathbb{1}) = \underbrace{(L(F)\xi_i)}_{\mu_i\xi_i}\otimes\underbrace{(J_n\mathbb{1})}_{n\mathbb{1}} = \mu_i n\xi_i\otimes\mathbb{1}$$

$$\Big(D(F)\otimes(nI_n - J_n)\Big)(\xi_i\otimes\mathbb{1}) = \\ = (D(F)\xi_i)\otimes\underbrace{\Big((nI_n-J_n)\otimes\mathbb{1}\Big)}_{=0} = 0$$

where in the first equation we used the fact that $(0,\mathbb{1})$ is an eigenpair of Laplacian matrix $L(H)$. Combining previous results we have:

$$L(F[H])(\xi_i\otimes\mathbb{1}) = \mu_i n(\xi_i\otimes\mathbb{1})$$

i.e. $(\mu_i n, \xi_i\otimes\mathbb{1})$ for $i=1,\dots,m$ are eigenpairs of $L(F[H])$.

Now, we consider vectors $e_i\otimes\zeta_j$ and once again do the analysis term-by-term as following:

$$(I_m\otimes L(H))(e_i\otimes\zeta_j) = (I_m e_i)\otimes(L(H)\zeta_j) = \eta_j e_i\otimes\zeta_j$$

Since $\zeta_1 = \frac{1}{\sqrt{n}}\mathbb{1}$ and $\zeta_j\perp\zeta_1$ i.e. $\mathbb{1}^T\zeta_j = 0$ for $j=2,\dots,n$, we have that:

$$(L(F)\otimes J_n)(e_i\otimes\zeta_j) = (L(F)e_i)\otimes\underbrace{(J_n\zeta_j)}_{=0} = 0$$

for any $j=2,\dots,n$, and similarly:

$$\Big(D(F)\otimes(nI_n - J_n)\Big)(e_i\otimes\zeta_j) = \\ = (D(F)e_i)\otimes\underbrace{\Big((nI_n-J_n)\zeta_j\Big)}_{n\zeta_j} = d(u_i)n(e_i\otimes\zeta_j)$$

From previous results follows that:

$$L(F[H])(e_i\otimes\zeta_j) = (\eta_j + d(u_i)n)(e_i\otimes\zeta_j)$$

i.e. $(\eta_j + d(u_i)n, e_i\otimes\zeta_j)$, for $i=1,\dots,m$, $j=2,\dots,n$ are eigenpairs of $L(F[H])$. □

Corollary 1. *Given graphs F and H as in theorem II.2, the algebraic connectivity of graph $F[H]$ is:*

$$\lambda_2 = a(F[H]) = \min\{a(H)+\delta(F)n, a(F)n\} \tag{9}$$

where $\delta(F)$ is the minimum degree of vertices in F.

III. Problem formulation

We will consider dynamics of continuous time-invariant linear systems of the following form:

$$\dot{x}_1(t) = -L(F)x_1(t) \tag{10}$$
$$\dot{x}_2(t) = -L(H)x_2(t) \tag{11}$$
$$\dot{x}(t) = -L(F[H])x(t) \tag{12}$$

with first two equations corresponding to the dynamics on factor graphs F and H, and the last equation describing dynamics in the lexicographic product network. We will assume that $x(0) = x_1(0)\otimes x_2(0)$ in order to develop results in the following section. However, note that initial value vector defined in this manner belongs to $m+n$-dimensional space and hence does not cover all possible initial conditions [6].

We will also consider this dynamical system in the context of control. We assume that exterior control inputs are applied to some of the nodes in the graph, called leaders. Other nodes that do not have exterior inputs are called followers. In this case, we consider the following dynamical systems:

$$\begin{aligned}\dot{x}_1(t) &= -L(F)x_1(t) + B_1u_1(t)\\ \dot{x}_2(t) &= -L(H)x_2(t) + B_2u_2(t)\\ \dot{x}(t) &= -L(F[H])x(t) + Bu(t)\end{aligned} \tag{13}$$

with $B = B_1\otimes B_2$. If the nodes $i_1, i_2, \dots, i_p$ are leaders of graph F then $B_1 = \begin{bmatrix} e_{i_1}, & e_{i_2}, \dots, & e_{i_p}\end{bmatrix}$. Note that we can put $u(t)\equiv 0$ on follower nodes, and fill matrix B with zero vectors on positions corresponding to the follower nodes. In this way matrices B and L are of the same dimensionality, a convention that will be used in the following discussion.

IV. Results

In this section we consider three main problems in the lexicographic product networks: we start with decomposing trajectories in product graphs into trajectories from factor graphs; then we consider controllability of these graphs and at the end we compare convergence rate in these graphs with the convergence rate in Cartesian product graphs.

A. Trajectory decomposition

We show that directly solving linear time-invariant system (12) is not the most appropriate way to determine trajectories of such system. Namely, matrix $L(F[H])$ is of size $mn\times mn$ which makes calculation of exponential of such matrix difficult for high values of m and n. However, we show that such dynamics can be decomposed into dynamics on factor graphs with Laplacian matrices of sizes $m\times m$ and $n\times n$. We discuss practical implications of this theorem in §V.

Before stating and proving the theorem, we present a result proven in [6], which will also be useful for lexicographic product graphs.

Lemma IV.1. *(Proposition 7 in [6]) Let A be an arbitrary matrix, $B\in\mathbb{R}^{n\times n}$ a symmetric matrix with set of eigenvalues and associated eigenvectors $\{(\lambda_i, u_i)\}_{i=1}^n$. Then, for any $\beta\in\mathbb{R}$ holds:*

$$e^{\beta(A\otimes B)t} = \sum_{i=1}^{n} e^{\lambda_i\beta tA}\otimes u_iu_i^T \tag{14}$$

Similarly, for a symmetric $m\times m$ matrix A with eigenpairs $\{(\eta_i, v_i)\}_{i=1}^m$ we obtain:

$$e^{\beta(A\otimes B)t} = \sum_{i=1}^{m} v_iv_i^T\otimes e^{\eta_i\beta tB} \tag{15}$$

Theorem IV.2. *Let F be a connected graph with m nodes, and let H be an arbitrary graph with n nodes. Moreover, let $x_1(t)$ and $x_2(t)$ be solutions of the dynamical systems* (10) *and* (11)*, respectively. Then, system* (12) *has the following solution:*

$$x(t) = \frac{1}{n} x_1(nt) \otimes \Big(J_n x_2(t)\Big) + \Big(e^{-ntD(F)} x_1(0)\Big) \otimes \Big((I_n - \frac{1}{n}J_n) x_2(t)\Big) \quad (16)$$

under assumption that initial condition of differential equation (12) *satisfies* $x(0) = x_1(0) \otimes x_2(0)$.

Proof. First, we claim that all terms of $L(F[H])$ from (8) commute with each other. In order to avoid cumbersome expressions we skip this part of the proof, but interested reader can obtain it using mixed-product property of Kronecker product and properties of Laplacian and identity matrices. Now, using the fact that for commutative matrices A and B holds that $e^{A+B} = e^A e^B$, we can write solution of the linear time-invariant system (12) as:

$$\begin{aligned} x(t) &= e^{-tL(F[H])} x(0) = \\ &= e^{-t[D(F)\otimes(nI_n - J_n) + L(F)\otimes J_n + I_m \otimes L(H)]} x(0) = \\ &= e^{-t[D(F)\otimes(nI_n - J_n)]} e^{-t[L(F)\otimes J_n]} e^{-t[I_m \otimes L(H)]} x(0) \end{aligned} \quad (17)$$

In order to keep expressions short, we will consider above equation term by term.

Identity matrix I_m is symmetric with eigenpairs $\{(1, e_i)\}_{i=1}^m$ and hence applying equation (15) and mixed-product rule (1) gives:

$$\begin{aligned} e^{-t[I_m \otimes L(H)]} x(0) &= \sum_{i=1}^m \Big(e_i e_i^T \otimes e^{-tL(H)}\Big)(x_1(0) \otimes x_2(0)) = \\ &= \sum_{i=1}^m \Big(e_i e_i^T x_1(0)\Big) \otimes \Big(e^{-tL(H)} x_2(0)\Big) = \\ &= \sum_{i=1}^m \Big(e_i e_i^T x_1(0)\Big) \otimes x_2(t) = \\ &= x_1(0) \otimes x_2(t) \end{aligned}$$

Now let's combine this obtained result with the second term in equation (17). We apply equation (14) and expand symmetric matrix J_n with eigenpairs $\{(\lambda_i, u_i)\}_{i=1}^n$ as follows:

$$\begin{aligned} & e^{-t[L(F)\otimes J_n]}(x_1(0) \otimes x_2(t)) = \\ &= \sum_{i=1}^n \Big(e^{-\lambda_i t L(F)} \otimes u_i u_i^T\Big)(x_1(0) \otimes x_2(t)) = \\ &= \sum_{i=1}^n \Big(e^{-\lambda_i t L(F)} x_1(0)\Big) \otimes \Big(u_i u_i^T x_2(t)\Big) = \\ &= \sum_{i=1}^n x_1(\lambda_i t) \otimes \Big(u_i u_i^T x_2(t)\Big) \end{aligned}$$

Note that for eigenvalues of J_n holds that $\lambda_1 = n$ with $u_1 = \frac{1}{\sqrt{n}}\mathbb{1}$ and $\lambda_2 = \cdots = \lambda_n = 0$. At the end, we insert obtained expression in (17) and apply equation (14) once again. This time we expand matrix $nI_n - J_n$ with eigenpairs $\{(\eta_i, v_i)\}_{i=1}^n$ to obtain:

$$\begin{aligned} x(t) &= e^{-t[D(F)\otimes(nI_n - J_n)]} \sum_{i=1}^n x_1(\lambda_i t) \otimes \Big(u_i u_i^T x_2(t)\Big) = \\ &= \sum_{j=1}^n e^{-\eta_j t D(F)} \otimes v_j v_j^T \sum_{i=1}^n x_1(\lambda_i t) \otimes \Big(u_i u_i^T x_2(t)\Big) = \\ &= \sum_{i=1}^n \sum_{j=1}^n \Big(e^{-\eta_j t D(F)} \otimes v_j v_j^T\Big)\Big(x_1(\lambda_i t) \otimes \Big(u_i u_i^T x_2(t)\Big)\Big) = \\ &= \sum_{i=1}^n \sum_{j=1}^n \Big(e^{-\eta_j t D(F)} x_1(\lambda_i t)\Big) \otimes \Big(v_j v_j^T u_i u_i^T x_2(t)\Big) \end{aligned}$$

For matrix $nI_n - J_n$ holds that $\eta_1 = 0$ with $v_1 = \frac{1}{\sqrt{n}}\mathbb{1}$ and $\eta_2 = \cdots = \eta_n = n$. Since eigenvectors are orthonormal, and $u_1 = v_1$ we have that $v_j^T u_i = 0$ for $j = 1$, $i > 2$ or in the case $i = 1, j > 2$. Separating summands for $i = 1$ and $j = 1$ and using orthogonality of eigenvectors we obtain:

$$\begin{aligned} x(t) &= \Big(e^{-\eta_1 t D(F)} x_1(\lambda_1 t)\Big) \otimes \Big(v_1 v_1^T u_1 u_1^T x_2(t)\Big) + \\ &+ \sum_{i=2}^n \sum_{j=2}^n \Big(e^{-\eta_j t D(F)} x_1(\lambda_i t)\Big) \otimes \Big(v_j v_j^T u_i u_i^T x_2(t)\Big) = \\ &= x_1(nt) \otimes \Big(\frac{1}{n} J_n x_2(t)\Big) + \\ &+ \sum_{i=2}^n \sum_{j=2}^n \Big(e^{-ntD(F)} x_1(0)\Big) \otimes \Big(v_j v_j^T u_i u_i^T x_2(t)\Big) = \\ &= \frac{1}{n} x_1(nt) \otimes \Big(J_n x_2(t)\Big) + \\ &+ \Big(e^{-ntD(F)} x_1(0)\Big) \otimes \Big(\sum_{i=2}^n \sum_{j=2}^n v_j v_j^T u_i u_i^T x_2(t)\Big) \end{aligned}$$

Double sum in the second term can be expressed as:

$$\begin{aligned} \sum_{i=2}^n \sum_{j=2}^n v_j v_j^T u_i u_i^T &= \sum_{j=2}^n v_j v_j^T \sum_{i=2}^n u_i u_i^T = \\ &= \Big(\sum_{j=1}^n v_j v_j^T - v_1 v_1^T\Big)\Big(\sum_{i=1}^n u_i u_i^T - u_1 u_1^T\Big) \\ &= (I_n - \frac{1}{n}J_n)(I_n - \frac{1}{n}J_n) = I_n - \frac{1}{n}J_n \end{aligned}$$

giving result we wanted to prove. □

Corollary 2. *Fundamental matrix* $\Phi(t)$ *of linear time-invariant system* (12) *is given by:*

$$\Phi(t) = e^{-ntL(F)} \otimes \Big(\frac{1}{n} J_n e^{-tL(H)}\Big) + e^{-ntD(F)} \otimes \Big((I_n - \frac{1}{n}J_n) e^{-tL(H)}\Big) \quad (18)$$

Proof. From linear time-invariant systems (10) and (11) and equation (16) we obtain:

$$\begin{aligned} x(t) &= \Big(e^{-ntL(F)} x_1(0)\Big) \otimes \Big(\frac{1}{n} J_n e^{-tL(H)} x_2(0)\Big) + \\ &+ \Big(e^{-ntD(F)} x_1(0)\Big) \otimes \Big((I_n - \frac{1}{n}J_n) e^{-tL(H)} x_2(0)\Big) \end{aligned}$$

and then using the mixed-product property of Kronecker product (1) we obtain:

$$x(t) = \Bigg[e^{-ntL(F)} \otimes \Big(\frac{1}{n}J_n e^{-tL(H)}\Big)\Big) +$$

$$+ e^{-ntD(F)} \otimes \Big((I_n - \frac{1}{n}J_n)e^{-tL(H)}\Big)\Bigg] \underbrace{\Big(x_1(0) \otimes x_2(0)\Big)}_{x(0)}$$

Noting that terms inside the big brackets correspond exactly to fundamental matrix $\Phi(t)$ finishes the proof. □

At the end we briefly comment practical usefulness of these results. Given $n \times n$ matrix calculating exponential of such matrix has complexity $\mathcal{O}(n^3)$ in general. Hence, direct calculation of exponential of $L(F[H])$ would have complexity $\mathcal{O}(m^3n^3)$. On the other hand, determining Kronecker product of matrices of sizes $m \times m$ and $n \times n$ has complexity $\mathcal{O}(m^2n^2)$. Using previous theorem we could determine dynamics of system (12) with computational complexity of $\mathcal{O}(m^3 + n^3 + m^2n^2)$, where the first two term come from matrix exponentials of factor Laplacians, and the second one from Kronecker products. As the number of nodes in graphs increase, evidently our approach will stay applicable for much higher values of m and n. We enrich this discussion in §V when we consider few numerical examples.

B. Controllability

In this section we examine controllability of a dynamical system defined on lexicographic product network $F[H]$, given by equation (13). We mention that controllability conditions can be discussed not only for symmetric Laplacian matrices, but for a more general symmetry preserving system matrices [4]. In that case, since system matrix is not necessarily symmetric it may not be diagonalizable, leading to more complicated analysis. Therefore, we consider only undirected graphs in this analysis which are always diagonalizable. This further implies that all eigenvalues have unitary geometric multiplicity and also that all eigenvectors are linearly independent.

Our main theorem is derived from Popov-Belevitch-Hautus (PBH) test, which states that system (A, B) is controllable if and only if there exists no left eigenvector of A which is orthogonal to all columns of B [16].

Before giving general controllability condition, we consider controllability of system (13) under assumption that $L(F[H])$ does not have repeated eigenvalues. In this way we will divide discussion into two parts, which will hopefully make analysis more comprehensible.

1) L(F[H]) does not have repeated eigenvalues: Let us first mention a straightforward result that for any two matrices A and B holds: $A \otimes B = 0 \iff A = 0 \vee B = 0$. Now we recall from Theorem II.2 that $L(F[H])$ has eigenvectors of the form $\xi_i \otimes \mathbb{1}$ and $e_i \otimes \zeta_j$. Since all of the eigenvalues of $L(F[H])$ are unique, by applying PBH test we get that $(L(F[H]), B_1 \otimes B_2)$ is controllable if and only if the following coditions hold:

$$(\xi_i \otimes \mathbb{1})^T (B_1 \otimes B_2) = (\xi_i^T B_1) \otimes (\mathbb{1}^T B_2) \neq 0 \tag{19}$$

for $i = 1, \dots, m$, and:

$$(e_i \otimes \zeta_j)^T (B_1 \otimes B_2) = (e_i^T B_1) \otimes (\zeta_j^T B_2) \neq 0 \tag{20}$$

for $i = 1, \dots, m$ and $j = 2, \dots, n$.

Let's first discuss equation (20). Since this equation must be nonzero for $i = 1, \dots, m$ it follows that B_1 must be identity matrix I_m, i.e. all nodes of F have to be leaders.

Now note that the condition from equation (19) that $\mathbb{1}^T B_2 \neq 0$ is equivalent to $\zeta_1^T B_2 \neq 0$ since $\zeta_1 = \mathbb{1}$. Hence combining that condition with condition from equation (20) that $\zeta_j^T B_2 \neq 0$ for $j = 2, \dots, n$, we obtain that we must have $\zeta_j^T B_2 \neq 0$ for $j = 1, \dots, n$ i.e. that $(L(H), B_2)$ must be controllable.

At the end, we consider condition that $\xi_i^T B_1 \neq 0$ for $i = 1, \dots, m$, but since we already determined $B_1 = I_m$, this condition is always satisfied. We conclude that $(L(F[H]), B_1 \otimes B_2)$ is controllable if and only if both of the following conditions hold:

- $B_1 = I_m$, i.e. all nodes of F are leaders
- $(L(H), B_2)$ is controllable

2) General case: Let's now consider the case when $L(F[H])$ might have repeated eigenvalues. Determining controllability of the system (13) in the case when matrix $L(F[H])$ has repeated eigenvalues is in general more involved. The reason being that repeated eigenvalues form an eigenspace in which every vector is an eigenvector of given matrix [4], [5]. Hence, instead of checking whether there exists an eigenvector orthogonal to matrix B, in this case we must consider all vectors from eigenspace formed by eigenvectors of repeated eigenvalues.

For all repeated eigenvalues of matrix $L(F[H])$ we can find indices $i_1, i_2, \dots, i_p, j_1, j_2, \dots, j_q$ and $k_1, k_2, \dots, k_q$ for some nonnegative integers p and q such that:

$$\begin{aligned} n\lambda_{i_1} &= n\lambda_{i_2} = \dots = n\lambda_{i_p} = \\ &= \eta_{j_1} + nd(u_{k_1}) = \dots = \eta_{j_q} + nd(u_{k_q}) \end{aligned} \tag{21}$$

Note that matrix $L(F[H])$ might have multiple repeated eigenvalues and then for every repeated eigenvalue we can find set of indices such that equation (21) holds for that eigenvalue.

Theorem IV.3. *Let system be given by* (13)*. Then* $(L(F[H]), B_1 \otimes B_2)$ *is controllable if and only if:*

1) $B_1 = I_m$
2) $(L(H), B_2)$ *is controllable*
3) *for every repeated eigenvalue of* $L(F[H])$*, there exists an index* $k^* \in \{k_1, \dots, k_q\}$ *such that:*

$$\mathrm{Null}(B_2) \cap \mathrm{Span}(V_{k^*}) = \emptyset \tag{22}$$

where:

a) $V_{k^*} = \{\mathbb{1}\}$ *if* $p > 0$ *and* $q = 0$
b) $V_{k^*} = \{\zeta_{j_1^*}, \zeta_{j_2^*}, \dots, \zeta_{j_q^*}\}$ *if* $p = 0$ *and* $q > 0$
c) $V_{k^*} = \{\mathbb{1}, \zeta_{j_1^*}, \zeta_{j_2^*}, \dots, \zeta_{j_q^*}\}$ *if* $p > 0$ *and* $q > 0$

where $j_1^*, j_2^*, j_q^* \subseteq \{j_1, j_2, \dots, j_q\}$ *are indices of eigenvalues* η_j *in equation* (21) *which have second* *term equal to $nd(u_{k^*})$.*

Proof. From previous section 1) we know that in the case when $L(F[H])$ has no repeated eigenvalues, first two conditions of theorem are both sufficient and necessary. Eigenvectors corresponding to repeated eigenvalues must satisfy the same conditions (19) and (20), since they are eigenvectors of $L(F[H])$. Hence, in the rest of the proof we will consider only condition 3).

Let's fix an arbitrary repeated eigenvalue of $L(F[H])$. For this eigenvalue we can form equalities of the form (21) for appropriately chosen indices. Now, note from Theorem II.2 that eigenvectors of $L(F[H])$ corresponding to eigenvalues in (21) have the following form:

$$\xi_{i_1} \otimes \mathbb{1}, \dots, \xi_{i_p} \otimes \mathbb{1}, e_{k_1} \otimes \zeta_{j_1}, \dots, e_{k_q} \otimes \zeta_{j_q}$$

Since all these eigenvectors have the same eigenvalue, then any general eigenvector of the following form:

$$v = \sum_{r=1}^{p} c_r \xi_{i_r} \otimes \mathbb{1} + \sum_{t=1}^{q} \tilde{c}_t e_{k_t} \otimes \zeta_{j_t} \tag{23}$$

for some nonzero constants $c_1, \dots, c_p$ and $\tilde{c}_1, \dots, \tilde{c}_q$, is also an eigenvector of $L(F[H])$ with the same eigenvalue. Hence, from PBH test we know that for any vector of the form (23) must hold that $v^T(B_1 \otimes B_2) \neq 0$ in order to have controllable system. Substituting $B_1 = I_m$ into this condition we get:

$$[v_{1:n}^T \; v_{n+1:2n}^T \cdots v_{(m-1)n+1:mn}^T]^T \begin{bmatrix} B_2 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & B_2 \end{bmatrix} \neq 0$$

which implies that there must exist at least one index k^* such that $v_{(k^*-1)n+1:k^*n}^T B_2 \neq 0$. Now, note from equation (23) that vector $v_{(k^*-1)n+1:k^*n}$ has the following form:

$$\sum_{r=1}^{p} c_r \xi_{i_r}^{(k^*)} \mathbb{1} + \sum_{t=1}^{q} \tilde{c}_t \zeta_{j_t^*} \tag{24}$$

where $j_1^*, \dots, j_q^*$ are indices j_t for which $k_t = k^*$. Noting that first term can be written as $c\mathbb{1}$ for some constant c, we have that:

$$v_{(k^*-1)n+1:k^*n}^T \in \mathrm{Span}\{\mathbb{1}, \zeta_{j_1^*}, \dots, \zeta_{j_q^*}\} \tag{25}$$

Hence we conclude that the system is controllable if and only if in addition to conditions 1) and 2) holds that:

$$\mathrm{Null}(B_2) \cap \mathrm{Span}\{\mathbb{1}, \zeta_{j_1^*}, \dots, \zeta_{j_q^*}\} = \emptyset \tag{26}$$

for at least one index k^* and for every repeated eigenvalue of $L(F[H])$.

Considering cases $p = 0, q > 0$ and $p > 0, q = 0$ is straightforward from discussion above and leads to developing conditions 3) a) and 3) b). □

Let's illustrate derived results in the following simple example.

Example IV.1. *Let adjacency matrices of graphs F and H be given by:*

$$A(F) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A(H) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \tag{27}$$

Then, the Laplacian matrices are given by:

$$L(F) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad L(H) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \tag{28}$$

Furthermore, assume that:

$$B_1 = I_2, \quad B_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{29}$$

One could find easily eigenvalues of matrices $L(F)$ and $L(H)$ which are equal to $(0, 2)$ and $(0, 1, 3)$, respectively. From Theorem II.2 we find that $L(F[H])$ has eigenvalues $(0, 4, 4, 6, 6, 6)$.

One can check that both $(A(F), B_1)$ and $(A(H), B_2)$ are controllable. Hence, we should only check condition 3) in Theorem IV.3 for repeated eigenvalues. Let's consider triple eigenvalue $\lambda(F[H]) = 6$. According to Theorem II.2 this eigenvalue corresponds to the eigenvectors $\xi_2 \otimes \mathbb{1}$, $e_1 \otimes \zeta_3$ and $e_2 \otimes \zeta_3$, where $\xi_2 = [-\frac{\sqrt{2}}{2}, \; \frac{\sqrt{2}}{2}]^T$ and $\zeta_3 = [-0.4082, \; -0.4082, \; 0.8165]^T$. In this example we have that $k^* \in \{1, 2\}$, but since in the both cases $j^* = 3$, condition (22) comes down to:

$$\mathrm{Null}(B_2) \cap \mathrm{Span}(\{1, \zeta_3\}) = \emptyset \tag{30}$$

Let's fix $c_1 = 0.4082$ and $c_2 = 1$. Then $c_1\mathbb{1} + c_2\zeta_3 \in \mathrm{Null}(B_2)$ and hence condition 3) of Theorem IV.3 does not hold. Hence we conclude that $(L(F[H]), B_1 \otimes B_2)$ in this case is not controllable. Note that there is no need to check condition 3) for the second repeated eigenvalue $\lambda(F[H]) = 4$ since one uncontrollable mode is enough to claim that the system is not controllable.

C. Convergence rate

In this section we discuss convergence rate of lexicographic product graphs. We will be especially interested in algebraic connectivity of these graphs, equal to the second smallest eigenvalue of $L(F[H])$. This eigenvalue and its associated eigenvector were first examined in [17], and are known as Fiedler eigenpair. Algebraic connectivity is a good descriptor of the system's behavior - namely, it is a measure of robustness, stability as well as convergence rate of the system [18]. Here we will consider algebraic connectivity only as a measure of convergence rate. Recall (9) where we have expressed algebraic connectivity $\lambda_2(L(F[H]))$ in terms of $\lambda_2(L(F))$ and $\lambda_2(L(H))$.

In order to develop results in this section, we will consider the special case of unitary weighted graphs, i.e. let

$A_{ij} \in \{0, 1\}$. According to [17], the algebraic connectivity of a connected graph G with n nodes is bounded as follows:

$$2\Big(1 - \cos\Big(\frac{\pi}{n}\Big)\Big) \leq \lambda_2(G) \leq n \tag{31}$$

where the lower bound holds when G is a path, and the upper bound holds for complete graphs. From spectrum characterization in Theorem II.1 we know that lexicographic product graph $F[H]$ has two types of eigenvalues: $\lambda_i n$ and $\eta_j + d(u_i)n$. In the case when F is not connected, graph $F[H]$ is also not connected and hence $\lambda_2(F[H]) = 0$ [1]. Hence we assume that F is connected implying that $d(u_i) \geq 1$ for every i.

Let's first determine the lower bound on $\lambda_2(F[H])$. Since every eigenvalue η_j of H is nonnegative and the minimum degree of vertices in connected graph F is $\delta(F) \geq 1$, we have that $\eta_j + d(u_i)n \geq n$. Assuming that m is a big number, from equation (31) we obtain:

$$\lambda_2(F) \geq 2(1 - \cos\Big(\frac{\pi}{m}\Big)) \approx 2(1 - (1 - \frac{1}{2!}\Big(\frac{\pi}{m}\Big)^2)) = \frac{\pi^2}{m^2} \tag{32}$$

$$\implies n\lambda_2(F) \geq \pi^2 \frac{n}{m^2} \tag{33}$$

Under assumption that m is big, we have $\pi^2 \frac{n}{m^2} << n$ and hence we conclude that $\lambda_2(F[H]) = \pi^2 \frac{n}{m^2}$ and is achieved by choosing F to be a path, and H can be an arbitrary graph. Note that if we replaced n by mn in equation (31) we would get better lower bound on $\lambda_2(F[H])$. However, in order for $F[H]$ to be path, H should be only one node and hence $F[H]$ would not be an actual lexicographic product of two graphs.

Let's now discuss the upper bound on $\lambda_2(F[H])$. From (31) we have $\lambda_2(F[H]) \leq mn$. Combining the following inequalities $\lambda_2(F) \leq m$ and $\lambda_2(H) \leq n$, $d(u_i) \leq m - 1$, we obtain $n\lambda_2(F) \leq nm$ and $\eta_2 + d(u_i)n \leq n + (m-1)n = mn$. Hence we have that $\lambda_2(F[H]) \leq mn$ with equality when both F and H are complete graphs.

Using similar reasoning as above one could obtain the following bound for the case of Cartesian product graphs:

$$\min\Big\{\frac{\pi^2}{m^2}, \frac{\pi^2}{n^2}\Big\} \leq \lambda_2(F[H]) \leq m + n \tag{34}$$

This implies that bounds for algebraic connectivity of the lexicographic product graphs are approximately n times larger than those for Cartesian product graphs. Moreover, we should expect that doubling number of the nodes in the graphs, increases convergence rate on lexicographic graphs four times, whereas only two times for Cartesian graphs. We check this conclusion experimentally in the following section.

V. Numerical Experiments

In this section we illustrate theory developed so far on a few experimental graphs. In all of the experiments initial conditions $x_1(0)$ and $x_2(0)$ are randomly chosen, and we set $x(0) = x_1(0) \otimes x_2(0)$.

For the lexicographic product graph from the Figure 1 we formed system of the form (12). Obtained trajectories are shown in the Figure 2. As expected, trajectories converge to

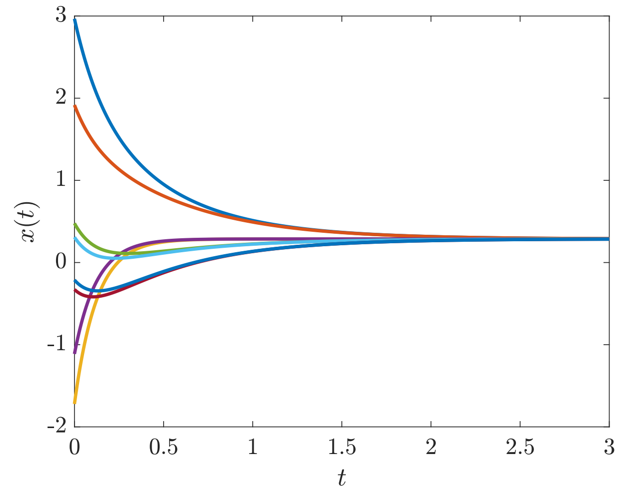

Fig. 2. Converging to consensus for lexicographic product graph from Figure 1.

consensus value given by product of averages of $x_1(0)$ and $x_2(0)$.

In the following experiments we will use graphs with randomly chosen weights. The results presented next are from practical point of view not general. However, we expect that our theoretical results developed so far hold true even in these randomly generated graphs.

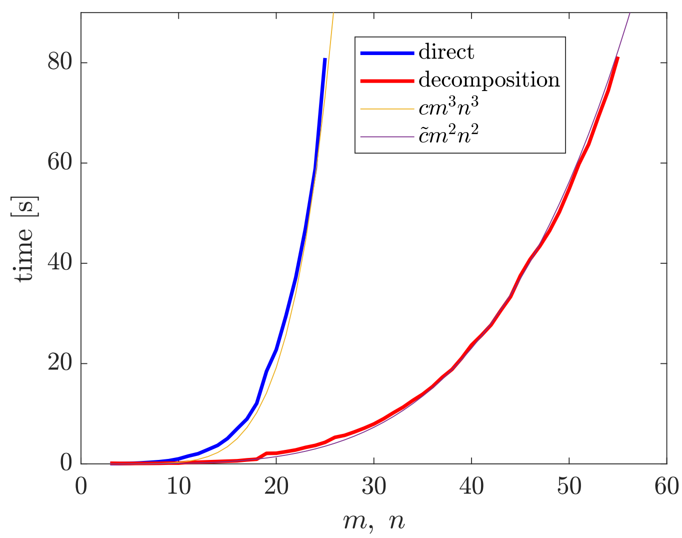

Fig. 3. Time needed for determing trajectories in lexicographic product networks using proposed method (decomposition) and direct calculation.

Let's illustrate time complexity needed for determining trajectories in lexicographic product graphs. Recall from section IV.A that our approach of trajectory decomposition has computational complexity $\mathcal{O}(m^3 + n^3 + m^2n^2)$, whereas the straight-forward approach of determining matrix exponential has complexity $\mathcal{O}(m^3n^3)$. In this experiment we assumed that $m = n$, and determined trajectories $x(t)$ have been determined in 1000 time points, in order to obtain average time. In Figure 3 are shown obtained results, and we compare them with power functions of suitable degree. As expected, in the case of direct exponential calculation time complexity grows like m^3n^3, whereas our approach has complexity growing like m^2n^2 in the setting $m = n$.

In the last set of experiments, we examine convergence rate of lexicographic product graphs depending on the number of vertices. Afterwards, we compare obtained results with the convergence rate of Cartesian product graphs. In both simulations we choose for simplicity $m = n$ and consider two cases $m = n = 5$ and $m = n = 10$. We determine convergence time as time when distances from initial to consensus values become 10% of the initial distances.

In Figures 4 and 5 are shown results for lexicographic and Cartesian product networks, respectively. All trajectories for one experiment are shown in the same color. Convergence times are represented as vertical dashed lines. For lexicographic product network the ratio of the algebraic connectivities in the two cases is 4.4 and the ratio of the convergence times is 5.5. In the case of Cartesian product networks, ratio of algebraic connectivities is 2.7, and ratio of convergence times is 2.5. This is approximately in accordance with our results

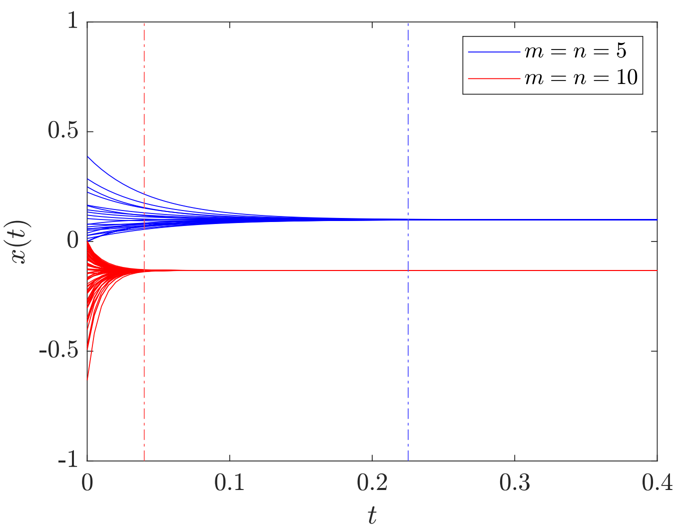

Fig. 4. Trajectories with convergence times in lexicographic product networks.

developed in section IV.C that convergence rate grows as mn in the case of lexicographic products, and only as $m+n$ for Cartesian product graphs. Note however that, in contrast to discussion in IV.C, here we used general, weighted graphs, showing that results from IV.C have wider applicability.

VI. Conclusion

We have developed controllability conditions of lexicographic product networks depending on the controllability of its factor graphs and their eigenvectors. We proved that trajectories of such graphs can be decomposed into two lower-dimensional terms, which makes calculation more manageable. We discussed and compared convergence rates of these graphs and showed that they have higher convergence rates than Cartesian product graphs with the same factor graphs. We

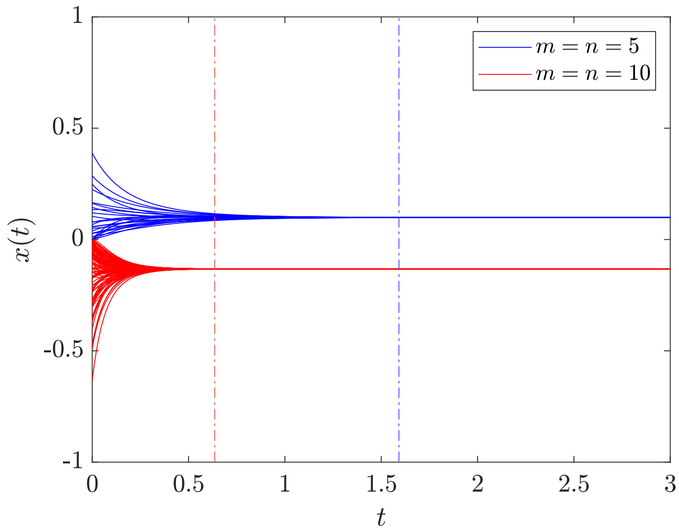

Fig. 5. Trajectories with convergence times in Cartesian product networks.

provided a few numerical simulations to check validity of our claims on arbitrary graphs.

It is needed to expand this analysis to directed graph products, as well as on system matrices that are not Laplacian. Also, further analysis could lead to developing more practical criteria for checking condition 3) in Theorem IV.3 depending on eigenvectors of graph H and matrix B_2. Finally, the ultimate goal is finding a suitable application where this analysis can be used for system control.

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